TEL AVIV UNIVERSITY

Chasing Positive Bodies

Roie Levin

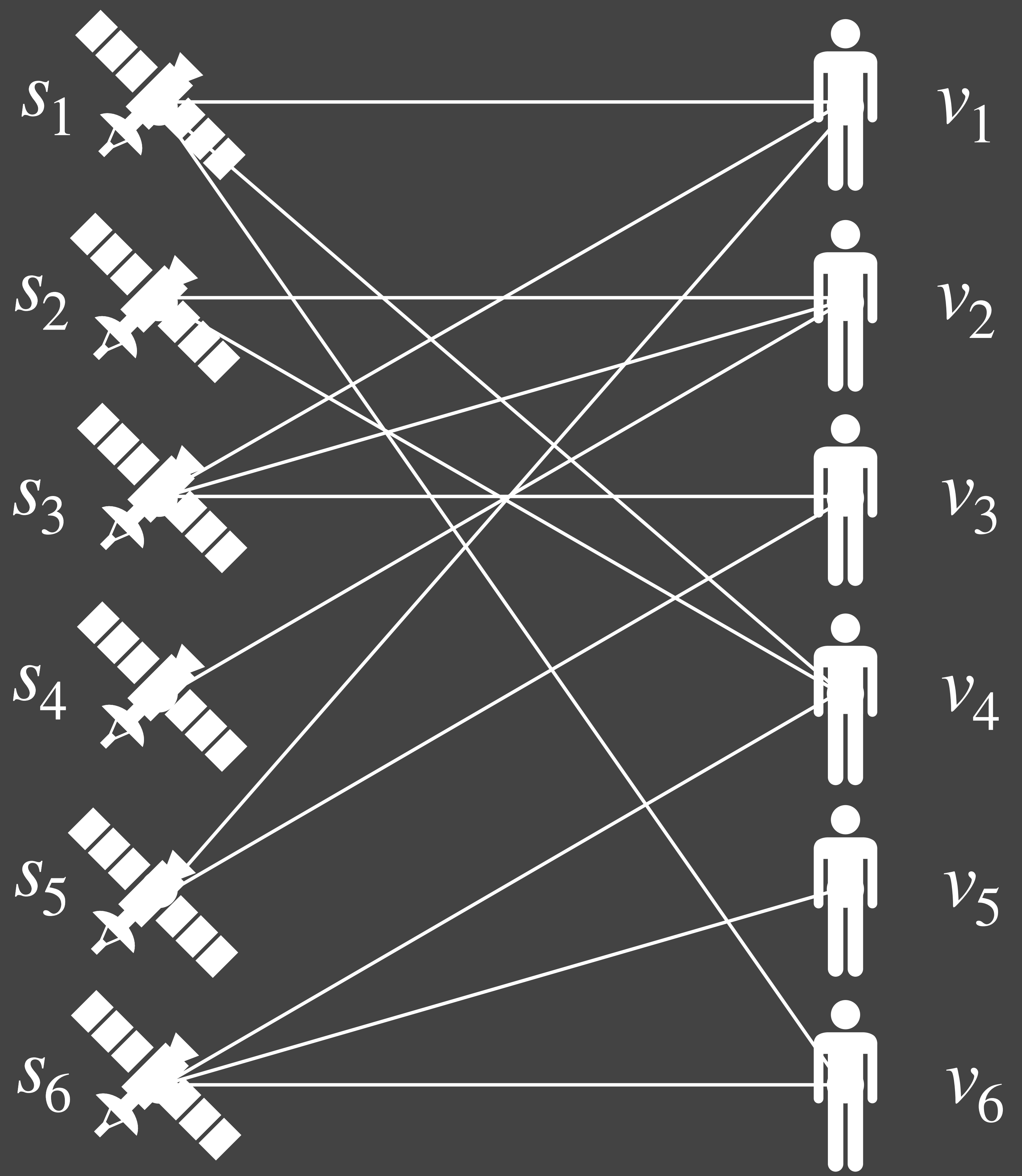
Sayan Bhattacharya
(U. of Warwick)

Niv Buchbinder
(Tel Aviv U.)

Thatchaphol Saranurak
(U. of Michigan)

Introduction

Motivating Problem



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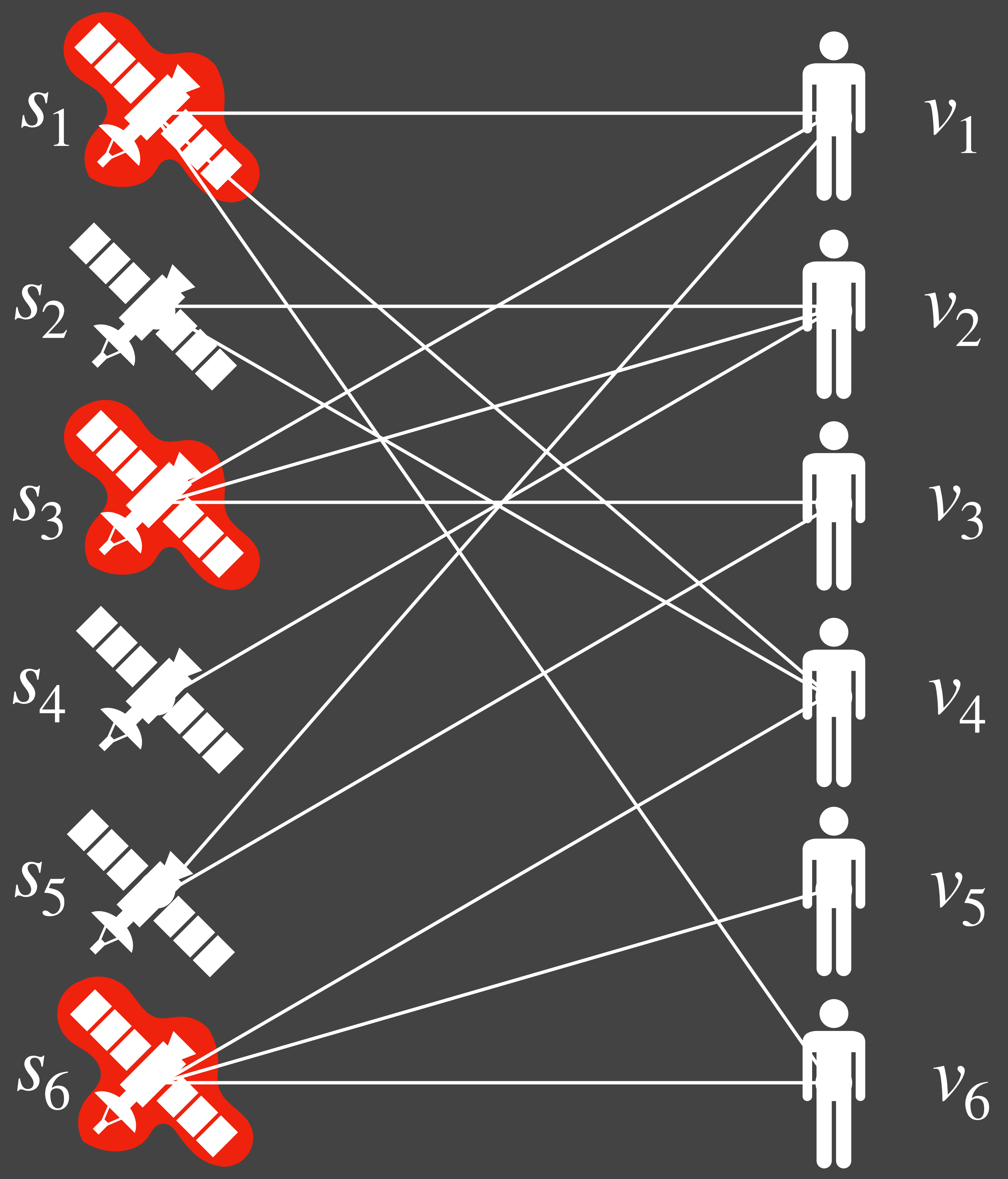

