signals - sheet 5 cont. properties :

$x(t) \xrightarrow{fs} a_K$

* $y(t) = x(t-t_0)$
  ↳ $b_K = e^{-jK\omega_0 t_0} a_K$

* $y(t) = x(-t)$
  ↳ $b_K = a_{-K}$

* $y(t) = \frac{d}{dt} x(t)$
  ↳ $b_K = a_K \cdot jK\omega_0$

* $y(t) = \underset{\downarrow}{x_1(t)}\ \underset{\downarrow}{x_2(t)}$
  $\qquad\quad a_K \quad b_K$
  ↳ $C_K = \sum_{l=-\infty}^{\infty} a_l b_{K-l}$

* $x(t) = \sum_{K=-\infty}^{\infty} a_K e^{j\omega_0 K t}$
  ↳ $a_K = \frac{1}{t_0} \int_{T_0} x(t) e^{-jK\omega_0 t} dt$

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signals - sheet 5

P1 $x(t) = 2 + \cos(\frac{2\pi}{3}t) + 4\sin(\frac{5\pi}{3}t)$
find w_0, a_k
and,
$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$

* $w_0 = \frac{2\pi}{T_0}$ $T_0 = \frac{2\pi}{w_0}$

* $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$

* $a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jkwt} dt$

$x(t) = 2 + \cos(\frac{2\pi}{3}t) + 4\sin(\frac{5\pi}{3}t)$

$= 2 + \frac{1}{2}(e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t}) + \frac{4}{2j}(e^{j\frac{5\pi}{3}t} - e^{-j\frac{5\pi}{3}t})$

$\therefore w_0 = \frac{\pi}{3}$

$x(t) = 2e^{j0\frac{\pi}{3}t} + \frac{1}{2}e^{j2\frac{\pi}{3}t} + \frac{1}{2}e^{-j2\frac{\pi}{3}t} + \frac{4}{2j}e^{j5\frac{\pi}{3}t} - \frac{4}{2j}e^{-j5\frac{\pi}{3}t}$

$a_0 = 2$

$a_2 = a_{-2} = \frac{1}{2}$

$a_{-5} = \frac{-2}{j}$ $a_5 = \frac{2}{j}$

$\boxed{P_3}$ $x(t) = \begin{cases} 1.5 & 0 \leq t \leq 1 \\ -1.5 & 1 \leq t \leq 2 \end{cases}$

$w_0 = \pi$, find a_K.

$T_0 = \frac{2\pi}{w_0} = \frac{2\pi}{\pi} = 2$

$$a_K = \frac{1}{T_0} \int_{T_0} x(t) e^{-JKw_0 t} \, dt$$

$$= \frac{1}{2} \int_0^2 x(t) e^{-JKw_0 t}$$

$$= \frac{1}{2} \int_0^1 1.5 e^{-JKw_0 t} dt + \frac{1}{2} \int_1^2 -1.5 e^{-JKw_0 t} dt$$

$$= \frac{1.5}{2} \frac{e^{-JKw_0 t}}{-JKw_0} \Big|_0^1 + \frac{-1.5}{2} \frac{e^{-JKw_0 t}}{-JKw_0} \Big|_1^2$$

$$= \frac{1.5}{-2JKw_0} \left[e^{-JKw_0 t} \Big|_0^1 - e^{-JKw_0 t} \Big|_1^2 \right]$$

$$= \frac{-3}{4JKw_0} e^{-JKw_0} - e^0 - e^{-2JKw_0} + e^{-JKw_0}$$

$$= \frac{-3}{4JKw_0} 2e^{-JKw_0} - 1 - e^{-2JKw_0}$$

$$= \frac{3}{4JKw_0} \left[-2e^{-JKw_0} + 1 + e^{-2JKw_0} \right.$$

$$* \; x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\hookrightarrow a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j\omega_0 kn}$$

problem 3

$x_1[n] \longrightarrow a_k$

$a_0 = a_3 = \frac{1}{2} a_2 - a_1 = 1$

$x_2[n] \longrightarrow b_k$

$b_0 = b_1 = b_2 = b_3 = 1$

$g[n] = x_1[n] \, x_2[n]$

$c_k = ?$

$b_{-1} = b_{-1+\boxed{4}} = b_3$

$b_{-2} = b_{-2+\boxed{4}} = b_2$

$b_{-3} = b_{-3+\boxed{4}} = b_1$

fundmental period $\boxed{N=4}$

solution:

$$c_k = \sum_{l<N>} a_l \, b_{k-l} = \sum_{l=0}^{3} a_l \, b_{k-l}$$

$$c_0 = a_l \, b_{-l} = a_0 b_0 + a_1 b_{-1} + a_2 b_{-2} + a_3 b_{-3} = \boxed{5}$$

problem 4 :

a) $x(t) = e^{-t}$ $\quad -1 < t < 1 \quad T_0 = 2$

$a_k = ?$

solution:

$$a_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0} \int_{-1}^{1} e^{-t} e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0} \int_{-1}^{1} e^{-t(1+jk\omega_0)}\, dt$$

$$= \frac{1}{2} \left(\frac{e^{-t(1+jk\omega_0)}}{-(1+jk\omega_0)} \right) \Bigg|_{-1}^{1} = \frac{1}{2} \left[\frac{e^{-1-jk\omega_0} - e^{1+jk\omega_0}}{-(1+jk\omega_0)} \right.$$

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b) $x(t) = \begin{cases} \sin(\pi t) & 0 \leq t \leq 2 \\ 0 & 2 \leq t \leq 4 \end{cases}$ $\quad T = 4$

