Control of an Inverted Pendulum

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Abstract. The balancing of an inverted pendulum by moving a cart along a horizontal track is a classic problem in the area of control. This paper will describe two methods to swing a pendulum attached to a cart from an initial downwards position to an upright position and maintain that state. A nonlinear heuristic controller and an energy controller have been implemented in order to swing the pendulum to an upright position. After the pendulum is swung up, a linear quadratic regulator state feedback optimal controller has been implemented to maintain the balanced state. The heuristic controller outputs a repetitive signal at the appropriate moment and is finely tuned for the specific experimental setup. The energy controller adds an appropriate amount of energy into the pendulum system in order to achieve a desired energy state. The optimal state feedback controller is a stabilizing controller based on a model linearized around the upright position and is effective when the cart-pendulum system is near the balanced state. The pendulum has been swung from the downwards position to the upright position using both methods and the experimental results are reported.

1. INTRODUCTION

The inverted pendulum system is a standard problem in the area of control systems. They are often useful to demonstrate concepts in linear control such as the stabilization of unstable systems. Since the system is inherently nonlinear, it has also been useful in illustrating some of the ideas in nonlinear control. In this system, an inverted pendulum is attached to a cart equipped with a motor that drives it along a horizontal track. The user is able to dictate the position and velocity of the cart through the motor and the track restricts the cart to movement in the horizontal direction. Sensors are attached to the cart and the pivot in order to measure the cart position and pendulum joint angle, respectively. Measurements are taken with a quadrature encoder connected to a MultiQ-3 general purpose data acquisition and control board. Matlab/Simulink is used to implement the controller and analyze data.

The inverted pendulum system inherently has two equilibria, one of which is stable while the other is unstable. The stable equilibrium corresponds to a state in which the pendulum is pointing downwards. In the absence of any control force, the system will naturally return to this state. The stable equilibrium requires no control input to be achieved and, thus, is uninteresting from a control perspective. The unstable equilibrium corresponds to a state in which the pendulum points strictly upwards and, thus, requires a control force to maintain this position. The basic control objective of the inverted pendulum problem is to maintain the unstable equilibrium position when the pendulum initially starts in an upright position. The control objective for this project will focus on starting from the stable equilibrium position (pendulum pointing down), swinging it up to the unstable equilibrium position (pendulum upright), and maintaining this state.

2. MODELLING

A schematic of the inverted pendulum is shown in Figure 1.

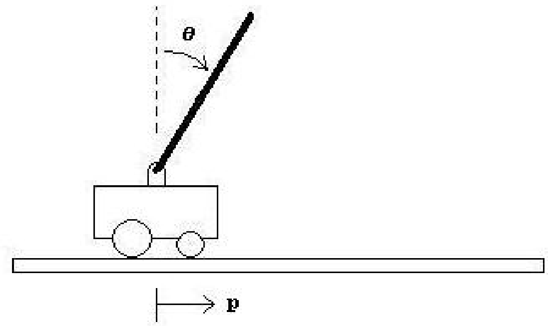

Figure 1. Inverted Pendulum Setup

A cart equipped with a motor provides horizontal motion of the cart while cart position, p, and joint angle, θ, measurements are taken via a quadrature encoder.

By applying the law of dynamics on the inverted pendulum system, the equations of motion are:

$$\ddot{p}\left(M - \frac{m_p l\cos^2(\theta)}{L}\right) = \frac{K_m K_g}{Rr}V - \frac{K_m^2 K_g^2}{Rr^2}\dot{p} - \frac{m_p \lg}{L}\cos(\theta)\sin(\theta) + m_p l\sin(\theta)(\dot{\theta})^2 \quad (1)$$

$$\ddot{\theta}\left(L - \frac{m_p l\cos^2(\theta)}{M}\right) = g\sin(\theta) - \frac{m_p l(\dot{\theta})^2}{L}\cos(\theta)\sin(\theta) - \frac{\cos(\theta)}{M}\left(\frac{K_m K_g}{Rr}V - \frac{K_m^2 K_g^2}{Rr^2}\dot{p}\right), \quad (2)$$

where m_c is the cart mass, m_p is the pendulum mass, I is the rotational inertia, l is the half-length of the pendulum, R is the motor armature resistance, r is the motor pinion radius, K_m is the motor torque constant, and K_g is the gearbox ratio. Also, for simplicity,

$$M = m_c + m_p \qquad L = \frac{I + m_p l^2}{m_p l} \quad (3)$$

and note that the relationship between force, F, and voltage, V, for the motor is:

$$F = \frac{K_m K_g}{Rr}V - \frac{K_m^2 K_g^2}{Rr^2}\dot{p}. \quad (4)$$

Let the state vector be defined as:

$$x = \begin{pmatrix} p & \dot{p} & \theta & \dot{\theta} \end{pmatrix}^T. \quad (5)$$

