

# Round eliminator: a tool for automatic speedup simulation

## 1 Introduction

This tool is an implementation of the main result of [1], that states that, under some assumptions, given a problem  $\Pi$  having a distributed complexity of exactly  $T$  rounds, it is possible to automatically compute a new problem  $\Pi'$  that has a distributed complexity of exactly  $T - 1$  rounds.

This document gives a high level overview of how (and when) it is possible to use this tool to obtain, in a semi-automatic way, lower and upper bounds on the complexity of distributed problems. We will first see, in Section 2, what are the assumptions under which the result of [1] can be applied, which kind of problems can be fed into the program, and how to properly encode them. Then, in Section 3 we will see some basic examples of how to use the program. In Sections 4 and 5 we will see how to use this tool to get proper lower and upper bounds. Finally, in Section 6 we will see some additional things left out in the previous sections. Do not be scared by the length of this guide, it is full of pictures!

## 2 Assumptions

**Infinite regular trees** Consider an infinite 2-colored tree, where all white nodes have the same degree  $\Delta$ , and all black nodes have the same degree  $\delta$ . This is the most general family where the technique applies, and notice that it also includes the family of  $\Delta$ -regular trees (without the 2-coloring), by setting  $\delta = 2$  and treating black nodes as edges. This implies, for example, that we cannot apply this technique to get a lower bound for a problem that is trivial (0 rounds solvable) on trees, and that, if we use this technique to get an upper bound, the result does not directly apply to general graphs.

**Anonymous networks** The theorem of [1] assumes that the network is anonymous and that algorithms are deterministic. That is, if we get a lower bound using this technique, it does not directly apply to the LOCAL model of distributed computing, but to a much weaker setting. Usually this is not an issue: we can lift lower bounds obtained with this technique to lower bounds that apply to the LOCAL model (see e.g. [2]), also for randomized algorithms.

**Local Checkability** This is the most important assumption: our problem must be *locally checkable*. That is, there should be a set of constraints that, if they are satisfied on each node, then the solution is globally valid. In order to use the round eliminator on a problem, the only thing that we need to type is the set of local constraints. In particular, a problem  $\Pi$  is described by a triple  $(L, W, B)$ :

- $L = \{l_1, \dots, l_{|L|}\}$  is a set of labels.

- $W$  represents the white constraint. That is,  $W$  is the set of allowed white configurations. Each configuration is a multiset of size  $\Delta$ , where each element is in  $L$ .
- $B$  represents the black constraint. That is,  $B$  is the set of allowed black configurations. Each configuration is a multiset of size  $\delta$ , where each element is in  $L$ .

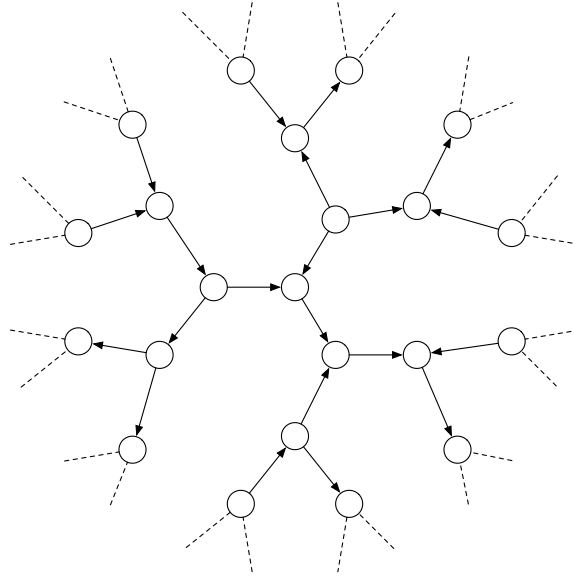
The assumption is that there is a *white* algorithm that solves the problem. That is, white nodes are active entities, and they must output a label  $l \in L$  on each incident edge (they can output different labels on different incident edges). The output is valid if the following holds:

- For each white node, consider the multiset of size  $\Delta$  containing the labels that appear on its incident edges. This multiset must be in  $W$ .
- For each black node, consider the multiset of size  $\delta$  containing the labels that appear on its incident edges. This multiset must be in  $B$ .

Note that black nodes are passive, that is, they do not output anything. We can imagine black nodes as entities that just propagate messages that white nodes send between each other. It is easier to understand this if we consider the case where black nodes have degree 2, that is, when they represent edges. In this case we have white nodes that communicate with each other, and messages pass through edges, that are passive entities.

The idea is that if we apply the technique on a problem that *white* nodes can solve in exactly  $T$  rounds, then we get a problem that can be solved in  $T - 1$  rounds by *black* nodes, that is, we save one round but we reverse the roles of active and passive. By repeating the same procedure we would get a third problem, that requires  $T - 2$  rounds where the active entities are again the *white* nodes.

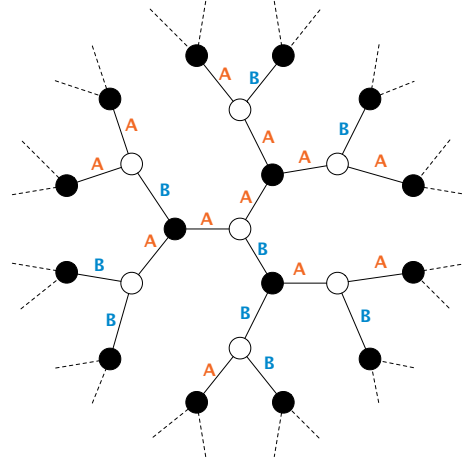
**Example: Bipartite Sinkless Orientation (BSO)** Sinkless orientation requires to orient each edge such that there are no sinks. That is, orient each edge such that each node has at least one outgoing edge. A solution looks like the following:



It is easy to see that this problem is locally checkable: each edge must be oriented, and if each node satisfies that it has at least one outgoing edge then the solution is globally valid. We will now see how to encode this problem in the 2-colored case, in particular in the case where  $\Delta = \delta = 3$ .

We need to define the set of labels  $L$ , and the set of white and black constraints,  $W$  and  $B$ . We can define  $L$  as a set containing 2 labels **A** and **B**. The label **A** will represent an edge oriented from black to white, and the label **B** will represent an edge oriented from white to black. Note that **A** and **B** are just two arbitrary names.

We now need to define the constraints. White nodes must satisfy to have at least one outgoing edge, that is an edge oriented from white to black, that is an edge labeled **B**. The other edges are unconstrained, that is, they can be oriented either ways, and thus be labeled either **A** or **B**. Thus, we can define  $W = \{\{\mathbf{B}, \mathbf{A}, \mathbf{A}\}, \{\mathbf{B}, \mathbf{A}, \mathbf{B}\}, \{\mathbf{B}, \mathbf{B}, \mathbf{B}\}\}$ . Notice that the order in the set, and in each multiset, does not matter: the set  $W$  is just the set containing all possible multisets that contain at least one **B**. Similarly, the black nodes must have at least one incident edge labeled **A**. Thus, we define  $B = \{\{\mathbf{A}, \mathbf{A}, \mathbf{A}\}, \{\mathbf{A}, \mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}, \mathbf{B}\}\}$ . A correct solution looks like the following:



In order to feed the set of constraints to the round eliminator, we use a more compact notation. In order to express the problem  $\Pi = (L, W, B)$  we will omit  $L$  (it can be inferred from  $W$  and  $B$ ), and we will write the set  $W$  as follows (each horizontal line is a tuple of  $W$ ):

<b>B</b>	<b>A</b>	<b>A</b>
<b>B</b>	<b>A</b>	<b>B</b>
<b>B</b>	<b>B</b>	<b>B</b>

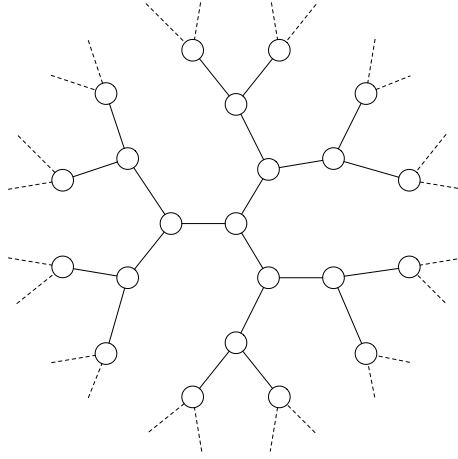
Similarly, we will write the set  $B$  as follows:

<b>A</b>	<b>A</b>	<b>A</b>
<b>A</b>	<b>A</b>	<b>B</b>
<b>A</b>	<b>B</b>	<b>B</b>

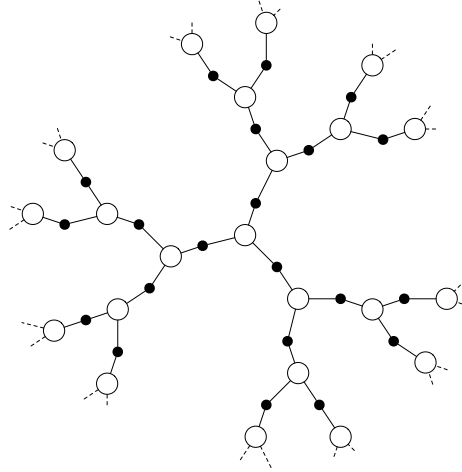
Also we will use an even more compact notation, that is, the set  $W$  will be written as  $\boxed{\mathbf{B} \ \mathbf{AB} \ \mathbf{AB}}$ , while the set  $B$  will be written as  $\boxed{\mathbf{A} \ \mathbf{AB} \ \mathbf{AB}}$ . This compact notation should be interpreted as follows: any choice that we make on each group is allowed. For example, from  $\boxed{\mathbf{B} \ \mathbf{AB} \ \mathbf{AB}}$ , we are forced to pick **B** from the first group, but we can pick either **A** or **B** from the second and the third. Also, these two notations can be mixed. For example,  $B$  could be written as follows:

A	A	AB
A	B	B

**Example: Non-bipartite Sinkless Orientation (SO)** We will now see an alternative way to encode the sinkless orientation problem, that may help to understand how to encode problems defined on non-2-colored graphs. We will assume  $\Delta = 3$  and  $\delta = 2$ . White nodes will represent proper nodes, while black nodes will represent edges. In fact, consider a 3 regular tree:



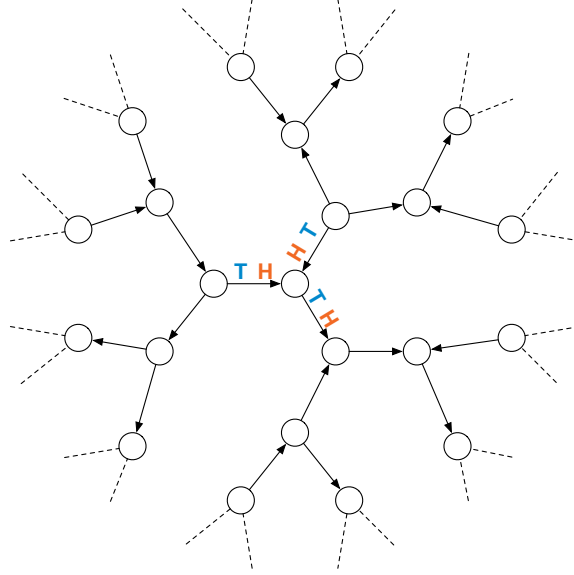
Let us add a black node in the middle of each edge, obtaining the following graph:



It should be clear that given an algorithm that runs on one graph, we can easily simulate it on the other. That is, these two graphs are basically equivalent.

As in the bipartite case, we want to orient edges of the original graph such that no node is a sink. In this case we want no sinks in the original graph (the one before adding the black nodes), but we need to define the constraints on the second graph (the one where we can apply the theorem of [1]). Thus, we can think of white constraints as *node constraints*, and black constraints as *edge constraints*.

We will have two labels, **H** and **T**, that represent, respectively, the head and the tail of an oriented edge. An edge must be properly oriented, that is,  $B = \{\{\mathbf{H}, \mathbf{T}\}\}$ . In other words, if a white node writes **H** on its incident edge, the other white node that is incident to the same edge must write **T**. In other words, each black node is a neighbor of exactly two white nodes, and it must see different labels. Also, white constraints must satisfy that each white node is incident to at least one **T**, that is  $W = \{\{\mathbf{T}, \mathbf{H}, \mathbf{H}\}, \{\mathbf{T}, \mathbf{H}, \mathbf{T}\}, \{\mathbf{T}, \mathbf{T}, \mathbf{T}\}\}$ . A solution looks like the following:



A compact way of writing the constraints is the following. Node constraints are  $\boxed{\mathbf{T} \mathbf{H} \mathbf{T} \mathbf{H} \mathbf{T}}$ , while edge constraints are  $\boxed{\mathbf{H} \mathbf{T}}$ . If we are given a solution for the problem encoded in this way, we can obtain an output for the sinkless orientation problem by orienting the edges as indicated by **T** and **H**. Also, given an edge orientation, we can easily produce a solution for the problem that we encoded. Thus, the problem that we encoded is equivalent to SO.

**Example: Maximal Independent Set (MIS)** While sinkless orientation is a problem that we could either define on 2-colored graphs or on non-2-colored graphs, for MIS it really only makes sense to define it on non-2-colored graphs. In fact, nothing stops us from defining it on 2-colored graphs, but we would obtain a 0-round solvable problem.

Let us see how to define this problem on 3-regular trees with node and edge constraints, that is, let us assume that  $\Delta = 3$  and  $\delta = 2$ . One may think that we could use 2 labels, one used by nodes that are in the MIS, and the other used by nodes that are not in the MIS. This is actually not possible [2], and the problem is that in order to express *maximality*, we need to somehow ensure that if a node is not in the MIS, then at least one neighbor is in the MIS, and this requires an additional label.

The idea is that we use 3 labels, **M**, **O** and **P**. Being in the MIS is encoded by having **M** on each incident edge. Nodes that are not in the MIS will need to *point* to one neighbor that is in the MIS, that is, they will output **P** on one edge and **O** on all the others. That is,  $W$  is defined as follows:

<b>M</b>	<b>M</b>	<b>M</b>
<b>P</b>	<b>O</b>	<b>O</b>

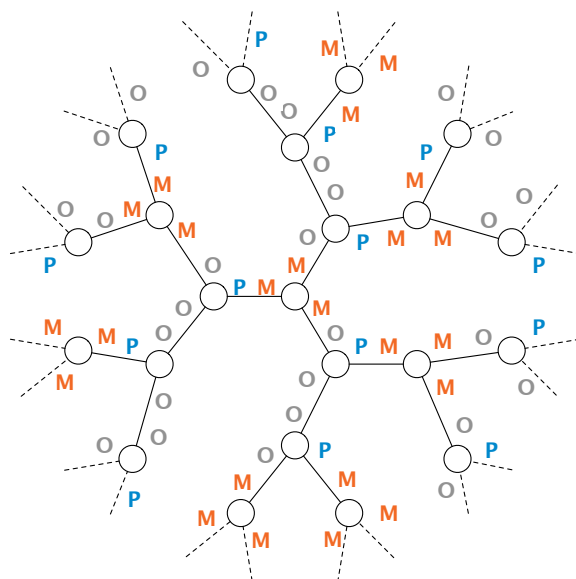
Let us now define edge constraints. The idea is that a pointer must point to a node in the MIS, so, on an edge, **P** can be compatible only with **M**. Also two neighboring nodes must not be both in the MIS, so **M** must *not* be compatible with itself. Finally, if a node is not in the MIS, and it already pointed to some neighbor with a **P**, then all other neighbors can be anything, so **O** can be compatible with both **O** and **M**. This gives the following set  $B$ .

<b>M</b>	<b>P</b>
<b>O</b>	<b>OM</b>

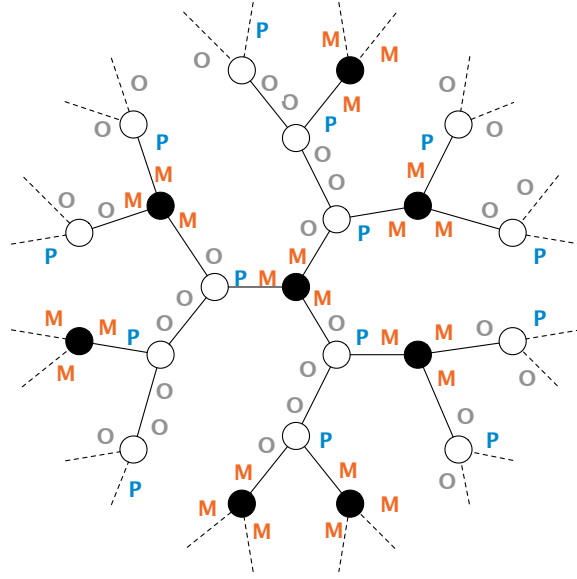
Please note that there is no unique way of writing these constraints. In fact, an alternative way is the following:

<b>M</b>	<b>OP</b>
<b>O</b>	<b>O</b>

An example of a solution is the following.



This labeling represents the following MIS, where nodes of the MIS are colored black.



Note that what we defined is not exactly the MIS problem. In fact, given an MIS, it requires exactly one round for nodes not in the MIS to know where to point, and thus output a valid solution. Nevertheless, since the complexity of the two problems differs by at most 1 round, if we prove a lower bound for one problem, we directly get a lower bound for the other as well.

**Example: Bipartite Maximal Matching (BMM)** We have a bipartite 2-colored graph, and we want to choose a subset of edges that are independent, such that this set cannot be extended. Again, 2 labels are not enough to encode the problem: if a node has no incident edges in the matching, then *all* neighbors must have one edge in the matching, so we need to use pointers like in the MIS case. The white constraints are defined as follows.

<b>M</b>	<b>O</b>	<b>O</b>
<b>P</b>	<b>P</b>	<b>P</b>

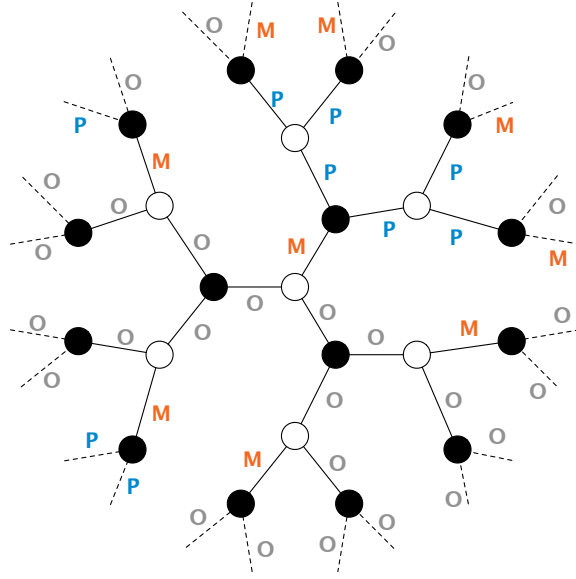
That is, a white node either outputs **M** on an edge (the edge in the matching) and **O** on all the others, or it outputs **P** on all incident edges (all black neighbors must be matched).

Let us now define the black constraints. The idea is that a black node can accept a pointer only if it has an edge in the matching, that is, **P** can appear only if **M** appears as well. Also, two **M**s cannot appear together. Thus, the black constraint is defined as follows.