$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$a_k = ?$

solution:

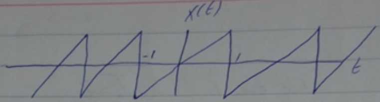
$$\sin(\pi t) = \frac{1}{2j} e^{j\pi t} - \frac{1}{2j} e^{-j\pi t}$$

$$a_2 = \frac{1}{2j}$$

$$a_{-2} = \frac{-1}{2j}$$

c)

$x(t) = t \quad -1 \le t \le 1$



$a_k = ?$

solution:

$T_0 = 2$

$$a_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{2} \int_{-1}^{1} t\, e^{-jk\omega_0 t}\, dt$$ hard to solve, so we use the derivative property

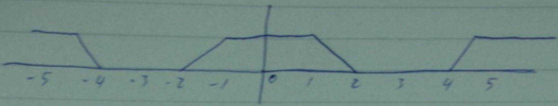
$$\frac{d\,x(t)}{dt} = 1$$

$$b_k = \frac{1}{T_0} \int_{-1}^{1} (1)\, e^{-jk\omega_0 t}\, dt$$

$$b_k = \frac{1}{2(jk\omega_0)} \left[e^{-jk\omega_0} - e^{jk\omega_0} \right]$$

$$a_k = \frac{b_k}{jk\omega_0} = \boxed{\frac{1}{2(jk\omega_0)^2} \left(e^{-jk\omega_0} - e^{jk\omega_0} \right)}$$

c)



$$x(t) = \begin{cases} t+2 & -2 \le t \le -1 \\ 1 & -1 \le t \le 1 \\ -(t-2) & 1 \le t \le 2 \end{cases}$$

$$\frac{d\,(x(t))}{t} = \begin{cases} 1 \\ 0 \\ 1 \end{cases}$$

then we $a_k = \dfrac{b_k}{jk\omega_0}$

$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \end{cases} \qquad T_0 = 2 \qquad a_0 = ?$$

$$a_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_0 = \frac{1}{2} \int_{T_0} x(t)\, dt$$

$$a_0 = \frac{1}{2} \int_0^1 t\, dt + \int_1^2 2-t\, dt$$

$$= \frac{1}{2}\left(\frac{t^2}{2}\Big|_0^1 + 2t - \frac{t^2}{2}\Big|_1^2\right) = \frac{1}{2}\left(\frac{1}{2} + 2 - 2 + \frac{1}{2}\right)$$

$$= \boxed{\frac{1}{2}}$$

sheet 5 cont.

problem 6:

a) $a_k = \cos\left(\frac{k\pi}{4}\right) + \sin\left(\frac{3k\pi}{4}\right)$ $\quad N = 8$

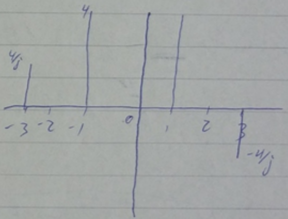
$w_0 = \frac{2\pi}{N} = \frac{2\pi}{8} = \pi/4$

$a_k = 1/2\, e^{jk\frac{\pi}{4}} + 1/2\, e^{-jk\frac{\pi}{4}} + \frac{1}{2j} e^{j\frac{3k\pi}{4}} - \frac{1}{2j} e^{-j\frac{3k\pi}{4}}$

$a_k = \frac{1}{N} \sum_{k=-\infty}^{\infty} x[n] e^{-jkw_0 n}$

$= \frac{1}{8}\left(4e^{j\frac{k\pi}{4}} + 4^{-j\frac{k\pi}{4}} + \frac{4}{j} e^{j\frac{3k\pi}{4}} - \frac{4}{j} e^{j\frac{3k}{4}\pi} \right)$

$x[-1] = 4$
$x[1] = 4$
$x[-3] = 4/j \quad x[3] = -4/j$



problem 7:

$$x[n] = \begin{cases} 1 & 0 \le n \le 7 \\ 0 & 8 \le n \le 9 \end{cases} \qquad \boxed{N = 10}$$

let $g[n] = x[n] - x[n-1]$
find a_k, $g[n]$.

a) $N_g = \dfrac{N_1 N_2}{GCD(N_1, N_2)} = \boxed{10}$

b) $x[n] - x[n-1] = \begin{cases} 1 & n = 0 \\ -1 & n = 8 \end{cases} \qquad N = 10$

$$b_k = \frac{1}{N} \sum_{N}^{\infty} g[n]\, e^{-jk\omega_0 n}$$

$$= \frac{1}{10} \sum_{k=0}^{9} g[n]\, e^{-jk\omega_0 n}$$

$$= \frac{1}{10} e^{-jk\omega_0 0} + \frac{1}{10} e^{-jk\omega_0 8}$$

$$= \boxed{\frac{1}{10} + \frac{1}{10} e^{-jk8 \frac{\pi}{5}}}$$

$g[n] = x[n] - x[n-1]$

$$b_k = a_k - \left(a_k e^{(jk\omega_0(-1))}\right)$$

$$= a_k \left(1 - e^{-jk\omega_0}\right)$$

$$a_k = \frac{b_k}{1 - e^{-jk\omega_0}}$$

problem 8:

$x(t) \longrightarrow a_k \quad \text{period} = T$

a) $x(t-t_0) + x(t+t_0)$

$$\boxed{a_k e^{-jk\omega_0 t_0} + a_k e^{jk\omega_0 t_0}}$$

b) $Ev\{x(t)\} \Rightarrow$ even part of $x(t)$

$$\frac{x(t) + x(-t)}{2} = \boxed{\frac{a_k + a_{-k}}{2}}$$