Chapter 2 Matrix Algebra

In the theory of equations, the simplest equations are the linear ones. More generally, any equation of the form

$$\mathbf{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$$

is called a linear equation, where $\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}$ and $\mathbf{b}$ are constants, and $\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n}$ are unknowns. Often, several linear equations need to be considered at once, in which case we can refer to a system of linear equations. The next two chapters are concerned with giving a detailed introduction to the theory and solution techniques for such systems.

$$\begin{aligned}
x_1 \qquad &+4x_3 = 0 \\
x_1 + x_2 \; &-3x_3 = 4 \\
3x_1 + x_2 \; &+5x_3 = -3 \\
3x_1 - x_2 &+19x_3 = 0
\end{aligned}$$

We see that this system is completely determined by the array of numbers (matrix)

Section 2.1 Matrix Notations

Notations.

A general matrix A of **size** $m \times n$ is written

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \qquad \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

If $m = n$, A is called a **square matrix.**

Equal Matrices

Two matrices A and B are equal, written $A = B$, if

1. They both have the same size, $m \times n$.
2. All corresponding elements in the matrices are equal: $a_{ij} = b_{ij}$ for all i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$.

Row Vectors and Column Vectors

Two matrices A and B are equal, written $A = B$, if
A $1 \times n$ matrix is called a **row** $n-$**vector**. An $n \times 1$ matrix is called a **column** $n-$**vector.** The elements of a row or column n-vector are called the **components** of the vector

Some Specific Matrices

In an $n \times n$ matrix A, **Diagonal Enteries:** are $a_{11}, a_{22}, \cdots, a_{nn}$

-Diagonal Matrix : $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}_{n\times n}$ **-Identity Matrix :** $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$

Lower Triangular Matrix

An nn matrix $A = [a_{ij}]$ is said to be **lower triangular** if $a_{ij} = 0$ whenever $i < j$ (zeros everywhere above (i.e., "northeast of") the main diagonal),

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ \vdots & a_{22} & 0 & 0 & 0 \\ a_{i1} & \cdots & a_{ii} & 0 & 0 \\ \vdots & & \vdots & & 0 \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

and it is said to be upper triangular if $a_{ij} = 0$ whenever $i > j$ (zeros everywhere below (i.e., "southwest of") the main diagonal).

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ 0 & a_{22} & \vdots & & \vdots \\ 0 & 0 & a_{ii} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & a_{nn} \end{bmatrix}_{n\times n}$$

THE TRANSPOSE OF A MATRIX

Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Symmetric Matrix

A square matrix A satisfying $A^T = A$ is called a **symmetric matrix.**

If A satisfying $A^T = -A$, is called a **skew-symmetric** (or anti-symmetric)

Section 2.2 Matrix Agebra

Sums and Scalar Multiplication

$A + B =$ $\qquad\qquad$ $rA =$

Properties of Sum and Scalar Multiplication

Let A, B, and C be matrices of the same size, and let r and $s$be scalars.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $r(A + B) = rA + rB$
4. $(r + s)A = rA + sA$
5. $r(sA) = (rs)A$

MATRIX MULTIPLICATION

If A is an $m \times n$ matrix and B is an $n \times p$ then the product AB is an $m \times p$ matrix. The entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)ij$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Example 1. Compute AB where $B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 4 \end{bmatrix}$.

Properties of Matrix Multiplication

Let A, B, and C be matrices of the same size, and let r be any scalar.

1. $(AB)C = A(BC)$ (associative law of multiplication)
2. $A(B + C) = AB + AC$ (left distributive law)
3. $(B + C)A = BA + CA$ (right distributive law)
4. $(rAB) = (rA)B = A(rB)$
5. $I_m A = A = AI_n$ (identity for matrix multiplication)

WARNINGS:

- In general, $AB \neq BA$.
- $AB = AC \not\Rightarrow B = C$.
- $AB = \mathbf{0} \not\Rightarrow A = \mathbf{0}$ or $B = \mathbf{0}$.

Properties of A^T

Let A, B, and C be matrices of the same size, and let r be any scalar.

1. $(A^T)^T = A$.
2. $(A + B)^T = A^T + B^T$.
3. $(rA)^T = rA^T$.
4. $(AB)^T = B^T A^T$.