Instituto Superior Tecnico
Departamento de Matemática
Computational Methods In Finance
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Computational project One

European and American Put options

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1 European option using method of lines and Runge-Kutta-4

1.1 Task presentation

The task at hand is to price a European put option using the Black-Sholes equation and numerical means. The Black-Sholes equation describes the price evolution of a European option as a function of the underlying asset price S and time t. Given a sufficiently large upper bound S^*, we define the computational domain as:

$$\mathcal{R}_V^T = \{(S,t), \quad 0 < S < S^*, \quad 0 \leq t \leq T\} \tag{1}$$

For a European option, the Black-Scholes equation is given by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2}$$

where:

- $V(S,t)$ represents the option price at asset price S and time t,
- K is the strike price for the option
- σ is the volatility of the underlying asset,
- r is the risk-free interest rate,
- T is the time to maturity.

As this work tries to solve the problem numerically using the method of lines, we perform a transformation and change of variables $U(S,t) = V(S, T-t)$ to obtain the following alternative problem:

$$\frac{\partial U}{\partial t} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} - rU, \quad (S,t) \in \mathcal{R}_V^T \tag{3}$$

with these boundary conditions:

$$U(S,0) = u_0(S), \quad S \in [0, S^*] \tag{4}$$
$$U(0,t) = u_a(t), \quad t \in [0,T] \tag{5}$$
$$U(S^*,t) = u_b(t), \quad t \in [0,T] \tag{6}$$

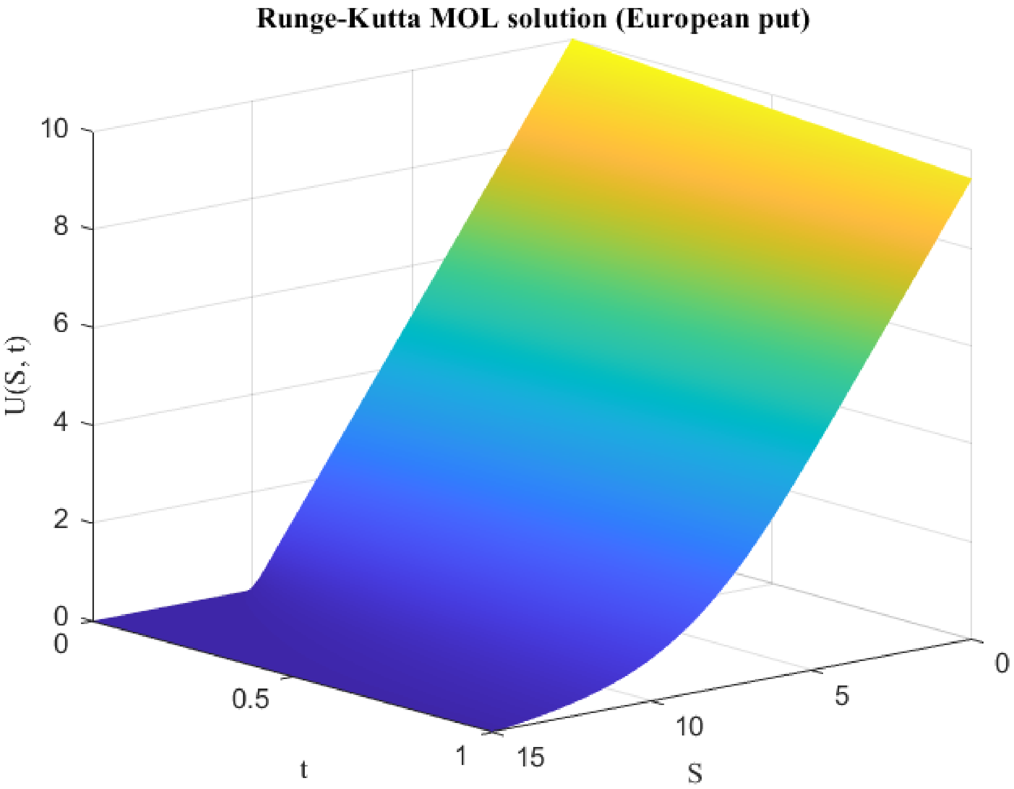
Considering Task 1(b), the variable values and boundary conditions are the following:

- variables: σ =0.3; r=0.06; K= 10; T=1 and S^*=15
- boundary and initial condition:
  - $u_a = K \cdot e^{-r \cdot t}$
  - $u_b = 0$
  - $U_0(S) = max(K - S, 0)$

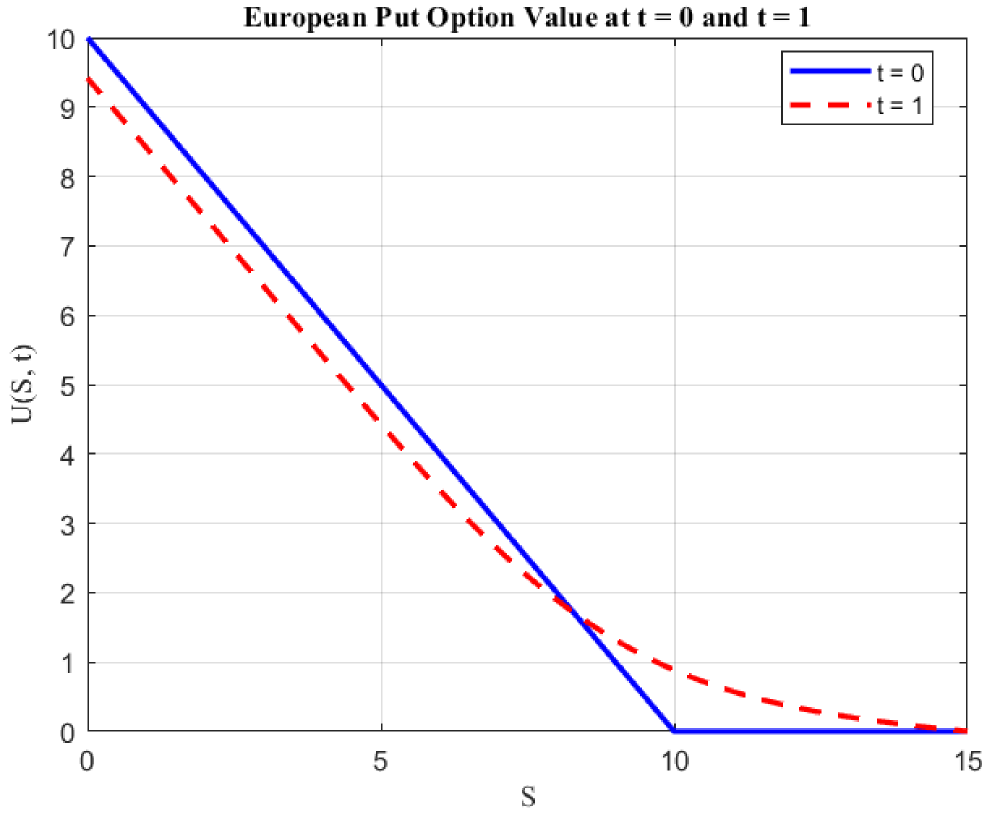
After presenting the task at hand, we discretize the spatial domain S into a grid $S_i, i = 1, 2, ..., N$ and perform Runge-Kutta method of fourth order to obtain a numerical solution to the problem. The exact calculatoins and formulas used in this method are too extensive for this report and can be studied in the code attached.

1.2 Solution

After solving the problem using the Runge-Kutta method, we obtain the following graphic solution.



The three-dimensional graph shows space and time as independent variables and the value of the option with reversed time (similar to the lecture slides). For $t = 0$ we can observe the value of the option with regard to the underlying stock price at maturity. Particularly interesting from a finance perspective is the value of the option at time $t = 1$, as we can observe the fair value at "current" time. These values are displayed in the next graphic.



1.3 Explain findings

From a finance perspective, all values of $U(S,t)$ can be interpreted as the fair price of the put option, for which neither the seller nor the buyer can benefit from arbitrage gains and gives equal information to all market participants. Evaluating the solution at t=1, the model provides the market with a unanimous fair price for options for all prices of the underlying stock at the current point in time.

