Probability Distribution

Discrete and Continuous Distributions

Probability Distribution

- Introduction to Distribution
- Classification
- Discrete Distributions
- Continuous Distributions
- Binomial Distribution
- Bernoulli Distribution
- Poisson Distribution
- Normal Distribution
- Standard Distribution

What is a Probability Distribution ?

Definition:
A distribution is a function which shows the possible values for a random variable and how often they occur.

In statistics, when we use the term distribution, we usually mean Probability Distribution.

Distribution ~ Probability Distribution

What is a Probability Distribution ?

Example:

Lets say, you have a single fair die (with 6 sides of course) numbered from 1 to 6. We roll the die:

probability of getting 1 = 1/6 ~ 0.17
probability of getting 2 = 1/6 ~ 0.17
probability of getting 3 = 1/6 ~ 0.17
probability of getting 4 = 1/6 ~ 0.17
probability of getting 5 = 1/6 ~ 0.17
probability of getting 6 = 1/6 ~ 0.17

We have an equal chance of getting all of the 6 outcomes.
Probability of getting any other result = 0

Lets tabulate this.

What is a Probability Distribution ?

Example:

Lets say, you have a single fair die (with 6 sides of course) numbered from 1 to 6. We roll the die:

Outcome	Probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
7 or other	0

What is a Probability Distribution ?

Lets Generalize this:

The distribution of an event not only consist of the outcomes that can be observed (outcome from 1 to 6), but it is made up of all the possible Outcomes (i.e. outcome for 7 and others as well).

Thus,

Probability Distribution of Rolling a die:

Outcome	Probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
All else	0

What is a Probability Distribution ?

Sum of Probabilities in a Distribution:
We are sure that we have exhausted all possible outcomes, when the sum of probabilities of all the outcomes in a distribution equals to 1 (or 100%).
i.e.
The sum of probabilities in a distribution is always equal to 1
Lets add the probabilities in the below distribution

Outcome	Probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
All else	0

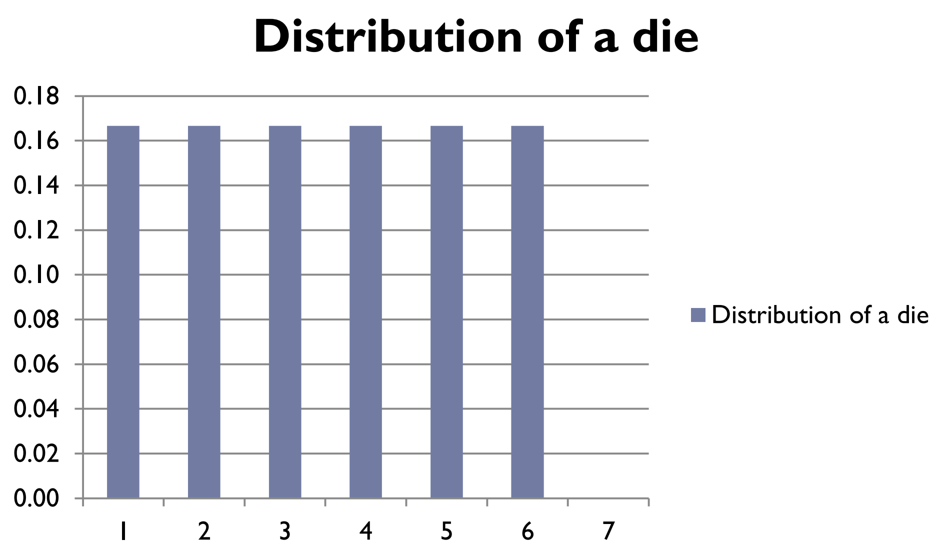
1/6 +1/6 +1/6 +1/6 +1/6 +1/6+0 = 1

The total is 1,
Hence the given table is the probability Distribution of a rolling die.

What is a Probability Distribution ?

Representation of Probability Distribution:
Each probability distribution has a visual representation.
It is a graph describing the likelihood of occurrence of every event.

