

VIP Cheatsheet: First-order ODE

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Introduction

□ **Differential Equations** – A differential equation is an equation containing derivatives of a dependent variable y with respect to independent variables x . In particular,

- Ordinary Differential Equations (ODE) are differential equations having one independent variable.
- Partial Differential Equations (PDE) are differential equations having two or more independent variables.

□ **Order** – An ODE is said to be of order n if the highest derivative of the unknown function in the equation is the n^{th} derivative with respect to the independent variable.

□ **Linearity** – An ODE is said to be linear only if the function y and all of its derivatives appear by themselves. Thus, it is of the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y + b(x) = 0$$

Direction Field Method

□ **Implicit form** – The implicit form of an ODE is where y' is not separated from the remaining terms of the ODE. It is of the form:

$$F(x, y, y') = 0$$

Remark: Sometimes, y' cannot be separated from the other terms and the implicit form is the only one that we can write.

□ **Explicit form** – The explicit form of an ODE is where y' is separated from the remaining terms of the ODE. It is of the form:

$$y' = f(x, y)$$

□ **Direction field method** – The direction field method is a graphical representation for the solution of ODE $y' = f(x, y)$ without actually solving for $y(x)$. Here is the procedure:

- Determine the values (x_i, y_i) that form the grid.
- Compute the slope $f(x_i, y_i)$ for each point of the grid.
- Report the associated vector for each point of the grid.

Separation of variables

□ **Separable** – An ODE is said to be separable if it can be written in the form:

$$f(x, y) = g(x)h(y)$$

□ **Reduction to separable form** – The following table sums up the variable changes that allow us to change the ODE $y' = f(x, y)$ to $u' = g(x, u)$ that is separable.

Original form	Change of variables	New form
$y' = f\left(\frac{y}{x}\right)$	$u \triangleq \frac{y}{x}$	$u'x + u = f(u)$
$y' = f(ax + by + c)$	$u \triangleq ax + by + c$	$\frac{u' - a}{b} = f(u)$

Equilibrium

□ **Characterization** – In order for an ODE to have equilibrium solutions, it must be (1) autonomous and (2) have a value y^* that makes the derivative equal to 0, i.e:

$$(1) \quad \frac{dy}{dt} = f(y) \quad \text{and} \quad (2) \quad \exists y^*, \frac{dy^*}{dt} = f(y^*) = 0$$

□ **Stability** – Equilibrium solutions can be classified into 3 categories:

- Unstable: solutions run away with any small change to the initial conditions.
- Stable: any small perturbation leads the solutions back to that solution.
- Semi-stable: a small perturbation is stable on one side and unstable on the other.

Linear first-order ODE technique

□ **Standard form** – The standard form of a first-order linear ODE is expressed with $p(x), r(x)$ known functions of x , such that:

$$y' + p(x)y = r(x)$$

Remark: If $r = 0$, then the ODE is homogenous, and if $r \neq 0$, then the ODE is inhomogeneous.

□ **General solution** – The general solution y of the standard form can be decomposed into a homogenous part y_h and a particular part y_p and is expressed in terms of $p(x), r(x)$ such that:

$$y = y_h + y_p \quad \text{with} \quad y_h = Ce^{-\int p dx} \quad \text{and} \quad y_p = e^{-\int p dx} \times \int [re^{\int p dx}] dx$$

Remarks: Here, for any function p , the notation $\int p dx$ denotes the primitive of p without additive constant. Also, the term $e^{-\int p dx}$ is called the basis of the ODE and $e^{\int p dx}$ is called the integrating factor.

□ **Reduction to linear form** – The one-line table below sums up the change of variables that we apply in order to have a linear form:

Name, setting	Original form	Change	New form
Bernoulli, $n \in \mathbb{R} \setminus \{0, 1\}$	$y' + p(x)y = q(x)y^n$	$u \triangleq y^{1-n}$	$u' + (1-n)p(x)u = (1-n)q(x)$

Existence and uniqueness of an ODE

Here, we are given an ODE $y' = f(x, y)$ with initial conditions $y(x_0) = y_0$.