2 American option using Crank-Nicolson and PSOR methods

2.1 Task presentation

In this part, the objective is to price an American put option by solving the Black-Scholes variational inequality numerically. Unlike European options, American options can be exercised at any time before maturity, which leads to a free boundary problem.

To numerically solve the American option pricing problem, we formulate it as a *linear complementarity problem (LCP)*, stemming from the nature of early exercise constraints.

The payoff function $\varphi(S)$ for an American put option is given by:

$$\varphi(S) = \max(K - S, 0)$$

We define the Black-Scholes operator:

$$\mathcal{L}_{BS}(V) = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

The variational inequality satisfied by the option price $V(S,t)$ is given by:

$$\begin{cases} \frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) = 0 & \text{if } V > \varphi(S), \\ \frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) \leq 0 & \text{if } V = \varphi(S). \end{cases}$$

To simplify the problem into an initial value form, we introduce the change of variables $\tau = T - t$ and set $U(S,\tau) = V(S, T - \tau)$. This leads to the LCP formulation:

$$\begin{cases} \frac{\partial U}{\partial \tau}(S,\tau) + \mathcal{L}_{BS}(U(S,\tau)) \geq 0, \\ U(S,\tau) \geq \varphi(S), \\ \left(\frac{\partial U}{\partial \tau}(S,\tau) + \mathcal{L}_{BS}(U(S,\tau))\right)(U(S,\tau) - \varphi(S)) = 0. \end{cases}$$

This system ensures that the solution stays above the payoff function and matches it where early exercise is optimal.

We discretize the problem in time using the *Crank-Nicolson method*, which is second-order accurate and unconditionally stable. For spatial discretization, we use central finite differences. At each time step, we obtain a linear system of the form:

$$AU^{j+1} = b,$$

where U^j is the known solution at time step j, and U^{j+1} is the unknown to be solved.

We define the obstacle vector:

$$G = [\varphi(S_1), \varphi(S_2), \ldots, \varphi(S_{N_S-1})]^T,$$

and rewrite the LCP as:

$$\begin{cases} (W - G)^T(AW - b) = 0, \\ AW - b \geq 0, \\ W \geq G, \end{cases}$$

where W is the approximation of U^{j+1}. Introducing:

$$X = W - G, \quad Y = AW - b,$$

we obtain:

$$\begin{cases} AX = Y - AG, \\ X^T Y = 0, \\ X \geq 0, \quad Y \geq 0, \end{cases}$$

which is a *standard form of linear complementarity problem* and can be solved using the *Projected Successive Over-Relaxation (PSOR)* method.

The PSOR iteration is defined as follows for each component $i = 1, 2, ..., N_S - 1$:

$$X_i^{(n+1)} = \max\left(0, X_i^{(n)} + \omega \cdot \frac{\tilde{b}_i - \sum_{j=1}^{i-1} a_{ij} X_j^{(n+1)} - \sum_{j=i}^{N} a_{ij} X_j^{(n)}}{a_{ii}}\right),$$

where ω is the relaxation parameter and $\tilde{b} = b - AG$.

