Bootcamp : Maths

Baptiste Rouger

September 15, 2016

Contents

1 Session 1

1.1 Calculus

Calculus : study of changes.

2 types of calculus : differential and integral calculus.

Differential : distance $\rightarrow$ speed
Integration : speed $\rightarrow$ distance

1.1.1 Derivative

Definition
The derivative is the slope of the tangent line on a point.
Definition : The derivative of f at x_0 is the slope of the line tangent to the graph of f at x_0.

$$\lim_{\Delta x \to 0}; \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0)$$

Some derivatives $f(x) = \frac{1}{x}(x > 0)$
$\frac{\Delta f}{\Delta x} = \frac{\frac{1}{x_0 + \Delta x} - \frac{1}{x_0}}{\Delta x} = -\frac{1}{x_0^2 + x_0 \Delta x}$

$$f'(x_0) = \lim_{\Delta x \to 0}; -\frac{1}{x_0^2 + x_0 \Delta x} = -\frac{1}{x_0^2}$$

$f(x) = x^n$

$$f'(x) = \lim_{\Delta x \to 0}; \frac{(x + \Delta x)^n - x^n}{\Delta x} = \lim_{\Delta x \to 0}; \frac{(\binom{n}{0} x^n \Delta x^0 + \binom{n}{1} x^n \Delta x^1 + ...) - x^n}{\Delta x} = nx^{n-1}$$

$(f + g)'(x) = f'(x) + g'(x)$

Continuity
Notations : $f'(x) = \frac{dy}{dx} = \frac{df}{dx} = Df = \frac{d}{dx} f =$ y'

Example : 400 ft , $y = 400 - 16t^2$
$400 - 16t^2 = 0$
$t = 5$
On average, -80 ft per second. But we want to know the instantaneous speed when the pumpkin hits the ground :
$f'(x) = -32t$

$f'(5) = -160ft.s^{-1}$

$f(x)$ is continuous at x_0 if

$$lim_{x \to x_0}; f(x) = f(x_0) \Leftrightarrow \lim_{x \to x_0}; f(x) - f(x_0) = 0$$

Theorem : If f is differentiable at x_0 then f is continuous at x_0

$$\lim_{x \to x_0}; f(x) - f(x_0) = \lim_{x \to x_0}; \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) = f'(x_0).0 = 0$$

Derivatives of composed functions

$$\begin{aligned}
(u+v)'(x) &= \lim_{\Delta x \to 0}; \frac{(u+v)(x+\Delta x) - (u+v)(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0}; \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x} \\
&= u'(x) + v'(x)
\end{aligned}$$

$$\begin{aligned}
(uv)' &= u'v + v'u \\
(\frac{u}{v})' &= \frac{u'v - v'u}{v^2}
\end{aligned}$$

$$\lim_{x \to 0}; \frac{\sin x}{x} = 1$$
$$\lim_{x \to 0}; \frac{\cos x - 1}{x} = 0$$

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\sin x \\
&= \lim_{x+\Delta x} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} \\
&= \lim_{x+\Delta x} \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x} \\
&= \lim_{x+\Delta x} \frac{\sin(\cos \Delta x - 1)}{\Delta x} + \lim_{x+\Delta x} \frac{\cos x \sin \Delta x}{\Delta x} \\
&= 0 + \cos x
\end{aligned}$$

Chain rules $\quad y = f(g(t))$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dx}$$

$y = f(x) = \sin x$
$x = g(t) = t^2$
$y = sin(t^2)$

$$\frac{d}{dt}(\sin t^2) = (\frac{dy}{dx})(\frac{dx}{dt}) = (\cos x)(2t) = \cos(t^2)2t$$

$f(g(x))' = g'(x)f'(g(x))$

Implicit differntiation $y = f(x)$
$x^2 + y^2 = 1$

$$\begin{aligned}
\frac{d}{dx}(x^a) &= ax^{a-1} \\
a &= m/n; y = x^{m/n}; y^n = x^m \\
ny^{n-1}.\frac{dy}{dx} &= mx^{m-1} \\
\frac{dy}{dx} &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} \\
&= \frac{m}{n}\frac{x^{m-1}}{x^{m-1^{n-1}}} \\
&= \frac{m}{n}\frac{x^{m-1}}{x^{\frac{m(n-1)}{n}}} \\
\frac{m}{n}x^{\frac{m}{n}-1} &= ax^{a-1}
\end{aligned}$$

$$\cos a + b = \sin a \cos b + \cos a \sin b \ \sin a + b = \cos a \cos b + \sin a \sin b$$

We can note derivative with D^n with n the degree of the derivative.
$D^2x^n = n(n-1)x^{n-2}$ so generalizing :

$$\begin{aligned}
D^{n-1}x^n &= n(n-1)...2x^{n-(n-1)} \\
&= n!x
\end{aligned}$$

For a bacterial populaiton, y_0 is the initial site of the population, t is the time in days. The bactrias double every days.
$y(t) = 2^t y_0$