Finally, we linearize the system about the unstable equilibrium $(0\ 0\ 0\ 0)^T$. Note that $\theta = 0$ corresponds to the pendulum being in the upright position. The linearization of the cart-pendulum system around the upright position is:

$$\begin{aligned}\dot{x} &= Ax + BV \\ y &= Cx\end{aligned} \tag{6}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{M - \frac{m_p l}{L}}\frac{K_m^2 K_g^2}{Rr^2} & -\frac{g m_p l}{L\left(M - \frac{m_p l}{L}\right)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{M\left(L - \frac{m_p l}{M}\right)}\frac{K_m^2 K_g^2}{Rr^2} & \frac{g}{L - \frac{m_p l}{M}} & 0 \end{pmatrix} \tag{7}$$

$$B = \begin{pmatrix} 0 \\ \frac{1}{M - \frac{m_p l}{L}}\frac{K_m K_g}{Rr} \\ 0 \\ -\frac{1}{M\left(L - \frac{m_p l}{M}\right)}\frac{K_m K_g}{Rr} \end{pmatrix} \; ; \qquad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Finally, by substituting the parameter values that correspond to the experimental setup:

$$\begin{aligned}\begin{pmatrix} \dot{p} \\ \ddot{p} \\ \dot{\theta} \\ \ddot{\theta} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -15.14 & -3.04 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 37.23 & 31.61 & 0 \end{pmatrix}\begin{pmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} 0 \\ 3.39 \\ 0 \\ -8.33 \end{pmatrix} V \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{pmatrix}.\end{aligned} \tag{8}$$

This system will allow us to design a controller to balance the inverted pendulum around the point of linearization.

3. STABILIZING CONTROLLER DESIGN

The controller design approach for this project is broken up into two components. The first part involves the design of an optimal state feedback controller for the linearized model that will stabilize the pendulum around the upright position. The second part involves the design of a controller that swings the pendulum up to the unstable equilibrium. When the pendulum approaches the linearized point, the control will switch to the stabilizing controller which will balance the pendulum around the upright position.

The state feedback controller responsible for balancing the pendulum in the upright position is based on a Linear Quadratic Regulator (LQR) design using the linearized system. In a LQR design, the gain matrix K for a linear state feedback control law $u = -Kx$ is found by minimizing a quadratic cost function of the form

$$J = \int_0^\infty x(t)^T Qx(t) + u(t)^T Ru(t)dt, \tag{9}$$

where Q and R are weighting parameters that penalize certain states or control inputs.

The weighting parameters chosen in the design of the optimal state feedback controller are:

$$\begin{aligned}Q &= \begin{pmatrix} 10000 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ R &= 1.\end{aligned} \tag{10}$$

Based on this design, the controller gain matrix for the linearized system is:

$$K = \begin{pmatrix} -99.0916 & -64.4448 & -180.6568 & -30.5668 \end{pmatrix}. \tag{11}$$

By using this K and the control law $u = -Kx$, the system is stabilized around the linearized point (pendulum upright). Since this control law is based on the linearized system, the state feedback optimal controller is only effective when the pendulum is near the upright position.

4. STATE ESTIMATION

For the inverted pendulum experimental setup, not all the state variables are available for measurement. In fact, only the cart position, p, and the pendulum angle, θ, are directly measured. This means that the cart velocity and the pendulum angular velocity are not immediately available for use in any control schemes beyond just stabilization. Thus, an observer is relied upon to supply accurate estimations of the states at all cart-pendulum positions.

A linear full state observer can be implemented based on the linearized system derived earlier. This observer is simple in design and provides accurate estimation of all the states around the linearized point. The observer is implemented by using a duplicate of the linearized system dynamics and adding in a correction term that is simply a gain on the error in the estimates. The observer gain matrix is determined by an LQR design similar to that used to determine the gain of the optimal state feedback stabilizing controller. In this case, the weighting parameters are chosen to be:

$$\begin{aligned}Q &= \begin{pmatrix} 10000 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ R &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned} \tag{12}$$

Based on this design, the observer gain matrix is:

$$L = \begin{pmatrix} .9999 & 0 & 0 & 0 \\ 0 & -.0015 & .9999 & .0490 \end{pmatrix}. \tag{13}$$