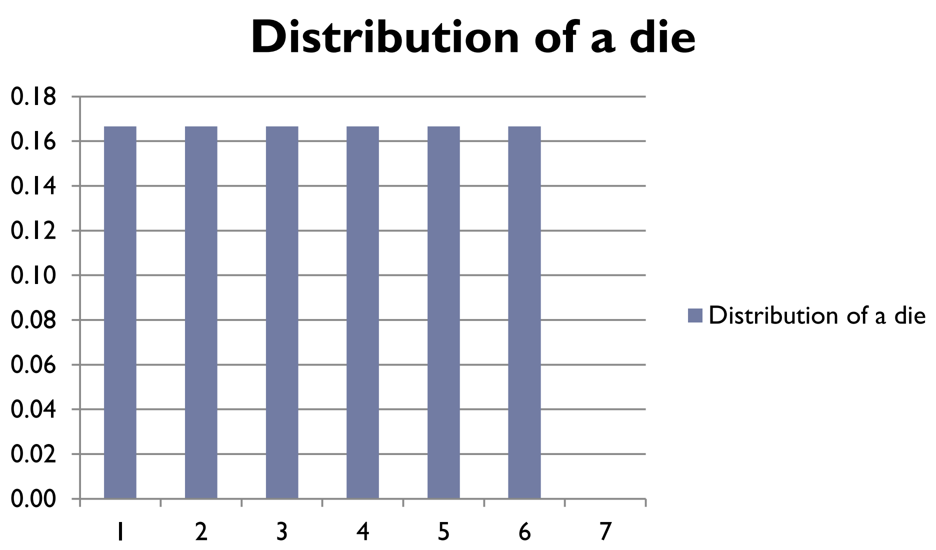
Outcome	Probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
All else	0



What is a Probability Distribution ?

Representation of Probability Distribution:
It is crucial to understand that the distribution is defined by the underlying probabilities and not the graph.
The graph is just a visual representation of distribution.

Outcome	Probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
All else	0



Classification

Classification

Broadly we can categorize distribution as:

- Discrete Distribution
  - Discrete Uniform Distribution (e.g. Bernoulli)
  - Discrete Non-Uniform Distribution (e.g. Binomial)
- Continuous Distribution
  - Continuous Uniform Distribution
  - Continuous Non-Uniform Distribution (Normal, Standard etc.)

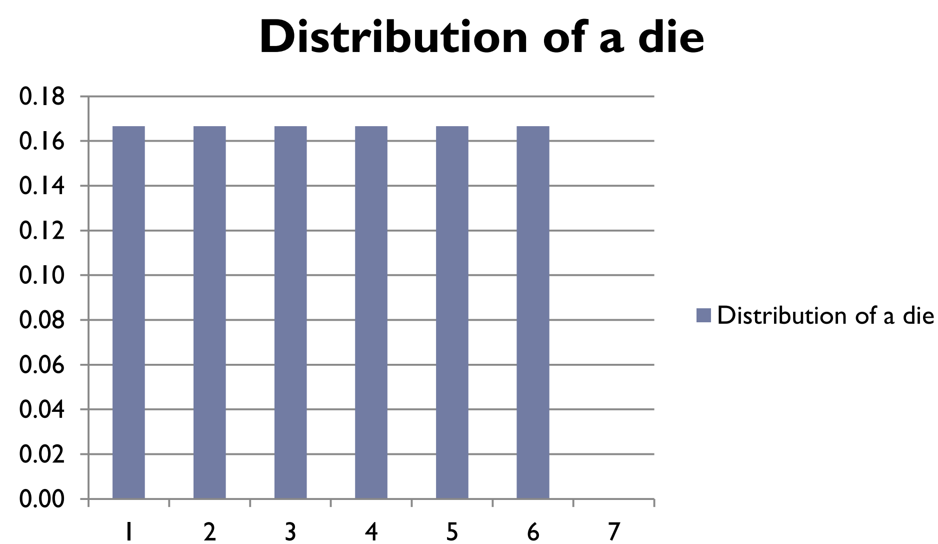
Discrete – Uniform Distribution

Consider the distribution of rolling a single fair die:

- It is indeed a ***'Discrete Distribution'***
- Also, all the outcomes have equal chance of occurring. Which makes It a ***'Uniform Distribution'***

Hence the probability distribution of rolling a single fair die is a ***'Discrete Uniform Distribution'***

