

Bootcamp : Physics

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Part I

Brownian motion

Diffusion give rise to *material transport*. It gives a quantitative understanding of many phenomenon (permeability...).

1 Historical aspects :

1834 : Clapeyron proposed a phenomenological equation for the ideal gas : $PV = nRT$,
 $PV = N.K_B.T$ (N : number of particles, K_B Boltzman constant $1.38 \times 10^{-23} \frac{kg.m}{K.s^2}$)

1828 : Brown : studied pollen that was moving

1905 : Einstein : $\langle x^2 \rangle = 2DT$ (D : diffusion coefficient)

1908 : Perrin : confirmed the Einstein guess

2 Random-walk

Simplest model that describes Brownian motion. A particle has the same probability to move to the right and to the left. After two jumps, there is one way to 2 ($\frac{1}{4}$) and -2 ($\frac{1}{4}$), and 2 ways to 0 ($\frac{1}{2}$).

$$\langle a \rangle = \frac{1}{m} \sum_{i=1}^m a_i = \sum_{i=1}^m p_i a_i$$
$$\langle a^2 \rangle = \frac{1}{m} \sum_{i=1}^m a_i^2$$

$\langle a \rangle$ the average position, with m the number of realisations, i the i-th realisation and a_i the i-th realisation.
In our example : $\langle a \rangle = \frac{1}{4}(2 + -2 + 0 + 0) = 0$ and $\langle a^2 \rangle = \frac{1}{4}(2^2 + -2^2 + 0^2 + 0^2) = 2$

In the 3 jumps case :

- 1 way to 3
- 1 way to -3
- 3 ways to 1
- 3 ways to -1

$$\langle a \rangle = \frac{1}{8}(3 + -3 + 3 \times 1 + 3 \times -1) = 0 \text{ and } \langle a^2 \rangle = \frac{1}{8}(3^2 + -3^2 + (3 \times 1)^2 + (3 \times -1)^2) = 3$$

$$N - jumps = \langle a \rangle = 0 \text{ and } \langle a^2 \rangle = N$$

If N is large enough, distribution of position is a binomial distribution.

If we say that the boxes have a l length, then $x = a \times l$, t_0 is the time delay between successive jumps. So $\frac{t}{t_0} = N$

$$\langle x \rangle = \langle a \rangle l = 0$$

$$\langle x^2 \rangle = \langle a^2 \rangle l^2$$

$$= l^2 N$$

$$= t \frac{l^2}{t_0}$$

$$= 2t \frac{l^2}{2t_0}$$

$$\text{So } D = \frac{l^2}{2t_0}$$

Example :

- $D = 100\mu m^2/s$
- $l_{bacteria} = 1\mu m$
- $l_{cell} = 20\mu m$
- $l_{arm} = 1m$

$$t_{bacteria} = \frac{l^2}{2D} = \frac{(1\mu m)^2}{2(100\mu m^2/s)} = 5ms$$

$$t_{cell} = \frac{l^2}{2D} = \frac{(20\mu m)^2}{2(100\mu m^2/s)} = 2s$$

$$t_{arm} = \frac{l^2}{2D} = \frac{(1m)^2}{2(100\mu m^2/s)} = 300years$$

Q : How long does it take to tRNA to find a ribosome ?

- $N_{tRNA} = 200000$ molecules / bacteria
- the length of the bacteria is around $1\mu m$ long
- The volume of the bacteria : $V = 1\mu m^3$
- $C_{tRNA} = \frac{200000}{1}$
- $d = \sqrt[3]{\frac{1}{C_{tRNA}}} \approx 0.016\mu m$
- So $D = 100\mu m/s^2 \rightarrow t = \frac{1^2}{2D} \approx 1\mu s$
- But ribosomes synthesize at 40 nucleotides per seconds, the the synthesis is about of 2 seconds for a 80 nucleotides tRNA.

So the ribosomes binds quickly to the tRNA, and takes longer to synthesize.

What is the concentration ? $C = \frac{\text{Number of particles}}{\text{Volume}} = 200000 \text{ molec}/\mu m$. Although, the concentration is not uniform in the bacteria.

