Bootcamp : Maths

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Contents

1 Session 1

1.1 Calculus

Calculus : study of changes.

2 types of calculus : differential and integral calculus.

Differential : distance $\rightarrow$ speed
Integration : speed $\rightarrow$ distance

1.1.1 Derivative

Definition
The derivative is the slope of the tangent line on a point.
Definition : The derivative of f at x_0 is the slope of the line tangent to the graph of f at x_0.

$$\lim_{\Delta x \to 0} ; \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0)$$

Some derivatives $f(x) = \frac{1}{x} (x > 0)$

$\frac{\Delta f}{\Delta x} = \frac{\frac{1}{x_0 + \Delta x} - \frac{1}{x_0}}{\Delta x} = -\frac{1}{x_0^2 + x_0 \Delta x}$

$$f'(x_0) = \lim_{\Delta x \to 0} ; -\frac{1}{x_0^2 + x_0 \Delta x} = -\frac{1}{x_0^2}$$

$$f(x) = x^n$$

$$f'(x) = \lim_{\Delta x \to 0} ; \frac{(x + \Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0} ; \frac{\left(\binom{n}{0} x^n \Delta x^0 + \binom{n}{1} x^n \Delta x^1 + ...\right) - x^n}{\Delta x} = nx^{n-1}$$

$$(f + g)'(x) = f'(x) + g'(x)$$

Continuity
Notations : $f'(x) = \frac{dy}{dx} = \frac{df}{dx} = Df = \frac{d}{dx} f = $ y'

Example :   400 ft , $y = 400 - 16t^2$
$400 - 16t^2 = 0$
$t = 5$
On average, -80 ft per second. But we want to know the instantaneous speed when the pumpkin hits the ground :
$f'(x) = -32t$

2

$$f'(5) = -160 ft.s^{-1}$$

$f(x)$ is continuous at x_0 if

$$lim_{x \to x_0}; f(x) = f(x_0) \Leftrightarrow \lim_{x \to x_0} ; f(x) - f(x_0) = 0$$

Theorem : If f is differentiable at x_0 then f is continuous at x_0

$$\lim_{x \to x_0} ; f(x) - f(x_0) = \lim_{x \to x_0} ; \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) = f'(x_0).0 = 0$$

Derivatives of composed functions

$$\begin{aligned}(u + v)'(x) &= \lim_{\Delta x \to 0} ; \frac{(u + v)(x + \Delta x) - (u + v)(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} ; \frac{u(x + \Delta x) + v(x + \Delta x) - u(x) - v(x)}{\Delta x} \\ &= u'(x) + v'(x)\end{aligned}$$

$$(uv)' = u'v + v'u$$
$$(\frac{u}{v})' = \frac{u'v - v'u}{v^2}$$

$$\lim_{x \to 0}; \frac{\sin x}{x} = 1$$
$$\lim_{x \to 0}; \frac{\cos x - 1}{x} = 0$$

$$\begin{aligned}f'(x) &= \frac{d}{dx} \sin x \\ &= \lim_{x + \Delta x} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \\ &= \lim_{x + \Delta x} \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x} \\ &= \lim_{x + \Delta x} \frac{\sin(\cos \Delta x - 1)}{\Delta x} + \lim_{x + \Delta x} \frac{\cos x \sin \Delta x}{\Delta x} \\ &= 0 + \cos x\end{aligned}$$

Chain rules $y = f(g(t))$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dx}$$

$$y = f(x) = \sin x$$
$$x = g(t) = t^2$$
$$y = \sin(t^2)$$

$$\frac{d}{dt}(\sin t^2) = (\frac{dy}{dx})(\frac{dx}{dt}) = (\cos x)(2t) = \cos(t^2)2t$$

$$f(g(x))' = g'(x)f'(g(x))$$

Implicit differntiation $y = f(x)$
$$x^2 + y^2 = 1$$

$$\frac{d}{dx}(x^a) = ax^{a-1}$$
$$a = m/n; y = x^{m/n}; y^n = x^m$$
$$ny^{n-1}.\frac{dy}{dx} = mx^{m-1}$$
$$\frac{dy}{dx} = \frac{m}{n}\frac{x^{m-1}}{y^{n-1}}$$
$$= \frac{m}{n}\frac{x^{m-1}}{x^{m-1^{n-1}}}$$
$$= \frac{m}{n}\frac{x^{m-1}}{x^{\frac{m(n-1)}{n}}}$$
$$\frac{m}{n}x^{\frac{m}{n}-1} = ax^{a-1}$$

$$\cos a + b = \sin a \cos b + \cos a \sin b \ \sin a + b = \cos a \cos b + \sin a \sin b$$

We can note derivative with D^n with n the degree of the derivative.
$D^2 x^n = n(n-1)x^{n-2}$ so generalizing :

$$D^{n-1}x^n = n(n-1)...2x^{n-(n-1)}$$
$$= n!x$$

For a bacterial populaiton, y_0 is the initial site of the population, t is the time in days. The bactrias double every days.
$$y(t) = 2^t y_0$$