Outcome	Probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
All else	0

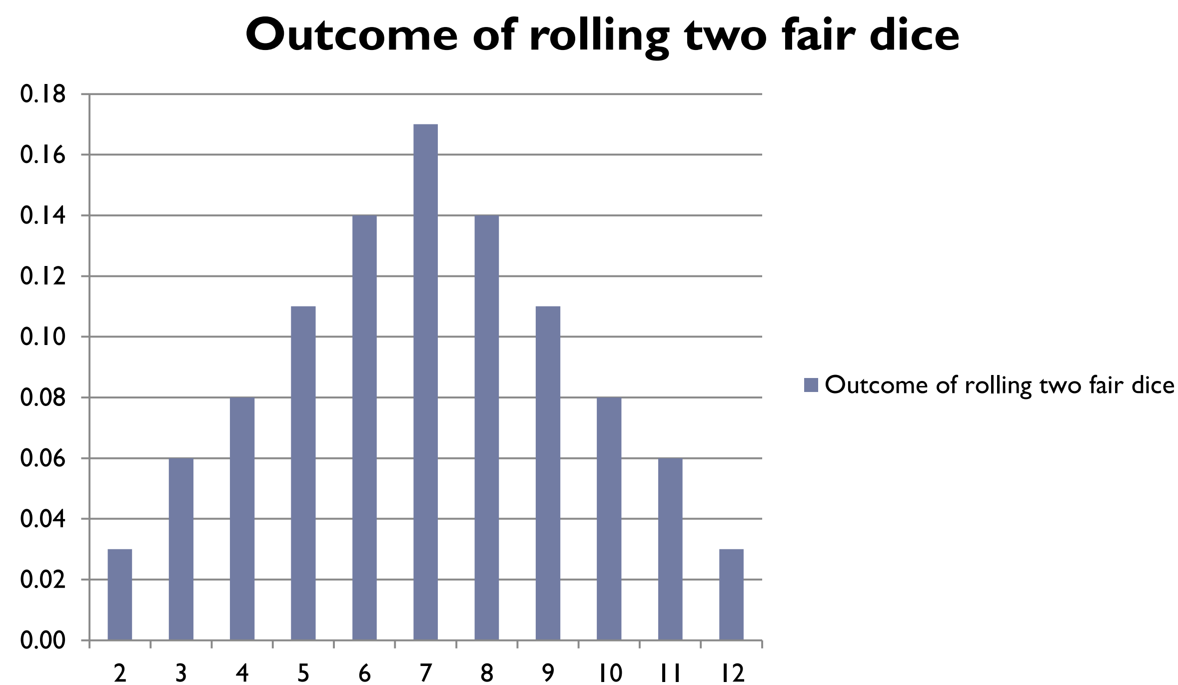


Discrete Non-Uniform Distribution

Consider the distribution of rolling a two fair dice:
Lets calculate the probability of each outcome.

Probability = Sum of outcomes / Total Possibilities

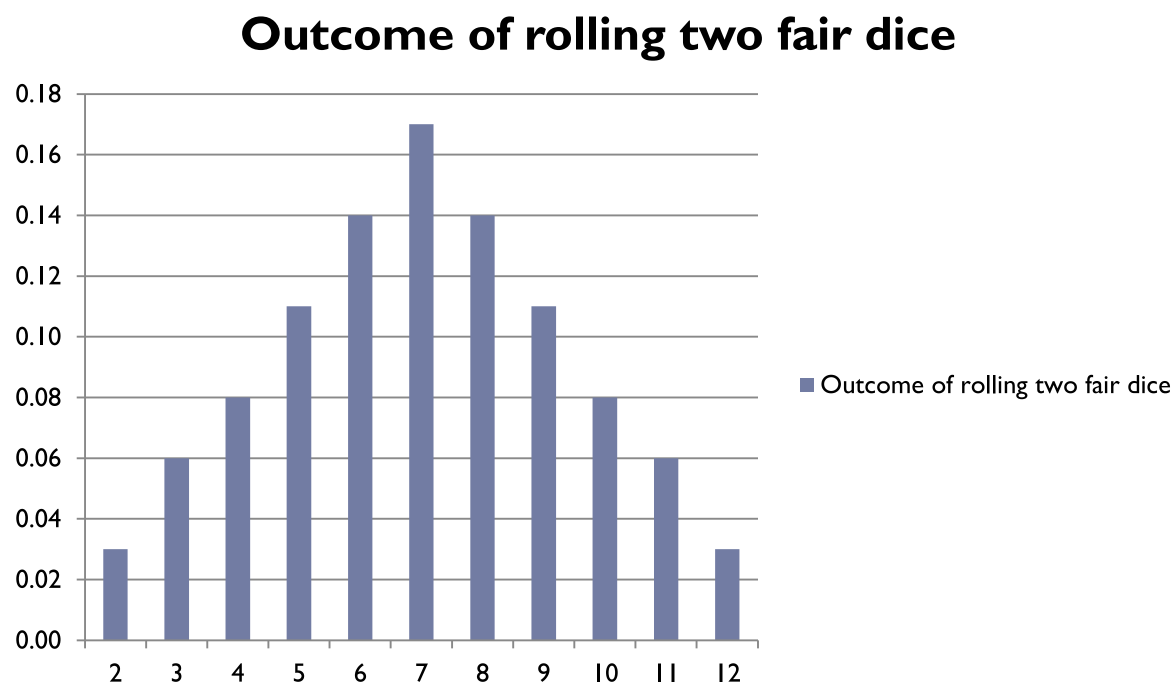
Outcome	Probability
2	1/36 ~ 0.03
3	2/36 ~ 0.06
4	3/36 ~ 0.08
5	4/36 ~ 0.11
6	5/36 ~ 0.14
7	6/36 ~ 0.17
8	5/36 ~ 0.14
9	4/36 ~ 0.11
10	3/36 ~ 0.08
11	3/36 ~ 0.06
12	1/36 ~ 0.03
All else	0



