

Bootcamp : Maths

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1 Session 1

1.1 Calculus

Calculus : study of changes.

2 types of calculus : differential and integral calculus.

Differential : distance \rightarrow speed

Integration : speed \rightarrow distance

1.1.1 Derivative

Definition

The derivative is the slope of the tangent line on a point.

Definition : The derivative of f at x_0 is the slope of the line tangent to the graph of f at x_0 .

$$\lim_{\Delta x \rightarrow 0}; \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0)$$

Some derivatives $f(x) = \frac{1}{x} (x > 0)$

$$\frac{\Delta f}{\Delta x} = \frac{\frac{1}{x_0 + \Delta x} - \frac{1}{x_0}}{\Delta x} = -\frac{1}{x_0^2 + x_0 \Delta x}$$

$$f'(x_0) = \lim_{\Delta x \rightarrow 0}; -\frac{1}{x_0^2 + x_0 \Delta x} = -\frac{1}{x_0^2}$$

$$f(x) = x^n$$

$$f'(x) = \lim_{\Delta x \rightarrow 0}; \frac{(x + \Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \rightarrow 0}; \frac{((\binom{n}{0})x^n \Delta x^0 + (\binom{n}{1})x^n \Delta x^1 + \dots) - x^n}{\Delta x} = nx^{n-1}$$

$$(f + g)'(x) = f'(x) + g'(x)$$

Continuity

Notations : $f'(x) = \frac{dy}{dx} = \frac{df}{dx} = Df = \frac{d}{dx}f = y'$

Example : 400 ft , $y = 400 - 16t^2$

$$400 - 16t^2 = 0$$

$$t = 5$$

On average, -80 ft per second. But we want to know the instantaneous speed when the pumpkin hits the ground :

$$f'(x) = -32t$$

$$f'(5) = -160 \text{ ft.s}^{-1}$$

$f(x)$ is continuous at x_0 if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0) \Leftrightarrow \lim_{x \rightarrow x_0} f(x) - f(x_0) = 0$$

Theorem : If f is differentiable at x_0 then f is continuous at x_0

$$\lim_{x \rightarrow x_0} f(x) - f(x_0) = \lim_{x \rightarrow x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \right) = f'(x_0) \cdot 0 = 0$$

Derivatives of composed functions

$$\begin{aligned} (u+v)'(x) &= \lim_{\Delta x \rightarrow 0} \frac{(u+v)(x+\Delta x) - (u+v)(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x} \\ &= u'(x) + v'(x) \end{aligned}$$

$$\begin{aligned} (uv)' &= u'v + v'u \\ \left(\frac{u}{v}\right)' &= \frac{u'v - v'u}{v^2} \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x}{x} &= 1 \\ \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} &= 0 \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{d}{dx} \sin x \\ &= \lim_{x+\Delta x} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} \\ &= \lim_{x+\Delta x} \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x} \\ &= \lim_{x+\Delta x} \frac{\sin(\cos \Delta x - 1)}{\Delta x} + \lim_{x+\Delta x} \frac{\cos x \sin \Delta x}{\Delta x} \\ &= 0 + \cos x \end{aligned}$$

Chain rules $y = f(g(t))$

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

$$y = f(x) = \sin x$$

$$x = g(t) = t^2$$

$$y = \sin(t^2)$$

$$\frac{d}{dt}(\sin t^2) = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right) = (\cos x)(2t) = \cos(t^2)2t$$

$$f(g(x))' = g'(x)f'(g(x))$$

Implicit differentiation $y = f(x)$

$$x^2 + y^2 = 1$$

$$\frac{d}{dx}(x^a) = ax^{a-1}$$

$$a = m/n; y = x^{m/n}; y^n = x^m$$

$$ny^{n-1} \cdot \frac{dy}{dx} = mx^{m-1}$$

$$\frac{dy}{dx} = \frac{m}{n} \frac{x^{m-1}}{y^{n-1}}$$

$$= \frac{m}{n} \frac{x^{m-1}}{x^{\frac{m-1}{n}n}}$$

$$= \frac{m}{n} \frac{x^{m-1}}{x^{\frac{m(n-1)}{n}}}$$

$$\frac{m}{n} x^{\frac{m}{n}-1} = ax^{a-1}$$

$$\cos a + b = \sin a \cos b + \cos a \sin b \sin a + b = \cos a \cos b + \sin a \sin b$$

We can note derivative with D^n with n the degree of the derivative.
 $D^2x^n = n(n-1)x^{n-2}$ so generalizing :

$$D^{n-1}x^n = n(n-1)...2x^{n-(n-1)}$$

$$= n!x$$

For a bacterial populaiton, y_0 is the initial site of the population, t is the time in days. The bactrias double every days.
 $y(t) = 2^t y_0$