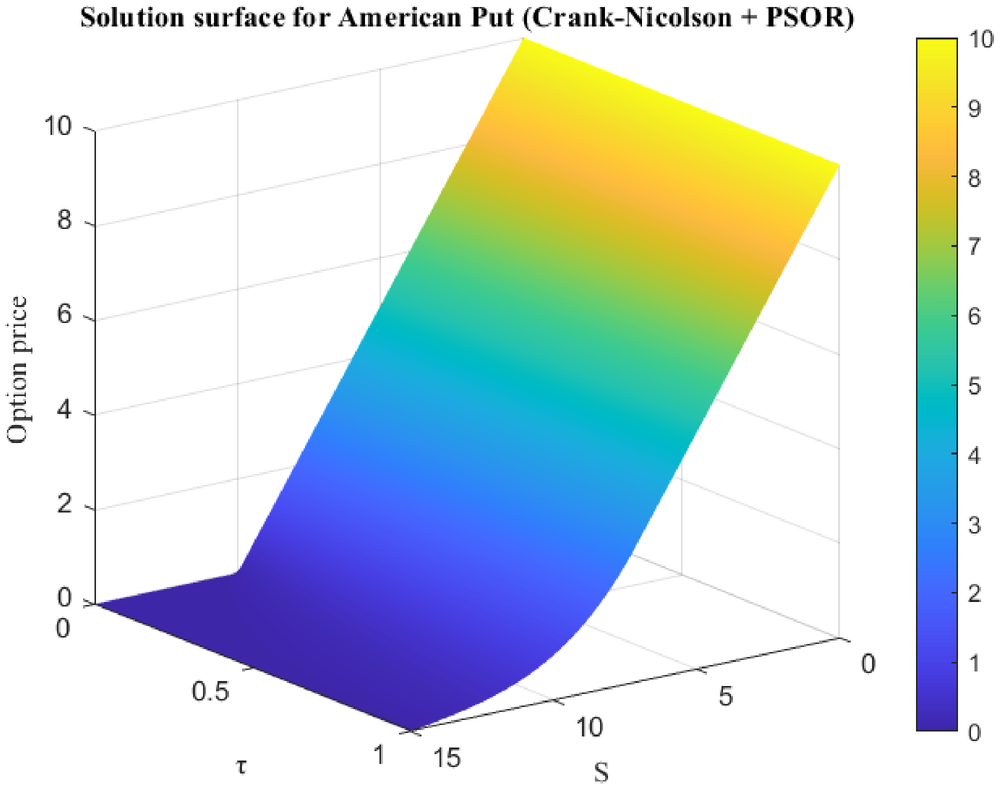
The algorithm iterates until convergence, ensuring that the constraints of the variational inequality are satisfied at each time step.

2.2 Solution

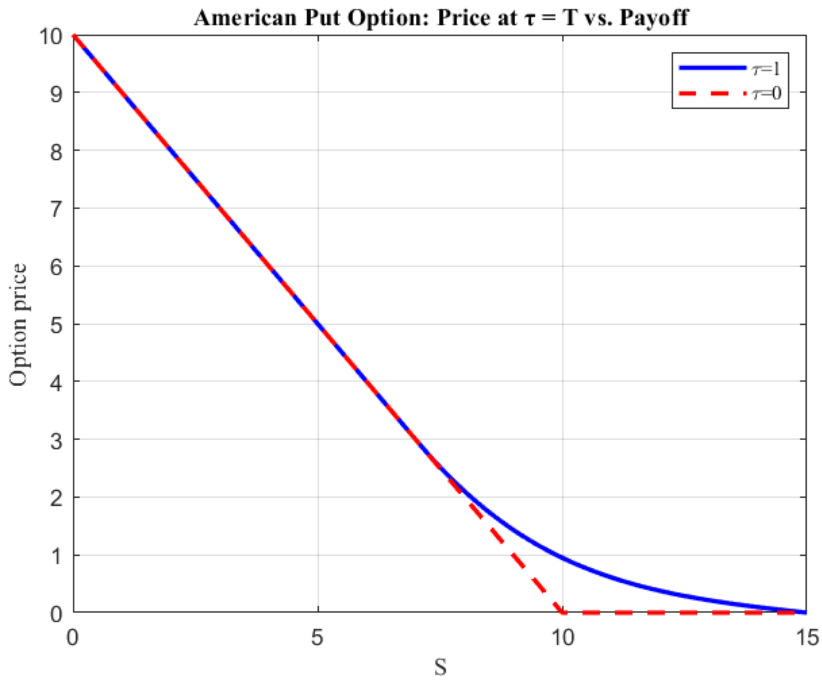
The parameters for this task are:

- $\sigma = 0.3$, $r = 0.06$, $K = 10$, $T = 1$, and $S^* = 15$.