Discrete Non-Uniform Distribution

Looks like the probability of most of the outcomes are different.
This is a ***'Discrete Non-Uniform Distribution'***

Outcome	Probability
2	1/36 ~ 0.03
3	2/36 ~ 0.06
4	3/36 ~ 0.08
5	4/36 ~ 0.11
6	5/36 ~ 0.14
7	6/36 ~ 0.17
8	5/36 ~ 0.14
9	4/36 ~ 0.11
10	3/36 ~ 0.08
11	3/36 ~ 0.06
12	1/36 ~ 0.03
All else	0



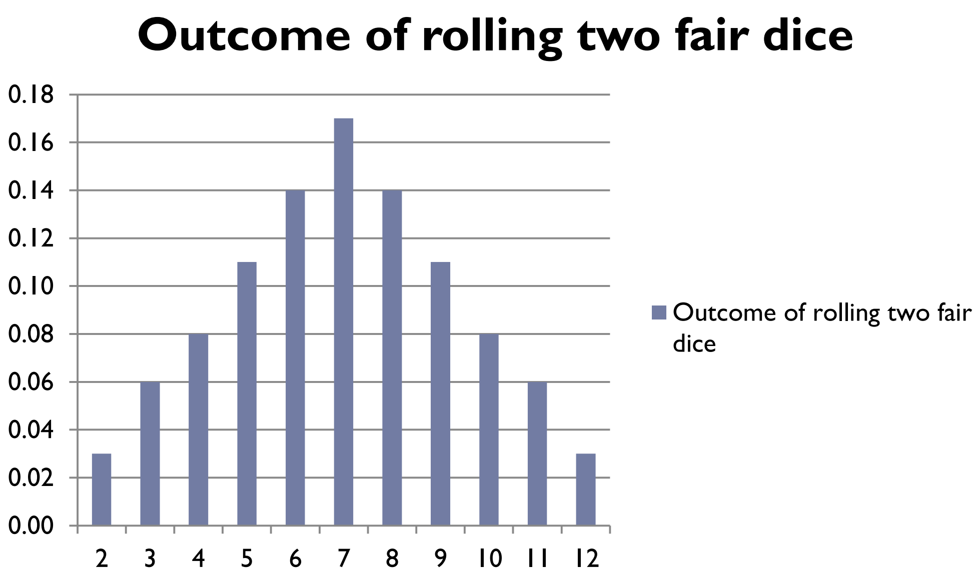
Discrete Distribution

Probability Mass Function (PMF) - *Related to Discrete Distributions*
The PMF is a probability measure that gives us probabilities of the possible values for a random variable.

For our Rolling die discrete distributions, PMF is

$$f(x) = \frac{1}{n}$$

where,
n= Total outcomes



Continuous Distribution

Think about data such as temperature, distance, time, mass etc. that can be measured to several decimal points.
In that case, what will be the number of outcomes ?

$$f(x) = \frac{1}{n}$$

Thus when n (Total outcomes) becomes very large i.e. n -> ∞ , $f(x)$ becomes smaller and smaller.

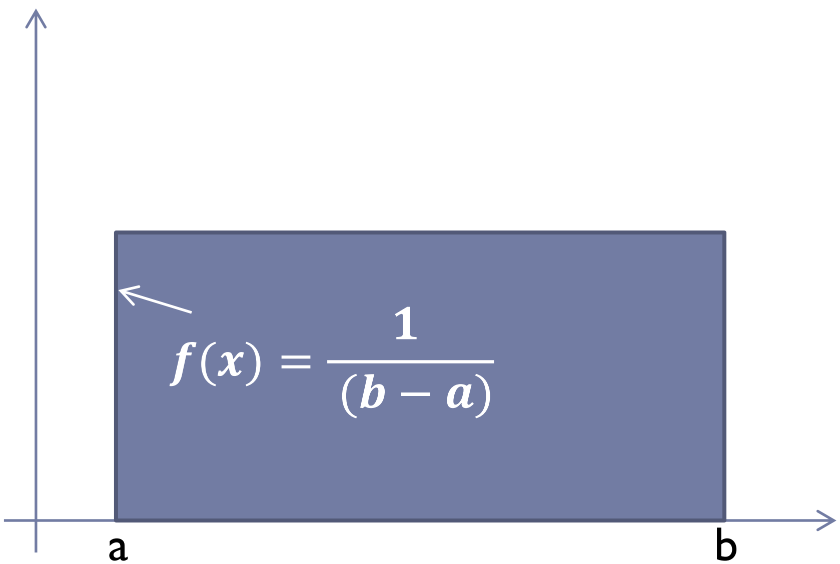
$$f(x) = \frac{1}{\infty} \sim 0$$

That means, as we increase the number of outcomes, the probability tends to become zero at that specified outcome.

