Trevor Oshiro
MAE263B

Project 5 Report

Part 1: Equations of Motion Derivation

The attached files can be referenced in the appendix for the base theory used to generate these equations. MATLAB was used to automate the process for deriving the equations of motions for each method. The script and plots generated can be found within this repository ([Repository Link]()).

As the equations of motion are of longer length, below is the equation of motion generated with the physical values from later sections plugged in for the variables. Symbolic forms can be found when running the script, from the variables sLangEq (symbolic variable for equation generated from the Lagrangian method) and sNEuEq (symbolic variable for the equations generated with the Newton-Euler method).

```
- 0.2528*sin(t2)*dt2^2 - 0.5055*dt1*sin(t2)*dt2 + 1.2912*d2t1 + 0.3929*d2t2 - 5*sin(t2) + 0.5055*g*cos(t1 + t2) + 0.5055*d2t1*cos(t2) + 0.2528*d2t2*cos(t2) + 1.5165*g*cos(t1) + 10
                                        0.2528*sin(t2)*dt1^2 + 0.3929*d2t1 + 0.3929*d2t2 + 0.5055*g*cos(t1 + t2) + 0.2528*d2t1*cos(t2) + 10
```

Figure 1: Equation of motion from MATLAB

Plotting of Joint Kinematics

The trajectory for the joints was generated within the joint space through the parabolic blend method for ensuring zero velocity at the start and end of the manipulator's path. Below, plots of the generated trajectory can be found:

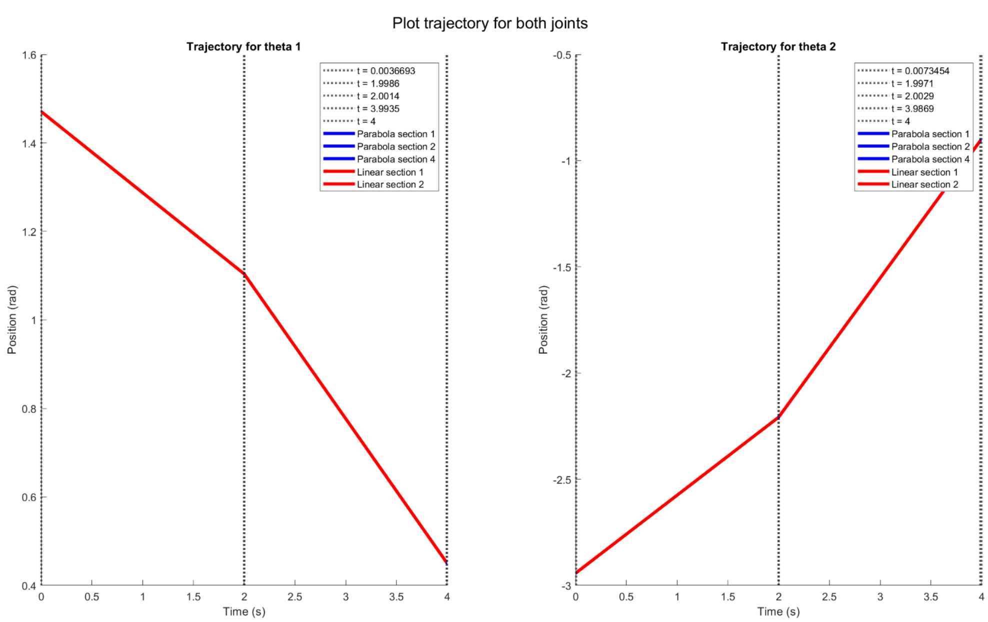


Figure 2: Plotted Trajectories

Below is also plots of the resulting velocity and acceleration values calculated within the joint space:

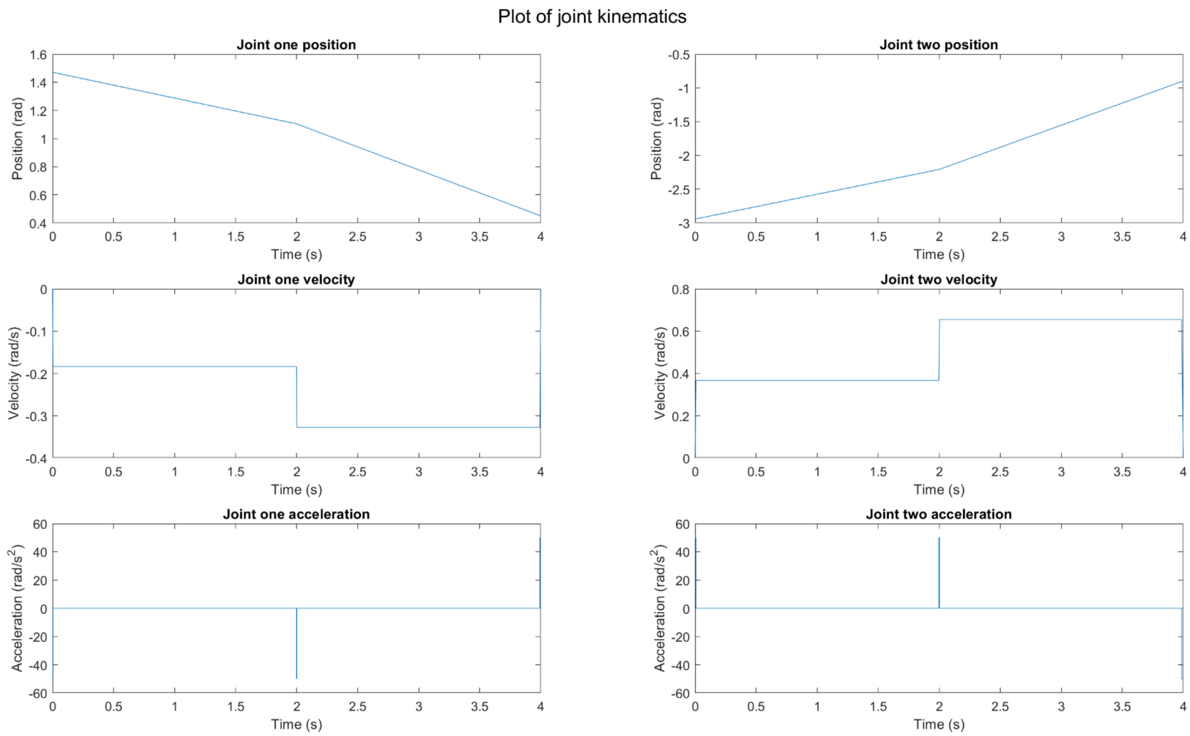


Figure 3: Plotted kinematics of the joints

Plotting of Task Space Kinematics and Torques

With the joint kinematic values calculated within the joint space, the kinematics in the task space was then calculated through the Jacobian and with forward kinematics. The jacobian used can be found below along with the plots generated for the kinematics in the task space:

```
[- L2*sin(t1 + t2) - L1*sin(t1), -L2*sin(t1 + t2)]
[  L2*cos(t1 + t2) + L1*cos(t1),  L2*cos(t1 + t2)]
[                             0,                0]
[                             0,                0]
[                             0,                0]
[                             1,                1]
```