Since the linear full state observer is based on the linearized system, it is only effective in estimating the state variables when the cart-pendulum system is near the upright position. Thus, a low-pass filtered derivative is used to estimate the two unmeasured states, cart velocity and pendulum angular velocity, when the system is not close to the unstable equilibrium. This method approximates the cart velocity and pendulum angular velocity by using a finite difference and then passing it through a low-pass filter. The following filter is chosen for this estimation method:

$$G(s) = \frac{50s}{s+50}. \tag{14}$$

The problems with such a method are that it introduces some delay and has a gain that is slightly less than one. The state estimates obtained from the filtered derivative, however, are reasonably accurate for the swing-up controllers implemented in this paper.

5. SWING-UP CONTROLLER DESIGN

Two different control schemes were implemented to swing the pendulum from the downwards position to the upright position. The first is a heuristic controller that provides a constant voltage in the appropriate direction and, thus, drives the cart back and forth along the track repeatedly. It will repeat this action until the pendulum is close enough to the upright position such that the stabilizing controller can be triggered to maintain this balanced state. The second scheme is an energy controller that regulates the amount of energy in the pendulum. This controller inputs energy into the cart-pendulum system until it attains the energy state that corresponds to the pendulum in the upright position. Similar to the heuristic control method, the energy control method will also switch to the stabilizing controller when the pendulum is close to the upright position. The switch that triggers the stabilizing controller in both cases is activated when the pendulum is within 5° of the upright position and the angular velocity is slower than 2.5 radians per second.

Heuristic Controller

The heuristic controller is a logic-based control design that determines the direction and the moment in time the cart should move depending on the state of the system. A specific voltage gain is applied to the cart motor based on results from repeated experimentation. This controller will make the cart drive forward or back whenever the pendulum crosses the downwards position and depending on the direction that the pendulum is swinging when it reaches the downwards position.

The logic-based control design is completely dependent on the pendulum angle, one of the available measured state variables. The control scheme will change the direction of the cart movement whenever the pendulum angle crosses the downwards position. Since this control design is based solely on the pendulum angle, the downwards position is the optimal moment in time to add energy to the pendulum by moving the cart in the appropriate direction. The direction the cart moves is the opposite sign of the pendulum angle immediately after it crosses the downwards position. When the direction of the cart movement is determined, a constant voltage gain is applied to the cart motor in that same direction until the pendulum returns to the downwards position. This control scheme will effectively move the cart back and forth along the track repeatedly until the pendulum swings close enough to the upright position.

It is important to note that the nature of this control scheme is that the same cart movement is applied regardless of whether the pendulum is above or below the horizontal axis (since the sign of the pendulum angle remains the same). The nature of the cart-pendulum system, however, is that the same cart movement that once added energy to the pendulum while it was below the horizontal axis now actually takes away energy from the pendulum. Eventually, the pendulum will reach a point where it can add no more energy to the pendulum system but it has yet to build enough energy to reach the upright position. To avoid this phenomenon, a switch has been implemented that changes the voltage input to the cart motor to 0 when the pendulum is 135° from the downwards position. As a result, the cart will not move to take energy away from the pendulum system when the pendulum is higher than 135°. This will allow the pendulum to simply return to the downwards position without losing anymore energy. When the pendulum crosses the downwards position again, the logic-based controller will be able to add more energy to the pendulum until it eventually approaches the upright position.

The voltage gain of this control scheme is determined by repeated experimentation. There is a direct correlation between the time it takes to swing the pendulum to its upright position and the magnitude of the voltage gain. A gain that is too high, though, may make the pendulum approach the upright position with too high a velocity and, thus, the stabilizing controller will be unable to balance the pendulum. On the other hand, a gain too low may not provide enough energy to the pendulum so that it can reach the upright position. Also, the reliability of the controller in performing the task varies depending on the gain selected. Thus, repeated experimentation is required to finely tune the gain so that the pendulum approaches the upright position with just the right amount of velocity and in a reasonable amount of time with a high success rate.

Energy Controller

The swinging up of a pendulum from the downwards position can also be accomplished by controlling the amount of energy in the system. The energy in the pendulum system can be driven to a desired value through the use of feedback control. By adding in enough energy such that its value corresponds to the upright position, the pendulum can be swung up to its unstable equilibrium. When the pendulum is close to the upright position, the stabilizing controller designed earlier can be triggered to catch the pendulum and balance it around the unstable equilibrium.