Lets visualize it

Continuous Distribution

Think of infinite number of bars placed side by side where area of each bar -> 0



Area = width x height
Total Area is always = 1
Hence,
1 = (b-a) x height

$$\boldsymbol{height = \frac{1}{(b-a)}}$$

Continuous Distribution

Probability Density Function (PDF) – *Related to Continuous Distribution*
A probability density function (PDF), or density of a continuous random variable, is a function that describes the relative likelihood for a random variable to take on a given value.

$$P(a \leq X \leq b) = \int_a^b f(x) d_x$$

where,

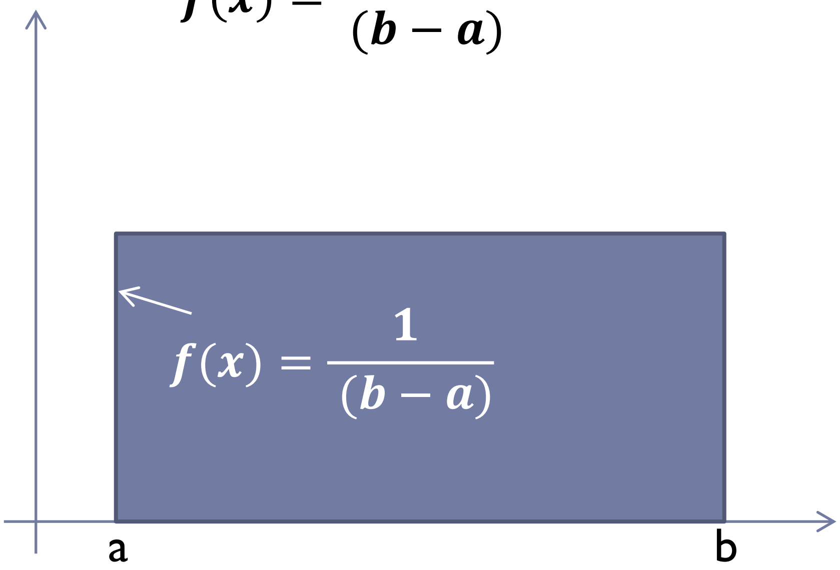
- [a , b] = Interval in which x lies.
- P(a ≤ X ≤ b) = probability that some value x lies within this interval.
- dx = b-a

In Continuous Distribution we calculate the probability distribution over a range and not a specific outcome.

Continuous Distribution

Probability Density Function (PDF) for our example is:

$$f(x) = \frac{1}{(b-a)}$$



Also, this is a ***'Continuous Uniform Distribution'***

We will see the Continuous Non Uniform Distributions in details.

Binomial Distribution

Binomial Distribution

The probability distribution of a **binomial random variable** is called a binomial distribution.

"A binomial random variable is the number of success (x) in n repeated trials of a binomial experiment."

In simple terms:
A binomial distribution is a type of distribution that has only two possible outcomes (the prefix 'bi' means 'two')

e.g. A coin toss has only two possible outcomes, Heads or Tails

Binomial Distribution

Following are the characteristics of a binomial experiment:

- The experiment contains ‘n’ repeated trials.
- Each trial is independent; the outcome of one does not affect the outcome of the other trials.
- Each trial can result in just two possible outcomes. We call on of those outcomes as ‘Success’ and other ‘Failure’.
- The probability of ‘Success’, denoted by ‘p’ is the same on every trial.

Binomial Distribution

Example: Consider the following statistical experiment.

You flip a fair coin 2 times and count the number of times the coin lands on head.

This is a binomial experiment because:

- The experiment consist of repeated trials - Flipping coin 2 times
- Each trial results in just 2 possible outcomes – Heads or Tails
- The probability of Success is constant – 0.5 on every trial
- The trial are independent – Getting Heads on one trial does not affect if we get Heads on the second trial or not.

The distribution of this experiment is a Binomial distribution.

Binomial Distribution

Probability Mass Function (PMF) :

$$f(k, n, p) = \Pr(k; n, p) = \Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Where,

k = Number of success that result from the experiment

n = Total number of trials in the experiment

p = Probability of success of individual trial

(1-p) = q = Probability of failure of an individual trial

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Distribution

Properties of Binomial Distribution:

The mean of distribution (μ_x) = n * p

The variance (σ^2_x) = μ_x * (1 – p)

The Standard Deviation (σ_x) = sqrt[σ^2_x]

Binomial Distribution

Example: Suppose a die is tossed 5 times. What is the probability of getting exactly two Fours ?