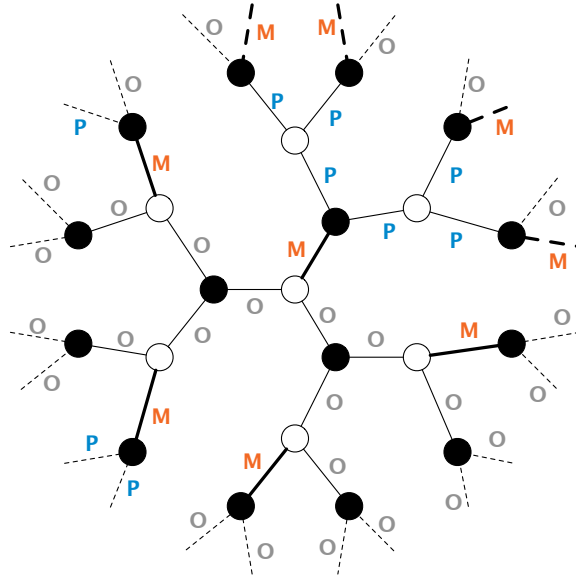
<b>M</b>	<b>OP</b>	<b>OP</b>
<b>O</b>	<b>O</b>	<b>O</b>

We can see that the independence of the edges is guaranteed by the fact that for both white and black nodes it holds that at most one incident edge is labeled with **M**. Then, maximality for white nodes is guaranteed by the fact that if a node has no incident edges labeled **M**, then it is outputting **P** everywhere, and these **P**s are accepted by black nodes only if they are matched. Similarly, maximality for black nodes is guaranteed by the fact that if a black node is unmatched, it writes **O** everywhere, and a white node accepts an **O** only if it also has an **M**.

An example of a solution is the following.



This labeling represents the following Maximal Matching.



**Example:  $\Delta + 1$  Vertex Coloring** This problem makes sense on non-2-colored graphs. Thus we will see an example where  $\Delta = 3$  and  $\delta = 2$ , that is, white nodes represent nodes and black nodes represent edges. We will use 4 labels, **R**, **G**, **B**, and **Y**, that correspond to the 4 possible colors. Nodes must output a color consistently, that is, they need to write the same color on each incident edge. That is,  $W$  is defined as follows.

<b>R</b>	<b>R</b>	<b>R</b>
<b>G</b>	<b>G</b>	<b>G</b>
<b>B</b>	<b>B</b>	<b>B</b>
<b>Y</b>	<b>Y</b>	<b>Y</b>

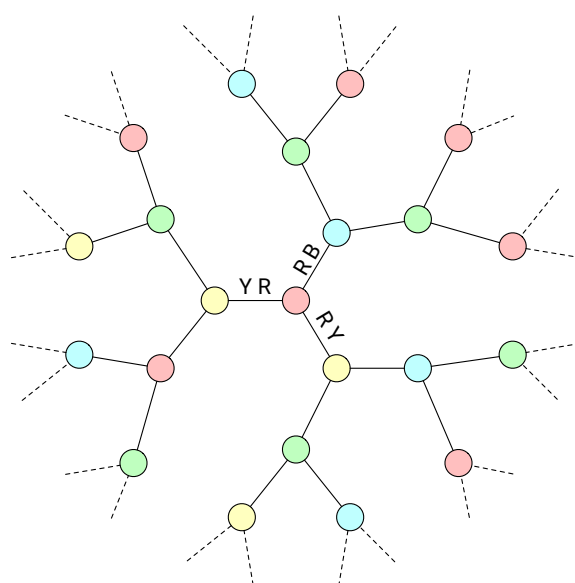


Then, edges must satisfy that if on one side there is one color, on the other side there must be a different color. Thus,  $B$  is defined as follows.

<b>R</b>	<b>GBY</b>
<b>G</b>	<b>RB Y</b>
<b>B</b>	<b>RGY</b>
<b>Y</b>	<b>RGB</b>

Note that some options are repeated multiple times (for example **R** with **B** on the first line, and **B** with **R** on the third line). This is an allowed input for the round eliminator and does not make any difference.

An example of a solution, with the corresponding coloring, is the following:



**Giving inputs to the round eliminator** By opening the round eliminator, we get the following interface.

Active

Passive

Start

Clear

There are two input areas, the left one corresponds to the constraints for the active side, that at the beginning corresponds to the white nodes, while the right one corresponds to the constraints for the passive side, that at the beginning corresponds to the black nodes. Also, there is a button, “Start”, that can be used to send the entered input to the program. For example, we can write **B AB AB** on the left, and **A AB AB** on the right, and press the button “Start”. This gives the BSO problem as input to the round eliminator. The program will parse this input and output some useful information about the problem.

Initial problem.

Close

The problem is NOT zero rounds solvable.

Active

Any choice satisfies previous Passive

AB AB B


Passive

Exists choice satisfying previous Active

AB AB A

Diagram

Strength of right side labels



Tools

Speedup, edit, simplifications, ...

Speedup Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

In particular, please note the following parts:

- The program detected that this problem is not 0 rounds solvable.
- The active and passive constraints are just printed back.
- The diagram.
- The tools, that will be explained in the next sections. They are necessary to perform operations on the problem.

The diagram is a useful part of the interface, that can be used to better understand the *strength* of the labels. For example, for the BSO problem, we can see that there is a direct edge from the label **B** to the label **A**. This means that, from the point of view of the passive side (the black constraints), each time the label **B** is allowed, the label **A** is allowed as well. In more detail, if some configuration containing a **B** is in the black constraint, we can replace **B** with **A** and obtain a new configuration that is also in the black constraint.

Sometimes we want to be more explicit about the labels, and instead of just using arbitrary letters, we prefer to give more sensible names. This can be done in the round eliminator by delimiting a label with parentheses. For example, while “**AB**” basically means “**A** or **B**”, we could write “(**AB**)” and this would represent a single label. To give another example, in order to encode BSO, we could use the two labels “**B->W**” and “**W->B**”, and give the following input to the round eliminator (please notice the two spaces between the three groups in each input box).

Active

(W->B) (B->W)(W->B) (B->W)(W->B)

Passive

(B->W) (B->W)(W->B) (B->W)(W->B)

Start Clear

### 3 Basics

In order to play with the round eliminator, it is not really necessary to know the details of [1], but once we get a nice result, for example a nice lower bound for BMM for some specific values of  $\Delta$  and  $\delta$ , we would like to get a general proof for any value of  $\Delta$  and  $\delta$ . For this purpose, it is necessary to know how the problem  $\Pi'$  (the problem that requires 1 round less to be solved) is defined as a function of  $\Pi$  (the original problem). Thus, we will now briefly and informally see how  $\Pi'$  is defined, and an example of how to compute  $\Pi'$  starting from  $\Pi$ .

**The problem  $\Pi'$**  We are given a problem  $\Pi = (L, W, B)$  having complexity  $T$  rounds, for some unknown value of  $T$ , where white nodes are the active entity. We want to compute a problem  $\Pi' = (L', B', W')$  of complexity  $T - 1$  rounds, where the black nodes are the active entity. Let us start with the new label set  $L'$ . The new labels are all the possible sets of old labels, excluding the empty set. That is,  $L' = 2^L \setminus \{\{\}\}$ . Then, the set  $B'$  is the *maximal* set satisfying the following:

$$\forall Y = \{Y_1, \dots, Y_d\} \in B', \forall (y_1, \dots, y_d) \in Y_1 \times \dots \times Y_d : \{y_1, \dots, y_d\} \in B$$

Finally, the set  $W'$  is the *maximal* set satisfying the following:

$$\forall X = \{X_1, \dots, X_\Delta\} \in W', \exists (x_1, \dots, x_\Delta) \in X_1 \times \dots \times X_\Delta : \{x_1, \dots, x_\Delta\} \in W$$

In other words, the new black constraints ask black nodes to output a set on each incident edge, and these sets must satisfy that for any choice that we make by picking one element for each set, this choice should be part of the original black constraints. Similarly, the new white constraints ask that the sets present on the incident edges of white nodes (given as output by the black nodes, since in the new problem white nodes are passive) satisfy that there exists a choice that is part of the original white constraints. The reason for why this problem has complexity exactly  $T - 1$  rounds can be found in [1].

**Manually computing  $\Pi'$**  We will now see an example of how to compute  $\Pi'$ , assuming that  $\Pi$  is the BSO problem. Recall that:

$$\begin{aligned} L &= \{\mathbf{A}, \mathbf{B}\} \\ W &= \{\{\mathbf{B}, \mathbf{A}, \mathbf{A}\}, \{\mathbf{B}, \mathbf{A}, \mathbf{B}\}, \{\mathbf{B}, \mathbf{B}, \mathbf{B}\}\} \\ B &= \{\{\mathbf{A}, \mathbf{A}, \mathbf{A}\}, \{\mathbf{A}, \mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}, \mathbf{B}\}\} \end{aligned}$$

First of all,  $L' = \{\{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\}$ . In order to compute  $B'$ , we start by listing all possible multisets using labels of  $L'$ , and then exclude the invalid ones. The candidates are:

$$\begin{array}{lll} \{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{A}\}\} & \{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}\} \\ \{\{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} \\ \{\{\mathbf{B}\}, \{\mathbf{B}\}, \{\mathbf{B}\}\} & \{\{\mathbf{B}\}, \{\mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} & \{\{\mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} \\ \{\{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} & & \end{array}$$

We now need to keep only candidates for which any possible choice is in the set  $\mathbf{B}$ . For example,  $\{\{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\}$  is invalid, since it is possible to pick  $\mathbf{B}$  from each subset, and  $\{\mathbf{B}, \mathbf{B}, \mathbf{B}\}$  is not in  $B$ . The valid multisets are:

$$\begin{array}{lll} \{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{A}\}\} & \{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}\} \\ \{\{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} \end{array}$$

Now, we could just rename the sets to obtain something more readable. For example, let  $\mathbf{X} = \{\mathbf{A}\}$ ,  $\mathbf{Y} = \{\mathbf{B}\}$ ,  $\mathbf{Z} = \{\mathbf{A}, \mathbf{B}\}$ . We would get that  $B'$  is  $\boxed{\mathbf{X} \mathbf{X} \mathbf{Y} \mathbf{Z} \mathbf{X} \mathbf{Y} \mathbf{Z}}$ . Instead, we will first get rid of more multisets, without changing the complexity of the problem. In fact, one can prove that if a multiset is not *maximal*, then it is not needed [1]. For example, consider the two multisets  $\{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}\}$  and  $\{\{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\}$ . We can find a perfect matching between the sets of the first and the second, such that each set of the first is included in its matched set. For this reason, we can safely ignore  $\{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}\}$ . By removing the non maximal multisets, we can notice that only one remains:

$$\{\{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\}$$

Notice that the set  $\{\mathbf{B}\}$  never appears, so we can actually redefine the set  $L'$  to contain only  $\{\mathbf{A}\}$  and  $\{\mathbf{A}, \mathbf{B}\}$ . By using the aforementioned renaming we get that the new black constraints  $B'$  are just  $\boxed{\mathbf{X} \mathbf{Z} \mathbf{Z}}$ .

