Example of Doing Two way ANOVA

1 Two Way Analysis of Variance by Hand

Error Decomposition

$$
\begin{aligned}
\underbrace{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}\left(Y_{ijk}-\bar{Y_{\cdot\cdot\cdot}}\right)^2}_{SS_{Total}} &= \underbrace{r\cdot b\cdot\sum_{i=1}^{a}\left(\bar{Y_{i\cdot\cdot}}-\bar{Y_{\cdot\cdot\cdot}}\right)^2}_{SS_A}+\underbrace{r\cdot a\cdot\sum_{j=1}^{3}\left(\bar{Y_{\cdot j\cdot}}-\bar{Y_{\cdot\cdot\cdot}}\right)^2}_{SS_B} \\
&\quad +\underbrace{r\times\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\bar{Y_{ij\cdot}}-\bar{Y_{i\cdot\cdot}}-\bar{Y_{\cdot j\cdot}}+\bar{Y_{\cdot\cdot\cdot}}\right)^2}_{SS_{A\times B}}+\underbrace{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}\left(Y_{ijk}-\bar{Y_{ij\cdot}}\right)^2}_{SS_{within}}
\end{aligned}
$$

ANOVA Table

Source	Degrees of Freedom	SS	MS	F
A	a-1	SS_A	MS_A	$MS_A \big/ MS_{within}$
B	b-1	SS_B	MS_B	$MS_B \big/ MS_{within}$
$A\times B$	(a-1)(b-1)	$SS_{A\times B}$	$MS_{A\times B}$	$MS_{A\times B} \big/ MS_{within}$
Within	ab(r-1)	SS_{within}	MS_{within}	
Total	abr-1	SS_{Total}		

Example Suppose you want to determine whether the brand of laundry detergent used and the temperature affects the amount of dirt removed from your laundry. To this end, you buy two different brand of detergent (“ Super” and “Best”) and choose three different temperature levels (“cold”, “warm”, and “hot”). Then you divide your laundry randomly into $6\times r$ piles of equal size and assign each r piles into the combination of (“Super” and “Best”) and (”cold”,”warm”, and “hot”). In this example, we are interested in testing Null Hypotheses

$$
\begin{aligned}
H_{0D} &: \text{The amount of dirt removed does not depend on the type of detergent} \\
H_{0T} &: \text{The amount of dirt removed does not depend on the temperature}
\end{aligned}
$$

One says the experiment has **two factors** (Factor Detergent, Factor Temperature) at $a = 2$(Super and Best) and $b = 3$(cold,warm and hot) **levels**. Thus there are $ab = 3 \times 2 = 6$ different combinations of detergent and temperature. With each combination you wash $r = 4$ loads. r is called the number of **replicates**. This sums up to $n = abr = 24$ loads in total. The amounts Y_{ijk} of dirt removed when washing sub pile k ($k = 1, 2, 3, 4$) with detergent i ($i = 1, 2$) at temperature j ($j = 1, 2, 3$) are recorded in Table 1.

	Cold	Warm	Hot
Super	4,5,6,5	7,9,8,12	10,12,11,9
Best	6,6,4,4	13,15,12,12	12,13,10,13

Solution :

	Cold	Warm	Hot	m_D
Super	4,5,6,5 (5)	7,9,8,12 (9)	10,12,11,9 (10)	8
Best	6,6,4,4 (5)	13,15,12,12 (13)	12,13,10,13 (12)	10
m_T	5	11	11	9

- SS_{within} and df_{within}

$$SS_{within} = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{4}\left(Y_{ijk}-\bar{Y_{ij\cdot}}\right)^2$$

$$
\begin{aligned}
&= (4-5)^2 + (5-5)^2 + (6-5)^2 + (5-5)^2 \\
&\quad +(7-9)^2 + (9-9)^2 + (8-9)^2 + (12-9)^2 \\
&\quad \cdots\cdots \\
&\quad +(12-12)^2 + (13-12)^2 + (10-12)^2 + (13-12)^2 \\
&= 38 \\
df_{within} &= (r-1) * a * b = 3 * 2 * 3 = 18 \\
MS_{within} &= SS_{within} \Big/ df_{within} = 38/18 = 2.1111
\end{aligned}
$$

