Section 1.Change of Variables

So far we have introduced techniques for solving separable and first-order linear differential equations. Clearly, most first-order differential equations are not of these two types. In this section, we consider two further types of differential equations that can be solved by using a change of variables to reduce them to one of the types we know how to solve. The key point to grasp in this section, however, is not the specific changes of variables that we discuss, but the general idea of changing variables in a differential equation. Further examples are considered in the exercises. We first require a preliminary definition.

Homogeneous of Degree Zero

A function $f(x, y)$ is said to be **homogeneous of degree zero** if

$$f(tx, ty) = f(x, y)$$

for all positive values of t for which (tx, ty) is in the domain of f.

The simplest nonconstant functions that are homogeneous of degree zero are

$$f(x, y) = \frac{y}{x} \text{ and } f(x, y) = \frac{x}{y}.$$

Example 1. Show that the equation

$$f(x, y) = \frac{x^2+3xy-y^2}{x^2+4y^2}$$

is homogeneous of degree zero.

Homogeneous first-order Differential Equation

If $f(x, y)$ is homogeneous of degree zero, then the differential equation

$$\frac{dy}{dx} = f(x, y)$$

is called a **homogeneous first-order differential equation**.

How to solve: Use the change of variable

$$V = \frac{y}{x} \text{ or equivalently } y = xV(x).$$

Example 2. Find a general solution to

$$\frac{dy}{dx} = \frac{4x-y}{x-4y}$$