Let us now see how to compute  $W'$ . Again, we can start from all the possible candidates:

$$\begin{array}{lll} \{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{A}\}\} & \{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}\} \\ \{\{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} \\ \{\{\mathbf{B}\}, \{\mathbf{B}\}, \{\mathbf{B}\}\} & \{\{\mathbf{B}\}, \{\mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} & \{\{\mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} \\ \{\{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} & & \end{array}$$

We now need to keep only candidates for which there exists a choice that is in the set  $W$ . In other words, we need to exclude all candidates for which it is not possible to pick at least one  $\mathbf{B}$ . The only bad candidate is  $\{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{A}\}\}$ . Thus, we get the following:

$$\begin{array}{lll} \{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{B}\}\} \\ \{\{\mathbf{A}\}, \{\mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} & \{\{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} & \{\{\mathbf{B}\}, \{\mathbf{B}\}, \{\mathbf{B}\}\} \\ \{\{\mathbf{B}\}, \{\mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} & \{\{\mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} & \{\{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} \end{array}$$

Now, since  $\{\mathbf{B}\}$  never appears in  $B'$ , it never makes sense to have it in  $W'$ , so we can remove all candidates containing  $\{\mathbf{B}\}$ . Thus, we get the following:

$$\{\{\mathbf{A}\}, \{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}\} \quad \{\{\mathbf{A}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\} \quad \{\{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}, \{\mathbf{A}, \mathbf{B}\}\}$$

By renaming, we get that the new white constraints  $W'$  are  $\boxed{\mathbf{Z} \mathbf{X} \mathbf{Z} \mathbf{X} \mathbf{Z}}$ .

**Automatically computing  $\Pi'$**  After having given in input a problem to the round eliminator, we can press the button “Speedup” to let the program compute the new problem  $\Pi'$  automatically. For example, starting from the BSO problem, thus by writing  $\boxed{\mathbf{B} \mathbf{A} \mathbf{B} \mathbf{A} \mathbf{B}}$  on the left and  $\boxed{\mathbf{A} \mathbf{A} \mathbf{B} \mathbf{A} \mathbf{B}}$  on the right, and then pressing “Start”, and then “Speedup”, we get the following result:

Performed speedup.

Close

The problem is NOT zero rounds solvable.

**Active (Before Renaming)**

Any choice satisfies previous Passive

**Active**

Any choice satisfies previous Passive

B   B   A

**Passive (Before Renaming)**

Exists choice satisfying previous Active

**Passive**

Exists choice satisfying previous Active

AB   AB   B

**Renaming**

Old and new labels

**Diagram**

Strength of right side labels

**Tools**

Speedup, edit, simplifications, ...

**Speedup** **Edit**

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

New Renaming

This is the same result that we obtained manually, just with a different renaming. By default, the sets are just renamed in alphabetical order. In fact, we can suggest to the round eliminator a different renaming, by pressing “New Renaming”. We can for example suggest the same renaming used before:

**New Renaming**

**Rename**

By pressing “Rename” we get the following result:

Renaming.

Close

The problem is NOT zero rounds solvable.

Active (Before Renaming)

Any choice satisfies previous Passive

Active

Any choice satisfies previous Passive

Z    Z    X

Passive (Before Renaming)

Exists choice satisfying previous Active

Passive

Exists choice satisfying previous Active

XZ    XZ    Z

Renaming

Old and new labels

Diagram

Strength of right side labels

Tools

Speedup, edit, simplifications, ...

Speedup Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

New Renaming

If we are interested in seeing the result before the automatic renaming, or in just seeing what is the renaming used by the program, we can press either “Active (Before Renaming)”, “Passive (Before Renaming)”, or “Renaming”. A set of labels will be represented by enclosing multiple labels in a box. We get the following:

Renaming.

Close

The problem is NOT zero rounds solvable.

Active (Before Renaming)

Any choice satisfies previous Passive

Active

Any choice satisfies previous Passive

AB

AB

A

Passive (Before Renaming)

Exists choice satisfying previous Active

Passive

Exists choice satisfying previous Active

A

AB

A

AB

AB

Renaming

Old and new labels

A

AB

x

z

Active

Any choice satisfies previous Passive

Passive

Exists choice satisfying previous Active

Diagram

Strength of right side labels

Tools

Speedup, edit, simplifications, ...

If we are interested in modifying the current problem, we can press the button “Edit”. By doing so, we can notice that the input areas (that can be found by scrolling again to the top) now contain the problem  $\Pi'$ :

**Active**

Z Z X

**Passive**

XZ XZ Z

Start
Clear

## 4 Lower Bounds

We have seen how to feed a problem to the round eliminator and how to get, by pressing a button, a problem that is exactly one round easier. It is time to see how to use this tool to get proper lower bounds.

**Bipartite Sinkless Orientation** Start from  $\Pi_0 = \text{BSO}$ , that is, by writing B AB AB on the left side, and A AB AB on the right side. Press “Start” and then “Speedup”. What is shown is the problem  $\Pi_1$ , that has complexity exactly one round less than BSO. On one side we have the constraint A B B, while on the other side we have the constraint B AB AB. Let us press again the button “Speedup”. We get the following:

Performed speedup.

Close

The problem is NOT zero rounds solvable.

Active (Before Renaming)

Any choice satisfies previous Passive

Passive (Before Renaming)

Exists choice satisfying previous Active

Renaming

Old and new labels

Active

Any choice satisfies previous Passive

B   B   A

Passive

Exists choice satisfying previous Active

AB   AB   B

Diagram

Strength of right side labels

B

A

Tools

Speedup, edit, simplifications, ...

Speedup
Edit

Simplifications
Harden v1
Harden v2
Automatic Lower Bound
Automatic Upper Bound
New Renaming

This is not an error, we got a problem  $\Pi_2$  that is exactly the problem  $\Pi_1$  on which we pressed the button! In other words,  $\Pi_1$  is a fixed point: by applying the round elimination technique on this problem we get back to the same problem. This is somehow a contradiction, how can  $\Pi_1$  and  $\Pi_2$  have the same complexity?  $\Pi_2$  should require exactly one round less than  $\Pi_1$ . The solution to this enigma is that, in order to apply the technique, we need to assume that  $T = o(\log n)$ , and a fixed point basically shows that under this assumption we get a contradiction. In other words, a fixed point given by a non 0-rounds solvable problem directly implies a lower bound of  $\Omega(\log n)$ .

Typically, this kind of lower bounds can be lifted to the LOCAL model to obtain an  $\Omega(\log n)$  lower bound for deterministic algorithms and an  $\Omega(\log \log n)$  lower bound for randomized algorithms (see e.g. the FDSO problem in [2]).

**Sinkless Orientation (SO)** Sometimes we cannot hope to get lower bounds on the bipartite case, since many problems are trivial given a bipartition. Also, it is not possible to get fixed points directly in the non bipartite case, since  $\delta = 2$  but typically  $\Delta > 2$ . So the resulting problem cannot be the same as the original problem, since at least the degrees differ. Nevertheless we can get something very similar to a fixed point, that is a cycle, and SO is an example of such behavior. Start with  $\Pi_0 = \text{SO}$ : **T HT HT** on the left side, and **H T** on the right side. Press “Start”. Press the button “Speedup” *three* times (each time on the new obtained problem). Note that the last problem that we get,  $\Pi_3$ , is different from the previous one,  $\Pi_2$ , but is the same as  $\Pi_1$ . That is, we did not get a fixed point, but something very similar. We have a problem  $\Pi_3$  that is 2 (half) rounds easier than itself! For the same reasoning used in the BSO case, this directly gives an  $\Omega(\log n)$  lower bound.

**Bipartite Sourceless and Sinkless Orientation (BSSO)** Fixed points are good. If we get a fixed point we get a lower bound of  $\Omega(\log n)$  for free. Do we get a fixed point for any problem that is  $\Omega(\log n)$ ? Unfortunately no, we do not. Understanding when and why we get fixed points is an open question. In the BSSO problem, each node must have at least one incoming *and* one outgoing edge. BSSO is an example of a problem for which we know a lower bound of  $\Omega(\log n)$  (since it is at least as hard as BSO), but it does not give a fixed point. It can be described by having both white and black constraint equal to **A B AB**. If we feed this problem to the round eliminator, and press “Speedup” multiple times (each time on the new obtained problem), we can see that we do *not* get a fixed point. Instead, each time we get a new problem that uses one label more than the previous one. Here is an example of the result after 8 steps:

Performed speedup.

Close

The problem is NOT zero rounds solvable.

Active (Before Renaming)

Any choice satisfies previous Passive

Passive (Before Renaming)

Exists choice satisfying previous Active

Renaming

Old and new labels

Active

Any choice satisfies previous Passive

J	J	A
J	I	B
J	H	C
J	G	D
J	F	E

Passive

Exists choice satisfying previous Active

ABCDEFGHJIJ	BCDEFGHIJ	J
ABCDEFGHJIJ	CDEFGHIJ	IJ
ABCDEFGHJIJ	DEFGHIJ	HIJ
ABCDEFGHJIJ	FGHIJ	EGHIJ

Diagram

Strength of right side labels

Tools

Speedup, edit, simplifications, ...

