Frequency: Ω rads/s; f Hz; ω rads $\quad \Omega = 2\pi f$; $\omega = \Omega T$

Impulse: $\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = f(a) \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$

Linear: $a x_1(t) + b x_2(t) \longrightarrow \boxed{H} \longrightarrow a y_1(t) + b\, y_2(t)$

Time Invariant: if $x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$, then $x(t-t_0) \longrightarrow \boxed{H} \longrightarrow y(t-t_0)$

BIBO stable: if $|x(t)| \le M_x < \infty$, then $|y(t)| \le M_y < \infty$

Convolution $x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau \qquad x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$

$x(t) * h(t) = h(t) * x(t) \qquad x(t) * \big(h_1(t) + h_2(t)\big) = x(t) * h_1(t) + x(t) * h_2(t)$

BIBO stable $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$, $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Difference Equations $\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$; $a_0 = 1$

Characteristic eqn $\sum_{k=0}^{N} a_k z^{-k} = 0$

Transient response $y_t[n] = \sum_{\ell=1}^{N} c_\ell (d_\ell)^n$, $n \ge 0$ distinct d_ℓ

Steady-State response $x[n] = z^n \longrightarrow y_s[n] = H(z)\, z^n$; $H(z) = \dfrac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$

Frequency Response $H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$

Impulse Response $h[n] \xleftrightarrow{DTFT} H(e^{j\omega})$ (partial fraction expansion)

terms $\dfrac{A_k}{1-\alpha_k e^{-j\omega}} \longleftrightarrow A_k \alpha_k^n u[n]$

Differential Equations $\sum_{k=0}^{N} a_k \dfrac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \dfrac{d^k}{dt^k} x(t)$

Characteristic eqn $a_N s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0 = 0$

Transient Response $y_t(t) = \sum_{\ell=1}^{N} c_\ell e^{\tilde{s}_\ell t}$

Steady-State Response $x(t) = e^{st} \longrightarrow y(t) = H(s) e^{st}$; $H(s) = \dfrac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$

Frequency Response $H(\Omega) = H(s)\big|_{s=j\Omega}$

Impulse Response $h(t) \xleftrightarrow{FT} H(\Omega)$ (partial fraction expansion)

terms $\dfrac{A_k}{j\omega - d_k} \longleftrightarrow A_k e^{d_k t} u(t)$

Fourier Series

Time Domain	Frequency Domain
$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\Omega_o t}$ Period T	$X[k] = \frac{1}{T}\int_{<T>} x(t) e^{-jk\Omega_o t} dt$ $\Omega_o = \frac{2\pi}{T}$
$x(t) = \begin{cases} 1, & \lvert t\rvert \le T_o \\ 0, & T_o < \lvert t\rvert \le T/2 \end{cases}$	$X[k] = \frac{\sin(k\Omega_o T_o)}{k\pi}$
$x(t) = e^{jp\Omega_o t}$	$X[k] = \delta[k-p]$
$x(t) = \cos(p\Omega_o t)$	$X[k] = \frac{1}{2}\delta[k-p] + \frac{1}{2}\delta[k+p]$
$x(t) = \sin(p\Omega_o t)$	$X[k] = \frac{1}{2j}\delta[k-p] - \frac{1}{2j}\delta[k+p]$
$x(t) = \sum_{p=-\infty}^{\infty} \delta(t-pT)$	$X[k] = \frac{1}{T}$