A_0 is the inital amount, n the number of years, the interest rate is 5%.
 $A(n) = A_0(1.05)^n$

$$\begin{aligned}
\frac{d}{dx}a^x &= \lim_{\Delta x \rightarrow 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{a^x(a^{\Delta x} - 1)}{\Delta x} \\
&= a^x \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x}
\end{aligned}$$

$$\text{with } \frac{d}{dx}a^x = a^x M(a)$$

$$\frac{d}{dx}a^x_{x=0} = a^0 M(a) = M(a)$$

$$\text{So: } \frac{d}{dx}e^x = e^x$$

$f(x) = 2^x$ initial function for which we want to compute the derivative.

$$f(kx) = 2^{kx} = (2^k)^x = b^x \quad (\text{with } b = 2^k)$$

$$\text{So } \frac{d}{dx}b^x = kf'(kx)$$

$$\frac{d}{dx}b^x_{x=0} = kf'(0) = M(2)k$$

If $y = e^x$ then $\ln y = x$

$$\begin{aligned}
&\frac{d}{dx} \ln x \\
w = \ln x &\Rightarrow e^w = x \\
\frac{d}{dx}e^w &= \frac{d}{dx}x = 1 \\
\frac{d}{dx}e^w \frac{dw}{dx} &= 1 \\
e^w \frac{dw}{dx} &= 1 \Rightarrow \\
\frac{dw}{dx} &= \frac{1}{e^w} = \frac{1}{x}
\end{aligned}$$

$$\frac{d}{dx}a^x = ?$$

$$a^x = (e^{\ln a})^x = e^{x \ln a}$$

$$\frac{d}{dx}a^x = e^{x \ln a} \cdot \ln a$$

$$= a^x \ln a$$

$$M(a) = \ln a$$

$$\begin{aligned}\frac{d}{dx}u &=? \\ \frac{d}{dx}\ln u &= \frac{d\ln u}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{u} \cdot \frac{du}{dx} \\ \frac{d}{dx}a^x &=?, u = a^x, \ln u = x \ln a \\ (\ln u)' &= \ln a \cdot \frac{u'}{u} \\ &= \ln a \cdot u' \\ &= \ln a \cdot u \\ (a^x)' &= a^x \ln a\end{aligned}$$

1.2 Differential equations

1.3 Antiderivatives

$$\int G(x) = g(x)dx \text{ and } G'(x) = g(x)$$

$$\int \sin x dx = \cos x + C$$

Find $G(x)$ such as $G'(x) = \sin x$

$$\begin{aligned}\int x^a dx &= \frac{x^{a+1}}{a+1} + C \\ \left(\frac{x^{a+1}}{a+1} + C\right)' &= \frac{1}{a+1} \cdot (a+1) \cdot x^a + 0 \\ &= x^a\end{aligned}$$

$$\begin{aligned}\text{If } a = -1, x^a &= x^{-1} = \frac{1}{x} \\ \int \frac{dx}{x} &= \ln |x| + C \\ (\ln(-x))' &= \frac{1}{-x} \cdot (-1) \\ &= \frac{1}{x}\end{aligned}$$

Theorem : If $F' = G'$, then $F(x) = G(x) + C$ for all x .
Proof : If $F' = G'$, then $(F - G)' = F' - G' = 0$

Hence, $F(x) - G(x)$ is a constant. Derivative is 0 \rightarrow function is constant (follows from the mean value theorem).

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad f \text{ differentiable for } a < x < b \quad f \text{ continuous for } a \leq x \leq b$$

$$\begin{aligned} \int x^3(x^4 + 2)^5 dx &= \\ \text{with } u = x^4 + 2 \text{ and } du = 4x^3 dx &= \int \frac{u^5}{4} du \\ &= \frac{1}{4} \cdot \frac{1}{6} u^6 + C \\ &= \frac{u^6}{24} + C \\ &= \frac{(x^4 + 2)^6}{24} + C \end{aligned}$$

Example

$$\begin{aligned} \int \frac{dx}{x} \ln x &= \int \frac{du}{u} \\ &= \ln |u| + C \\ &= \ln |\ln x| + C \\ \text{with } u = \ln x, \quad du &= \frac{1}{x} dx \end{aligned}$$

1.3.1 Differential equations

$$\begin{aligned} \frac{dy}{dx} &= f(x) \\ y &= \int f(x) dx \end{aligned}$$

Example :

$$\begin{aligned}
\frac{d}{dy}y + xy &= 0 \\
\Leftrightarrow \frac{dy}{dx} &= -xy \\
\Rightarrow \frac{dy}{y} &= -x dx \\
\int \frac{dy}{y} &= - \int x dx \\
\ln |y| &= -\frac{x^2}{2} + C \\
|y| &= e^{-\frac{x^2}{2} + C} \\
&= e^{-\frac{x^2}{2}} \cdot A \text{ with } A = e^C
\end{aligned}$$

Generalisation :

$$\begin{aligned}
\frac{dy}{dx} &= f(x)g(y) \\
\frac{dy}{g(y)} &= f(x)dx \\
H(y) &= F(x) + C \\
y &= H^{-1}(F(x) + C)
\end{aligned}$$

1.4 Linear Algebra