Figure 4: Un-reduced Jacobian used in mapping to the task space

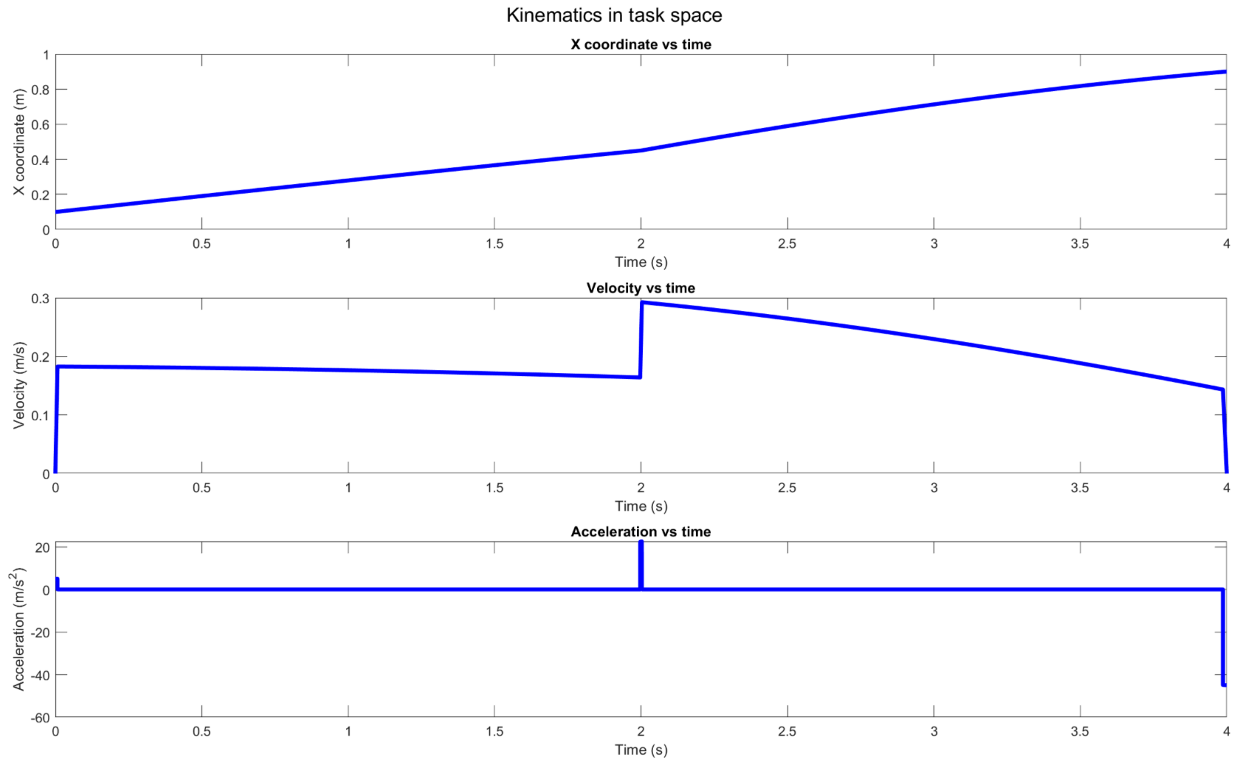


Figure 5: Plot of end-effector kinematics in task space

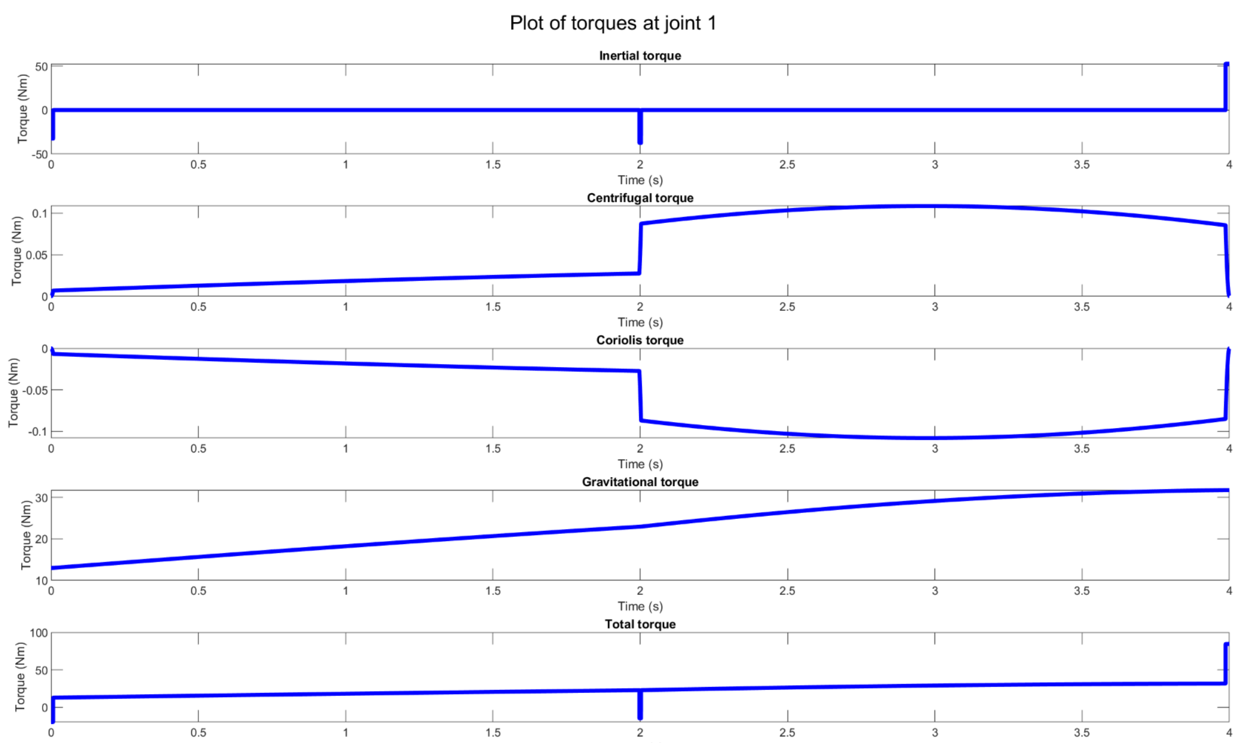


Figure 6: Joint 1 torques

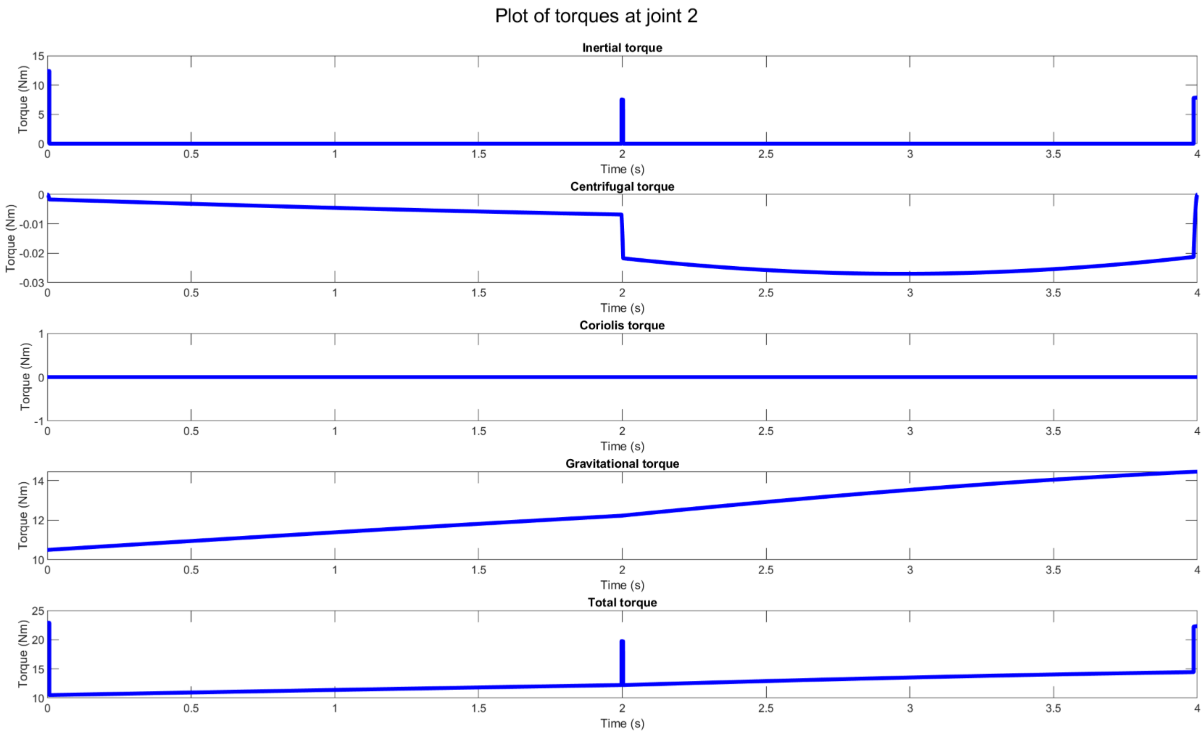


Figure 7: Joint 2 torques

As seen with the plots above, there is a consistent rise within the torque from gravity over the span of motion. The inertial torque acts when the acceleration is present during the parabolic blend areas of the trajectory motion, and significant rises/falls of the torque values in the coriolis and centrifugal terms are also seen within the parabolic blend region.

Plotting of Torques with Modified Effect of Gravity

The following equation was generated when modifying the gravity to act along the joint axis.

```
- 0.2528*sin(t2)*dt2^2 - 0.5055*dt1*sin(t2)*dt2 + 1.2912*d2t1 + 0.3929*d2t2 - 5*sin(t2) + 0.5055*d2t1*cos(t2) + 0.2528*d2t2*cos(t2) + 10
                                                   0.2528*sin(t2)*dt1^2 + 0.3929*d2t1 + 0.3929*d2t2 + 0.2528*d2t1*cos(t2) + 10
```

Figure 8: Equation of motion generated with the gravity along the joint axis

The following show plots of the torque values for each joint with the modified gravity applied:

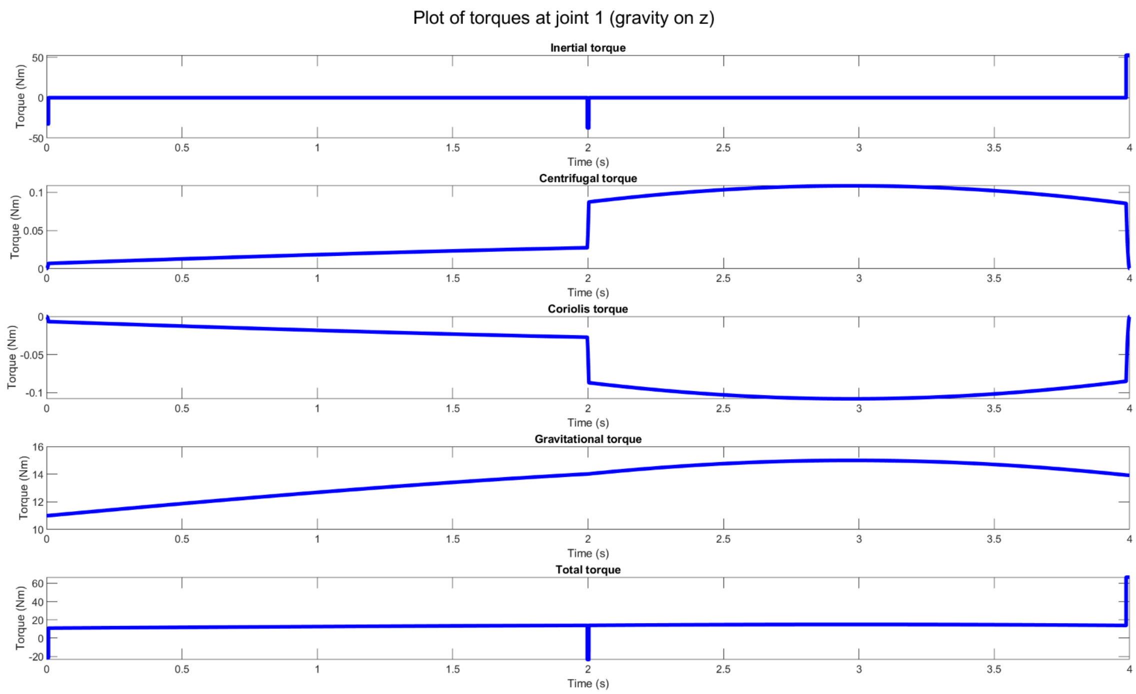


Figure 9: Joint 1 torque values with the gravity along the joint axis

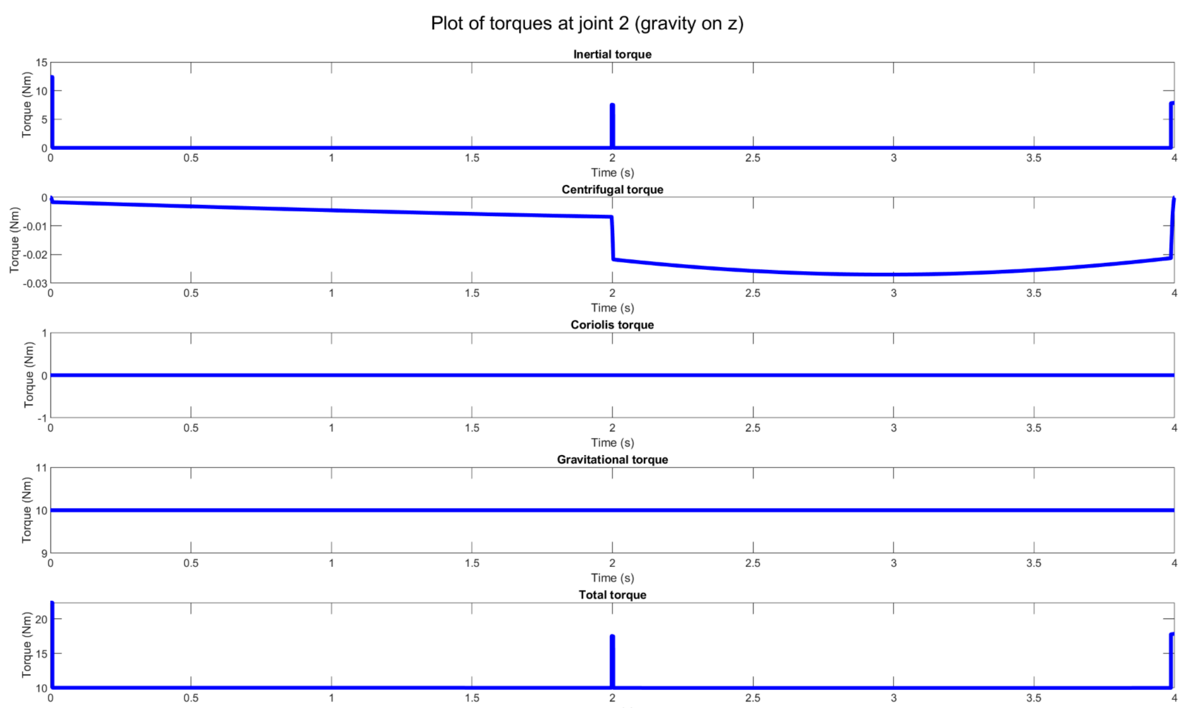


Figure 10: Joint 2 torque values with the gravity along the joint axis

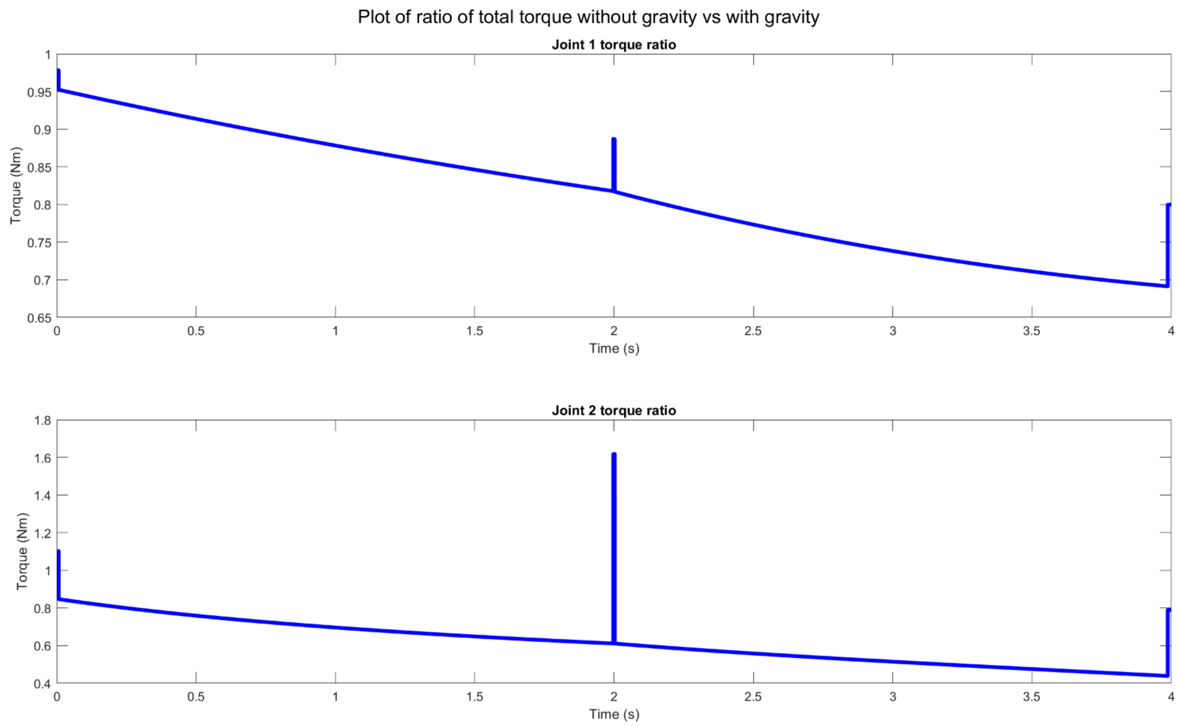
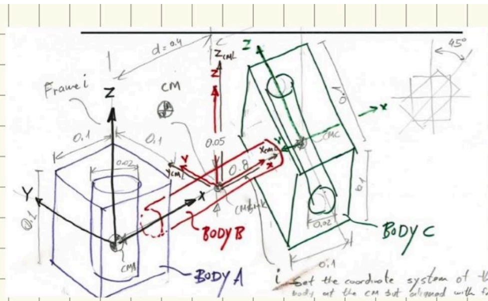