Property	$x(t) \xleftrightarrow{FS;\,\Omega_o} X[k] \quad y(t) \xleftrightarrow{FS;\,\Omega_o} Y[k]$
Linearity	$ax(t)+by(t) \xleftrightarrow{FS;\,\Omega_o} aX[k]+bY[k]$
Time Shift	$x(t-t_o) \xleftrightarrow{FS;\,\Omega_o} e^{-jk\Omega_o t_o} X[k]$
Frequency Shift	$e^{jk_o\Omega_o t} x(t) \xleftrightarrow{FS;\,\Omega_o} X[k-k_o]$
Scaling	$x(at) \xleftrightarrow{FS;\,a\Omega_o} X[k]$
Differentiation in Time	$\frac{d}{dt}x(t) \xleftrightarrow{FS;\,\Omega_o} jk\Omega_o X[k]$
Multiplication-Convolution	$x(t)y(t) \xleftrightarrow{FS;\,\Omega_o} \sum_{p=-\infty}^{\infty} X[p]Y[k-p]$
Parseval's Theorem	$\frac{1}{T}\int_{<T>} \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert X[k]\rvert^2$
Symmetry (for $x(t)$ real)	$X[-k] = X^*[k]$ $\text{Real}\{X[k]\} = \text{Real}\{X[-k]\}$ $\text{Imag}\{X[k]\} = -\,\text{Imag}\{X[-k]\}$ $\lvert X[k]\rvert = \lvert X[-k]\rvert$ $\arg\{X[k]\} = -\arg\{X[-k]\}$

Fourier Transform

Time Domain	Frequency Domain
$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$	$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$
$x(t) = \begin{cases} 1, & \lvert t\rvert < T \\ 0, & \text{otherwise} \end{cases}$	$X(\Omega) = \frac{2\sin(\Omega T)}{\Omega}$
$x(t) = \frac{1}{\pi t}\sin(Wt)$	$X(\Omega) = \begin{cases} 1, & \lvert\Omega\rvert < W \\ 0, & \text{otherwise} \end{cases}$
$x(t) = \delta(t)$	$X(\Omega) = 1$
$x(t) = 1$	$X(\Omega) = 2\pi\delta(\Omega)$
$x(t) = u(t)$	$X(\Omega) = \frac{1}{j\Omega} + \pi\delta(\Omega)$
$x(t) = e^{-at}u(t), \quad a > 0$	$X(\Omega) = \frac{1}{a+j\Omega}$
$x(t) = te^{-at}u(t), \quad a > 0$	$X(\Omega) = \frac{1}{(a+j\Omega)^2}$
$x(t) = e^{-a\lvert t\rvert}, \quad a > 0$	$X(\Omega) = \frac{2a}{a^2+\Omega^2}$
$x(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$	$X(\Omega) = e^{-\frac{\Omega^2}{2}}$

Property	$x(t) \xleftrightarrow{FT} X(\Omega) \quad y(t) \xleftrightarrow{FT} Y(\Omega)$
Linearity	$ax(t)+by(t) \xleftrightarrow{FT} aX(\Omega)+bY(\Omega)$
Time Shift	$x(t-t_o) \xleftrightarrow{FT} e^{-j\Omega t_o} X(\Omega)$
Frequency Shift	$e^{j\gamma t} x(t) \xleftrightarrow{FT} X(\Omega-\gamma)$
Scaling	$x(at) \xleftrightarrow{FT} \frac{1}{\lvert a\rvert} X\left(\frac{\Omega}{a}\right)$
Differentiation in Time	$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\Omega X(\Omega)$
Differentiation in Frequency	$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\Omega} X(\Omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Convolution-Multiplication	$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \xleftrightarrow{FT} X(\Omega)Y(\Omega)$
Multiplication-Convolution	$x(t)y(t) \xleftrightarrow{FT} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Gamma)Y(\Omega-\Gamma)d\Gamma$
Parseval's Theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$
Symmetry (for $x(t)$ real)	$X(-\Omega) = X^*(\Omega)$ $\text{Real}\{X(\Omega)\} = \text{Real}\{X(-\Omega)\}$ $\text{Imag}\{X(\Omega)\} = -\,\text{Imag}\{X(-\Omega)\}$ $\lvert X(\Omega)\rvert = \lvert X(-\Omega)\rvert$ $\arg\{X(\Omega)\} = -\arg\{X(-\Omega)\}$