A_0 is the inital amount, n the number of years, the interest rate is 5%.
$A(n) = A_0(1.05)^n$

$$
\begin{aligned}
\frac{d}{dx}a^x &= \lim \Delta x \to 0; \frac{a^{x+\Delta x} - a^x}{\Delta x} \\
&= \lim \Delta x \to 0; \frac{a^x(a^{\Delta x} - 1)}{\Delta x} \\
&= a^x \lim \Delta x \to 0; \frac{a^{\Delta x} - 1}{\Delta x}
\end{aligned}
$$

$$
\begin{aligned}
with \frac{d}{dx}a^x &= a^x M(a) \\
\frac{d}{dx}a^x_{x=0} &= a^0 M(a) = M(a) \\
So: \frac{d}{dx}e^x &= e^x
\end{aligned}
$$

$f(x) = 2^x$ initial function for wich we want to comute the derivative.
$f(kx) = 2^{kx} = (2^k)^x = b^x$ (with $b = 2^k$)
So $\frac{d}{dx}b^x = kf'(kx)$

$$
\frac{d}{dx}b^x_{x=0} = kf'(0) = M(2)k
$$

If $y = e^x$ then $\ln y = x$

$$
\begin{aligned}
&\frac{d}{dx}\ln x \\
w = \ln x \Rightarrow e^w &= x \\
\frac{d}{dx}e^w &= \frac{d}{dx}x = 1 \\
\frac{d}{dx}e^w \frac{dw}{dx} &= 1 \\
e^w \frac{dw}{dx} &= 1 \Rightarrow \\
\frac{dw}{dx} &= \frac{1}{e^w} = \frac{1}{x}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}a^x &=? \\
a^x &= (e^{\ln a})^x = e^{x \ln a} \\
\frac{d}{dx}a^x &= e^{x \ln a}. \ln a \\
&= a^x \ln a \\
M(a) &= \ln a
\end{aligned}
$$

$$\begin{aligned}
\frac{d}{dx}u &= ? \\
\frac{d}{dx}\ln u &= \frac{d\ln u}{du}.\frac{du}{dx} \\
&= \frac{1}{u}.\frac{du}{dx}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}a^x &= ?, u = a^x, \ln u = x\ln a \\
(\ln u)' &= \ln a.\frac{u'}{u} \\
&= \ln a.u' \\
&= \ln a.u \\
(a^x)' &= a^x \ln a
\end{aligned}$$

1.2 Differential equations

1.3 Antiderivatives

$\int G(x) = g(x)dx$ and $G'(x) = g(x)$

$$\begin{aligned}
\int \sin x dx &= \cos x + C \\
Find G(x) such as G'(x) &= \sin x
\end{aligned}$$

$$\begin{aligned}
\int x^a dx &= \frac{x^{a+1}}{a+1} + C \\
(\frac{x^{a+1}}{a+1} + C)' &= \frac{1}{a+1}.(a+1).x^a + 0 \\
&= x^a
\end{aligned}$$

$$\begin{aligned}
If a = -1, x^a = x^{-1} &= \frac{1}{x} \\
\int \frac{dx}{x} &= \ln|x| + C \\
(\ln(-x))' &= \frac{1}{-x}.(-1) \\
&= \frac{1}{x}
\end{aligned}$$

Theorem : If $F' = G'$, then $F(x) = G(x) + C$ for all x.
Proof : If $F' = G'$, then $(F - G)' = F' - G' = 0$

Hence, $F(x)-G(x)$ is a constant. Derivative is 0 $\rightarrow$ function is constant (follows from the mean value theorem).

$$\frac{f(b)-f(a)}{b-a} = f'(c) \quad f\ differentiable\ for\ a\ <\ x\ <\ b \quad f\ continuous\ for\ a \leq x \leq b$$

$$\begin{aligned} \int x^3(x^4+2)^5 dx &= \\ with\ u = x^4+2\ and\ du = 4x^3dx &= \int \frac{u^5}{4} du \\ &= \frac{1}{4} . \frac{1}{6} u^6 + C \\ &= \frac{u^6}{24} + C \\ &= \frac{(x^4+2)^6}{24} + C \end{aligned}$$

Example

$$\begin{aligned} \int \frac{dx}{x \ln x} &= \int \frac{du}{u} \\ &= \ln |u| + C \\ &= \ln |\ln x| + C \\ with\ u = \ln x,\ du = \frac{1}{x} dx \end{aligned}$$

1.3.1 Differential equations

$\frac{dy}{dx} = f(x)$
$y = \int f(x) dx$

Example :

$$\begin{aligned}
\frac{d}{dy}y + xy &= 0 \\
\Leftrightarrow \frac{dy}{dx} &= -xy \\
\Rightarrow \frac{dy}{y} &= -xdx \\
\int \frac{dy}{y} &= -\int xdx \\
\ln|y| &= -\frac{x^2}{2} + C \\
|y| &= e^{-\frac{x^2}{2}+C} \\
&= e^{-\frac{x^2}{2}}.A \ with \ A = e^C
\end{aligned}$$

