# Lecture 7

**STAT 109: Introductory Biostatistics**

# Lecture 7: Bayes' Rule

Learning Outcomes:

1. State the **law of total probability**..
2. State **Bayes' rule**.

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**Law of Total Probability**

Suppose events $A_1, A_2, \dots, A_k$ are **mutually exclusive** (no two can occur together) and **exhaustive** (one of them must occur). Then for any event $B$,

$$P(B) = P(B \mid A_1)\, P(A_1) + P(B \mid A_2)\, P(A_2) + \cdots + P(B \mid A_k)\, P(A_k).$$

Note: Each term $P(B \mid A_i)\, P(A_i)$ is just $P(B \text{ and } A_i)$ by the multiplication rule (Lecture 6), so this law just says: split $B$ into the cases "$B$ and $A_1$," "$B$ and $A_2$," …, add their probabilities.

**Special case (two events):** Since $A$ and "not $A$" partition the sample space, then

$$P(B) = P(B \mid A)\, P(A) + P(B \mid \text{not } A)\, P(\text{not } A).$$

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**Bayes' Rule**

For events $A$ and $B$ with $P(B) > 0$,

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}$$

In words: the probability of $A$ given $B$ equals the probability of $B$ given $A$ times the probability of $A$, divided by the probability of $B$.

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**Optional Derivation of Bayes' Rule (using facts from Lecture 6)**

We use two facts from Lecture 6:

1. **Definition of conditional probability:** For $P(B) > 0$,

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}.$$

2. **Multiplication rule:** For any two events $A$ and $B$ (with $P(A) > 0$),

$$P(A \text{ and } B) = P(B \mid A)\, P(A).$$

**Derivation:** Start with the definition of $P(A \mid B)$. In the numerator, replace $P(A \text{ and } B)$ using the multiplication rule:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(B \mid A)\, P(A)}{P(B)}.$$

The right-hand side is Bayes' rule. So Bayes' rule follows directly from the definition of conditional probability and the multiplication rule.

**In practice,** the denominator $P(B)$ is often not given directly. We usually compute it using the **law of total probability** by partitioning the sample space (e.g., into "$A$" and "not $A$," or into disease / no disease) and writing $P(B)$ as a sum of terms $P(B \mid A_i)\, P(A_i)$.

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**Tree diagram (binary case: partition** $A$ / not $A$, then $B$ / not $B$)

First branch on $A$ vs not $A$; from each, branch on $B$ vs not $B$. Edge labels are the probabilities on each branch.

- **Path probabilities:** $P(A)\, P(B \mid A)$ and $P(A)\, P(\text{not } B \mid A)$ for the two paths through $A$; $P(\text{not } A)\, P(B \mid \text{not } A)$ and $P(\text{not } A)\, P(\text{not } B \mid \text{not } A)$ for the two through not $A$.
- $P(B) =$ sum of paths that end at $B$: $P(A)\, P(B \mid A) + P(\text{not } A)\, P(B \mid \text{not } A)$ (law of total probability).
- **Bayes:** $P(A \mid B) = (\text{path through } A \text{ to } B) \div P(B) = \dfrac{P(B \mid A)\, P(A)}{P(B)}$.