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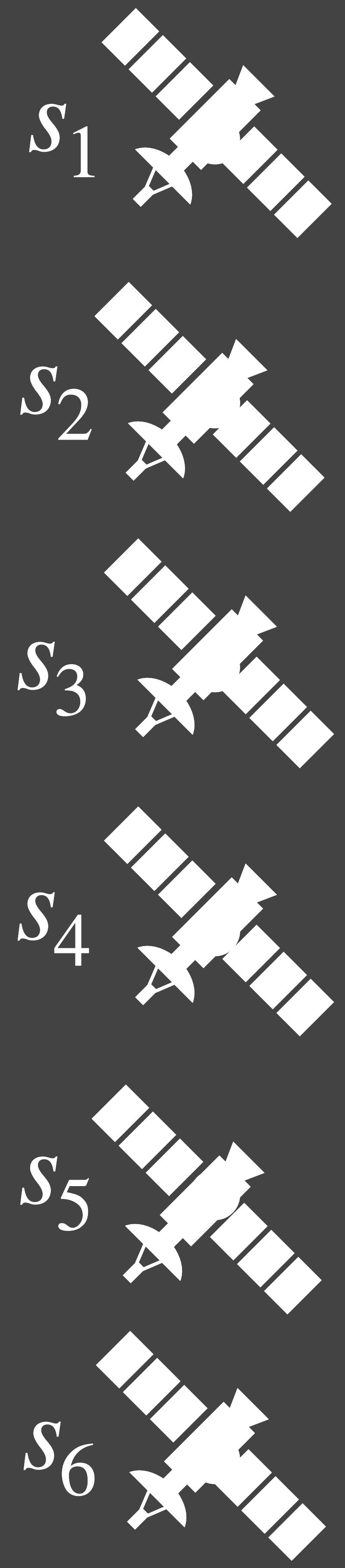

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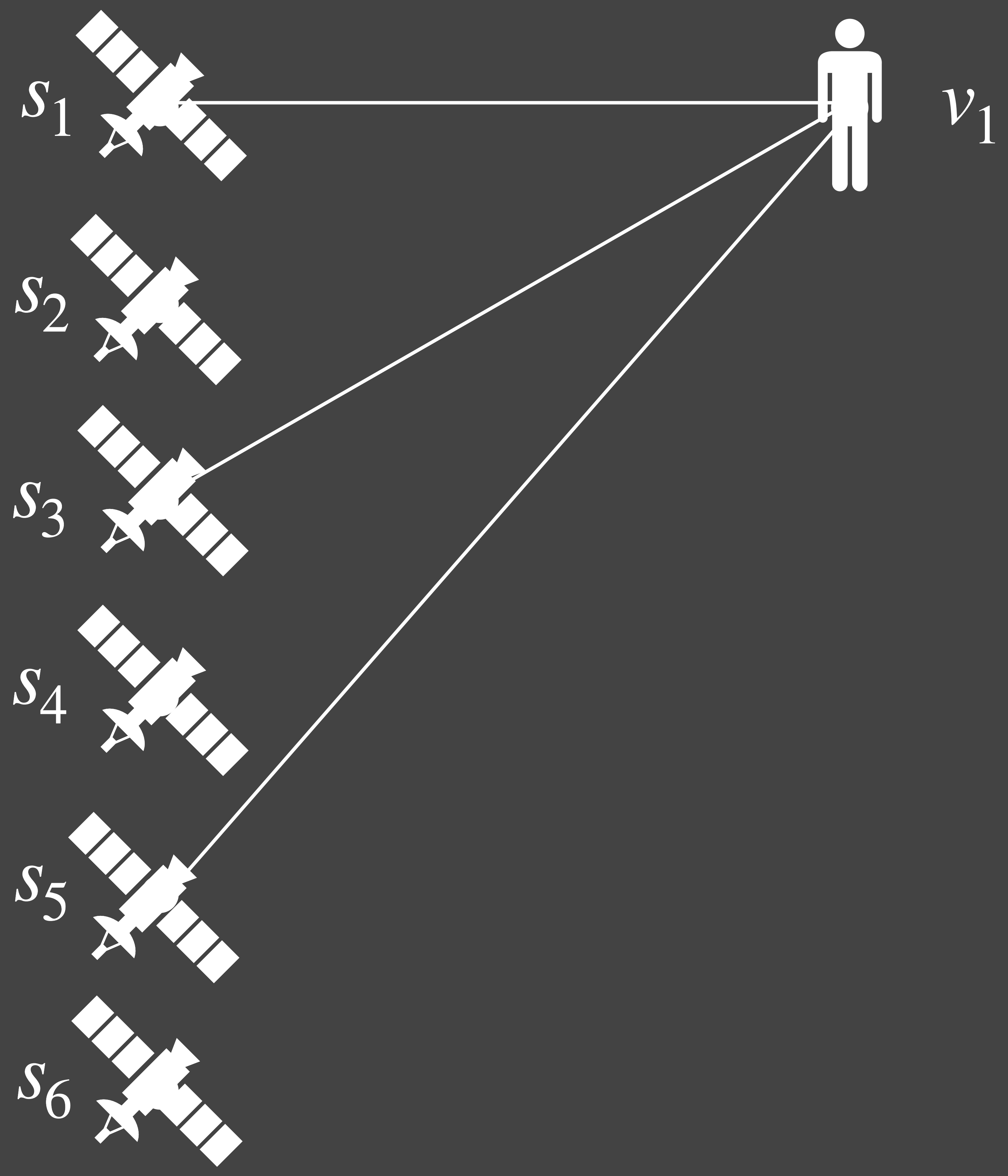

