Section 2.5 Gaussian Elimination

We now illustrate how elementary row operations applied to the augmented matrix of a system of linear equations can be used first to determine whether the system is consistent, and second, if the system is consistent, to find all of its solutions. In doing so, we will develop the general theory for solving linear systems of equations.

Therefore, starting with the augmented matrix for any linear system, we may apply elementary row operations to bring it to row-echelon form, and then solve the resulting simpler linear system. Let us illustrate with an example.

Example 1. Determine the solution set to

$$\begin{array}{rcrcrcr} 4x_1 & - & 3x_2 & + & 6x_3 & = & 2 \\ x_1 & - & 3x_2 & + & 6x_3 & = & 5 \\ -2x_1 & + & 3x_2 & - & 8x_3 & = & -6 \end{array}$$

Lemma: Uniq Solution

Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let $A^{\#}$ denote the augmented matrix of the system.

- If $rank(A) = rank(A^{\#}) = n$, then the system has a **unique solution.**
- If $rank(A) < rank(A^{\#})$, then the system is **inconsistent**

$$A^{\#} = \left[\begin{array}{cccccc|c} 1 & * & * & * & \cdots & * & * \\ 0 & 1 & * & * & \cdots & * & * \\ 0 & 0 & 1 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & * \end{array}\right]$$

Example 2. Determine the solution set to

$$\begin{array}{ccccl} x_1 & +x_2 & -x_3 & +x_4 = & 1 \\ 2x_1 & +3x_2 & +x_3 & = & 4 \\ 3x_1 & +5x_2 & +3x_3 & -x_4 = & 5 \end{array}$$

Lemma: Infinite Solutions and Free Variables

Consider the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$. Let $A^{\#}$ denote the augmented matrix of the system.

- If $rank(A^{\#}) < n$, then the system has **infinite number of solutions.**

$$A^{\#} = \left[\begin{array}{cccccc|c} 1 & * & * & * & \cdots & * & * \\ 0 & 1 & * & * & \cdots & * & * \\ 0 & 0 & 1 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic Variable= the unknown corresponding to the pivot column
Free Variable= the unknown corresponding to the non-pivot column

Example 3. Use Gaussian elimination to solve

$$\begin{array}{ccccc} x_1 & -2x_2 & +x_3 & -x_4 = & 3 \\ 3x_1 & +x_2 & +6x_3 & +11x_4 = & 16 \\ 2x_1 & -x_2 & +4x_3 & +4x_4 = & 9 \end{array}$$

Solution: A row-echelon form of the augmented matrix of the system is
$$\left[\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$