Propositional Logic Cheat Sheet

In this section, we summarize the syntax, semantics, and properties of propositional logic.

Syntax

We specify the syntax of valid propositional sentences p using the following grammar:

$$
\begin{array}{rcll}
p, q & ::= & A, B, C & \text{(atomic props)} \\
& \mid & \top & \text{(top, \textit{i.e.}, true)} \\
& \mid & \bot & \text{(bottom, \textit{i.e.}, false)} \\
& \mid & \neg p & \text{(negation)} \\
& \mid & p_1 \wedge p_2 & \text{(conjunction)} \\
& \mid & p_1 \vee p_2 & \text{(disjunction)} \\
& \mid & p_1 \longrightarrow p_2 & \text{(implication)} \\
& \mid & p_1 \leftrightarrow p_2 & \text{(biconditional)}
\end{array}
$$

Semantics

We specify the semantics of propositional logic in a natural deduction style, named thusly because it reflects the "natural" way that we might think about proving a statement in this logic. In natural deduction, we typically specify a pair of rules for each logical connective: an *introduction rule* introducing a use of the connective and an *elimination rule* describing how it is eliminated as an assumption. These rules are specified using inference rules of the form:

$$
\dfrac{A \qquad B \qquad C}{D}\ \textsc{Example}
$$

where we read the rule top-down as saying "if *premises* A, B, and C hold, then *conclusion* D also holds". When constructing a proof, it is beneficial to read the rules bottom-up, *i.e.*, "to show that conclusion D holds, we must show that the premises A, B, and C also hold".

The rules for conjunction are:

$$
\dfrac{p_1 \qquad p_2}{p_1 \wedge p_2}\ \wedge\text{-INTRO} \qquad \dfrac{p_1 \wedge p_2}{p_1}\ \wedge\text{-ELIM-LEFT} \qquad \dfrac{p_1 \wedge p_2}{p_2}\ \wedge\text{-ELIM-RIGHT}
$$

The rules for disjunction are:

$$
\dfrac{p_1}{p_1 \vee p_2}\ \vee\text{-INTRO-LEFT} \qquad \dfrac{p_2}{p_1 \vee p_2}\ \vee\text{-INTRO-RIGHT} \qquad \dfrac{p_1 \vee p_2 \qquad p_1 \longrightarrow q \qquad p_2 \longrightarrow q}{q}\ \vee\text{-ELIM}
$$

The elimination rule for disjunction states that if I know a disjunction holds and each of the disjunction implies a third proposition, then that third proposition must hold.

The rules for implication are:

$$\overset{\longrightarrow\text{-INTRO}}{\dfrac{\dfrac{\begin{array}{c}\underline{[x \,:\, p]}\\ \\ \vdots\end{array}}{q}}{p \longrightarrow q}} \qquad \overset{\longrightarrow\text{-ELIM}}{\dfrac{p \longrightarrow q \qquad p}{q}}$$

When introducing an implication, we go on to prove the consequent q given the antecedent p. In our proof of q we give our assumption a name x and then assert that $[x \,:\, p]$ holds in our system. When invoking this assumption, we simply assert it:

$$\overset{\text{ASSUMPTION}}{\dfrac{[x \,:\, p]}{p}}$$

Elimination of implication is simply the logical *modus ponens* rule.

A biconditional is simply a two-way implication, so rather than giving additional rules for biconditionals, we simply assert that it is equivalent to the conjunction of implications:

$$p_1 \longleftrightarrow p_2 \equiv p_1 \longrightarrow p_2 \wedge p_2 \longrightarrow p_1$$

Note that this means that when we prove a biconditional, we can break up the problem (via the $\wedge$-intro rule) into proving both sides of the biconditional given the other side.

There is only an introduction rule for $\top$ with no corresponding elimination rule:

$$\overset{\top\text{-INTRO}}{\dfrac{}{\top}}$$

Correspondingly, there is no introduction rule for $\bot$, only a curious elimination rule:

$$\overset{\bot\text{-ELIM}}{\dfrac{\bot}{p}}$$

This rule is also known as the *principle of explosion.* It states that whenever we are able to prove false, *i.e.*, arrive at a logical contradiction, we can prove anything.

With the rules for $\bot$ defined and a little bit of thought, we can realize negation as simply the combination of $\bot$ and an implication:

$$\neg p \equiv p \longrightarrow \bot$$

Finally, we introduce a pair of rules that encompass our intitution that either a proposition is true or false (so-called, *classical* logic):

$$\overset{\text{MIDDLE}}{\dfrac{}{p \wedge \neg p}} \qquad \overset{\text{CONTRADICTION}}{\dfrac{\dfrac{\begin{array}{c}\underline{[x \,:\, \neg p]}\\ \\ \vdots\end{array}}{\bot}}{p}}$$

The first rule is also called the *law of the excluded middle* which captures exactly our intitution that a proposition is either true or false. The second rule captures the idea of *proof by contradiction.* We may prove a statement true by assuming it is false and deriving a contradiction.