The system is defined such that the energy, E, is zero in the upright position. The energy of the pendulum can be written as

$$E = m_p gl\left(\frac{1}{2}\left(\frac{\dot{\theta}}{\omega_0}\right)^2 + \cos\theta - 1\right) \tag{15}$$

where

$$\omega_0 = \sqrt{\frac{m_p gl}{4I}} \tag{16}$$

and m_p is the mass of the pendulum, l is the half-length of the pendulum, g is the acceleration of gravity, and I is the rotational inertia. Thus, the energy in the pendulum is a function of the pendulum angle and the pendulum angular velocity. Note also that the energy corresponding to the pendulum in the downwards position is $-2m_p gl$. The goal of the control scheme is to add energy into the system until the value corresponds to the pendulum in the upright position.

The control law implemented to achieve the desired energy is

$$a = sat_V\left(k\left(E - E_0\right)\right)sign\left(\dot{\theta}\cos\theta\right), \tag{17}$$

where k is a design parameter and E_0 is the desired energy level. The control output, a, is the acceleration of the pivot which can be translated to a voltage input to the cart motor by using equation (4) and the fact that:

$$F \approx Ma \tag{18}$$

for the system. In this control scheme, the sat_V function is defined as the value for which the voltage supplied to the cart saturates. This controller essentially uses pendulum angle and pendulum angular velocity to determine the direction the cart should move at any point in time. A proportional controller that scales with the amount of energy still required to achieve the desired energy state dictates the amount of voltage applied to the cart motor. The value of the parameter V in sat_V dictates the maximum amount of control signal available and thus the maximum amount of energy increase to the pendulum system. The value of k determines how much the control favors using the maximum control input to achieve the desired energy state. This control is effective in increasing the energy of the pendulum to a desired value. When used as a swing-up control method, the desired value corresponds to the energy of the pendulum in its upright position. This will allow the switch to be triggered so that the stabilizing controller can be used to catch the pendulum and balance it around the unstable equilibrium point.

6. EXPERIMENTAL RESULTS

Results were gathered from the implementation of both swing-up control methods. Data were collected from experimental runs where each control scheme swings up the pendulum from an initially downwards position to an upright position and balances the pendulum around the unstable equilibrium point.

The heuristic controller was finely tuned to swing the pendulum by applying a constant voltage of 3.26 V. Repeated experimentation with this voltage gain showed that this controller was successful in swinging the pendulum to an upright position for the stabilizing controller to maintain the balanced state about 75% of the time. A plot of the controller output during an experimental run for the heuristic controller is shown in Figure 2.

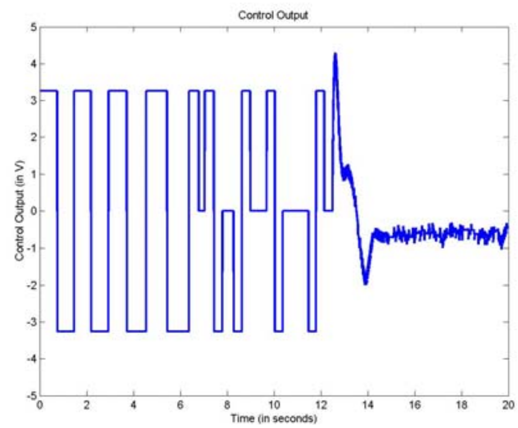

Figure 2. Plot of Control Output for the Heuristic Controller

It is important to note that the swing-up controller takes approximately 12.5 seconds to reach the upright position. The point at which the stabilizing controller catches the pendulum in the upright position is clearly displayed in the plot. Also, the control output to the cart motor alternates between 3.26 V and -3.26 V as determined by pendulum angle. At a little under 7 seconds, the control output also begins to output 0 V at small stretches of time since the pendulum angle is beyond 135° from the downwards position. Thus, it takes about another 5.5 seconds for the pendulum to get from beyond 135° from the downwards position to within 5° of the upright position.

The corresponding plot of the pendulum angle is shown in Figure 3. Each swing increases the pendulum angle slightly until the pendulum is close to its unstable equilibrium. The controller takes about 13 swings before the pendulum is close enough to the upright position for the stabilizing controller to catch it. The point in which the stabilizing controller is activated is discernible from the plot. Also, once activated, the pendulum angle remains fairly constant around the balanced position.