- $SS_{detergent}$ and $df_{detergent}$

$$
\begin{aligned}
SS_{detergent} &= r \cdot b \cdot \sum_{i=1}^{2} \left(\bar{Y_{i..}} - \bar{Y_{...}}\right)^2 \\
&= 4 \times 3 \times \left[(8-9)^2 + (10-9)^2\right] = 24 \\
df_{detergent} &= a - 1 = 1 \\
MS_{detergent} &= SS_{detergent} \Big/ df_{detergent} = 24 \Big/ 1 = 24
\end{aligned}
$$

- $SS_{temperature}$ and $df_{temperature}$

$$
\begin{aligned}
SS_{temperature} &= r \cdot a \cdot \sum_{j=1}^{3} \left(\bar{Y_{.j.}} - \bar{Y_{...}}\right)^2 \\
&= 4 \times 2 \times \left[(5-9)^2 + (11-9)^2 + (11-9)^2\right] = 24 \\
df_{temparature} &= b - 1 = 2 \\
MS_{temperature} &= SS_{temperature} \Big/ df_{temperature} = 192 \Big/ 2 = 81
\end{aligned}
$$

- $SS_{interaction}$ and $df_{interaction}$

$$
\begin{aligned}
SS_{interaction} &= r \times \sum_{i=1}^{2} \sum_{j=1}^{3} \left(\bar{Y_{ij.}} - \bar{Y_{i..}} - \bar{Y_{.j.}} + \bar{Y_{...}}\right)^2 \\
&= 4 \times \left[(5-8-5+9)^2 + (9-8-11+9)^2 + (110-8-11+9)^2 + \cdots + (12-11-10+9)^2\right] = 12 \\
df_{interaction} &= (a-1) \times (b-1) = 2 \times 1 = 2 \\
MS_{interaction} &= SS_{interaction} \Big/ df_{interaction} = 12 \Big/ 2 = 6
\end{aligned}
$$

- F-Test

$$
\begin{aligned}
MS_{detergent} \Big/ MS_{within} &\sim F\Big(df_{detergent}, df_{within}\Big) \\
MS_{temperature} \Big/ MS_{within} &\sim F\Big(df_{temperature}, df_{within}\Big) \\
MS_{interaction} \Big/ MS_{within} &\sim F\Big(df_{interaction}, df_{within}\Big)
\end{aligned}
$$

Two Way ANOVA in R

```
> wash=scan()
1: 4 5 6 5 7 9 8 12 10 12 11 9
13: 6 6 4 4 13 15 12 12
21: 12 13 10 13
25:
Read 24 items
> mean(wash)
[1] 9.083333
> deter=factor(c(rep(1,12),rep(2,12)))
> water=factor(rep(gl(3,4),2))
```

```
> water
 [1] 1 1 1 1 2 2 2 2 3 3 3 3 1 1 1 1 2 2 2 2 3 3 3 3
Levels: 1 2 3
> deter
 [1] 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2 2 2 2
Levels: 1 2
```

```
> tapply(wash,water,mean)
    1     2     3
 5.00 11.00 11.25
> tapply(wash,deter,mean)
        1         2
 8.166667 10.000000
> tapply(wash,deter:water,mean)
 1:1  1:2  1:3  2:1  2:2  2:3
 5.0  9.0 10.5  5.0 13.0 12.0
> anova(lm.deter)
Analysis of Variance Table

Response: wash
            Df  Sum Sq Mean Sq F value   Pr(>F)
deter        1  20.167  20.167  9.8108 0.005758 **
water        2 200.333 100.167 48.7297 5.44e-08 ***
deter:water  2  16.333   8.167  3.9730 0.037224 *
Residuals   18  37.000   2.056
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
> matrix(round(fitted(lm.deter),1),byrow=T,nrow=2)
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12]
[1,]    5    5    5    5    9    9    9    9 10.5  10.5  10.5  10.5
[2,]    5    5    5    5   13   13   13   13 12.0  12.0  12.0  12.0
> matrix(round(residuals(lm.deter),1),byrow=T,nrow=2)
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12]
[1,]   -1    0    1    0   -2    0   -1    3 -0.5   1.5   0.5  -1.5
[2,]    1    1   -1   -1    0    2   -1   -1  0.0   1.0  -2.0   1.0
> matrix(round(residuals(lm.deter),1)^2,byrow=T,nrow=2)
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12]
[1,]    1    0    1    0    4    0    1    9 0.25  2.25  0.25  2.25
[2,]    1    1    1    1    0    4    1    1 0.00  1.00  4.00  1.00
interaction.plot(water,deter,wash)
```