Solution:

$n = 5$, $k = 2$, $p = 1/6$

Binomial Probability = 5 Choose 2 * $(1/6)^{5}$ * $(1-1/6)^{5-2}$ = 0.161

Binomial Distribution

Cumulative Probability

A cumulative probability refers to the probability that the binomial variable falls within a specified range.
(e.g. is greater than or equal to a stated or lower limit
And less than or equal to a stated upper limit)

Example: Find the probability of getting 45 or fewer Heads in 100 toss of a coin.

Solution:

This would be the sum of individual probabilities.

P(X <= 45) = P(X=0) + P(X=1) + P(X=2) + ……………. + P(X=45)

We calculate the 46 individual probabilities from 0 to 45 using binomial formula and sum of those will give us the cumulative binomial probability.

P(X <= 45) = 0.184

Bernoulli Distribution

Bernoulli Distribution

A Bernoulli Distribution is a case of binomial distribution where

- Experiment has only two possible outcomes viz. Success, Failure
- Number of trial is ‘1’

(Or we can say Binomial Distribution is a case of n independent Bernoulli trials)

A random variable ‘X’ which has a Bernoulli distribution can take:

- value ‘1’ with the possibility of Success, say ‘p’,
- value ‘0’ with the possibility of Failure, say ‘1 – p’ or ‘q’

Thus the **Probability Mass Function (PMF):**

$P(X) = p^{x} * (1 - p)^{1-x}$

$$P(X) = p^{x} * q^{1-x}$$

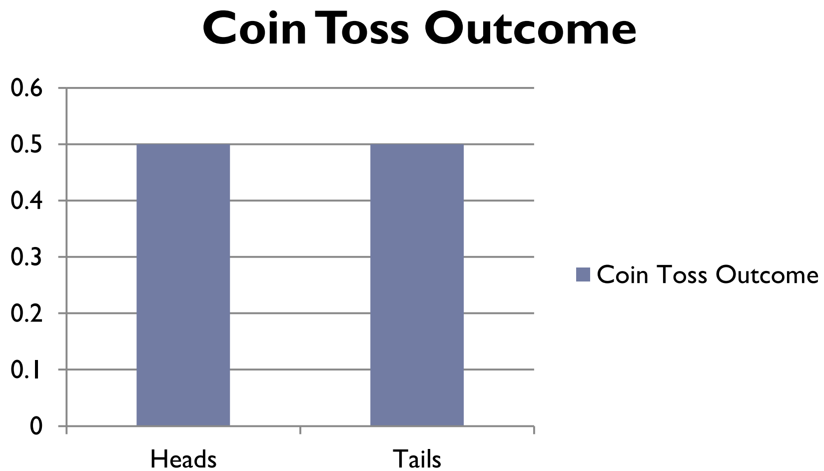
where, x is (1,0)

**Note ‘x’ is sometimes also denoted as ‘k’

Bernoulli Distribution

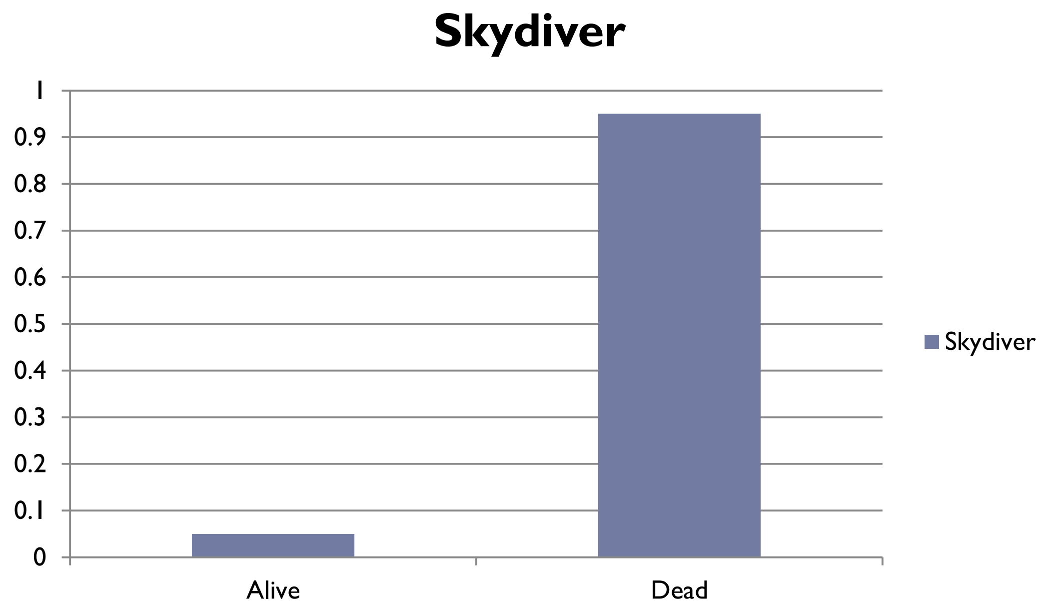
Properties of Bernoulli Distribution:
The mean of distribution (μ) = p
The variance (σ^2) = μ * (1 – p)
The Standard Deviation (σ) = sqrt[σ^2]

