2.6 The Inverse of a Matrix

This section investigate the matrix reciprocal, or multiplicative inverse of a nonzero number.

Recall that the multiplicative inverse of a number such as 5 is $\frac{1}{5} = 5^{-1}$. this inverse satisfies the equation

$$5^{-1} \cdot 5 = 1 \text{ and } 5 \cdot 5^{-1} = 1$$

Definition. An $n \times n$ matrix A is said to be **invertible** if there is another matrix C such that

$$\mathbf{CA} = \mathbf{I} \text{ and } \mathbf{AC} = \mathbf{I}.$$

Where I is the identity matrix.

- In this case C is the **inverse** of A and it is denoted by $\mathbf{A^{-1}}$.
- The inverse of a matrix is unique (why?) and we write

$$\mathbf{A^{-1}A} = \mathbf{I} \text{ and } \mathbf{AA^{-1}} = \mathbf{I}.$$

Example 1. Show that if $A^4 = 0$, then $I - A$ is invertible with

$$(I - A)^{-1} = I + A + A^2 + A^3.$$

How do we find the inverse of a matrix?
We start by 2×2 matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The quantity $ad - bc$ is called $det(A)$.

Example 1. Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Theorem 5. If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$

Proof.

Example 2. Use the inverse of the coefficient matrix to to solve the system.

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7 \end{aligned} .$$

Theorem 6. If A and B are both $n \times n$ invertible matrices, then

a) $(\mathbf{A^{-1}})^{-1} = \mathbf{A}$

b) $(\mathbf{AB})^{-1}) = \mathbf{B^{-1}A^{-1}}$

c) $(\mathbf{A^T})^{-1} = (\mathbf{A^{-1}})^{\mathbf{T}}$

Elementary Matrices

An **elementary matrix** is one that is obtained by performing one elementary roe operation on the identity matrix. Here are some examples.

The relation between the elementary matrices and elementary row operation.

General Facts.

- If an elementary row operation is performed on a matrix A the resulting matrix can be written as EA, where E is the correspondence elementary matrix.
- Each elementary matrix E is invertible.
- A is invertible if and only if A is row equivalent the the identity matrix $\mathbf{I}$.

How to find the inverse of a matrix of any size.

Apply elementary row operation on the augmented matrix $[\mathbf{A}\ \ \mathbf{I}]$ to get $[\mathbf{I}\ \ \mathbf{A^{-1}}]$.

Example 3. Find the inverse of the matrix if it exits.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$