Existence theorem – If $f(x, y)$ is continuous at all points in a rectangular region containing (x_0, y_0) , then $y' = f(x, y)$ has at least one solution $y(x)$ passing through (x_0, y_0) .

Remark: If the condition does not apply, then we cannot say anything about existence.

Uniqueness theorem – If both $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are continuous at all points in a rectangular region containing (x_0, y_0) , then $y' = f(x, y)$ has a unique solution $y(x)$ passing through (x_0, y_0) .

Remark: If the condition does not apply, then we cannot say anything about uniqueness.

Numerical methods for ODE - Initial value problems

In this section, we would like to find $y(t)$ for the interval $[0, t_f]$ that we divide into $N + 1$ equally-spaced points $t_0 < t_1 < \dots < t_N = t_f$, such that:

$$\frac{dy}{dt} = f(t, y) \quad \text{with} \quad y(0) = y_0$$

Error – In order to assess the accuracy of a numerical method, we define its local and global errors $\epsilon_{\text{local}}, \epsilon_{\text{global}}$ as follows:

$$\epsilon_{\text{local}} = |y^{\text{exact}}(t_n) - y^{\text{numerical}}(t_n)| \quad \text{and} \quad \epsilon_{\text{global}} = \sqrt{\frac{1}{N} \sum_{n=1}^N |y^{\text{exact}}(t_n) - y^{\text{numerical}}(t_n)|^2}$$

Remarks: If $\epsilon_{\text{local}} = O(h^k)$, then $\epsilon_{\text{global}} = O(h^{k-1})$. Also, when we talk about the 'error' of a method, we refer to its global error.

Taylor series – The Taylor series giving the exact expression of y_{n+1} in terms of y_n and its derivatives is:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y'''_n + \dots = \sum_{k=0}^{+\infty} \frac{h^k}{k!} y_n^{(k)}$$

We can also have an expression of y_n in terms of y_{n+1} and its derivatives:

$$y_n = y_{n+1} - hy'_{n+1} + \frac{h^2}{2}y''_{n+1} - \frac{h^3}{6}y'''_{n+1} + \dots = \sum_{k=0}^{+\infty} \frac{(-h)^k}{k!} y_{n+1}^{(k)}$$

Stability – The stability analysis of any ODE solver algorithm is performed on the model problem, defined by:

$$y' = \lambda y \quad \text{with} \quad y(0) = y_0 \quad \text{and} \quad \lambda < 0$$

which gives $y_n = y_0 \sigma^n$, for which h verifies the condition $|\sigma(h)| < 1$.

Euler methods – The Euler methods are numerical methods that aim at estimating the solution of an ODE:

Type	Update formula	Error	Stability condition
Forward Euler	$y_{n+1} = y_n + hf(t_n, y_n)$	$O(h)$	$h < \frac{2}{ \lambda }$
Backward Euler	$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$	$O(h)$	None

Runge-Kutta methods – The table below sums up the most commonly used Runge-Kutta methods:

Type	Method	Update formula	Error	Stability condition
RK1	Euler's	$y_{n+1} = y_n + hk_1$ where $k_1 = f(t_n, y_n)$	$O(h)$	$h < \frac{2}{ \lambda }$
RK2	Heun's	$y_{n+1} = y_n + h \left(\frac{1}{2}k_1 + \frac{1}{2}k_2 \right)$ where $k_1 = f(t_n, y_n)$ and $k_2 = f(t_n + h, y_n + hk_1)$	$O(h^2)$	$h < \frac{2}{ \lambda }$

System of linear ODEs

Definition – A system of n first order linear ODEs

$$\begin{cases} y'_1 = a_{11}y_1 + \dots + a_{1n}y_n \\ \vdots \\ y'_n = a_{n1}y_1 + \dots + a_{nn}y_n \end{cases}$$

can be written in matrix form as:

$$\vec{y}' = A\vec{y}$$

$$\text{where } A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \text{ and } \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

System of homogeneous ODEs – The resolution of the system of 2 homogeneous linear ODEs $\vec{y}' = A\vec{y}$ is detailed in the following table:

Case	Eigenvalues \leftrightarrow Eigenvectors	Solution
Real distinct eigenvalues	$\lambda_1 \leftrightarrow \vec{\eta}_{\lambda_1}$ $\lambda_2 \leftrightarrow \vec{\eta}_{\lambda_2}$	$\vec{y} = C_1 \vec{\eta}_{\lambda_1} e^{\lambda_1 t} + C_2 \vec{\eta}_{\lambda_2} e^{\lambda_2 t}$
Double root eigenvalues	$\lambda \leftrightarrow \vec{\eta}$ $\vec{\rho}$ s.t. $(A - \lambda I)\vec{\rho} = \vec{\eta}$	$\vec{y} = [(C_1 + C_2 t)\vec{\eta} + C_2 \vec{\rho}]e^{\lambda t}$
Complex conjugate eigenvalues	$\alpha + i\beta \leftrightarrow \vec{\eta}_R + i\vec{\eta}_I$ $\alpha - i\beta \leftrightarrow \vec{\eta}_R - i\vec{\eta}_I$	$\vec{y} = C_1 (\cos(\beta t)\vec{\eta}_R - \sin(\beta t)\vec{\eta}_I) e^{\alpha t} + C_2 (\cos(\beta t)\vec{\eta}_I + \sin(\beta t)\vec{\eta}_R) e^{\alpha t}$

VIP Cheatsheet: Second-order ODE

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General case

□ **General form** – The general form of a second-order ODE can be written as a function F of x, y, y' and y'' as follows:

$$F(x, y, y', y'') = 0$$

□ **Methods of resolution** – The table below summarizes the general tricks to apply when the ODE has the following classic forms:

Old form	Trick	New form
$F(x, y', y'') = 0$	$y' \triangleq u, \quad y'' = \frac{du}{dx}$	$G(x, u, \frac{du}{dx}) = 0$
$F(y, y', y'') = 0$	$y' \triangleq u, \quad y'' = u \frac{du}{dy}$	$G(y, u, \frac{du}{dy}) = 0$
$F(y', y'') = 0$	$y' \triangleq u, \quad y'' = \frac{du}{dx}$ $y' \triangleq u, \quad y'' = u \frac{du}{dy}$	Missing- y approach $G(u, \frac{du}{dx}) = 0$ Missing- x approach $G(u, \frac{du}{dy}) = 0$

□ **Standard form of a linear ODE** – The standard form of a second-order linear ODE is expressed with p, q and r known functions of x such that:

$$y'' + p(x)y' + q(x)y = r(x)$$

for which the total solution y is the sum of a homogeneous solution y_h and a particular solution y_p :

$$y = y_h + y_p$$

Remark: if $r = 0$, then the ODE is homogeneous (and we have $y_p = 0$). If $r \neq 0$, then the ODE is said to be inhomogeneous.

□ **Linear dependency** – Two functions y_1, y_2 are said to be *linearly dependent* if $\frac{y_2}{y_1} = C$ constant. Conversely, they are *linearly independent* if $\frac{y_2}{y_1} \neq C$.

Linear homogeneous – Variable coefficients

□ **Method of reduction of order** – Let y_1 be a solution to the equation $y'' + p(x)y' + q(x)y = 0$. By noting C_1, C_2 constants, the global solution y_h is written as:

$$y_h = C_1 y_1 + C_2 y_1 \int \frac{e^{-\int p dx}}{y_1^2} dx$$

Remark: Here, for any function p , the notation $\int p dx$ denotes the primitive of p without additive constant.

Linear homogeneous – Constant coefficients

□ **General form** – The general form of a linear homogeneous second-order ODE with a, b, c constant coefficients is:

$$ay'' + by' + cy = 0$$

□ **Resolution** – Based on the types of solution of the characteristic equation $a\lambda^2 + b\lambda + c = 0$, and by noting $\Delta = b^2 - 4ac$ its discriminant, we distinguish the following cases:

Name	Case	Roots	Solution
Two distinct real roots	$\Delta > 0$	$\lambda_1 = \frac{-b + \sqrt{\Delta}}{2a}$ $\lambda_2 = \frac{-b - \sqrt{\Delta}}{2a}$	$y_h = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
Double real root	$\Delta = 0$	$\lambda = -\frac{b}{2a}$	$y_h = [C_1 + C_2 x] e^{\lambda x}$
Complex conjugate roots	$\Delta < 0$	$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ where $\alpha = -\frac{b}{2a}$ and $\beta = \frac{\sqrt{ \Delta }}{2a}$	$y_h = [C_1 \cos(\beta x) + C_2 \sin(\beta x)] e^{\alpha x}$