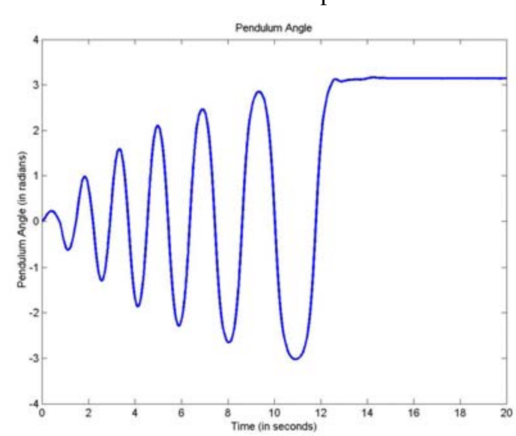

Figure 3. Plot of the Pendulum Angle for the Heuristic Controller

The energy controller is implemented with the design parameter, k, chosen to be 6.5. Also, as a result of the friction in the cart-pendulum system and the approximation made in equation (18), the desired energy was offset to a value slightly higher than 0. The appropriate offset can be determined through experimentation. In these experiments, the offset is raised to E_0 = 0.70. Repeated experimentation on the energy controller showed that this controller was reliable at least 90% of the time. A plot of the controller output during an experimental run using the energy controller is shown in Figure 4.

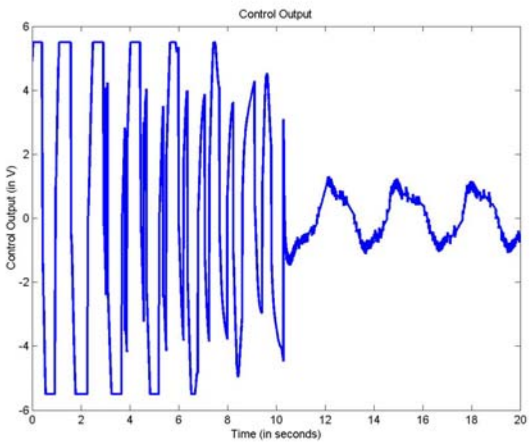

Figure 4. Plot of the Control Output for the Energy Controller

It is important to note that the energy control takes approximately 10 seconds to reach the upright position. The control output initially alternates between 5.5 V and -5.5 V since it attempts to increase the energy of the system as quickly as it possibly can by using its maximum control output (in this case, the saturation is defined to be at 5.5 V). When the pendulum is close to the upright position, the control output starts to decrease in magnitude since the control output is based on the difference between the energy of the system and the desired value. As with the heuristic controller, the point at which the stabilizing controller is activated is clearly discernible on the plot.

The corresponding plot of the pendulum angle for the energy controller is shown in Figure 5. Note that with each swing the pendulum angle is increased slightly. This controller takes about 12 swings before the pendulum is close to the upright position. It is easy to see that the stabilizing controller is able to catch the pendulum and balance it once the energy controller successfully swings the pendulum to the upright position.

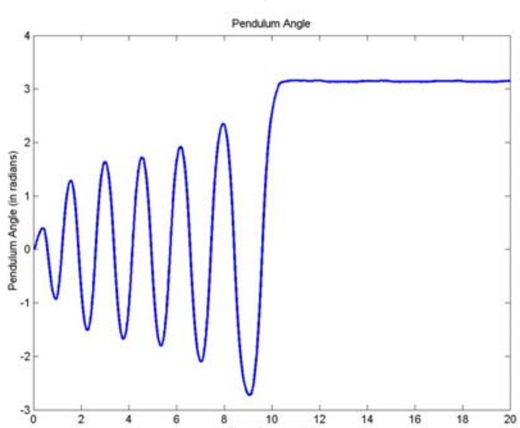

Figure 5. Plot of the Pendulum Angle for the Energy Controller

7. CONCLUSIONS

Two swing-up control schemes have been implemented that will switch to a stabilizing controller when the pendulum is near the upright position in order to balance the pendulum. Both controllers are capable of successfully swinging a pendulum from an initially downwards position to the upright position and balancing the pendulum around that point. The energy control happens to be more robust and reliable than the heuristic controller in successfully swinging the pendulum to the upright position. As the data indicates, the energy controller is also slightly faster than the heuristic controller implemented. Another advantage in the energy controller is that it is capable of reaching the upright position even if it runs out of track length and begins to run into the walls at the end of the track. The heuristic controller implemented in this paper, on the other hand, will immediately fail once the cart hits the end of the track. Both swing-up methods still require multiple swings to reach the upright position and also require a stabilizing controller to catch the pendulum in the upright position. Overall, it is seen that the energy controller is more convenient to swing up a pendulum to its unstable equilibrium than the heuristic controller. It has been shown, however, that both controllers can be effective in swinging a pendulum to the upright position from the downwards position.

8. REFERNCES

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