Figure 11: Ratio of torque between values with the gravity along the joint axis and gravity about the y axis

Seen within figure 11, the ratio of the torques within joint 1 remained below 1 throughout the path, while the torque ratio had spikes above 1 for joint 2. This is likely due to the sudden increase in acceleration within the parabolic blend areas of the trajectory. As the gravity doesn't counteract the other sources of torque during the motion, the net torque needed to move the joint increases. This then can make the torque required for the parabolic blend areas higher when gravity is along the joint axis, as opposed to the gravity acting downwards in the motion of the joints.

The overall ratio of torque was seen to be decreasing overall as time moved forward during the trajectory. As the arm reaches more extension outwards during the trajectory motion, gravity has a larger impact on the overall torque (when acting downwards along the y axis). This then makes the torque when gravity acts along the joint axis significantly less when the manipulator reaches out further during motion. Similarly, the ratio is closer to 1 at the beginning of motion. This is because the manipulator is more collapsed, reducing the torque induced from gravity and making the torque values more similar to each other.

Part 2: Inertia Tensor Calculation



$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$m_b = \rho V = \left(\frac{2700\,kg}{m^3}\right)(0.1\,m)^3 = 2.7\,kg$$

$$m_{bp} = \rho V = \left(\frac{2700\,kg}{m^3}\right)\left(\pi(0.01\,m)^2(0.1\,m)\right) = 8.482 \cdot 10^{-2}\,kg$$

$$m_c = \rho V = \left(\frac{2700\,kg}{m^3}\right)\left(\pi(0.025\,m)^2(0.8\,m - 0.1\,m)\right) = 3.711\,kg$$

$$I = I_A + I_B + I_C$$

I_{xx}

$$I_{A_{xx}} = \left[\frac{1}{12} m_b (0.1\,m^2 + 0.1\,m^2)\right] - \left[\frac{1}{12} m_{bp}\left(3(0.01\,m)^2 + 0.1\,m^2\right)\right] = 4.427 \cdot 10^{-3}\,kg\,m^2$$

$$I_{B_{xx}} = \frac{1}{2}\left(m_c (0.025\,m)^2\right) = 1.160 \cdot 10^{-3}\,kg\,m^2$$

$$I_{C_{xx}} = I_A$$

$$\rightarrow I_{xx} = I_{A_{xx}} + I_{B_{xx}} + I_{C_{xx}} = 1.001 \cdot 10^{-2}\,kg\,m^2$$

I_{yy}

$$I_{A_{yy}} = \left[\frac{1}{12} m_b (0.1\,m^2 + 0.1\,m^2) + m_b (0.4\,m^2)\right] - \left[\frac{1}{12} m_{bp}\left(3(0.01\,m)^2 + 0.1\,m^2\right) + m_{bp}(0.4\,m^2)\right]$$
$$= 0.423\,kg\,m^2$$

$$I_{B_{yy}} = \frac{1}{12} m_c \left(3(0.025\,m)^2 + 0.7\,m^2\right) = 0.152\,kg\,m^2$$

$$I_{C_{yy}} = \left[\frac{1}{12} m_b (0.1\,m^2 + 0.1\,m^2) + m_b (0.4\,m^2)\right] - \left[\frac{1}{2} m_{bp} (0.01\,m)^2 + m_{bp}(0.4\,m^2)\right]$$
$$= 0.423\,kg\,m^2$$

$$I_{yy} = I_{A_{yy}} + I_{B_{yy}} + I_{C_{yy}} = 0.998\,kg\,m^2$$

$$I_{yy} = I_{zz}$$

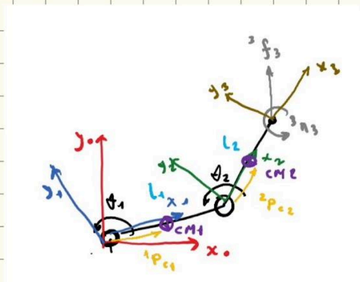
$$\hookrightarrow I = \begin{bmatrix} 1.001 \cdot 10^{-2} & 0 & 0 \\ 0 & 0.998 & 0 \\ 0 & 0 & 0.998 \end{bmatrix} kg\,m^2$$

Figure 12: Inertia tensor calculations

As the body was made of standard shapes, the moment of inertia for the link could be directly calculated analytically. The inertia about the y and z axis were the same, so the inertia about the x axis was a different value.

