Lecture 16

STAT 109: Introductory Biostatistics

Lecture 16: Normal Approximation to the Binomial and One-Proportion Test by Hand

Learning outcomes

- **Use the CLT to approximate the Binomial Distribution with the Normal Distribution.**
- **State the Empirical Rule for the normal distribution.**
- **Draw a rejection region by hand to test a null hypothesis for one proportion/probability p.**

Normal approximation to the binomial (CLT)

When $X \sim \text{Binomial}(n, p)$ with n large enough, the distribution of X is approximately **normal** with the same mean and standard deviation:

- **Mean:** $\mu = np$
- **Standard deviation:** $\sigma = \sqrt{n\,p\,(1-p)}$

So we can approximate $P(X \le x)$ or $P(X \ge x)$ using a normal curve with that μ and σ.

Conditions for the approximation to be reasonable: Both the expected number of successes and the expected number of failures under the null are at least 10:

- $n\,p \ge 10$
- $n\,(1-p) \ge 10$

This keeps the binomial from being too skewed so the normal curve fits reasonably well.

Empirical Rule (normal distribution)

For a **normal distribution** with mean μ and standard deviation σ:

- About **68%** of the distribution lies within **1** standard deviation of the mean: between $\mu - \sigma$ and $\mu + \sigma$.
- About **95%** lies within **2** standard deviations: between $\mu - 2\sigma$ and $\mu + 2\sigma$.
- About **99.7%** lies within **3** standard deviations: between $\mu - 3\sigma$ and $\mu + 3\sigma$.

Visualize with the One Proportion applet

Use the **One Proportion** applet to see the binomial distribution and its normal approximation

One Proportion applet

Try different values of n and p: when $np \ge 10$ and $n(1-p) \ge 10$, the normal curve overlays the binomial bars closely. When those conditions fail, the binomial is skewed and the normal fit is poor.

Drawing a rejection region by hand

To test a null hypothesis about a proportion (e.g. $H_0 : p = 0.5$ for a fair coin):

1. Under H_0, identify n and p; compute $\mu = np$ and $\sigma = \sqrt{np(1-p)}$.
2. Sketch a **normal curve** centered at μ with spread σ (or mark μ, $\mu \pm \sigma$, $\mu \pm 2\sigma$ using the Empirical Rule).
3. Mark the **observed value** of x (the count) on the horizontal axis.
4. **Shade the rejection region**: the tail(s) in the direction of H_a (e.g. for $H_a : p > 0.5$, shade the right tail; for $H_a : p \neq 0.5$, shade both tails).
5. The **p-value** is the area of the shaded region under the normal curve (approximate). If that area is small, reject H_0; if it is large, fail to reject H_0.

Example 1: Testing a coin (reject H_0)

Research question: Is the coin fair? We test $H_0 : p = 0.5$ vs $H_a : p \neq 0.5$, where $p =$ probability of heads on a single flip.

Data: We flip the coin **100** times and observe **35** heads.

- n and p under H_0: $n = 100$, $p = 0.5$.
- **Check conditions:** $np = 50 \geq 10$ and $n(1-p) = 50 \geq 10$. OK to use the normal approximation.
- **Mean and SD:** $\mu = 100 \times 0.5 = 50$, $\sigma = \sqrt{100 \times 0.5 \times 0.5} = 5$.
- **Observed:** $x = 35$ heads. Under a normal curve with mean 50 and SD 5, 35 is **3 standard deviations** below the mean ($z = (35-50)/5 = -3$). The Empirical Rule says only about 0.15% of the distribution is below $\mu - 3\sigma$, so the **left tail** (and the symmetric right tail for a two-sided test) is very small.
- **Rejection region:** Shade both tails (since $H_a : p \neq 0.5$). The observed 35 falls far into the left tail.
- **p-value:** The area in the left tail (and the symmetric right tail) is very small. So the **p-value is small**.
- **Conclusion: Reject** H_0. The data are inconsistent with a fair coin; we have evidence that $p \neq 0.5$ (here, fewer heads than expected).

Example 2: Testing a die (fail to reject H_0)

Research question: Is the die fair for the face "6"? We test $H_0 : p = 1/6$ vs $H_a : p \neq 1/6$, where $p =$ probability of a 6 on a single roll.

Data: We roll the die **60** times and observe **12** sixes.

- n and p under H_0: $n = 60$, $p = 1/6$.
- **Check conditions:** $np = 60 \times (1/6) = 10 \geq 10$ and $n(1-p) = 60 \times (5/6) = 50 \geq 10$. OK to use the normal approximation.
- **Mean and SD:** $\mu = 60 \times (1/6) = 10$, $\sigma = \sqrt{60 \times (1/6) \times (5/6)} = \sqrt{50/6} \approx 2.89$.
- **Observed:** $x = 12$ sixes. Under a normal curve with mean 10 and SD 2.89, $z = (12-10)/2.89 \approx 0.69$. So 12 is a bit **above** the mean, within about one standard deviation. Most of the distribution lies between $\mu - 2\sigma$ and $\mu + 2\sigma$ (about 95%); 12 is well inside that range.
- **Rejection region:** For $H_a : p \neq 1/6$, we would shade both tails. The observed 12 is **not** in the tails; it's near the center.
- **p-value:** The area in the tails (values as or more extreme than 12) is **large** — 12 is a typical outcome when $p = 1/6$.
- **Conclusion: Fail to reject** H_0. The data are consistent with a fair die; we do not have evidence that $p \neq 1/6$.

Summary

Example	H_0	n	Observed x	μ	σ	Location of x	p-value	Conclusion
Coin	$p = 0.5$	100	35 heads	50	5	Far in left tail	Small	Reject H_0
Die	$p = 1/6$	60	12 sixes	10	≈ 2.89	Near center	Large	Fail to reject H_0

Use the One Proportion applet to enter these n and p, see the normal overlay, and confirm the rejection regions and conclusions.