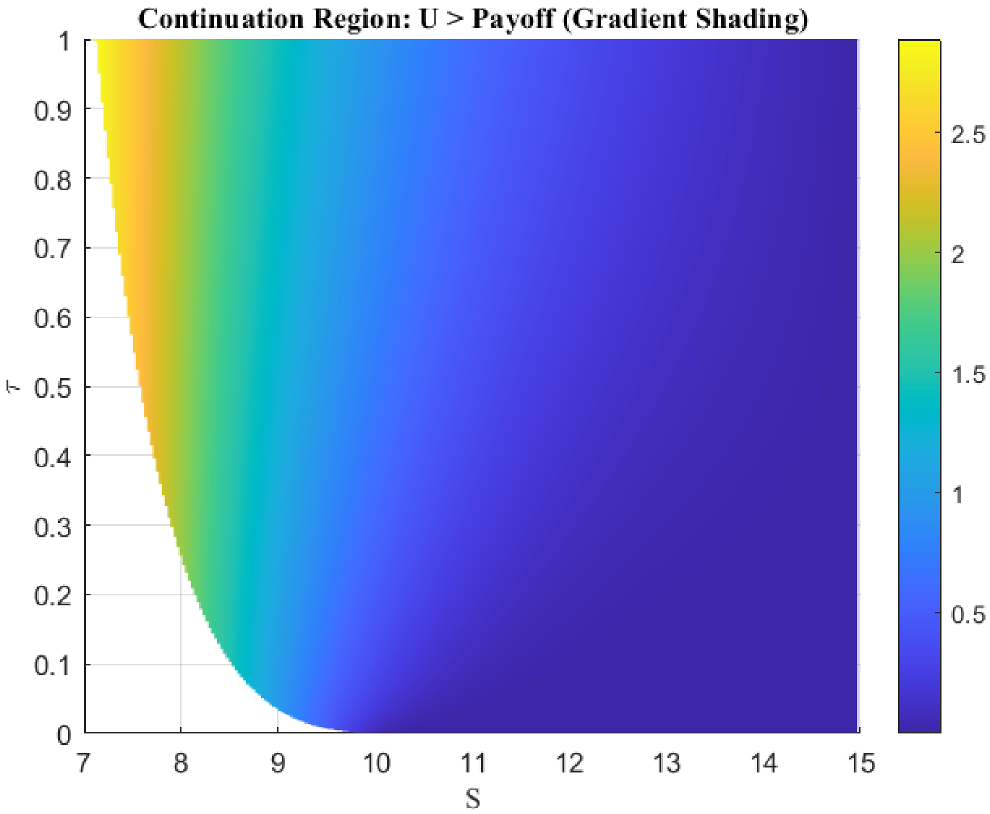
In this section, we implement this method in MATLAB and use it to compute and plot the price of the American put option within the continuation region. As we use a reversed-time model ($\tau = T - t$), the results may graphically differ from the ones seen in the lecture. First, we compute the fair value of the option regarding an American put for the continuation region for all values of S and τ. The solution to this can be found in the first graphic.



Plotting this solution in two dimensions for the initial value ($\tau = 0$) and at maturity ($\tau = 1$), we can observe the most important characteristic of an American option: The fact, that the value of the option is always greater or equal compared to the payoff. These results can be seen in the next graphic.



Last, to combine our findings for solving the model for the American put option, we calculate the plot for the full range of values of the option in both the stop and continuation region. This plot is shown in the last graphic.



The last graph shows the overall value of the option considering reversed time (therefore it looks different to the lecture slides).

2.3 Explain findings

The numerical results confirm the key properties of American put options. Most importantly, the computed option price remains above or equal to the payoff $\varphi(S) = \max(K - S, 0)$, reflecting the early exercise premium inherent in American-style options. This behavior reflects the presence of an early exercise premium, which is the main distinguishing feature of American options compared to European ones. The ability to exercise the option at any time before maturity adds value. The 2D slices at $\tau = 0$ and $\tau = 1$ illustrate that the option price matches the payoff at maturity and exceeds it beforehand, particularly in the continuation region. The final top-down plot captures both the continuation and stopping regions across time and stock prices. This allows for a visual identification of the free boundary, the point at which the holder becomes indifferent between holding and exercising the option.