A special case: the Euler-Cauchy equation

□ **General form** – The Euler-Cauchy equation is a special case of linear homogeneous ODEs and has the following general form, where each $a_i \in \mathbb{R}$ is a constant coefficient:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = 0$$

□ **Second-order case** – For $n = 2$, by noting $y = x^m$, the ODE provides the indicial equation:

$$am^2 + (b-a)m + c = 0$$

with discriminant $\Delta = (b-a)^2 - 4ac$ and where the resolution of the ODE depends on the cases summarized in the table below.

Name	Case	Roots	Solution
Two distinct real roots	$\Delta > 0$	$m_1 = \frac{-b + a + \sqrt{\Delta}}{2a}$ $m_2 = \frac{-b + a - \sqrt{\Delta}}{2a}$	$y_h = C_1 x^{m_1} + C_2 x^{m_2}$
Double real root	$\Delta = 0$	$m = -\frac{b-a}{2a}$	$y_h = [C_1 + C_2 \ln x] x^m$
Complex conjugate roots	$\Delta < 0$	$m_1 = \alpha + i\beta$ $m_2 = \alpha - i\beta$ <p>where $\alpha = -\frac{b-a}{2a}$ and $\beta = \frac{\sqrt{ \Delta }}{2a}$</p>	$y_h = [C_1 \cos(\beta \ln x)$ $+ C_2 \sin(\beta \ln x)] x^\alpha$

Linear inhomogeneous – Variable coefficients

□ **Wronskian** – Given y_1 and y_2 the two solutions of the homogeneous equation, we define the Wronskian W as follows:

$$W = y_1 y_2' - y_2 y_1'$$

□ **Method of Variation of Parameters** – The particular solution y_p of the inhomogeneous ODE is given by:

$$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

Linear inhomogeneous – Constant coefficients

□ **Undetermined coefficients method** – The particular solution y_p of the inhomogeneous ODE $ay'' + by' + cy = r(x)$ is determined from the correspondence table below:

Form of r	Form of y_p
C	A
$x^n, n \in \mathbb{N}^*$	$A_0 + A_1 x + \dots + A_n x^n$
$e^{\gamma(x)}$	$A e^{\gamma x}$
$\cos(\omega x)$ or $\sin(\omega x)$	$A \cos(\omega x) + B \sin(\omega x)$
$x^n e^{\gamma x} \cos(\omega x)$ or $x^n e^{\gamma x} \sin(\omega x)$	$(A_0 + A_1 x + \dots + A_n x^n) \cos(\omega x) e^{\gamma x} +$ $(B_0 + B_1 x + \dots + B_n x^n) \sin(\omega x) e^{\gamma x}$

Remark: all new constants are determined after plugging back y_p into the ODE.

□ **Modification rule** – If the particular solution y_p picked from the above table matches either y_1 or y_2 , then has to be multiplied by the lowest power of x such that it is no more the case.

□ **Sum rule** – If $r(x)$ is a sum of functions of the first column of the above table, then y_p is the sum of its associated particular solutions.

VIP Cheatsheet: Applications

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Physics Laws

□ **Gravitational force** – A mass m is subject to the gravitational force \vec{F}_g , which is expressed with respect to \vec{g} of magnitude $9.81 \text{ m} \cdot \text{s}^{-2}$ and directed towards the center of the Earth, as follows:

$$\vec{F}_g = m\vec{g}$$

□ **Spring force** – A spring of constant k and of relaxed position \vec{x}_0 attached a mass m of position \vec{x} has a force \vec{F}_s expressed as follows:

$$\vec{F}_s = -k(\vec{x} - \vec{x}_0)$$

□ **Friction force** – The friction force F_f of constant coefficient β applied on a mass of velocity \vec{v} is written as:

$$\vec{F}_f = -\beta\vec{v}$$

□ **Mass moment of inertia** – The mass moment of inertia of a system of mass m_i located at distance r_i from point O , expressed in point O is written as:

$$J_0 = \sum_i m_i r_i^2$$

□ **Torque** – The torque \vec{T} of a force \vec{F} located at \vec{r} from the reference point O is written as:

$$\vec{T} = \vec{r} \times \vec{F}$$

□ **Newton's second law** – A mass m of acceleration \vec{a} to which forces \vec{F}_i are applied verifies the following equation:

$$m\vec{a} = \sum_i \vec{F}_i$$

– In the 1-D case along the x axis, we can write it as $mx'' = \sum_i F_i$.