Speedup

Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

New Renaming



How can we prove lower bounds for this kind of problems? One way would be to characterize all problems  $\Pi_i$ , and prove that none of them is 0 rounds solvable. While this may be doable for BSSO, this may be very challenging for more complicated problems. An alternative way to get a lower bound for a problem  $\Pi$  is to try to relax the problem (that is, make it easier) obtaining a new problem  $\Pi'$ , and hope that  $\Pi'$  gives a fixed point. This would directly imply a lower bound for the harder problem  $\Pi$ . For example, it is easy to do so for BSSO: we can relax it to BSO! More generally, what we can always do, is to add more configurations to the black and white constraints, and this can only make the problem easier. In particular, if we add  $\boxed{\mathbf{B\ B\ B}}$  to the white constraint of BSSO, and add  $\boxed{\mathbf{A\ A\ A}}$  to the black constraint, we get the BSO problem, for which we know a lower bound.

**Making a problem too easy** Can we always get a fixed point for problems that are  $\Omega(\log n)$  by relaxing them? We do not know! What we know is that if we make the wrong relaxations we get a problem that is too easy, and thus we do not get a lower bound. For example, for BSSO, if instead of adding  $\boxed{\mathbf{A\ A\ A}}$  to one side and  $\boxed{\mathbf{B\ B\ B}}$  to the other we add  $\boxed{\mathbf{A\ A\ A}}$  to both sides, we get a 0 rounds solvable problem (in fact, if all nodes output  $\boxed{\mathbf{A\ A\ A}}$  we get a solution that satisfies both white and black constraint). The round eliminator would inform us about that:

Initial problem.

Close

The problem is zero rounds solvable.

Active

Any choice satisfies previous Passive

AB AB A


Passive

Exists choice satisfying previous Active

AB AB A

Diagram

Strength of right side labels



Tools

Speedup, edit, simplifications, ...

Speedup

Edit

Simplifications

Harden v1

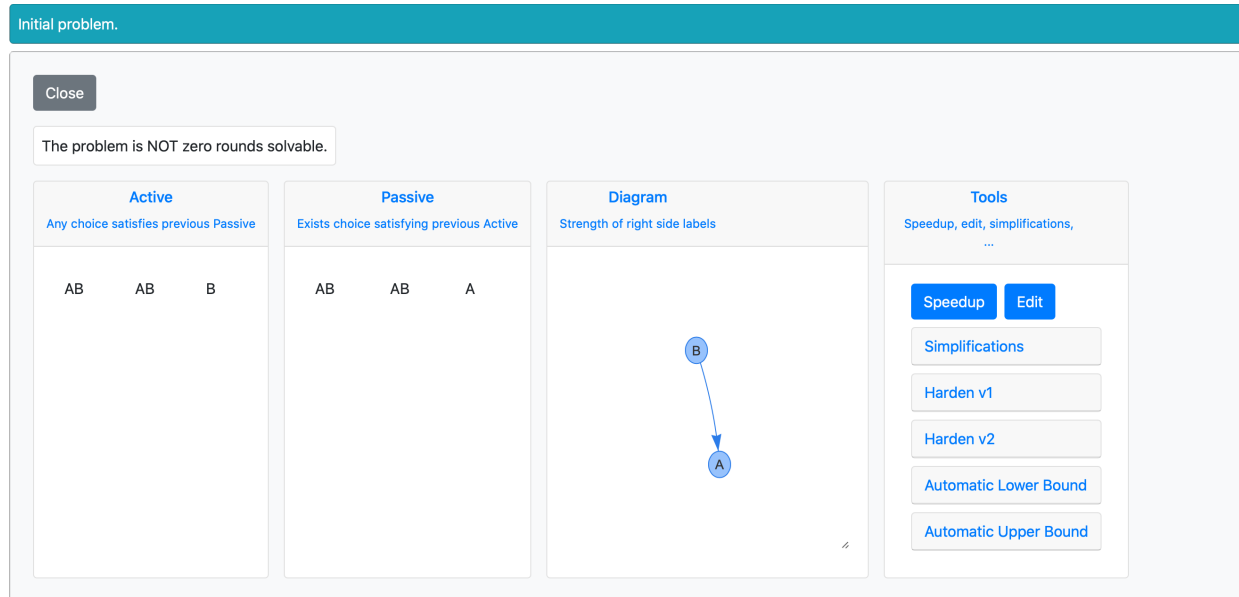
Harden v2

Automatic Lower Bound

Automatic Upper Bound

Thus, if we are aiming to prove an  $\Omega(\log n)$  lower bound, one possible way is to try to relax the problem and hope to get a fixed point, while trying to avoid to relax it too much and get a trivial problem.

**Diagram** Before seeing more techniques that can be used to prove lower bounds, we need to understand the diagram associated to a problem. For example, by giving BSO as input, we get the following:



If we look at the diagram, we can see that there is a direct edge from the label **B** to the label **A**. This means that, on the passive side (the black constraint), each time the label **B** is allowed, the label **A** is allowed as well. In more detail, if some configuration containing a **B** is in the black constraint, we can replace **B** with **A** and obtain a new configuration that is also in the black constraint. This diagram is important for multiple reasons.

We know that the labels of the new problem are a subset of all possible sets of old labels. We know something more: one can prove that we do not have to consider all possible sets, we only have to consider *right closed subsets* with regards to the diagram of the original problem [1]. In other words, if on the passive side a label **B** is allowed each time a label **A** is allowed, then the sets of the new problem will satisfy that if a set contains **B**, then it must contain **A** as well. That's the reason why, by applying the round elimination technique on the BSO problem, we do *not* get the set  $\{\mathbf{B}\}$  as a new label. Instead, it disappears, and we only get  $\{\mathbf{A}\}$  and  $\{\mathbf{A}, \mathbf{B}\}$ . Thus, we can somehow control the number of labels of the new problem, by trying to limit the number of right closed subsets of the old problem.

Another reason why we care about the diagram is the following. Consider the following example:

Initial problem.

Close

The problem is NOT zero rounds solvable.

The following labels could be merged without changing the complexity of the problem: BC

Active

Any choice satisfies previous Passive

ABC      B      A

Passive

Exists choice satisfying previous Active

BC      ABC      BC

Diagram

Strength of right side labels

Tools

Speedup, edit, simplifications, ...

Speedup

Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

This is an arbitrary artificial problem that is at least as hard as BSO. We can notice that there is a bidirectional edge between the labels  $B$  and  $C$ , and the round eliminator suggests that these labels are equivalent. Why are they equivalent? It is true that, from the point of view of black nodes, each time  $B$  is allowed,  $C$  is allowed as well, and vice versa, but the same is not true for the white nodes. A white node can have 2 edges labeled  $B$ , but only one labeled  $C$ . Nevertheless, if we define a new problem where we “identify” the labels  $B$  and  $C$ , this will have the same complexity as the original problem. The idea is that, since black nodes do not really care whether white nodes write  $B$  or  $C$  (remember that white nodes are the only ones writing outputs), then in 0 rounds we could transform the solution of one problem to a solution for the other problem (and this really works in both ways). This merging of labels can be achieved by pressing “Simplifications” and then either the button “ $B \rightarrow C$ ” or the button “ $C \rightarrow B$ ”. We obtain a problem that uses fewer labels, and it has the same complexity as the original one. This is the best thing we could wish for, since, as we will see later, having few labels helps in speedup process.

**Guaranteed Fixed Point** Should we just try to randomly add allowed configurations until we get a fixed point? Well, that may work, but it may also require a lot of time. The main issue is that we really do not know why and when we get fixed points. We know something, though. We know that the labels of the new problem can only be *right closed subsets* with regards to the diagram. This in turn implies that, if a problem uses  $k$  labels, and these  $k$  labels form a diagram that is a directed path, we get the following: the number of new labels is also upper bounded by  $k$ , since that is the number of right closed subsets in the directed path. The new diagram may not look like a directed path anymore, but one can prove that by applying the round elimination technique again, what we get as result is again a problem having at most  $k$  labels and a directed path as diagram. In other words, these problems behave really nicely using the round elimination technique. That is, they do not *grow*. We can press “Speedup” as many times as we want, and we will always obtain a problem with at most  $k$  labels. Since their number of labels is bounded, by keeping pressing “Speedup”, we cannot get a new problem forever: at some point we must get an old problem, that is, we get a cycle. Basically, for these problems we are guaranteed to find a fixed point. Does this mean that all problems of this kind are  $\Omega(\log n)$ ? Unfortunately not: during this process we may obtain a

0 rounds solvable problem, and get a fixed point only after this. An example is the problem having constraint “A AB AB” on both sides: if we feed this into the round eliminator we notice two things:

- It is 0 rounds solvable.
- The diagram is a directed path of length 2 (that is equal to the number of labels).

Initial problem.

Close

The problem is zero rounds solvable.

Active

Any choice satisfies previous Passive

AB    AB    A


Passive

Exists choice satisfying previous Active

AB    AB    A

Diagram

Strength of right side labels



Tools

Speedup, edit, simplifications, ...

Speedup
Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

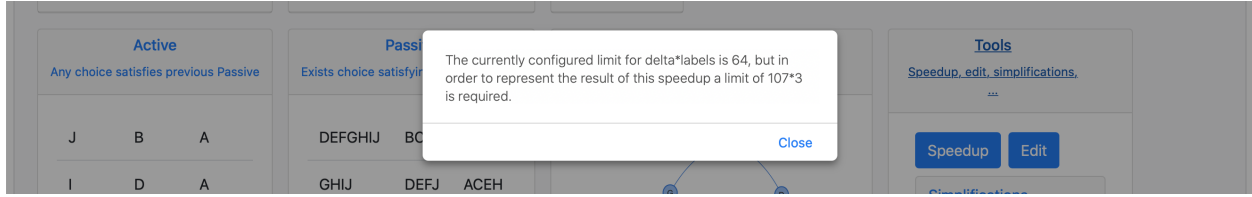
By pressing “Speedup” multiple times, we can notice that  $\Pi_3 = \Pi_2$ , that is, we got a fixed point, but it is a fixed point given by a 0 round problem, thus we got no lower bound for free.