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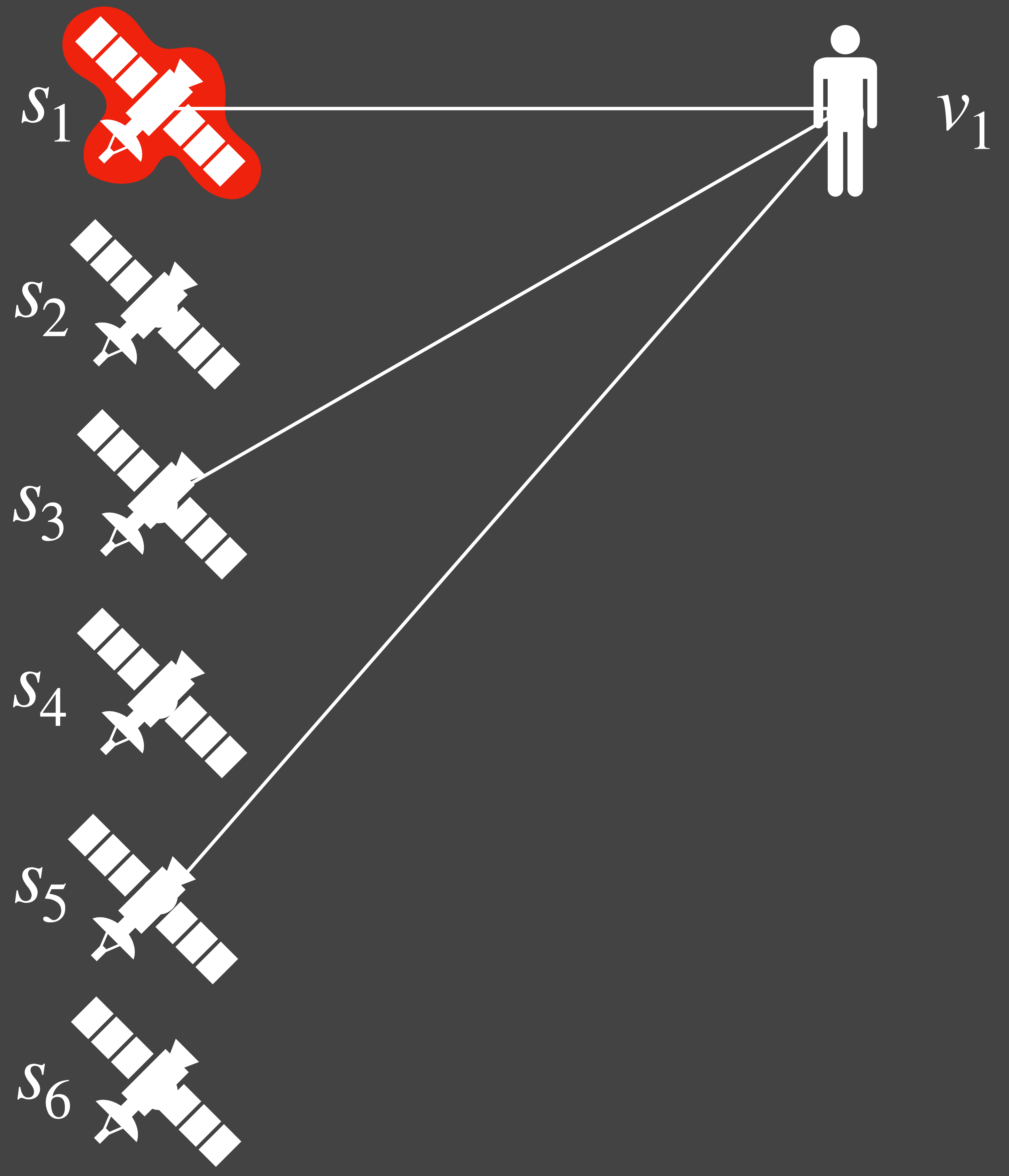

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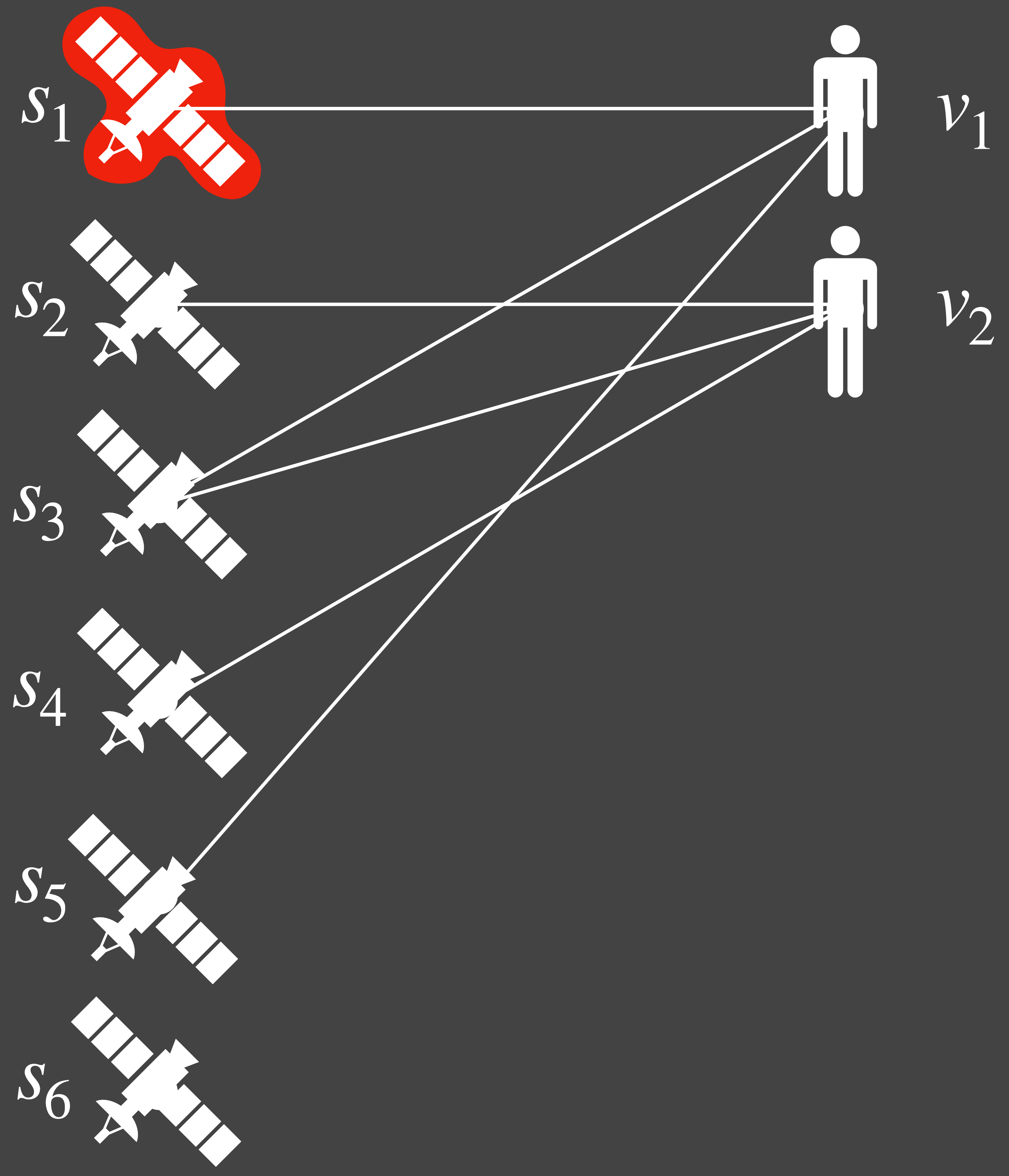

