Section 2.3 Terminology for Systems of Linear Equations

Terminology.

General System of Equations

The general $m \times n$ system of linear equations is of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{n1}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

where the **system coefficients** a_{ij} and the **system constants** b_j are given scalars and $x_1, x_2, \cdots, x_n$ denote the unknowns in the system

The System is:

1. **homogeneous** if $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$
2. **non-homogeneous** if at least on of $b_i \neq 0$.

Solution to a System

By a **solution** to a system of linear equation (we mean an ordered $n-$tuple of scalars, $(c_1, c_2, ..., c_n)$, which, when substituted for $x_1, x_2, \cdots, x_n$ into the left-hand side of system, yield the values on the right-hand side. The set of all solutions to system is called the **solution set** to the system.

A system of equations that has at least one solution is said to be **consistent**, whereas a system that has no solution is called **inconsistent.**

Example 1. Determine if the following is consistent.

$$\begin{array}{rcrl} x_1 & + & x_2 = & 3 \\ 3x_1 & - & 2x_2 = & -1 \end{array}$$

$$\begin{array}{rcrl} 2x_1 & + & 3x_2 = & 3 \\ 6x_1 & - & 9x_2 = & -1 \end{array}$$

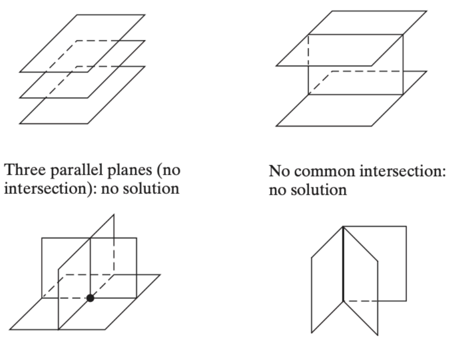
Returning to the general discussion of a system, there are some fundamental questions that we will consider:

- Does the system have a solution (consistent)?
- If the answer is yes, then how many solutions are there?
- How do we determine all of the solutions?

An idea to answer the the above uestions

Now, consider the special case of a system of three equations in three unknowns.

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{array}$$



Three parallel planes (no intersection): no solution

No common intersection: no solution

Planes intersect at a point: a unique solution

Planes intersect in a line: an infinite number of solutions

The Matrix Equation Ax=b

We associate two matrices to a system of linear equations

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & & & & & & & \vdots \\
a_{n1}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}$$

1. The matrix of **coefficents** $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

2. The **augmented matrix** $A^{\#} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{1n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$

Vector form of a system $\mathbf{Ax} = \mathbf{b}$.