UNIVERSITÀ
DEGLI STUDI
DI PADOVA

Encoding BN in WMC

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Problem Definition

Bayesian Network

Definition 1: A Bayesian network on a set of random variables $\boldsymbol{X} = \{X_1, \dots, X_n\}$ is a pair $\mathcal{B} = (G, Pr)$ is a pair composed of a directed acyclic graph $G = ([n], E)$ (where $[n] = \{1, \dots, n\}$) and Pr specifies the conditional probabilities

$$\Pr\left(X_i = x_i \mid \boldsymbol{X}_{\mathrm{par}(i)} = \boldsymbol{x}_{\mathrm{par}(i)}\right)$$

for every $X_i \in \boldsymbol{X}$. $\mathcal{B}$ uniquely define the join distribution on $\boldsymbol{X}$

$$\Pr(\boldsymbol{X} = \boldsymbol{x}) = \prod_{i=1}^{n} \Pr\left(X_i = x_i \mid \boldsymbol{X}_{\mathrm{par}(i)} = \boldsymbol{x}_{\mathrm{par}(i)}\right)$$

Problem Definition

Weighted Model Counting

Definition 2: For every set of propositional variables $\mathcal{P}$ and weight function $w : 2^{\mathcal{P}} \rightarrow \mathbb{R}^{+}$. the weighted model counting of a propositional formula ϕ with propositional variables in $\mathcal{P}$ w.r.t, w, is defined as:

$$\text{WMC}(\phi, w) = \sum_{\mathcal{I}:\mathcal{P}\rightarrow\{0,1\}} \mathcal{I}(\phi) \cdot w(\mathcal{I})$$

Problem Definition

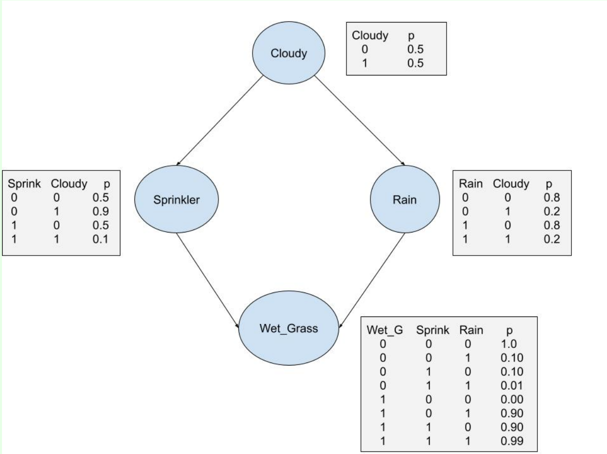
Conditional Probability Queries

Definition 3: Given a Bayesian Network $\mathcal{B}$ on a set of propositional variables $\mathcal{P}$ a conditional probability queries is an expression of the form $\mathsf{Pr}_{\mathcal{B}}(\phi \mid \psi)$, where ψ is called the evidence, and ϕ the query. The answer to this query is:

$$\frac{\mathsf{Pr}_{\mathcal{B}}(\phi \wedge \psi)}{\mathsf{Pr}_{\mathcal{B}}(\phi)} = \frac{\sum_{\mathcal{I} \models \phi \wedge \psi} \mathsf{Pr}_{\mathcal{B}}(\mathcal{I})}{\sum_{\mathcal{I} \models \psi} \mathsf{Pr}_{\mathcal{B}}(\mathcal{I})}$$

Problem Definition

Example



Cloudy	p
0	0.5
1	0.5

Sprink	Cloudy	p
0	0	0.5
0	1	0.9
1	0	0.5
1	1	0.1

Rain	Cloudy	p
0	0	0.8
0	1	0.2
1	0	0.8
1	1	0.2

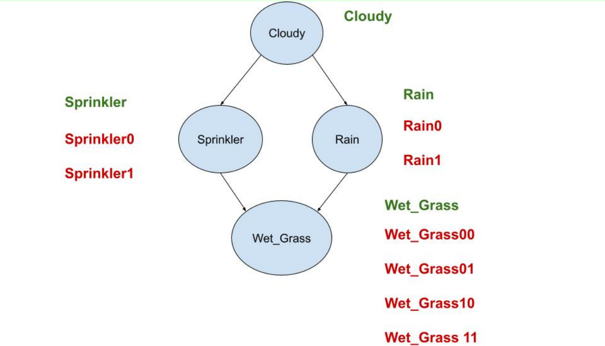
Wet_G	Sprink	Rain	p
0	0	0	1.0
0	0	1	0.10
0	1	0	0.10
0	1	1	0.01
1	0	0	0.00
1	0	1	0.90
1	1	0	0.90
1	1	1	0.99

Encoding Procedure

Define propositional variables

For every node p with $k > 0$ parents introduce 2^k new propositional variables p_b for $\boldsymbol{b} \in \{0, 1\}^k$.

Example

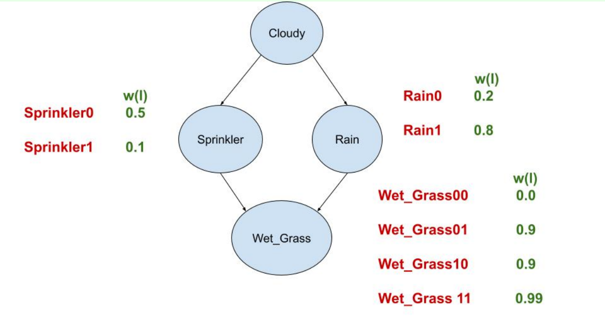


Encoding Procedure

Define the weights of propositional variables

$$w_{\mathcal{B}}\left(p_{\boldsymbol{b}}\right) \triangleq \Pr(p = 1 \mid \mathrm{par}(p) = \boldsymbol{b})$$