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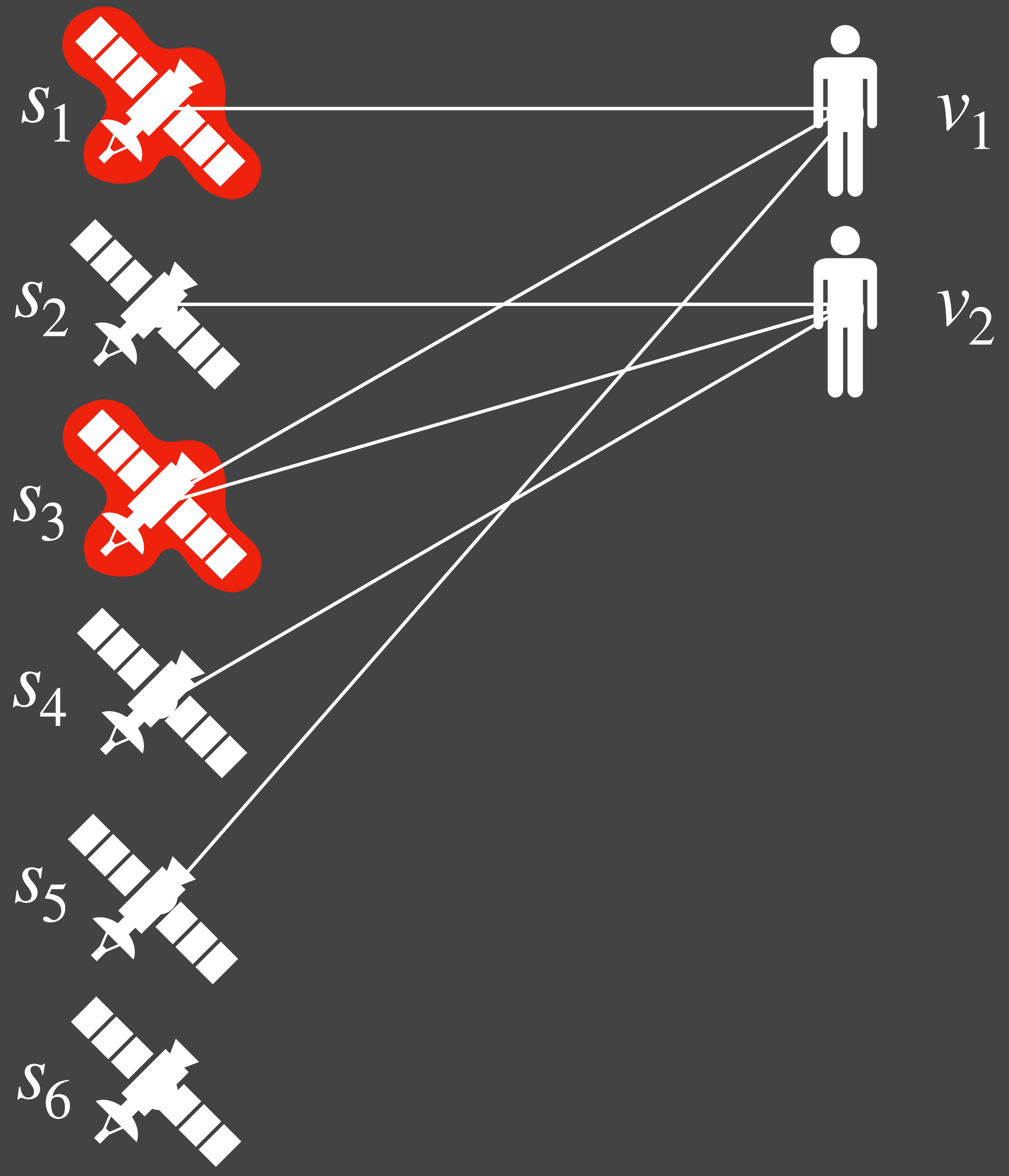

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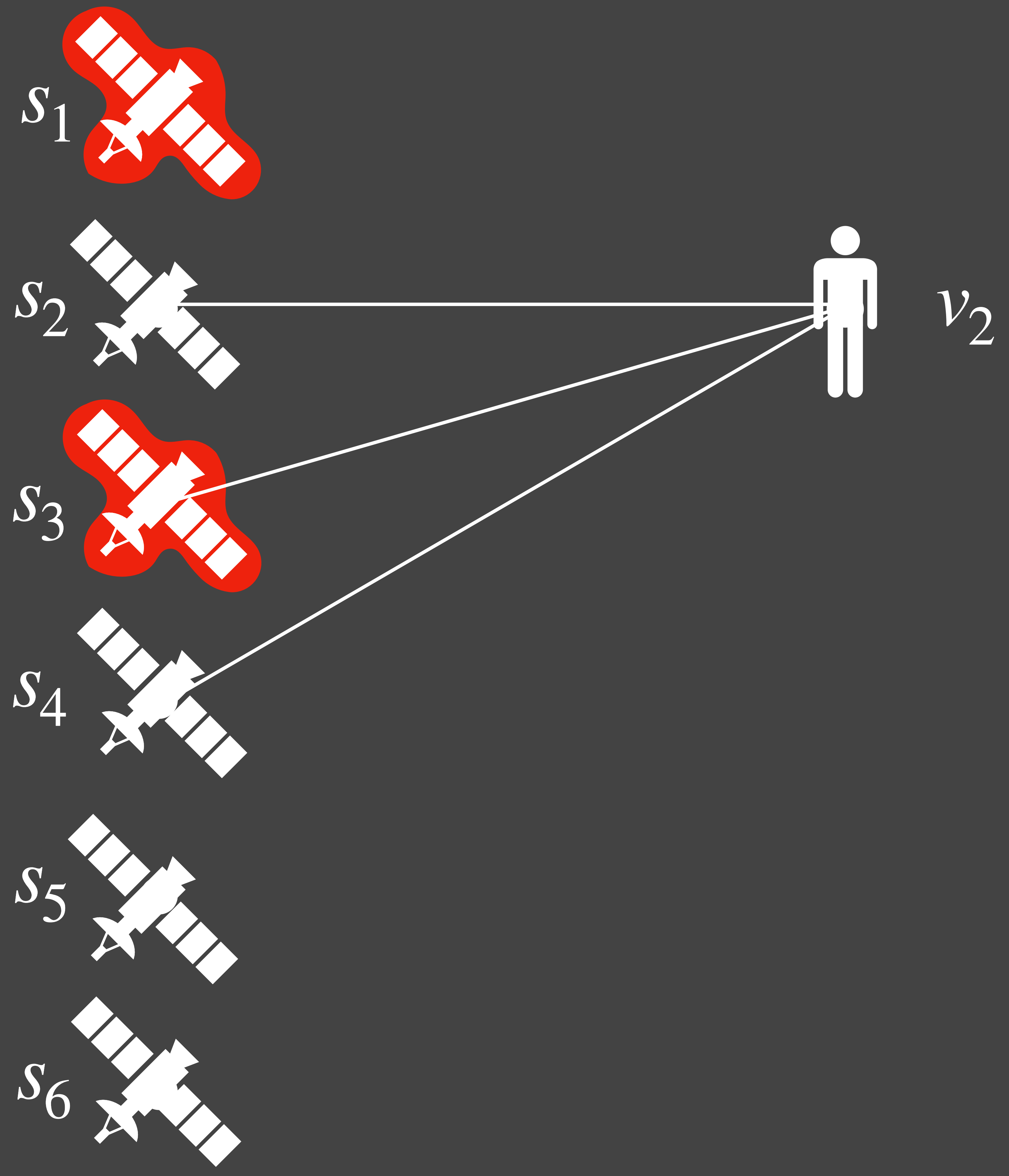