Can we get fixed points only for problems having a path as diagram? No, there are others, and an example can be found in [2], where a problem called FDSO is defined. That problem gives a fixed point, but the diagram is a square. Understanding why we get fixed points is an open question, and answering this question may really help in proving lower bounds.

**Bipartite  $2\Delta - 2$  edge coloring** We now know a class of problems that are guaranteed to give a fixed point (sometimes a useless one, though). Can we use this knowledge to prove a lower bound for problems that are *not* in this class? An example of a problem that does not give a fixed point, and that is hard to handle, is the bipartite  $2\Delta - 2$  edge coloring, that can be encoded (for  $\Delta = \delta = 3$ ) by using the following constraint for both white and black nodes.

<b>A</b>	<b>B</b>	<b>CD</b>
<b>A</b>	<b>C</b>	<b>D</b>
<b>B</b>	<b>C</b>	<b>D</b>

In fact, by feeding this problem to the round eliminator, and by pressing “Speedup” twice, we get an error, saying that the resulting problem is too big to be handled by the program, as it contains 107 different labels. While the round eliminator can be configured to handle more labels, it may be hopeless to try to read a huge resulting problem and try to understand it.



Instead, we could try a different approach. We could try to add allowed configurations and hope for a fixed point. We now know a class of problems that are guaranteed to give a fixed point, the ones where the diagram contains all labels and forms a directed path. What we could try to do is to add configurations, in such a way that the diagram changes shape, and becomes one that looks like the desired path. Let us recap:

- We have a problem that uses 4 labels, its diagram does not contain any edge.
- We want to obtain a diagram that is a directed path of 4 nodes.

One way to do that is the following: we assign an arbitrary order to the labels, such as  $\mathbf{A} < \mathbf{B} < \mathbf{C} < \mathbf{D}$ . We now process labels in order. We relax the constraint such that each time  $\mathbf{A}$  is allowed,  $\mathbf{B}$  is allowed as well, thus, each time there is an  $\mathbf{A}$ , we add a  $\mathbf{B}$ . This will ensure that the edge  $\mathbf{A} \rightarrow \mathbf{B}$  is in the diagram. We repeat the same procedure for the other labels. Each time  $\mathbf{B}$  appears, we add the label  $\mathbf{C}$ . Each time the label  $\mathbf{C}$  appears, we add the label  $\mathbf{D}$ . This procedure must be done only on the passive side (since the diagram depends only on that). Thus, we obtain the following problem. The white constraint is the same as before, while the black constraint is the following:

ABCD	BCD	CD
ABCD	CD	D
BCD	CD	D

By performing this simplification, we are sure that the diagram contains a path  $\mathbf{A} \rightarrow \mathbf{B} \rightarrow \mathbf{C} \rightarrow \mathbf{D}$ . It may actually contain more edges, in the case where, by adding allowed configurations, we made two labels interchangeable (from the point of view of the passive side). This is not an issue, the guaranteed fixed point still applies. This is what we get by feeding the new problem to the round eliminator.

Initial problem.

Close

The problem is NOT zero rounds solvable.

The following labels could be merged without changing the complexity of the problem: CD

Active

Any choice satisfies previous Passive

CD	B	A
D	C	A
D	C	B

Passive

Exists choice satisfying previous Active

CD	BCD	ABCD
----	-----	------

Diagram

Strength of right side labels

Tools

Speedup, edit, simplifications, ...

Speedup
Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

We can note that, at least at this point, the problem is still not 0 rounds solvable. Also, we can note that the diagram is not exactly a path, since labels **C** and **D** became equivalent. The round eliminator actually warns about that. We could merge these labels, but it is not necessary. We can now press “Speedup” multiple times, and note that we get a fixed point, and it is the same fixed point that we obtained with BSO. By generalizing this reasoning for all values of  $\Delta$ , we get that  $2\Delta - 2$  edge coloring requires  $\Omega(\log n)$ , even if nodes are 2-colored.

**Simplifications** Sometimes we know that a problem can be solved in  $o(\log n)$ , so it does not make sense to try to get a fixed point. Sometimes we just want to prove some  $\Omega(f(\Delta))$  lower bound, for some unknown function  $f$  that we want to infer. For example, we may want to prove a linear in  $\Delta$  lower bound for some local problem. One thing that we could try to do is to press the “Speedup” button  $\Delta$  times and see if the result that we get is a problem that cannot be solved in 0 rounds. Unfortunately, it often happens that the problem that we get after pressing “Speedup” is much larger than the original one. In fact, the new number of labels can be exponentially larger than the number of labels of the original problem, in the worst case. This creates multiple issues:

- The round eliminator could become too slow, if we give it a problem with too many labels.
- Even if we get some meaningful result, it could be so large that it is hopeless to try to read it, understand it, and try to generalize it for any value of  $\Delta$ .

Thus, one way to solve this issue is to try to reduce the number of labels. If we want to prove a lower bound, we cannot just remove some labels, as it would make the problem harder. The only thing that we can do is to try to reduce the number of labels by making the problem easier. Thus we need to find a way to get rid of labels by adding allowed configurations to the black and white constraints. The idea is to *merge* labels. Merging labels means that we add allowed configurations such that each time one label is allowed, we allow the other as well. In this way, we obtain two labels that are equivalent, thus we can just remove one without making the problem harder. How to find the *right* labels to merge? It is not easy, and sometimes we have some large problem that we cannot understand, and the only thing that we can do is to try to merge labels randomly and hope

for the best. Sometimes instead, the problem that we have can be interpreted in some meaningful manner, and we can try to find two labels that we think are similar enough, such that if we merge them, we get a problem that is not much easier.

In order to merge labels on the round eliminator, we have two ways. One is to press the “Edit” button, modify the constraints as described, and press “Start” again. Another is to press “Simplifications” and press the button that describes the desired merging. For example, consider the following (artificial) problem:

Initial problem.

Close

The problem is NOT zero rounds solvable. It can be solved in zero rounds given a 2 coloring.

Active

Any choice satisfies previous Passive

A	A	X
M	M	M
P	U	U

Passive

Exists choice satisfying previous Active

MX	PUX
U	AMUX

Diagram

Strength of right side labels

Tools

Speedup, edit, simplifications, ...

Speedup
Edit

Simplifications

Available simplifications:

A → M
A → U
M → X
P → U
P → X

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

By pressing the button “ $A \rightarrow M$ ”, the round eliminator merges the labels **A** and **M** and gets rid of the label **A**. The result that we get is the following:

Performed simplification A → M

Close

The problem is NOT zero rounds solvable. It can be solved in zero rounds given a 2 coloring.

Active

Any choice satisfies previous Passive

M	M	X
M	M	M
P	U	U

Passive

Exists choice satisfying previous Active

MX	PUX
U	MUX

Diagram

Strength of right side labels

Tools

Speedup, edit, simplifications, ...

Speedup
Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

Not all pairs of labels are listed in the list of possible mergings. The only pairs that are currently listed are the ones that have an edge in the diagram. There are multiple reasons:

- Usually, they seem to be the best candidates.
- On the diagram, if a label can be reached from another label in multiple steps, merging these two labels would actually make equivalent all the labels on the directed path connecting the two labels. This usually gives a problem that is much easier than the original one. If we want to obtain this result, we can just press all buttons related to the edges of the path.
- Usually merging labels that are not connected from a directed path makes the problem trivial. This is not always the case, and some times these kind of mergings can help to get fixed points. This kind of mergings are planned to be added soon.

We still do not understand what are the right simplifications that we should do, but we have some examples of lower bounds that give some ideas about strategies that sometimes work [3]. The best that we can hope for is to do simplifications such that each time we get the *same* number of labels, with problems that are all similar to each other. That is, we start from a problem with  $k$  labels, we press “Speedup” and we get some large problem, we make some simplifications and we get a problem with  $k$  labels. We press “Speedup” again and we get again some large problem, we make the *same* simplifications done before and we get a problem with  $k$  labels again, and we repeat. Usually this gives a nice problem family, that can be described in a compact way, and makes it easy to prove properties on this family. One trick that we can use to get this nice family is to try to make the diagram look the same at each step. This somehow captures the case in which we have two problems that are similar to each other. This does not always work, and sometimes we just get 0 rounds solvable problems, but sometimes it works, and it gives an easy way to prove lower bounds.

**Bipartite Maximal Matching (BMM)** This is an example of a problem for which we can prove a nice lower bound easily, if we do the right simplifications. We will see how, by just looking at the diagram and by trying to make it look the same at each step, we can easily prove a  $2\Delta - 1$  lower bound for this problem. Notice that this problem can be actually solved in  $2\Delta - 1$  rounds, so we really get a tight result here. We will see an example where  $\Delta = 4$ , but it can be generalized. Thus, we will start with the problem where the white and black constraints are the following.

M	O	O
P	P	P

M	OP	OP
O	O	O

Let us start by pressing “Start” and then “Speedup” twice. We get the following.



Performed speedup.

Close

The problem is NOT zero rounds solvable.

Active (Before Renaming)  
Any choice satisfies previous Passive

Passive (Before Renaming)  
Exists choice satisfying previous Active

Renaming  
Old and new labels

Active  
Any choice satisfies previous Passive

Passive  
Exists choice satisfying previous Active

Diagram  
Strength of right side labels

Tools  
Speedup, edit, simplifications, ...

F	E	E	A
E	E	E	B
D	C	C	C

CDEF	CDEF	CDEF	BDF
EF	ABCDEF	EF	EF

Speedup

Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

New Renaming

We will now do the only guess that is needed to get the lower bound, that is, we will merge the labels **D** and **F** by pressing the “**D** → **F**” button. We get the following.

Performed simplification D→F

Close

The problem is NOT zero rounds solvable.