– In the rotational case, around point O , we can write it as $J_0\theta'' = \sum_i T_i$.

Spring-mass system

□ **Free undamped motion** – A free undamped spring-mass system of mass m and spring coefficient k follows the ODE $x'' + \frac{k}{m}x = 0$, which can be written as a function of the natural frequency ω as:

$$x'' + \omega^2 x = 0 \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}}$$

□ **Free damped motion** – A free damped spring-mass system of mass m , of spring coefficient k and subject to a friction force of coefficient β follows the ODE $x'' + \frac{\beta}{m}x' + \frac{k}{m}x = 0$, which can be written as a function of the damping parameter λ and the natural frequency ω as:

$$x'' + 2\lambda x' + \omega^2 x = 0 \quad \text{with} \quad \lambda = \frac{\beta}{2m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}$$

which has the following cases summed up in the table below:

Condition	Type of motion
$\lambda > \omega$	Over damped
$\lambda = \omega$	Critically damped
$\lambda < \omega$	Under damped

□ **Forcing frequency** – A forcing function $F(t)$ is often modeled with a periodic function of the form $F(t) = F_0 \sin(\gamma t)$, where γ is called the forcing frequency.

□ **Forced undamped motion** – A forced undamped spring-mass system of mass m and spring coefficient k follows the ODE $x'' + \frac{k}{m}x = F_0 \sin(\gamma t)$, which can be written as a function of the natural frequency ω as:

$$x'' + \omega^2 x = F_0 \sin(\gamma t) \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}}$$

which has the following cases summed up in the table below:

Condition	Type of motion
$\gamma \neq \omega$	General response
$\gamma \approx \omega$	Beats
$\gamma = \omega$	Resonance

□ **Forced damped motion** – A forced damped spring-mass system of mass m , of spring coefficient k and subject to a friction force of coefficient β follows the ODE $x'' + \frac{\beta}{m}x' + \frac{k}{m}x = F_0 \sin(\gamma t)$, which can be written as a function of the damping parameter λ and the natural frequency ω as:

$$x'' + 2\lambda x' + \omega^2 x = F_0 \sin(\gamma t) \quad \text{with} \quad \lambda = \frac{\beta}{2m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}$$

Boundary Value Problems

□ **Types of boundary conditions** – Given a numerical problem between 0 and L , we distinguish the following types of boundary conditions:

Name	Boundary values
Dirichlet	$y(0)$ and $y(L)$
Neumann	$y(0)$ and $y'(L)$
Robin	$y(0)$ and $\alpha y(L) + \beta y'(L)$

□ **Numerical differentiation** – The table below sums up the approximation of the derivatives of y at point x_j , knowing the values of y at each point of a uniformly spaced set of grid points.

Order of derivative	Name	Formula	Order of error
First derivative	Forward difference	$y'_j = \frac{y_{j+1} - y_j}{h}$	$O(h)$
	Backward difference	$y'_j = \frac{y_j - y_{j-1}}{h}$	$O(h)$
	Central difference	$y'_j = \frac{y_{j+1} - y_{j-1}}{2h}$	$O(h^2)$
Second derivative	Central difference	$y''_j = \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}$	$O(h^2)$

□ **Direct method** – The direct method can solve linear ODEs by reducing the problem to the resolution of a linear system $Ay = f$, where A is a tridiagonal matrix.

□ **Shooting method** – The shooting method is an algorithm that can solve ODEs through an iterative process. It uses a numerical scheme, such as Runge-Kutta, and converges to the right solution by iteratively searching for the missing initial condition $y'(0)$.

Remark: in the linear case, the shooting method converges after the first two initial guesses.