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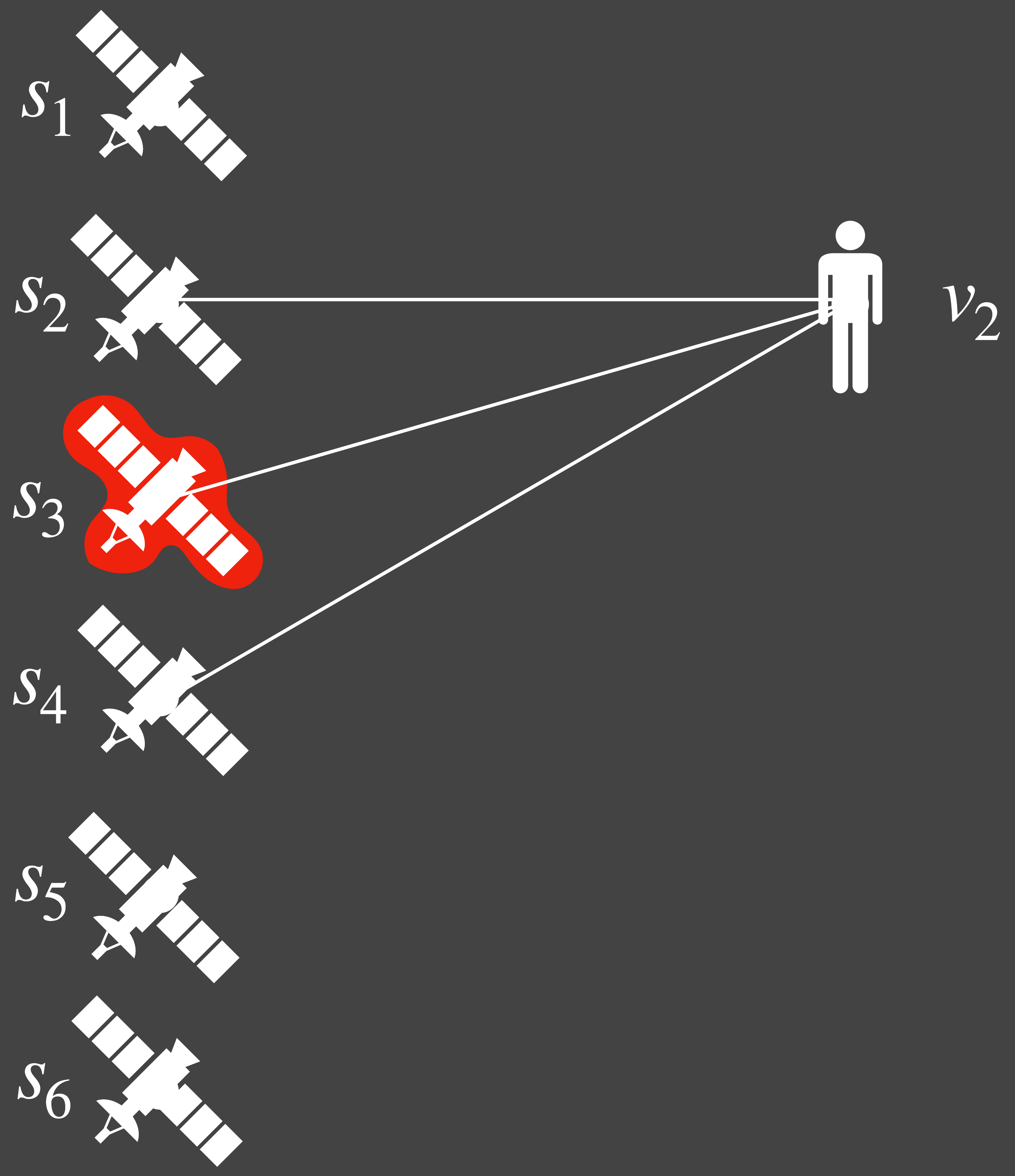

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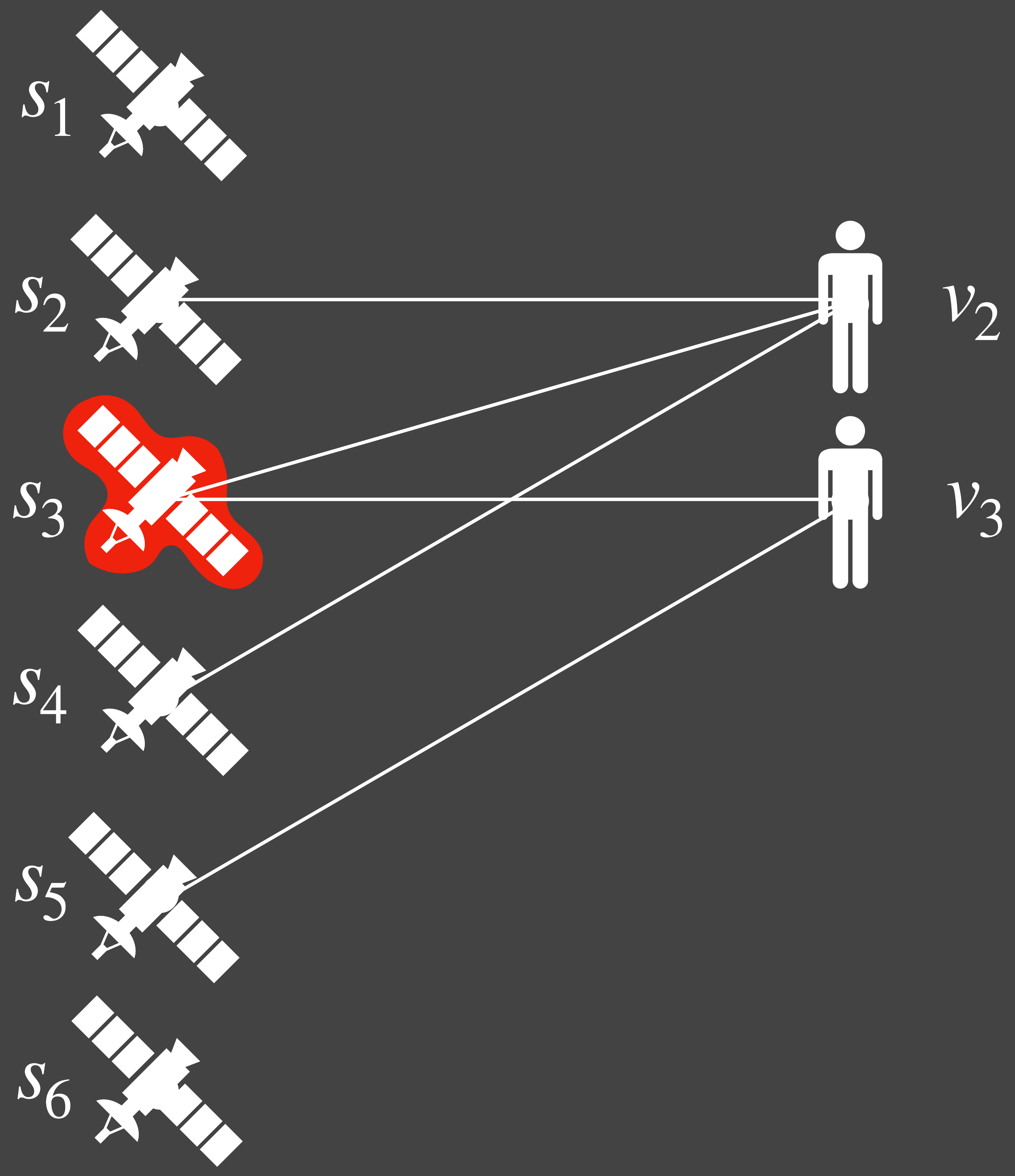