Example: Lets say, We toss a single coin. The occurrence of Heads denoted Success and the Tails denoted Failure.
Hence: P(Heads) = 0.5, P(Tails) = 0.5



Bernoulli Distribution

However, the probability of success and Failure need not be equally likely. Lets say, a skydiver jumps out of airplane without parachute **Hence,** In this case, the probability of him/her landing safe and alive is, may be 5% (0.05) while the failure can be 0.95.



Poisson Distribution

Poisson Distribution

The Poisson distribution is the discrete probability distribution of the number of events occurring in a given time period, given the average number of times the event occurs over that time period.

Following are the characteristics of a Poisson experiment:

- Outcome of an event/trial can be either Success or Failure. i.e. only two outcomes.
- The average number of Success (μ or λ) in a specified range is known.
- The trial are independent of each other. i.e Outcome of one trial does not affect the outcome of subsequent ones.
- The Probability that an event occur in a given length of time does not change through the time.

  i.e. say, if there are r cars crossing a signal during an hours of a day, then more or less 5 cars shall cross the signal during the rest of the hours for the day

Poisson Distribution

If those four conditions/properties hold true, then the number of events in a fixed unit of time (or continuum) has a Poisson distribution.

The Poisson situation is most often invoked for rare events.

The Probability Mass Function:

$$\boxed{P(k \text{ events in interval}) = e^{-\lambda}\frac{\lambda^k}{k!}}$$

Where,

k = takes value 0,1,2,3,….., ∞ (sometimes also denoted as ‘x’)

λ = Average number of events per interval

e = is the number 2.71828

Poisson Distribution

Properties of Poisson Distribution:

The mean of distribution (λ) = μ

The variance (σ^2) = λ

The Standard Deviation (σ) = sqrt[σ^2]

Poisson Distribution has some hint of right skewness , but it depends on the value of λ:

- When λ is larger -> Distribution will be close to symmetric
- When λ is small (close to zero) -> Distribution will be right skewed

Note:

Binomial Distribution tends toward the Poisson distribution as

$n \to \infty$, $p \to 0$ and $n*p$ = constant

i.e. The Poisson distribution with $\lambda=n*p$ closely approximate to binomial distribution if ‘n’ is large and ‘p’ is small.

Poisson Distribution

Example:
The mean number of people arriving per hour at a shopping center is 18. Find the probability that the number of customers arriving in an hour is 20.

Solution:
r = 20
λ = 18
e = 2.7183
Find P(20)

$$P(20) = \frac{18^{20} e^{-18}}{20!} = 0.0798$$

There is almost an 8% chance that twenty people will arrive in an hour.

Normal Distribution

Normal Distribution

“ The normal distribution is a probability function that describes how the values of a variable are distributed.“

- The normal distribution is the most important probability distribution in statistics because it fits many natural phenomena.
  For example, heights, blood pressure, measurement error, and IQ scores follow the normal distribution.

- It is also known as the **Gaussian distribution** or the **bell curve**.

Normal Distribution

Often times, data is described as being ‘normal’ (In statistical sense);
But what does that mean ?

Say, we consider some natural or man-made events
Natural: Human Height, Weight, Temperature, blood-pressure etc.
Man-Made: Financial data, sales etc.

For these measures, and many more, the average (mean) tends to be very frequent or closely spaced while measures away from mean are less frequent.

i.e. The data in most of the phenomenon tends to clump around the mean

Normal Distribution

Properties of Normal Distribution Curve:

1. Lower tail and Upper tail
2. Probability Area: We are interested in everything underneath the curve.
3. Symmetric: Left area is equal to right when we talk about perfect theoretical normal distribution.
4. Mean = Median = Mode and lie in exact center of the graph.
5. Top point represents the mode

Normal Distribution

Parameters of Normal Distribution:

The normal distribution the two parameters viz. mean and standard deviation which gives the curve its shape and is denoted as follows:

$$X \sim N(\mu, \sigma^2)$$

where,

μ = Mean

σ = Standard Deviation

Significance of Parameters:

Mean give the curve its position on the axis

Standard Deviation gives the curve its shape.

Normal Distribution

Probability Density Function (PDF) :

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

where,
μ = Mean
σ^2 = Variance

Standard Distribution

Standard Distribution

Standard Distribution or Standard Normal Distribution or Z Distribution is a Normal Distribution with

- Mean of distribution = 0
- Standard Deviation = 1

Standard Distribution is the process of transforming, also called as Standardization, with the mean of zero and standard deviation of one.

