1.7 Modeling Problems

In this chapter we concentrate on examples of models that involve first-order differential equations. In studying these models, the following broad outline of the process may be helpful.

Formulate the problem

Develop the model

Test the model

Mixing Problems

Many complicated processes can be broken down into distinct stage and entire system modeled by describing the interactions between the various stages. Such systems are called **compartmental** and geometrically depicted by **block diagram**.

The basic one-compartment system consists of a function $x(t)$ that presents the amount of a substance in the compartment at the time t, and an input rate at which the substance enters the compartment, and an output rate at which the substance leave the compartment.

Because the derivative of x with respect to t can be interpreted as the rate of change in the amount of the substance in the compartment, the one-compartment system suggests

$$\frac{dx}{dt} = \text{input rate -output rate}$$

as a mathematical model for the process.

Mixing Problems

$$\begin{aligned} A(t) &= \text{the amount of the substance in the tank at the time } t \\ V(t) &= \text{the volume of the fluid in the tank} \\ c(t) &= \text{concentration at the time } t \end{aligned}$$

$$\text{concentration} = \frac{\text{amount of mass}}{\text{volume}}, c = \frac{A}{V} \Rightarrow A = c \cdot V$$

We have $V_1 = r_1 \cdot \Delta t$ and $v_2 = r_2 \cdot \Delta t$ where r_1, r_2 are the rate of flow in and flow out, moreover,

$$\begin{aligned} \Delta v &= V_1 - V_2 \\ &= r_1 \cdot \Delta t - r_2 \cdot \Delta t \\ &= (r_1 - r_2)\Delta t \end{aligned}$$

, then

$$\frac{dV}{dt} = r_1 - r_2 \Rightarrow V = (r_1 - r_2)t + V_\circ.$$

On the other hand, $A = cV$ and

$$\begin{aligned} \Delta A &= c_1 V_1 - c_2 V_2 \\ &= c_1 r_1 \Delta t - c_2 r_2 \Delta t. \end{aligned}$$

This implies that

$$\frac{dA}{dt} = r_1 c_1 - r_2 \frac{A(t)}{V(t)},$$

and then we have a linear differential equation

$$\frac{dA}{dt} + \frac{r_2}{(r_1 - r_2)t + V_\circ} A = r_1 c_1$$

Example 1. Consider a large tank holding 1000 L of pure water into which a brine solution of salt beings to flow at a rate of 6 L/min. The solution inside the tank is kept well stirred and is flowing out of the tank at a rate of 6 L/min. If the concentration of salt in the brine entering the tank is 0.1 kg/L, determine when the concentration of salt in the tank will reach 0.05 kg/L.

Example 2. A tank initially has 600 L of a solutions with 1500 grams of salt. A solution with 5 g/L flows into the tank at a rate of 6L/min. The mixture flows out at a rate of 3 L/min. Find the concentration of the solution after 1 hour.

Solution: $V_\circ = V(0) = 600, A_\circ = A(0) = 1500, c_1 = 5, r_1 = 6, r_2 = 3$

Example 3. A bucket contains 10 L of water and to it is being added a salt solution that contains 0.3 kg of salt per liter. This salt solution is being poured in at the rate of 2 L/min. The solution is being thoroughly mixed and drained off. The mixture is drained at the same rate so that the bucket contains 10 L at all times. How much salt is in the bucket after 5 min?