Basic Discrete-Time Fourier Transform Pairs

Time Domain	Frequency Domain
$x[n] = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\omega})e^{j\omega n}d\omega$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
$x[n] = \begin{cases} 1, & \lvert n\rvert \le M \\ 0, & \text{otherwise} \end{cases}$	$X(e^{j\omega}) = \frac{\sin[\omega(\frac{2M+1}{2})]}{\sin(\frac{\omega}{2})}$
$x[n] = \alpha^n u[n], \quad \lvert\alpha\rvert < 1$	$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$
$x[n] = \delta[n]$	$X(e^{j\omega}) = 1$
$x[n] = u[n]$	$X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \pi\sum_{p=-\infty}^{\infty}\delta(\omega - 2\pi p)$
$x[n] = \frac{1}{\pi n}\sin(Wn), \quad 0 < W \le \pi$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(e^{j\omega})$ is 2π periodic
$x[n] = (n+1)\alpha^n u[n]$	$X(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}$

Discrete-Time Fourier Transform Properties

Property	$x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) \quad y[n] \xleftrightarrow{DTFT} Y(e^{j\omega})$
Linearity	$ax[n] + by[n] \xleftrightarrow{DTFT} aX(e^{j\omega}) + bY(e^{j\omega})$
Time Shift	$x[n-n_o] \xleftrightarrow{DTFT} e^{-j\omega n_o}X(e^{j\omega})$
Frequency Shift	$e^{j\gamma n}x[n] \xleftrightarrow{DTFT} X(e^{j(\omega-\gamma)})$
Differentiation in Frequency	$-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\omega}X(e^{j\omega})$
Convolution-Multiplication	$\sum_{k=-\infty}^{\infty} x[k]y[n-k] \xleftrightarrow{DTFT} X(e^{j\omega})Y(e^{j\omega})$
Multiplication-Convolution	$x[n]y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\gamma})Y(e^{j(\omega-\gamma)})d\gamma$
Parseval's Theorem	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$
Symmetry (for $x[n]$ real)	$X(e^{-j\omega}) = X^*(e^{j\omega})$ $\text{Real}\{X(e^{j\omega})\} = \text{Real}\{X(e^{-j\omega})\}$ $\text{Imag}\{X(e^{j\omega})\} = -\text{Imag}\{X(e^{-j\omega})\}$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Transform Pairs - Periodic Signals

Periodic Time Domain Signal	Fourier Transform
$x(t) \xleftrightarrow{FS;\,\Omega_o} X[k]$	$X(\Omega) = 2\pi\sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_o)$
$x(t) = \cos(p\Omega_o t)$	$X(\Omega) = \pi\delta(\Omega + p\Omega_o) + \pi\delta(\Omega - p\Omega_o)$
$x(t) = \sin(p\Omega_o t)$	$X(\Omega) = \frac{\pi}{j}\delta(\Omega - p\Omega_o) - \frac{\pi}{j}\delta(\Omega + p\Omega_o)$
$x(t) = e^{jp\Omega_o t}$	$X(\Omega) = 2\pi\delta(\Omega - p\Omega_o)$
$x(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT_s)$	$X(\Omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\frac{2\pi}{T_s})$

Periodic Time Domain Signal	Discrete-Time Fourier Transform
$x[n] = \cos(\omega_o n), \quad \lvert\omega_o\rvert < \pi$	$X(e^{j\omega}) = \pi\delta(\omega + \omega_o) + \pi\delta(\omega - \omega_o), \quad \lvert\omega\rvert < \pi$
$x[n] = \sin(\omega_o n), \quad \lvert\omega_o\rvert < \pi$	$X(e^{j\omega}) = \frac{\pi}{j}\delta(\omega - \omega_o) - \frac{\pi}{j}\delta(\omega + \omega_o), \quad \lvert\omega\rvert < \pi$
$x[n] = e^{j\omega_o n}, \quad \lvert\omega_o\rvert < \pi$	$X(e^{j\omega}) = 2\pi\delta(\omega - \omega_o), \quad \lvert\omega_o\rvert < \pi$

Sampling

if $x(t) \xleftrightarrow{FT} X(\Omega)$, $x[n] = x(nT_s)$

Then $x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$,

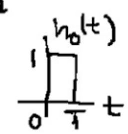
$x_s(t) \xleftrightarrow{FT} X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\Omega - k\frac{2\pi}{T_s})$,

and $x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) = X_s(\Omega)\Big|_{\Omega=\frac{\omega}{T_s}}$