A_0 is the inital amount, n the number of years, the interest rate is 5%.
$$A(n) = A_0(1.05)^n$$

$$\frac{d}{dx}a^x = \lim \Delta x \to 0; \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

$$= \lim \Delta x \to 0; \frac{a^x(a^{\Delta x} - 1)}{\Delta x}$$

$$= a^x \lim \Delta x \to 0; \frac{a^{\Delta x} - 1}{\Delta x}$$

$$with \frac{d}{dx}a^x = a^x M(a)$$

$$\frac{d}{dx}a^x_{x=0} = a^0 M(a) = M(a)$$

$$So: \frac{d}{dx}e^x = e^x$$

$f(x) = 2^x$ initial function for wich we want to comute the derivative.
$f(kx) = 2^{kx} = (2^k)^x = b^x$ (with $b = 2^k$)
So $\frac{d}{dx}b^x = kf'(kx)$

$$\frac{d}{dx}b^x_{x=0} = kf'(0) = M(2)k$$

If $y = e^x$ then $\ln y = x$

$$\frac{d}{dx}\ln x$$

$$w = \ln x \Rightarrow e^w = x$$

$$\frac{d}{dx}e^w = \frac{d}{dx}x = 1$$

$$\frac{d}{dx}e^w \frac{dw}{dx} = 1$$

$$e^w \frac{dw}{dx} = 1 \Rightarrow$$

$$\frac{dw}{dx} = \frac{1}{e^w} = \frac{1}{x}$$

$$\frac{d}{dx}a^x = ?$$

$$a^x = (e^{\ln a})^x = e^{x \ln a}$$

$$\frac{d}{dx}a^x = e^{x \ln a} . \ln a$$

$$= a^x \ln a$$

$$M(a) = \ln a$$

5

$$\frac{d}{dx}u = ?$$

$$\frac{d}{dx}\ln u = \frac{d\ln u}{du}.\frac{du}{dx}$$

$$= \frac{1}{u}.\frac{du}{dx}$$

$$\frac{d}{dx}a^x = ?, u = a^x, \ln u = x\ln a$$

$$(\ln u)' = \ln a.\frac{u'}{u}$$

$$= \ln a.u'$$

$$= \ln a.u$$

$$(a^x)' = a^x \ln a$$

1.2 Differential equations

1.3 Antiderivatives

$\int G(x) = g(x)dx$ and $G'(x) = g(x)$

$$\int \sin x\,dx = \cos x + C$$

$$Find G(x) such as G'(x) = \sin x$$

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C$$

$$(\frac{x^{a+1}}{a+1} + C)' = \frac{1}{a+1}.(a+1).x^a + 0$$

$$= x^a$$

$$If a = -1, x^a = x^{-1} = \frac{1}{x}$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$(\ln(-x))' = \frac{1}{-x}.(-1)$$

$$= \frac{1}{x}$$

Theorem : If $F' = G'$, then $F(x) = G(x) + C$ for all x.
Proof : If $F' = G'$, then $(F - G)' = F' - G' = 0$

Hence, $F(x) - G(x)$ is a constant. Derivative is $0 \to$ function is constant (follows from the mean value theorem).

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad f\ differentiable\ for\ a\ <\ x\ <\ b \quad f\ continuous\ for\ a \leq x \leq b$$

$$\int x^3 (x^4 + 2)^5 dx =$$

$$with\ u = x^4 + 2\ and\ du = 4x^3 dx = \int \frac{u^5}{4} du$$

$$= \frac{1}{4} \cdot \frac{1}{6} u^6 + C$$

$$= \frac{u^6}{24} + C$$

$$= \frac{(x^4 + 2)^6}{24} + C$$

Example

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u}$$

$$= \ln |u| + C$$

$$= \ln |\ln x| + C$$

$$with\ u = \ln x, \quad du = \frac{1}{x} dx$$

1.3.1 Differential equations

$\frac{dy}{dx} = f(x)$
$y = \int f(x) dx$

Example :

$$\frac{d}{dy}y + xy = 0$$

$$\Leftrightarrow \frac{dy}{dx} = -xy$$

$$\Rightarrow \frac{dy}{y} = -xdx$$

$$\int \frac{dy}{y} = -\int xdx$$

$$\ln|y| = -\frac{x^2}{2} + C$$

$$|y| = e^{-\frac{x^2}{2} + C}$$

$$= e^{-\frac{x^2}{2}}.A \text{ with } A = e^C$$

Generalisation :

$$\frac{dy}{dx} = f(x)g(y)$$

$$\frac{dy}{g(y)} = f(x)dx$$

$$H(y) = F(x) + C$$

$$y = H^{-1}(F(x) + C)$$

Examples :

$$\frac{dy}{dx} = (2x+5)^4$$

$$y = \int (2x+5)^4 dx$$

$$= \frac{1}{10}(2x+5)^5 + C \text{ If we know that } y(0) = 1$$

$$1 = \frac{1}{10}(2 \times 0 + 5)^5 + C$$

$$\Rightarrow C = 1 - \frac{5^5}{10}$$

$$\frac{dy}{dx} = (y+1)^{-1}$$

$$\frac{dy}{(y+1)^{-1}} = dx$$

$$\int (y+1)dy = \int dx$$

$$\frac{y^2}{2} + y = x + C$$

8

$$\frac{dy}{dx} = xy^2$$

$$\int y^{-2}\,dy = \int x\,dx$$

$$-y^{-1} = \frac{x^2}{2} + C$$

$$y = -\frac{1}{\frac{x^2}{2} + C}$$

$$\frac{dy}{dx} = 4xy \quad \text{With } y(1) = 3, \text{find } y(3)$$

$$\frac{dy}{y} = 4x\,dx$$

$$\ln|y| = \frac{4}{2}x^2 + C$$

for $y(1) = 3$ $2 + C = ln3$

$$C = \ln 3 - 2 \quad \text{So we have} : \ln|y| \qquad = 2 \times 3^2 + \ln 3 - 2$$

$$= 16 + \ln 3$$

$$y = e^{16 + \ln 3}$$

1.4 Linear Algebra