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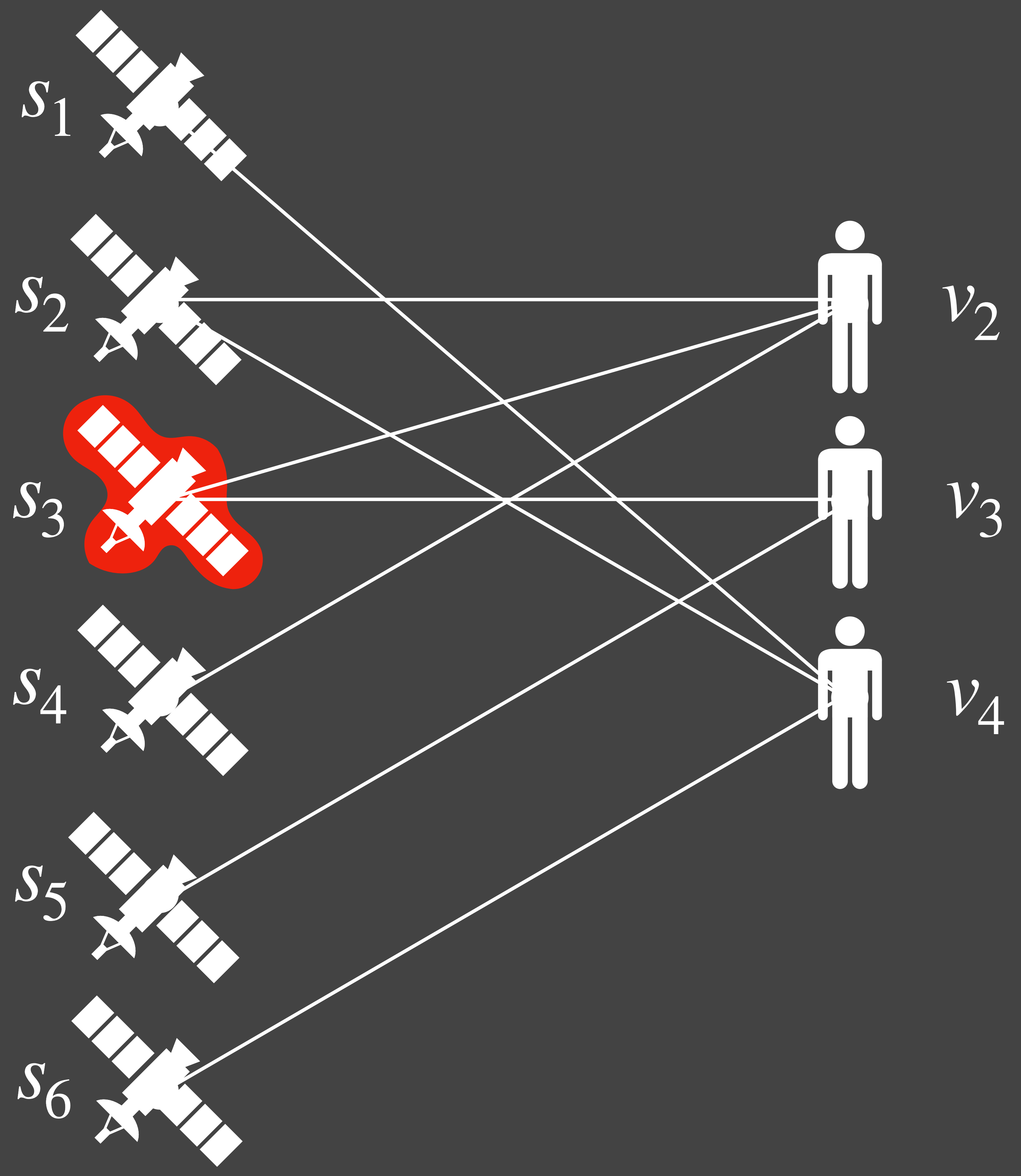