Example



	w(l)
Sprinkler0	0.5
Sprinkler1	0.1

	w(l)
Rain0	0.2
Rain1	0.8

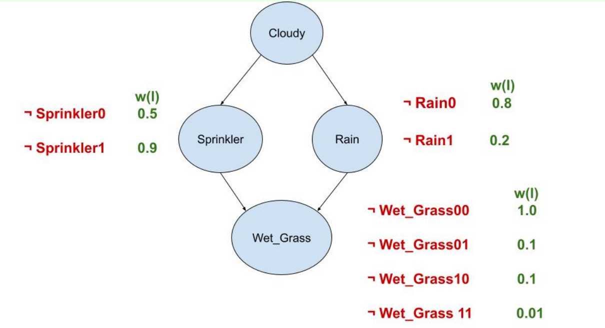
	w(l)
Wet_Grass00	0.0
Wet_Grass01	0.9
Wet_Grass10	0.9
Wet_Grass 11	0.99

Encoding Procedure

Define the weights of negative propositional variables

$$w\left(\neg p_{\boldsymbol{b}}\right) \triangleq 1 - w\left(p_b\right)$$

Example



	w(l)
¬ Sprinkler0	**0.5**
¬ Sprinkler1	**0.9**

	w(l)
¬ Rain0	**0.8**
¬ Rain1	**0.2**

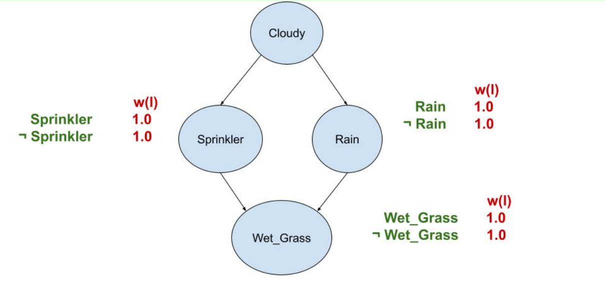
	w(l)
¬ Wet_Grass00	**1.0**
¬ Wet_Grass01	**0.1**
¬ Wet_Grass10	**0.1**
¬ Wet_Grass 11	**0.01**

Encoding Procedure

Define the weights of all the other literals

Set the weight of all the other literals to 1

Example



	w(l)
Sprinkler	1.0
¬ Sprinkler	1.0

	w(l)
Rain	1.0
¬ Rain	1.0

	w(l)
Wet_Grass	1.0
¬ Wet_Grass	1.0

Encoding Procedure

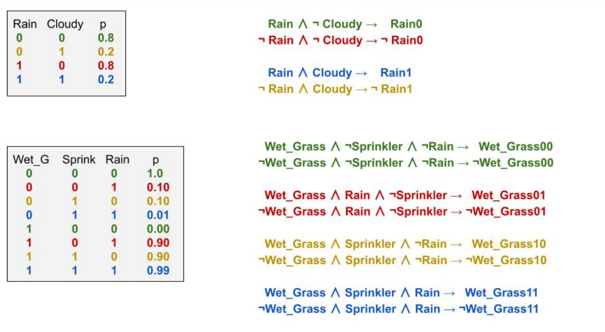
Connect new variables to the variables in the graph

$$p \wedge \left(\bigwedge_{\substack{i=1\\b_i=1}}^{k} p_i \wedge \bigwedge_{\substack{i=1\\b_i=0}}^{k} \neg p_i \right) \rightarrow p_{\boldsymbol{b}}$$

$$\neg p \wedge \left(\bigwedge_{\substack{i=1\\b_i=1}}^{k} p_i \wedge \bigwedge_{\substack{i=1\\b_i=0}}^{k} \neg p_i \right) \rightarrow \neg p_{\boldsymbol{b}}$$

Encoding Procedure

Example



Rain	Cloudy	p
0	0	0.8
0	1	0.2
1	0	0.8
1	1	0.2

Rain ∧ ¬ Cloudy → Rain0
¬ Rain ∧ ¬ Cloudy → ¬ Rain0

Rain ∧ Cloudy → Rain1
¬ Rain ∧ Cloudy → ¬ Rain1

Wet_G	Sprink	Rain	p
0	0	0	1.0
0	0	1	0.10
0	1	0	0.10
0	1	1	0.01
1	0	0	0.00
1	0	1	0.90
1	1	0	0.90
1	1	1	0.99

Wet_Grass ∧ ¬Sprinkler ∧ ¬Rain → Wet_Grass00
¬Wet_Grass ∧ ¬Sprinkler ∧ ¬Rain → ¬Wet_Grass00

Wet_Grass ∧ Rain ∧ ¬Sprinkler → Wet_Grass01
¬Wet_Grass ∧ Rain ∧ ¬Sprinkler → ¬Wet_Grass01

Wet_Grass ∧ Sprinkler ∧ ¬Rain → Wet_Grass10
¬Wet_Grass ∧ Sprinkler ∧ ¬Rain → ¬Wet_Grass10

Wet_Grass ∧ Sprinkler ∧ Rain → Wet_Grass11
¬Wet_Grass ∧ Sprinkler ∧ Rain → ¬Wet_Grass11

Final steps and CPQ's calculation

Additional Steps Required by SAT Solver

- Map all the propositional variables to integers.
- Convert implications to CNF.
- Calculate the weight for every interpretation that satisfies $\Phi_{\mathcal{B}}$.

Final steps and CPQ's calculation

CPQ's calculation

$$Pr(\phi \mid \psi) = \frac{\mathsf{WMC}(\Phi_{\mathcal{B}} \wedge \phi \wedge \psi, \mathbf{w_B})}{\mathsf{WMC}(\Phi_{\mathcal{B}} \wedge \psi, \mathbf{w_B})}$$

Demo