What is flux (j) ? Flux the number of particles that pass through an area A per unit of time. $j \propto 2 - 1 = 1$

$$\frac{\# \text{ of particles}}{\text{time step} \times \text{area of the plane}}$$

Example :

- $N_{H_2O} = 10^{10} \text{ molecules}$
- $T = 3000sec$
- $A = 6\mu m^2$

$$j = \frac{10^{10}}{3 \times 10^3 \times 6} = 10^6 \frac{\text{molecules}}{s \cdot \mu m^2}$$

Fick's Law

$$j = -D \frac{\partial c}{\partial x}$$

Darcy's Law

$$v = -d \frac{\partial P}{\partial x} \text{ : mobility coef}$$

Ohm's Law

$$\vec{j}_E = \sigma \vec{E} = -\sigma \frac{\partial V}{\partial x} \quad \text{sigma : electric conductivity, V : electric potential}$$

Fourier's Law

$$j_T = -K \frac{\partial T}{\partial x} \quad T : \text{temperature, K : thermal conductivity}$$

Random walk :

$$\begin{aligned}
 j &= \frac{\text{particles}(\rightarrow) - \text{particles}(\leftarrow)}{t_0 A} \\
 \text{particles}(\rightarrow) &= N(x) \times \frac{1}{2} = C(x) V_{Box} \frac{1}{2} = C(x) A l \frac{1}{2} \\
 \text{particles}(\leftarrow) &= N(x+l) \frac{1}{2} = C(x+l) A l \frac{1}{2} \\
 j &= \frac{C(x) \frac{l}{2} - C(x+l) \frac{l}{2}}{t_0} \\
 &= \frac{l^2}{2t_0} \left(\frac{C(x) - C(x+l)}{l} \right) \\
 &= -D \frac{\partial C}{\partial x}
 \end{aligned}$$

Law of mass conservation : $\#part@_{t+t_0} = \#part@_t + \#part_{entering} - \#part_{leaving}$
 $\#part_{entering} = j(x) A t_0$, $\#part_{leaving} = j(x+l) A t_0$
 $\#part@_{t+t_0} = C(x, t+t_0) l A$

$$\begin{aligned}
 (...) &= \left(\frac{C(x, t+t_0) - C(x, t)}{t_0} \right) l A t_0 \\
 &= \frac{\partial C}{\partial t} l A t_0 \\
 \frac{\partial C}{\partial t} &= -\frac{\partial j}{\partial x} \\
 \frac{\partial C}{\partial t} &= D \frac{\partial^2 C}{\partial x^2}
 \end{aligned}$$

$$\frac{\partial C}{\partial t} = 0, \frac{\partial^2 C}{\partial x^2} = 0, \frac{\partial C}{\partial t} = 0$$

If there is a concentration gradient :

$$C = -Kx \text{ with } K \text{ the slope of the concentration gradient}$$

This is a non-equilibrium state

$$\begin{aligned}
 \frac{\partial C}{\partial x} &= -K \\
 \frac{\partial C}{\partial t} &= 0 \\
 \frac{\partial^2 C}{\partial x^2} &= 0
 \end{aligned}$$

Example 3 : There is a film of water, and a drop of ink on it. We just take a plan to consider the diffusion. d is the size of the drop.

$$C(x, t) = \frac{N}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$$

Example 4 : There are certain molecules inside a cell. There is much more molecules outside the cell. There is a flux at the interface of the membrane.

$$\begin{aligned}
 j_M &= -P_m \Delta C \text{ with : } \Delta C = C_{ext} - C_{int} \\
 R &= 10\mu m; P_m = 20\mu m/s \\
 \frac{\partial C_i}{\partial t} &= \frac{j_m A}{V} = \frac{P_m A}{V} (C_e - C_i) \\
 \frac{\partial C_i}{\partial t} &= -\frac{P_m A}{V} C_i + \frac{P_m A}{V} C_e
 \end{aligned}$$

$$\begin{aligned}\tau &= \frac{V}{P_m A} = \frac{L^3}{\frac{L}{T} L^2} = T \\ &= \frac{\frac{4}{3}\pi R^3}{P_m 4\pi R^2} = \frac{R}{3P_m} = \frac{1}{6} \text{sec}\end{aligned}$$

$$\begin{aligned}\frac{\partial C_i}{\partial t} &= -\frac{1}{t}(C_i - C_e) \\ \rightarrow C_i(t) &= C_e - C_e e^{-t/\tau}\end{aligned}$$