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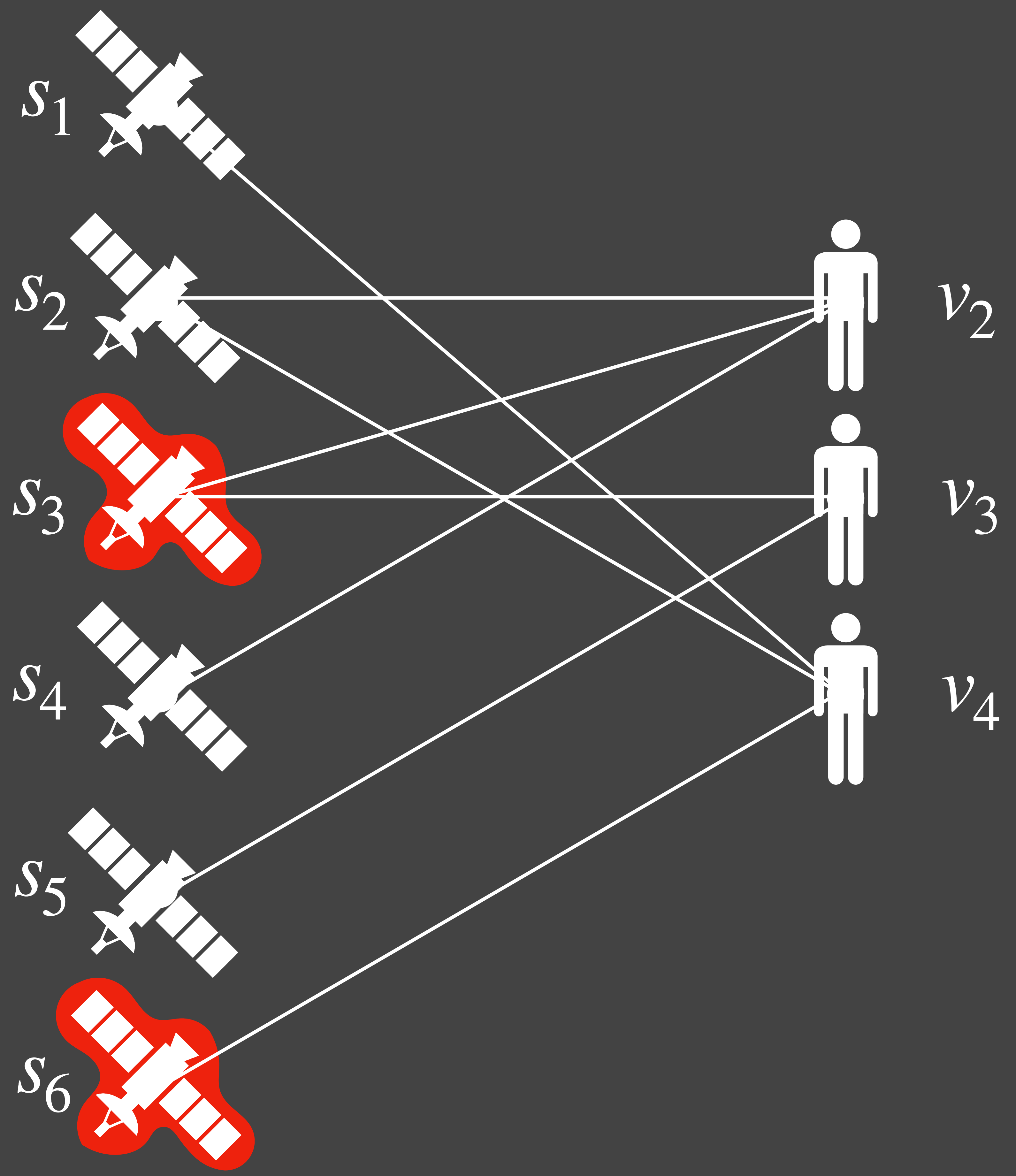

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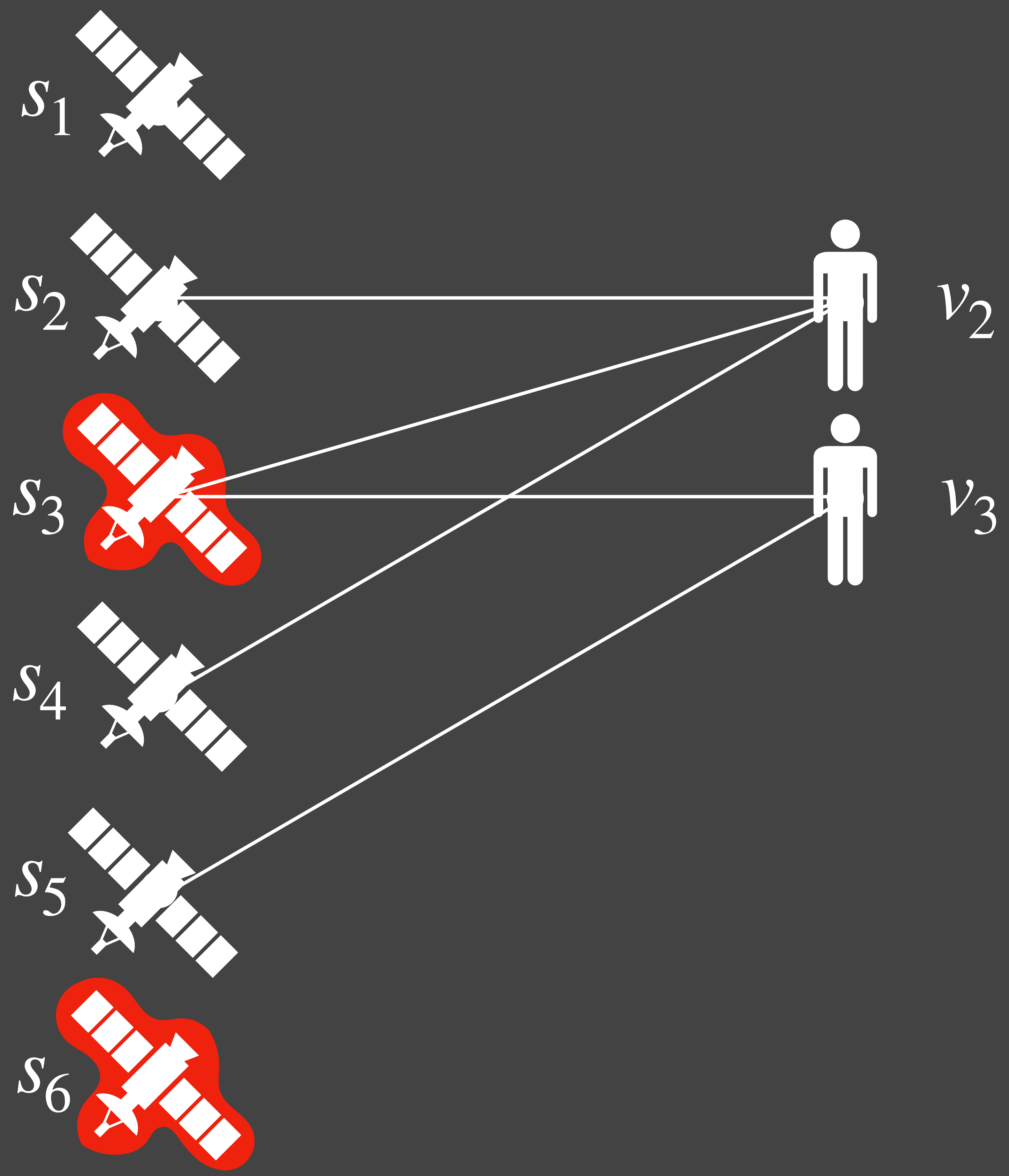