Active (Before Renaming)  
Any choice satisfies previous Passive

Passive (Before Renaming)  
Exists choice satisfying previous Active

Renaming  
Old and new labels

Active  
Any choice satisfies previous Passive

Passive  
Exists choice satisfying previous Active

Diagram  
Strength of right side labels

Tools  
Speedup, edit, simplifications, ...

F	E	E	A
E	E	E	B
F	C	C	C

CEF	CEF	CEF	BF
EF	ABCEF	EF	EF

Speedup

Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

New Renaming

We now press the “Speedup” button, and we get the following.

Performed speedup.

Close

The problem is NOT zero rounds solvable.

**Active (Before Renaming)**  
Any choice satisfies previous Passive

**Passive (Before Renaming)**  
Exists choice satisfying previous Active

**Renaming**  
Old and new labels

**Active**  
Any choice satisfies previous Passive

**Passive**  
Exists choice satisfying previous Active

**Diagram**  
Strength of right side labels

**Tools**  
Speedup, edit, simplifications, ...

F	E	E	A
E	E	E	B
G	C	C	C
E	D	C	C

CDEFG	ABCDEFG	CDEFG	G
CDEFG	CDEFG	CDEFG	BDFG
EFG	ABCDEFG	EFG	EFG

Speedup

Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

New Renaming

Notice that we now got a diagram that is very similar to what we got before the  $D \rightarrow F$  simplification. Thus, we can try to obtain the same diagram again. The idea is that by performing the simplification  $X \rightarrow Y$ , we basically push the node  $X$  of the diagram to  $Y$ . For example, if we now want to get the same rectangle that we had before, we have to press the button “ $F \rightarrow G$ ”. In fact we obtain the following:

Performed simplification  $F \rightarrow G$

Close

The problem is NOT zero rounds solvable.

**Active (Before Renaming)**  
Any choice satisfies previous Passive

**Passive (Before Renaming)**  
Exists choice satisfying previous Active

**Renaming**  
Old and new labels

**Active**  
Any choice satisfies previous Passive

**Passive**  
Exists choice satisfying previous Active

**Diagram**  
Strength of right side labels

**Tools**  
Speedup, edit, simplifications, ...

G	E	E	A
E	E	E	B
G	C	C	C
E	D	C	C

CDEG	ABCDEG	CDEG	G
CDEG	CDEG	CDEG	BDG
EG	ABCDEG	EG	EG

Speedup

Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

New Renaming

We can now notice that we have the same rectangular diagram that we had few steps back (up to

relabeling). What we did at that point was to do the simplification  $\mathbf{D} \rightarrow \mathbf{F}$ . Now, the labels that are in the same position of  $\mathbf{D}$  and  $\mathbf{F}$  are  $\mathbf{D}$  and  $\mathbf{G}$ . Thus, we can try to do the same simplification that we did before, by pressing the “ $\mathbf{D} \rightarrow \mathbf{G}$ ” button. By doing so, we get again a diagram that looks like a cycle of 5 nodes, and by pressing “Speedup” we again get the rectangular diagram:

Performed speedup.

Close

The problem is NOT zero rounds solvable.

**Active (Before Renaming)**  
Any choice satisfies previous Passive

**Active**  
Any choice satisfies previous Passive

F	E	E	A
E	E	E	B
F	C	C	C
E	D	C	C

**Passive (Before Renaming)**  
Exists choice satisfying previous Active

**Passive**  
Exists choice satisfying previous Active

CDEF	ABCDEF	CDEF	F
CDEF	CDEF	CDEF	BDF
EF	ABCDEF	CDEF	EF

**Renaming**  
Old and new labels

**Diagram**  
Strength of right side labels

**Tools**  
Speedup, edit, simplifications, ...

Speedup Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

New Renaming

We can now continue, each time we do the same simplification  $\mathbf{D} \rightarrow \mathbf{F}$ . We can notice that we can repeat this many times before getting a 0 rounds solvable problem. In particular, we get a 0 rounds solvable problem only after having pressed “Speedup” 7 times. This basically gives a 7 rounds lower bound for BMM when  $\Delta = 4$ . We can then try different values of  $\Delta$ , to see if this was just a coincidence, or if this sequence of simplifications actually works for different values of  $\Delta$ . By doing so, one can notice that this easy chain of simplifications actually works.

The next step is to write a formal proof. How to do that? It is now time to look at the problems that we got at each step. We can notice that they all look very similar. There is a common structure in them. Thus, we can try to write a parametrized family of problems, that, for any  $\Delta$  and  $i$ , it describes what is the problem that we get after  $i$  steps by performing speedups and simplifications. Once we have this family, we need to prove a generic round elimination result, that is, that if we start from one problem of the family and we apply the round elimination technique, we get a problem that is harder than the next problem of the family. This is basically the only step that we cannot do automatically, since it requires to prove what is the result of the round elimination technique on an infinite family of problems. A proof for the BMM problem will be online soon. An example of a self contained proof can be found in [2] for a problem called FDSO.

**Automatic lower bounds** In the BMM example, we had to guess a simplification, and then, by trying to keep the diagram the same, we got a nice lower bound. We do not actually need to perform manual guessing: the round eliminator can do that for us. Let us start again from the BMM problem for the case where  $\Delta = 4$ . This time let us press “Automatic Lower Bound”. We get the following.

Initial problem.

Close

The problem is NOT zero rounds solvable.

Active

Any choice satisfies previous Passive

U	U	U	M
P	P	P	P


Passive

Exists choice satisfying previous Active

PU	PU	PU	M
U	U	U	U

Diagram

Strength of right side labels



Tools

Speedup, edit, simplifications, ...

Speedup
Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Maximum number of iterations:

15

Maximum number of labels:

5

Automatic Lower Bound

Automatic Upper Bound

There are two parameters that we can tweak:

- Maximum number of iterations. You do not have to worry about this parameter. Just give it some value that is higher than what you think is the lower bound that you should get. This just gives a maximum on the number of rounds that we can get as lower bound. The reason is that sometimes we may get fixed points automatically, and the round eliminator does not detect them automatically. Thus it would not stop, and it would keep giving a better and better lower bound.
- Maximum number of labels. What the round eliminator does is the following. After each speedup step, it tries to merge labels until some number of labels is reached. In particular, if we set this limit to  $k$ , it tries all possible mergings that give at most  $k$  labels as result.

Let us leave the default values, 15 and 5, and press the “Automatic Lower Bound” button. After a while, we get the following:

Lower bound of 7 rounds.

The round eliminator automatically found the right simplifications to do to get a 7 rounds lower bound! We can then read the result, and we can see what simplifications the program did. What we find is that it basically did the same simplifications that we did manually before. If we try to do the same for the case where  $\Delta = 5$ , we automatically get a 9 rounds lower bound (it takes more time though). Thus, the only thing that we need to do is to read the simplifications that the program did, see what is the pattern, and try to generalize it.

## 5 Upper Bounds

Until now we have seen how to apply this technique to get lower bounds. This technique can be actually applied to get upper bounds! The only issue is that the upper bounds that we get are not general. That is, they only hold in a regular tree. Thus, if we get something with this technique, we then have to try to generalize it.

The idea is that, while for proving lower bounds we have to make the problem easier at each step and hope to not get a problem that is 0 round solvable, here we have to make the problem harder and hope to *get* a problem that is 0 rounds solvable. How to make the problem harder? It is very easy, just remove labels! In the round eliminator program there are two ways to do that. By pressing “Harden v1” we can select which labels we want to keep. By pressing “Harden v2” we can press a button to remove a label. This is what we see if we start from BMM.

Initial problem.

Close

The problem is NOT zero rounds solvable.

Active

Any choice satisfies previous Passive

U	U	U	M
P	P	P	P

Passive

Exists choice satisfying previous Active

PU	PU	PU	M
U	U	U	U

Diagram

Strength of right side labels

Tools

Speedup, edit, simplifications, ...

Speedup Edit

Simplifications

Harden v1

Please choose which labels should be kept.

☐ M
 ☐ P
 ☐ U

☒ Add diagram predecessors
 

Harden

Harden v2

Click on the label that you want to remove

M

P

U

☒ Add diagram predecessors
 

Automatic Lower Bound

Automatic Upper Bound

For example, let us press the button “**P**”. The result is the following.

Removed label P

Close

The problem is NOT zero rounds solvable.

**Active**

Any choice satisfies previous Passive

U    U    U    M


**Passive**

Exists choice satisfying previous Active

U	U	U	M
U	U	U	U

**Diagram**

Strength of right side labels



**Tools**

Speedup, edit, simplifications, ...

Speedup
Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

Often, by removing labels we get problems that are way harder than the original one. For example, we can notice that after removing the label **P**, if we press “Speedup” many times, we get a fixed point. It means that we made the problem way harder! While BMM can be solved in  $O(\Delta)$  rounds, this new problem requires  $\Omega(\log n)$ , that is the highest possible lower bound that we can get with this technique.

Are there ways to remove labels without making the problem too hard? Actually, yes, and it is the default option in the round eliminator program. Please note that there is an option called “Add diagram predecessors” in both “Harden v1” and “Harden v2”, that is enabled by default. The idea is that we do not just remove a label, but instead we replace it with its diagram predecessors. One can prove that this basically gives the easiest possible problem that one can get using only the remaining labels, but still harder than the original one. If we want to prove an upper bound, this is the best option: we reduce the number of labels, so the result after a speedup step does not get too large, but we make the problem as less hard as possible.

**Automatic Upper Bounds** Again, we do not really need to manually try all possible buttons, and the round eliminator can do that for us. For example, let us start from the BMM problem where  $\Delta = 4$ , and let us press “Automatic Upper Bound”. We get the following.

Initial problem.

Close

The problem is NOT zero rounds solvable.