3 Bias Random walk

This is the case where the probability of going left is not the same than to go right. $K_+ \neq K_-$

$$F = \gamma v \rightarrow v = \frac{F}{\gamma} \text{ avec } \gamma \text{ le coefficient de friction}$$

$$\begin{aligned}j_{New} &= vC \\ j &= -D \frac{\partial C}{\partial x} + vc\end{aligned}$$

$$\Delta x = v \Delta t \text{ and } N = C \Delta x A = c A \Delta t v$$

$$\begin{aligned}j &= \frac{\#_{part}}{time.area} = \frac{CvA\Delta t}{\Delta t A} \\ &= Cv = c \frac{F}{\gamma}\end{aligned}$$

$$j_{tot} = -D \frac{\partial C}{\partial x} + Cv$$

Equilibrium situation :

$$0 = -D \frac{\partial C}{\partial x} + C \frac{F}{\gamma}$$

With the force $F = cst$

$$\begin{aligned}\frac{\partial C}{\partial x} &= \frac{CF}{\gamma}; \frac{\partial C}{\partial x} = \frac{CF}{\gamma D} \\ \frac{\partial C}{\partial x} &= \frac{C}{\lambda} \text{ as } \lambda = \frac{\gamma D}{F} \\ \frac{\partial C}{\partial x} &= \frac{C}{\lambda}; \frac{\partial C}{C} = \frac{\partial x}{\lambda}\end{aligned}$$

$$\begin{aligned}\int \frac{\partial C}{C} &= \ln C - \ln C_0 = \ln \frac{C}{C_0} \\ \int \frac{\partial x}{\lambda} &= \frac{x}{\lambda}\end{aligned}$$

$$\begin{aligned}\text{So : } \ln \frac{C}{C_0} &= \frac{x}{\lambda} \\ C &= C_0 e^{x/\lambda}\end{aligned}$$

Same case as above, but with $F = -Kx$

$$j_{Tot} = 0 = -D \frac{\partial C}{\partial x} + C \frac{F}{\gamma} = -D \frac{\partial C}{\partial x} - \frac{CKx}{\gamma} = C$$

Solution :

$$D \frac{\partial C}{\partial x} = -\frac{CKx}{\gamma}; \frac{\partial C}{\partial x} = -\frac{CKx}{\gamma D}$$

$$\frac{dC}{dx} = \frac{-CKx}{\gamma D}; \frac{dC}{C} = -\frac{CKx}{\gamma D} dx;$$

$$\int \frac{dC}{C} = \ln C - \ln C_0$$

$$\int \frac{-Kx}{\gamma D} dx = -\frac{K}{\gamma D} \frac{x^2}{2}$$

$$\ln \frac{C}{C_0} = -\frac{Kx^2}{2\gamma D} \Rightarrow$$

$$C = C_0 e^{\frac{-Kx^2}{2\gamma D}}$$

We expect that : $\sigma^2 = \frac{\gamma D}{K}$

In a drop :

At a surface, the particles evaporate at a rate $\frac{1}{\tau}$

$$N_{surface} = C_0 A \lambda$$

$$j = \frac{N_{surface}}{A \tau} = \frac{C_0 \lambda}{\tau}$$

$$N(t + \Delta t) = N(t) - j A \Delta t$$

$$\Rightarrow \frac{dN}{dt} = -j A$$

$$\text{And : } C_0 = \frac{N}{V} = cst ; N = C_0 V$$

$$\rightarrow \frac{dN}{dt} = C_0 \frac{dV}{dt}$$

$$\frac{dV}{dt} = \frac{-j A}{C_0}$$

$$= \frac{C_0 \lambda A}{C_0 \tau}$$

$$= -\frac{\lambda A}{\tau}$$

$$\frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt} = A \frac{dR}{dt}$$

$$\frac{dR}{dt} = \frac{-\lambda}{\tau}$$

$$R(t) = R_0 \times -\frac{\lambda}{\tau} t$$