Appendix:

1a) Derive with Newton-Euler methods



$$I_x = \left(\frac{m}{2}\right)(r_i^2 + r_o^2)$$
$$I_y = \left(\frac{m}{12}\right)\left(3(r_i^2 + r_o^2) + \ell^2\right)$$

→ Start propagating from EE frame since force/torques applied there

$$^if_i = {}^i_{i+1}R\,{}^{i+1}f_{i+1} + {}^iF_i$$ ← From outwards

From outwards ↙

$$^in_i = {}^iN_i + {}^i_{i+1}R\,{}^{i+1}n_{i+1} + {}^iP_{C_i} \times {}^iF_i + {}^iP_{i+1} \times {}^i_{i+1}R\,{}^{i+1}f_{i+1}$$

$$\tau_i = {}^in^T_i\,{}^i\hat{Z}_i$$

$$^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1P_{c_1} = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \quad f_3 = {}^3f_3$$

$$^1_2T = \begin{bmatrix} c_2 & s_2 & 0 & L_1 \\ -s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2P_{c_2} = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \quad n_3 = {}^3n_3 \quad {}^0\dot{v}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} \quad {}^0w_0 = 0$$

$\boxed{i=0}$

$$^{i+1}\omega_{i+1} = {}^{i+1}_iR\,{}^i\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^1w_1 = {}^1_0R\,\cancel{{}^0w_0} + \dot{\theta}_1\,\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_iR\,{}^i\dot{\omega}_i + {}^{i+1}_iR\,{}^i\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^1\dot{w}_1 = {}^1_0R\,\cancel{{}^0\dot{w}_0} + {}^1_0R\,\cancel{{}^0w_0} \times \dot{\theta}_1\,{}^1\hat{Z}_1 + \ddot{\theta}_1\,{}^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}$$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_iR\left({}^i\dot{\omega}_i \times {}^iP_{i+1} + {}^i\omega_i \times ({}^i\omega_i \times {}^iP_{i+1}) + {}^i\dot{v}_i\right)$$

$$^1\dot{v}_1 = {}^1_0R\left(\cancel{{}^0\dot{w}_0} \times {}^0P_1 + \cancel{{}^0w_0} \times ({}^0w_0 \times {}^0P_1) + {}^0\dot{v}_0\right) = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix}$$

$$^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1}$$

$$^1\dot{v}_{c_1} = {}^1\dot{w}_1 \times {}^1P_{c_1} + {}^1w_1 \times ({}^1w_1 \times {}^1P_{c_1}) + {}^1\dot{v}_1$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \ddot{\theta}_1 \\ L_1 & 0 & 0 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \dot{\theta}_1 \\ L_1 & 0 & 0 \end{vmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ L_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \dot{\theta}_1 \\ 0 & L_1\dot{\theta}_1 & 0 \end{vmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} gs_1 - L_1\dot{\theta}_1^2 \\ gc_1 + L_1\ddot{\theta}_1 \\ 0 \end{bmatrix}$$

$$^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}}$$

$$^1F_1 = m_1\,{}^1\dot{v}_{c_1} = m_1 \begin{bmatrix} -L_1\dot{\theta}_1^2 + gs_1 \\ L_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} m_1gs_1 - m_1L_1\dot{\theta}_1^2 \\ m_1gc_1 + m_1L_1\ddot{\theta}_1 \\ 0 \end{bmatrix}$$

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}$$

$$^1N_1 = \overset{0}{\cancel{{}^{c_1}I_1}}\,{}^1\dot{w}_1 + {}^1w_1 \times \overset{0}{\cancel{{}^{c_1}I_1}}\,{}^1w_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{i=2}$

$$^2f_2 = {}^2_3R\,{}^3f_3 + {}^2F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{3x} \\ f_{3y} \\ f_{3z} \end{bmatrix} + {}^2F_2$$

$$^2n_2 = {}^2N_2 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_{3x} \\ n_{3y} \\ n_{3z} \end{bmatrix} + {}^2P_{c_2} \times {}^2F_2 + {}^2P_3 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{3x} \\ f_{3y} \\ f_{3z} \end{bmatrix}$$

Plug into MATLAB:

$\tau_1 =$ $(1.2912)\,\ddot{\theta}_1$ $(0.5055\,c_2\,\ddot{\theta}_1)$
$(0.3929)\,\ddot{\theta}_2$ $(0.2528\,c_2\,\ddot{\theta}_2)$ inertial

$(-0.2528)\,s_2\,\dot{\theta}_2^2$ centrifugal

$(-0.5055)\,s_2\,\dot{\theta}_1\dot{\theta}_2$ coriolos

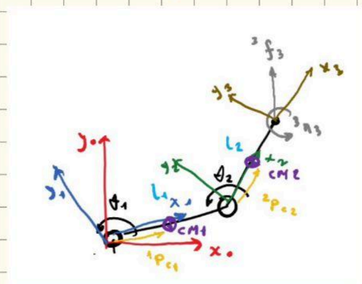
$(1.5165\,g\,c_1 + 10)$
$(-5\,s_2)$ gravitational
$(0.5055\,g\,c_{12})$