If $X(\Omega)=0$, $\lvert\Omega\rvert>B$, then $x(t)$ is uniquely represented by $x(nT_s)$ provided $\frac{2\pi}{T_s} > 2B$

Zero-Order Hold

$y_o(t) = \sum_{n=-\infty}^{\infty} y[n]\, h_o(t - nT)$



$Y_o(\Omega) = Y_s(\Omega)\, H_o(\Omega)$

$H_o(\Omega) = \frac{2e^{-j\Omega T/2}\sin(\Omega T/2)}{\Omega}$

DFT Sampling DTFT

$x[n] = 0, \; n<0, \; n>N-1$

$x[n] \xleftrightarrow{DFT;N} X[k] = X(e^{j\omega})\Big|_{\omega = k\frac{2\pi}{N}}$

Since $\omega = 2\pi f T_s$, $f_k = \frac{k}{NT_s}$ Hz

where $X[k]$ represents frequency f_k

Discrete Fourier Transform Properties

Property	$x[n] \xleftrightarrow{DFT;N} X[k] \quad y[n] \xleftrightarrow{DFT;N} Y[k]$
DFT Definition	$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}$
IDFT Definition	$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{jk\frac{2\pi}{N}n}$
Linearity	$ax[n] + by[n] \xleftrightarrow{DFT;N} aX[k] + bY[k]$
Time Shift	$x[n-m] \xleftrightarrow{DFT;N} e^{-jk\frac{2\pi}{N}m} X[k]$
Frequency Shift	$e^{j\frac{2\pi}{N}mn} x[n] \xleftrightarrow{DFT;N} X[k-m]$
Convolution-Multiplication	$x[n] \circledast y[n] \xleftrightarrow{DFT;N} X[k]Y[k]$
Circular Convolution	$x[n] \circledast y[n] = \sum_{m=-\infty}^{\infty} x[n] * y[n-mN]$
Symmetry (for $x[n]$ real)	$X[-k] = X^*[k]$ $\text{Real}\{X[k]\} = \text{Real}\{X[-k]\}$ $\text{Imag}\{X[k]\} = -\text{Imag}\{X[-k]\}$ $\lvert X[k]\rvert = \lvert X[-k]\rvert$ $\arg\{X[k]\} = -\arg\{X[-k]\}$

If $x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$ and $\tilde{X}[k] = X(e^{j\frac{2\pi}{N}k})$, then

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

where $\tilde{x}[n] \xleftrightarrow{DFT} \tilde{X}[k]$

If $x[n]$ is length L, $h[n]$ is length M, then

$$x[n] * h[n] = \underbrace{x[n] \circledN h[n]}_{\text{one period}}$$

for $N \geq L + M - 1$

DFT Approximation to FT

1) Sample in time - Sampling Thm
2) Window (truncate)
3) Sample in frequency

If $x[n]$ is length M_x, $y[n]$ is length M_y, then $x[n] * y[n]$ is length $M_x + M_y - 1$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Downarrow$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$= \sum_{k=0}^{N} \frac{A_k}{1 - d_k z^{-1}} \quad \text{for } d_k \text{ distinct and } M < N$$

c_k: zeros, d_k: poles

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} |e^{j\omega} - c_k|}{\prod_{k=1}^{N} |e^{j\omega} - d_k|}$$

$$= \left|\frac{b_0}{a_0}\right| \frac{\prod \text{distances to zeros}}{\prod \text{distances to poles}}$$

IIR (pole-zero) filters:

Butterworth: monotone decreasing pass and stop band

Chebyshev I: equiripple passband, monotone decreasing stop band

Chebyshev II: monotone decreasing passband, equiripple stop band

Elliptic: equiripple pass and stop band

Window-based FIR Design:

1) Ideal impulse response
2) Window impulse response
3) Time-shift to make causal

Length N Window	Mainlobe (rads)	Relative Sidelobe (dB)
Rectangular	$4\pi/N$	-13
Hamming	$8\pi/N$	-41