ME 331: Transformation

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2-D Matrix multiplication:

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u & x \\ v & y \end{bmatrix} = \begin{bmatrix} (au+bv) & (ax+by) \\ (cu+dv) & (cx+dy) \end{bmatrix}$$

Homogeneous representation of 2-D Translation by (d_x, d_y)

$$\begin{Bmatrix} x' \\ y' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}$$ Short notation for the translation matrix: $T = \begin{bmatrix} I & d \\ 0 & 1 \end{bmatrix}$

Homogeneous representation of 2-D (anti-clockwise) Rotation θ about the origin

$$\begin{Bmatrix} x' \\ y' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}$$ Short notation for the rotation matrix: $T = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$

Concatenation: (a) Rotate θ about the origin, followed by (b) Translation by d

$$T = \begin{bmatrix} I & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (I*R+d*0) & (I*0+d*1) \\ (0*R+1*0) & (0*0+1*1) \end{bmatrix} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

Example: A point (2, 1) is rotated 90 degrees about origin and then translated by (1,1). Find the final point

$$T = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 90 & -\sin 90 & 1 \\ \sin 90 & \cos 90 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}; \quad \begin{Bmatrix} x' \\ y' \\ 1 \end{Bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3 \\ 1 \end{Bmatrix}$$

Concatenation: (a) Translation by d , followed by (b) Rotate θ about the origin

$$T = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (R*I+0*0) & (R*d+0*1) \\ (0*I+1*0) & (0*d+1*1) \end{bmatrix} = \begin{bmatrix} R & Rd \\ 0 & 1 \end{bmatrix}$$

Example: A point (2, 1) is translated by (1,1), and then rotated 90 degrees about origin. Find the final point

Note: From previous example: $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Therefore $Rd = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$;

$$T = \begin{bmatrix} R & Rd \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}; \quad \begin{Bmatrix} x' \\ y' \\ 1 \end{Bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 3 \\ 1 \end{Bmatrix}$$

Generalized Rotation: Rotation about a point p

$$T = \begin{bmatrix} R & p - Rp \\ 0 & 1 \end{bmatrix}$$

Example: A point (2, 1) is rotated 90 degrees about the point (1, -1). Find the final point

$$p - Rp = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2 \end{Bmatrix};$$

Therefore, $T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, and the new point is $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$

Inversion: The inversion of a generic transformation matrix is given as follows

$$\text{If } T = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, \text{ then } T^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

Example: Suppose $T = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, then $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; d = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix};\ T^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Homogeneous representation of 3-D translation

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$ Short notation for the translation matrix: $T = \begin{bmatrix} I & d \\ 0 & 1 \end{bmatrix}$

Homogeneous representation of 3-D rotation α about the x axis

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$ Short notation for the rotation matrix: $T = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$

Homogeneous representation of 3-D rotation β about the y axis

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$ Short notation for the rotation matrix: $T = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$

Homogeneous representation of 3-D rotation θ about the z axis

$$\begin{Bmatrix} x' \\ y' \\ z' \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \\ 1 \end{Bmatrix}$$ Short notation for the rotation matrix: $T = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$

Mapping between coordinate systems: If a point is given in CS-W, then the same point is CS-A is given by

$p^A = T_W^A p^W$, where T_W^A is the transformation between the two coordinate system

The first column of T_W^A captures the x-axis orientation of CS-W with respect to CS-A

The second column of T_W^A captures the y-axis orientation of CS-W with respect to CS-A

The third column of T_W^A captures the origin of CS-W with respect to CS-A

Mapping between multiple coordinate systems: Given 3 CS: CS-A, CS-W and CS-N then

$p^A = T_W^A T_N^W p^N$, with the usual meaning

Suppose a CS-W is associated with a rigid body that is subject to a transformation, then

$$\left(T_N^W\right)' = T T_N^W$$

Inverse of coordinate transformations

$$T_A^W = \left(T_W^A\right)^{-1}$$