$\tau_2 =$ $(0.3929)\,\ddot{\theta}_1$ $(0.2528\,c_2)\,\ddot{\theta}_1$ inertial
$(0.3929)\,\ddot{\theta}_2$

$(0.2528\,s_2)\,\dot{\theta}_1^2$ centrifugal

$(0.5055\,g\,c_{12} + 10)$ gravitational

2) Lagrange Method:



$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0\\ s_1 & c_1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} c_2 & s_2 & 0 & L_1\\ -s_2 & c_2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0P_1 = \begin{bmatrix} 0\\0\\0 \end{bmatrix} $$

$$ {}^1P_{c1} = \begin{bmatrix} \frac{1}{2}L_1\\0\\0 \end{bmatrix}, \quad {}^0P_{c1} = {}^0_1T\,{}^1P_{c1} = \begin{bmatrix} \frac{1}{2}L_1c_1\\ \frac{1}{2}L_1s_2\\ 0\\ 1 \end{bmatrix} $$

$$ {}^0P_2 = {}^0_1T\,{}^1_2P = \begin{bmatrix} L_1c_1\\ L_1s_1\\ 0\\ 1 \end{bmatrix} $$

$$ {}^2P_{c2} = \begin{bmatrix} \frac{1}{2}L_2\\0\\0 \end{bmatrix}, \quad {}^0P_{c2} = {}^0_1T\,{}^1_2T\,{}^2P_{c2} = $$

$${}^0\omega_0 = 0$$

$$f_3 = {}^3f_3$$

$$n_3 = {}^3n_3$$

$$ {}^0\dot{v}_0 = \begin{bmatrix} 0\\g\\0 \end{bmatrix} $$

Moment of inertia:

$$ {}^iI_{ci} = \frac{1}{12} m L_i^2 \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0I_1 = {}^0_1R\,{}^2I_{c1}\,({}^0_1R)^T $$

$$ {}^0I_2 = {}^0_2R\,{}^2I_{c2}\,({}^0_2R)^T $$

Jacobian matrix:

$$ J^j_{v1} = {}^0Z_j \times ({}^0P_{ci} - {}^0P_j) \qquad J^j_{\omega i} = {}^0Z_j $$

i=1

$$ {}^0J_{v1} = \begin{bmatrix} {}^0Z_1 \times ({}^0P_{c1} - {}^0P_1) & 0 \end{bmatrix} $$

$$ {}^0J_{\omega 1} = \begin{bmatrix} {}^0Z_1 & 0 \end{bmatrix} $$

i=2

$$ {}^0J_{v2} = \begin{bmatrix} {}^0Z_1 \times ({}^0P_{c2} - {}^0P_1) & Z_2 \times ({}^0P_{c2} - {}^0P_2) \end{bmatrix} $$

$$ {}^0J_{\omega 2} = \begin{bmatrix} {}^0Z_1 & {}^0Z_2 \end{bmatrix} $$

Gravitational Vector:

$$ G_i = -\sum_{j=1}^{n} m_j g^T J^i_{vj} $$

i=1

$$ \rightarrow G_i = -\sum_{j=1}^{n} m_j g^T J^1_{vj} $$

Manipulator Inertia Matrix:

$$ M = \left(J_{v1}^T m_1 J_{v1} + J_{\omega 1}^T\,{}^1I_{c1}\,J_{\omega 1}\right) + \left(J_{v2}^T m_2 J_{v2} + J_{\omega 2}^T\,{}^2I_{2c}\,J_{\omega 2}\right) $$

velocity coupling vector:

$$ V_i = \sum_{j=1}^{2}\sum_{k=1}^{2}\left(\frac{\partial M_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{q_i}\right)\dot{q}_k\dot{q}_j $$

$$ = \dot{q}_1\sum_{k=1}^{n}\left(\frac{\partial M_{i1}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{1k}}{q_i}\right)\dot{q}_k + \dot{q}_2\sum_{k=1}^{n}\left(\frac{\partial M_{i2}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{2k}}{q_i}\right)\dot{q}_k $$

$$ = \dot{q}_1\left[\left(\frac{\partial M_{i1}}{\partial q_1} - \frac{1}{2}\frac{\partial M_{11}}{q_i}\right)\dot{q}_1 + \left(\frac{\partial M_{i1}}{\partial q_2} - \frac{1}{2}\frac{\partial M_{12}}{q_i}\right)\dot{q}_2\right] $$

$$ + \dot{q}_2\left[\left(\frac{\partial M_{i2}}{\partial q_1} - \frac{1}{2}\frac{\partial M_{21}}{q_i}\right)\dot{q}_1 + \left(\frac{\partial M_{i2}}{\partial q_2} - \frac{1}{2}\frac{\partial M_{22}}{q_i}\right)\dot{q}_2\right] $$

Overall equation: $\tau = Q - J^T F_e$