Generalisation :

$$\begin{aligned}
\frac{dy}{dx} &= f(x)g(y) \\
\frac{dy}{g(y)} &= f(x)dx \\
H(y) &= F(x) + C \\
y &= H^{-1}(F(x) + C)
\end{aligned}$$

Examples :

$$\begin{aligned}
\frac{dy}{dx} &= (2x+5)^4 \\
y &= \int (2x+5)^4 dx \\
&= \frac{1}{10}(2x+5)^5 + C \text{ If we know that } y(0) = 1 \\
1 &= \frac{1}{10}(2 \times 0 + 5)^5 + C \\
\Rightarrow C &= 1 - \frac{5^5}{10}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= (y+1)^{-1} \\
\frac{dy}{(y+1)^{-1}} &= dx \\
\int (y+1)dy &= \int dx \\
\frac{y^2}{2} + y &= x + C
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= xy^2 \\
\int y^{-2} dy &= \int x dx \\
-y^{-1} &= \frac{x^2}{2} + C \\
y &= -\frac{1}{\frac{x^2}{2} + C}
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= 4xy \text{With } y(1) = 3, \text{find } y(3) \\
\frac{dy}{y} &= 4xdx \\
\ln|y| &= \frac{4}{2}x^2 + C \\
\text{for } y(1) = 3 \quad 2 + C &= ln3 \\
C &= \ln 3 - 2 \text{So we have : } \ln|y| \qquad = 2 \times 3^2 + \ln 3 - 2 \\
&= 16 + \ln 3 \\
y &= e^{16 + \ln 3}
\end{aligned}$$

1.4 Linear Algebra

Part I
Matrices

2 Vectors

$\overrightarrow{v} = (v_1, v_2, v_3 ...)$.

$$\overrightarrow{u} + \overrightarrow{v} = \begin{pmatrix} x_u + x_v \\ y_u + y_v \end{pmatrix}$$

$$\overrightarrow{u} . \overrightarrow{v} = ||u|| \times ||v|| \times \cos\theta = (x_u x_v + y_u y_v)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

3 Matrices

$$matrix\ A = \begin{pmatrix} a_{11} & a_{12} & ... \\ a_{21} & a_{22} & ... \\ a_{31} & ... & \end{pmatrix}$$

Tu multiply 2 matrices A and B : we need : $m \times n$ and $n \times p$ and $A.B \neq B.A$

$$\begin{pmatrix} 7 & 4 \\ A & 2 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 \\ -1 & 0 & 8 \end{pmatrix} = \begin{pmatrix} 7.4 + 4. - 1 & 7.2 + 4.0 & 7.1 + 4.8 \\ 1.4 + 2. - 1 & 1.2 + 2.0 & 1.1 + 2.8 \end{pmatrix} = \begin{pmatrix} 24 & 14 & 39 \\ 2 & 2 & 17 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} ; B = \begin{pmatrix} 3 & 2 \\ -5 & 1 \\ 8 & 0 \end{pmatrix}$$

$$A.B = \begin{pmatrix} 20 & 4 \\ 35 & 13 \\ 53 & 22 \end{pmatrix}$$

4 Square matrices

The identity matrix is a matrix the is empty but for 1 on her diagonal.
$A.I = A; I.A = A$
Determinant of A :

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a.d - b.c$$

$$det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a. \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b. \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c. \begin{vmatrix} d & e \\ g & h \end{vmatrix} \text{ and then do it again...}$$

$$\overrightarrow{u} = \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}, \overrightarrow{v} = \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix}$$

$$\overrightarrow{u} \otimes \overrightarrow{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{x}n - \hat{y}m + \hat{z}p$$

With n,m and p the determinants of the matrix.

5 Eigenvalues and eigenvectors

$$A\overrightarrow{v} = k\overrightarrow{v}?$$

A matrix, $\overrightarrow{v}$ a vector and k a number.

$$\begin{aligned}
A\overrightarrow{v} &= k\overrightarrow{v} \\
A\overrightarrow{v} &= kI\overrightarrow{v} \\
A\overrightarrow{v} - kI\overrightarrow{v} &= 0 \\
(A - kI)\overrightarrow{v} &= 0 \\
\text{So : } det(A - kI) &= 0
\end{aligned}$$

$$\begin{aligned}
A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} &= \begin{vmatrix} -k & 1 \\ -2 & 3-k \end{vmatrix} \\
&= -k \times (3-k) - 1 \times -2 = (k-2)(k-1) = 0
\end{aligned}$$

So there is two eigenvalues : $k_1 = 2$ and $k_2 = 1$