Formula for Standardization:

$z = (X_i - \mu) / \sigma$

where,

X_i = each element of sample/population

Standard Distribution

Standard Distribution is the process of transforming, also called as Standardization, with the mean of zero and standard deviation of one.

Formula for Standardization:

$z = (X_i - \mu) / \sigma$

where,

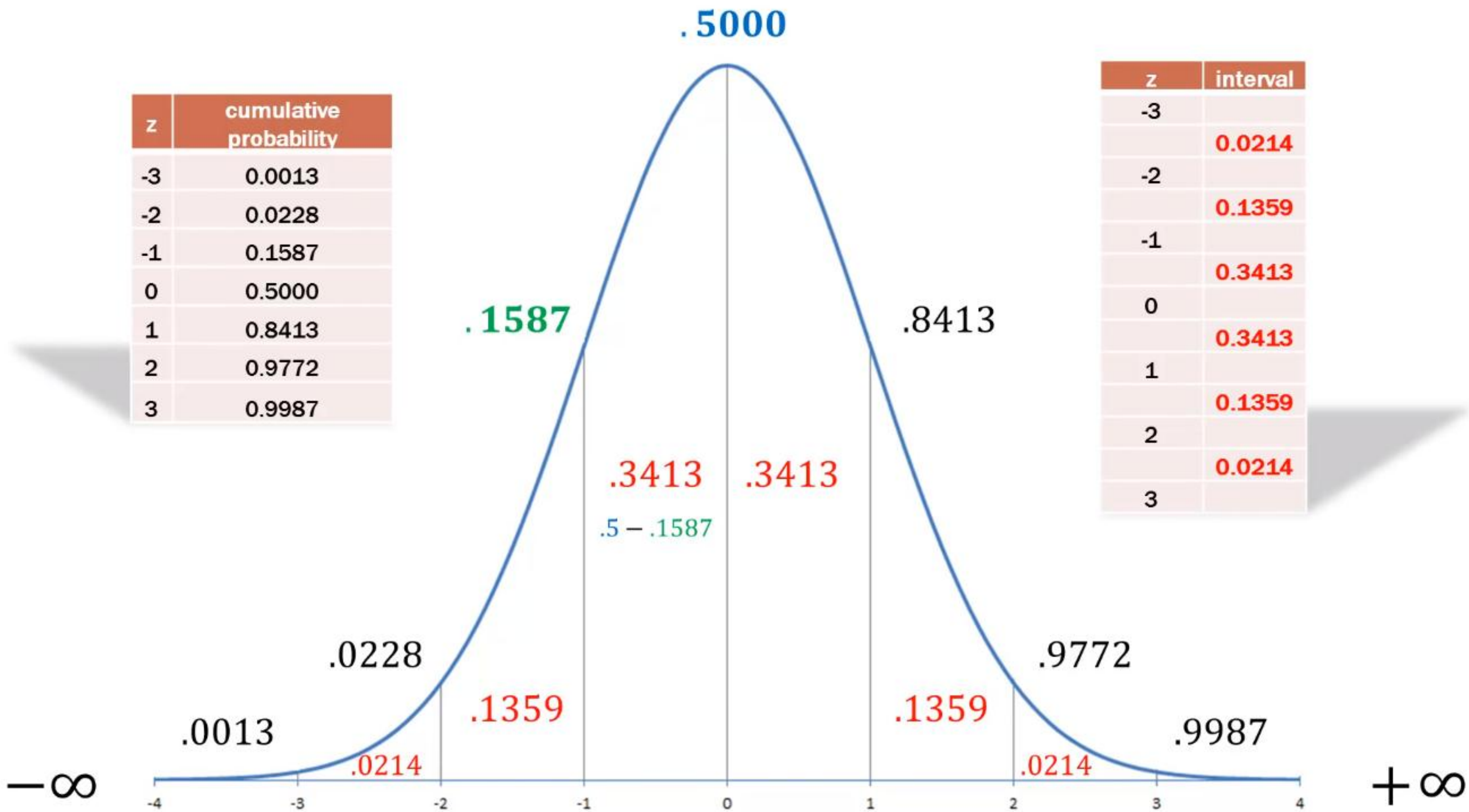
X_i = each element of sample/population

Formula for Z score:

$z = (X - \mu) / \sigma$ (Population)

$z = (X - \bar{x}) / s$ (Sample)

Standard Distribution



z	cumulative probability
-3	0.0013
-2	0.0228
-1	0.1587
0	0.5000
1	0.8413
2	0.9772
3	0.9987

z	interval
-3	
	0.0214
-2	
	0.1359
-1	
	0.3413
0	
	0.3413
1	
	0.1359
2	
	0.0214
3	

That’s all Folks