**Active**  
Any choice satisfies previous Passive

U	U	U	M
P	P	P	P

**Passive**  
Exists choice satisfying previous Active

PU	PU	PU	M
U	U	U	U

**Diagram**  
Strength of right side labels

**Tools**  
Speedup, edit, simplifications, ...

Speedup Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

Maximum number of iterations:  
5

Maximum number of labels:  
4

☒ Slow and Accurate

☐ Test

Automatic Upper Bound

Here again there are two parameters:

- Maximum number of iterations. Here you should put a value that represents your target complexity, that is, the number of rounds that you think are enough to solve the problem. By putting high values, the program gets slower.
- Maximum number of labels. Like in the automatic lower bound case, the round eliminator tries to limit the number of labels in all possible ways at each step. Increasing this value may help in finding an upper bound, but it may make the program slower.

Since we got a 7 rounds lower bound, let us try to get a 7 rounds upper bound. Also, we used a limit of 5 labels to get a lower bound, but let us be optimistic and try to get an upper bound with just 4 labels. Thus, let us put the values 7 and 4 and press “Automatic Upper Bound”. We get the following:

Upper bound of 7 rounds.

The round eliminator found a 7 rounds algorithm, meaning that BMM requires exactly 7 rounds if  $\Delta = 4$ . By scrolling down we can also see what labels got removed at each step. There are other options that can be used while trying to get an upper bound automatically:

- Slow and Accurate. This enables the “Add diagram predecessors” option while removing labels. This typically makes the result larger, and slows down the process, but it can actually help to get upper bounds (without this option we may get none, since the problem may always get too hard).

- Test. This is some experimental technique that sometimes can help to get upper bounds very fast. It usually does not work.

**Defective edge coloring** Very often, it is very hard to understand what is the actual algorithm that we get using this technique. We need to start from the 0 rounds solvable problem that we got, and go backwards and understand what is going on. There are some exceptions though. Sometimes we get very easy and a bit surprising algorithms. Consider the following problem: color the edges with 3 colors, such that each node has at most  $\lceil \frac{2}{3}\Delta \rceil$  edges colored with the same color. Let us consider the easiest possible case where  $\Delta = 3$ . Each color must be used at most twice. This problem can be encoded as follows:

A	B	C
A	A	BC
B	B	AC
C	C	AB

A	A
B	B
C	C

That is, edges have 3 possible colors, and we either use 3 different colors on each node, or we use one color twice and then a different one. Let us feed this problem to the round eliminator, and perform two steps of round elimination. We get the following:

Performed speedup.

Close

The problem is zero rounds solvable.

**Active (Before Renaming)**  
Any choice satisfies previous Passive

G	F	A
G	E	B
G	D	C
F	E	C

**Passive (Before Renaming)**  
Exists choice satisfying previous Active

ACEG	ACEG
BCFG	BCFG
DEFG	DEFG

**Renaming**  
Old and new labels

**Diagram**  
Strength of right side labels

**Tools**  
Speedup, edit, simplifications, ...

Speedup
Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

New Renaming

The round eliminator is telling us that this new problem is 0 rounds solvable, meaning that the original one is 2 rounds solvable. Actually, since we are in the non bipartite setting, we are counting *half* rounds, instead of full rounds. This means that we actually got a 1 round algorithm.

How to get an algorithm out of this? The first thing that we need to do is to identify which is the output that gives 0 rounds solvability. We can notice that there is a line, **F E C**, that satisfies the following: any pair of labels of this line is allowed in the edge constraint. For example, **F F**



appears on the edge constraint on the second line, while  $\boxed{\mathbf{E\ C}}$  appears on the first line. This means that if all nodes output  $\boxed{\mathbf{F\ E\ C}}$ , then we have a valid solution. By clicking “Renaming” we can see what these labels correspond to:

Renaming	
Old and new labels	
$\boxed{\mathbf{A}}$	A
$\boxed{\mathbf{B}}$	B
$\boxed{\mathbf{AB}}$	C
$\boxed{\mathbf{C}}$	D
$\boxed{\mathbf{AC}}$	E
$\boxed{\mathbf{BC}}$	F
$\boxed{\mathbf{ABC}}$	G

So what are nodes doing? They are just writing three different sets,  $\{\mathbf{A}, \mathbf{B}\}$ ,  $\{\mathbf{A}, \mathbf{C}\}$ , and  $\{\mathbf{B}, \mathbf{C}\}$ , on their 3 incident edges, in some arbitrary order. As we have seen when we manually computed the round elimination result on the BSO problem, these sets must satisfy a forall quantifier. This means that no matter what nodes write on an edge, the edge should be able to pick something that is valid. Let us see what is allowed on the previous step:

Performed speedup.

Close

The problem is NOT zero rounds solvable.

**Active (Before Renaming)**  
Any choice satisfies previous Passive

**Passive (Before Renaming)**  
Exists choice satisfying previous Active

**Renaming**  
Old and new labels

**Active**  
Any choice satisfies previous Passive

A	A
B	B
C	C

**Passive**  
Exists choice satisfying previous Active

C	B	A
A	BC	A
B	AC	B
C	AB	C

**Diagram**  
Strength of right side labels

**Tools**  
Speedup, edit, simplifications, ...

Speedup

Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

New Renaming

Easy, just the three colorings are allowed. So why these sets are good? Well, no matter how we choose the sets, their intersection is not empty! So we can always pick a common element and decide that it is the color of the edge. The only remaining thing to see is why this is good for the original problem. Nodes must have at most 2 edges with the same color, and since they wrote 3 different sets on the edges, it is not possible to pick the same color from all of them!

Thus, the algorithm is the following. Each node sends the set  $\{A, B\}$  to one arbitrary neighbor,  $\{B, C\}$  to another, and  $\{A, C\}$  to the remaining one. Nodes will then receive the sets sent from the neighbors, and for each edge they will pick the smallest element in the intersection between the set sent on that edge and the set received on the same edge. This algorithm requires just one round of communication, and by generalizing it to larger values of  $\Delta$  we get an algorithm that solves the desired problem.

## 6 Extra

**Input coloring** Sometimes we are interested on the following question: what is the complexity of a problem if we are given a  $\Delta + 1$  coloring in input? The round eliminator gives some hints about that, if used in the non bipartite setting (that is, when the black constraint has degree  $\delta = 2$ ). For example, starting from the MIS problem, we can see that the round eliminator warns us that this problem is 0 rounds solvable, if a 2 coloring is given:

### Initial problem.

Close

The problem is NOT zero rounds solvable. It can be solved in zero rounds given a 2 coloring.

Active

Any choice satisfies previous Passive

M	M	M
U	U	P

Passive

Exists choice satisfying previous Active

PU	M
U	U

Diagram

Strength of right side labels

Tools

Speedup, edit, simplifications, ...

Speedup

Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

Automatic Upper Bound

No surprise here, this problem is easy on 2 colored graphs. Let us press “Speedup” 4 times. We get the following:

### Performed speedup.

Close

The problem is NOT zero rounds solvable. It can be solved in zero rounds given a 3 coloring.

Active (Before Renaming)

Any choice satisfies previous Passive

Passive (Before Renaming)

Exists choice satisfying previous Active

Renaming

Old and new labels

Active

Any choice satisfies previous Passive

Q	A	A
S	E	B
P	C	C
N	D	D
K	E	E
F	F	F
R	R	G
O	H	H
M	I	I
J	J	J
L	L	L

Passive

Exists choice satisfying previous Active

BCDEFGHIJKLMNOPQRS	Q
GHIJKLMNOPQRS	FKNPQS
ACDEFGHIJKLMNOPQRS	RS
DEFIJKLMNOPQRS	OPQRS
LMNOPQRS	EFJKNOPQRS

Diagram

Strength of right side labels

Tools

Speedup, edit, simplifications, ...

Speedup

Edit

Simplifications

Harden v1

Harden v2

Automatic Lower Bound

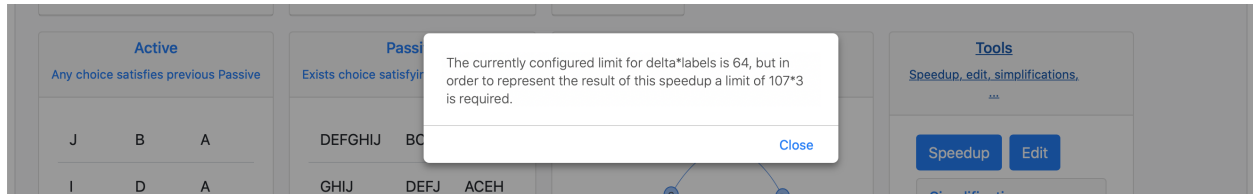
Automatic Upper Bound

New Renaming

In this case the round eliminator tells us that the problem that is 4 (half) rounds easier than MIS does not need a 2 coloring to be solved in 0 rounds, but that actually 3 colors are enough.

Also, in the automatic lower and upper bound features, we can specify how many colors we assume are given, and this influences the 0 rounds solvability.

**More labels** Sometimes, if the result of the round elimination technique gives a problem with too many labels, the round eliminator program gives an error.



What if we are actually interested in seeing this big result? In this case the error says that we need a limit of  $107 * 3$ , that is the number of labels of the new problem times  $\Delta$ , and that the current limit is 64. Thus, we need a limit of at least 321. To solve the issue just compute  $k = \lceil 321/64 \rceil = 6$ . Then, open in your browser <https://roundeliminator.github.io/re6/> (replace the last number in the URL with your value of  $k$ ). This is a different version of the round eliminator, compiled with a different limit. The downside of using this version is that every operation will be roughly 6 times *slower*.

**Constraints that depend on inputs** Sometimes we want to encode constraints such as “nodes that have in input a 1 must do something, while nodes that have in input a 0 must do something else”. It is actually possible to apply the round elimination technique also in this case! The program does not support this kind of constraints at the moment, but there is a plan to add this feature at some point.

## References

- [1] Sebastian Brandt. An Automatic Speedup Theorem for Distributed Problems. In *Proc. 38th ACM Symposium on Principles of Distributed Computing (PODC 2019)*, 2019.
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- [3] Alkida Balliu, Sebastian Brandt, Juho Hirvonen, Dennis Olivetti, Mikaël Rabie, and Jukka Suomela. Lower bounds for maximal matchings and maximal independent sets. In *60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019*, pages 481–497, 2019.