Exercise 1: Ramsey Model under Marshallian Externalities (20 points)

Consider a competitive economy. Time is continuous and indexed by $t \in \mathbb{R}$.

Households. There is a continuum of mass one of identical, infinitely lived households. Each household is endowed with $L > 0$ units of working time per period, which is supplied inelastically to the labor market. The typical household is assumed to solve the following problem

$$\max_{\{c(t)\}} \int_{t=0}^{\infty} \ln\left[c(t)\right] \cdot e^{-\rho \cdot t} \mathrm{d}t$$
$$\text{s.t. } \dot{a}(t) = r(t)a(t) + w(t)L - c(t),$$
$$a(0) = a_0$$

where $a(t)$ denotes wealth at time t, $r(t)$ the competitive interest rate, $w(t)$ the competitive wage rate per unit of working time, and $c(t)$ consumption, and $\rho > 0$ the discount rate. The No-Ponzi-Game condition is assumed to hold.

Firms. There is a continuum of mass one of identical final output firms. Each firm has access to the following production technology

$$Y(i,t) = K(i,t)^{\alpha}\left[A^{\beta} \cdot L(i,t)\right]^{1-\alpha},$$

where $Y(i,t)$ denotes final output of firm i at time t, $K(i,t)$ capital employed by firm i, $L(i,t)$ the amount of labor employed by firm i. The level of technology depends positively on the average (across firms) capital stock, i.e. $A = \int_0^1 K(i,t)di$.

<u>Assumption</u>: The technology parameters satisfy $0 < \alpha < 1$ and $\beta = 1$.

(a) Determine and describe the economy's steady state with regard to aggregate capital $K(t) = \int_{i=0}^{1} K(i,t)\, di$ and aggregate output $Y(t) = \int_{i=0}^{1} Y(i,t)\, di$. Explain every step of your derivation. [*Simplification*: You can take for granted the following Keynes-Ramsey rule: $\dot{c}(t) = c(t)(r(t) - \rho)$.] (*8 points*)

Exam 2021 1

$$\max_{K,L} K_{it}^{\alpha}(A^{\beta}L_{it})^{1-\alpha} - wL_{it} - rK_{it}$$

$$FOC_K \quad r = \alpha K_{it}^{\alpha-1}(A^{\beta}L_{it})^{1-\alpha}$$

$$r = \alpha K_t^{\alpha-1+\beta-\alpha\beta} L_t^{1-\alpha}$$

spillover: $A = \int_0^1 K_{it}\, di = K$
+ symm Eq.

$$FOC_L \quad w = (1-\alpha)K_t^{\alpha+\beta-\alpha\beta} L_t^{-\alpha}$$

a.) Solution 1

GE: $\dot{K}_t = Y_t - C_t$

$$\hat{K} = \frac{Y_t}{K_t} - \frac{C_t}{K_t} = \text{const}$$

requires: $\frac{Y_t}{K_t} = \text{const} \Rightarrow \hat{Y} = \hat{K}$

$\frac{C_t}{K_t} = \text{const} \Rightarrow \hat{C} = \hat{K}$

$$\Rightarrow \underbrace{\hat{Y} = \hat{C} = \hat{K} = \hat{A}}_{\text{Spill-over \& symmetric eq.: } A = \bar{K} = \int K_i \, di = K} = g$$

↳ In steady state, the aggregate output and the aggregate capital grow with the same growth rate g.

Solution 2: Zero growth steady-state

$$\hat{C} = r - \rho$$

$$= \alpha K_t^{\alpha - 1 + \beta - \alpha\beta} L_t^{1-\alpha} - \rho \overset{!}{=} 0$$

$$\Leftrightarrow \hat{K} = \left(\frac{\rho}{\alpha L_t^{1-\alpha}}\right)^{\frac{1}{\alpha - 1 + \beta - \alpha\beta}}$$

GE $\dot{K}_t = Y_t - C_t \qquad | A = K$

$$= K_t^{\alpha + \beta - \alpha\beta} L_t^{1-\alpha} - C_t \overset{!}{=} 0$$

$$\Rightarrow \hat{C}_t = \hat{K}_t^{\alpha + \beta - \alpha\beta} L_t^{1-\alpha}$$

$$\hat{C}_t = \left(\frac{\rho}{\alpha L_t^{1-\alpha}}\right)^{\frac{\alpha + \beta - \alpha\beta}{\alpha + \beta - \alpha\beta - 1}} L_t^{1-\alpha}$$

(b) Explain briefly whether the allocation of resources in the market economy represents a first-best allocation. Provide an economic reasoning. A brief verbal explanation suffices. (*6 points*)

b.) Allocation is not optimal because firms pay only their private MPC for capital and dont consider the positive spillover effects of capital

$$r_{priv} = \alpha K_t^{\alpha-1+\beta-\alpha\beta} L_t^{1-\alpha}$$

Aggregate inputs maximization

$$\max_{L,K} K_t^{\alpha+\beta-\alpha\beta} L_t^{1-\alpha} - wL_t - rK_t$$

$$FOC_K: \underbrace{(\alpha+\beta-\alpha\beta)}_{>0} K_t^{\alpha+\beta-\alpha\beta-1} L_t^{1-\alpha} = r_{agg}$$

$$r_{agg} > r_{priv}$$

(c) Determine the optimal tax / subsidy on capital. (*Hint*: The easiest way to do this is to compare the competitive interest rate to the social marginal product of capital.) (*6 points*)

c.) Capital should be subsidized because then price of capital for firms is lower and firms demand more capital.

$$(1-s) r_{priv} = \alpha K_t^{\alpha-1+\beta-\alpha\beta} L_t^{1-\alpha}$$

$$r^*_{priv} = \frac{1}{1-s}\left(\alpha K_t^{\alpha-1+\beta-\alpha\beta} L_t^{1-\alpha}\right)$$

$$r^*_{priv} \overset{!}{=} r_{agg}$$

$$\Rightarrow \frac{1}{1-s}\left(\alpha K_t^{\alpha-1+\beta-\alpha\beta} L_t^{1-\alpha}\right) = (\alpha+\beta-\alpha\beta) K_t^{\alpha+\beta-\alpha\beta-1} L_t^{1-\alpha}$$

$$\Rightarrow s^* = 1 - \frac{\alpha}{\alpha+\beta-\alpha\beta} = \frac{\beta-\alpha\beta}{\alpha+\beta-\alpha\beta}$$