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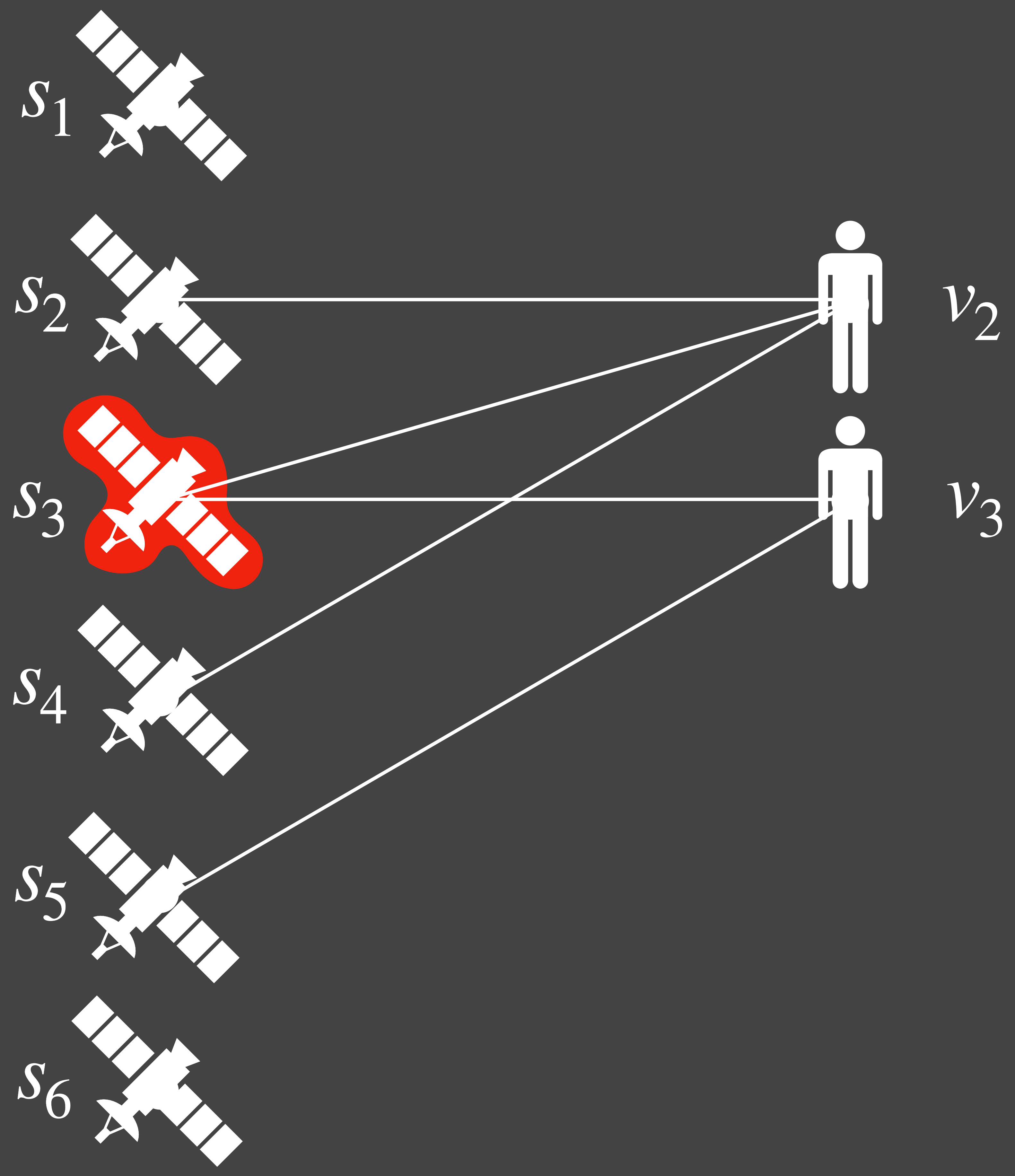

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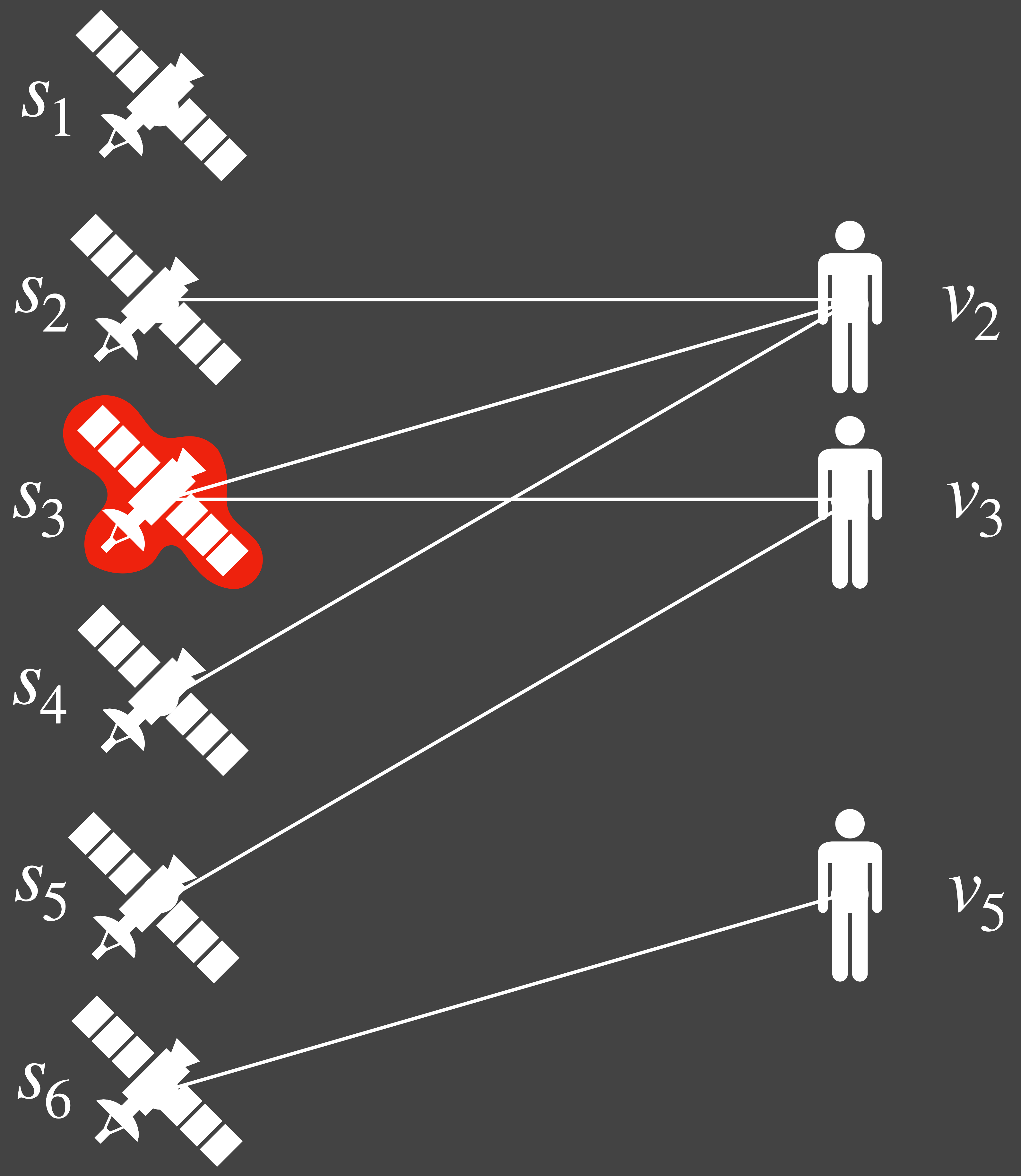

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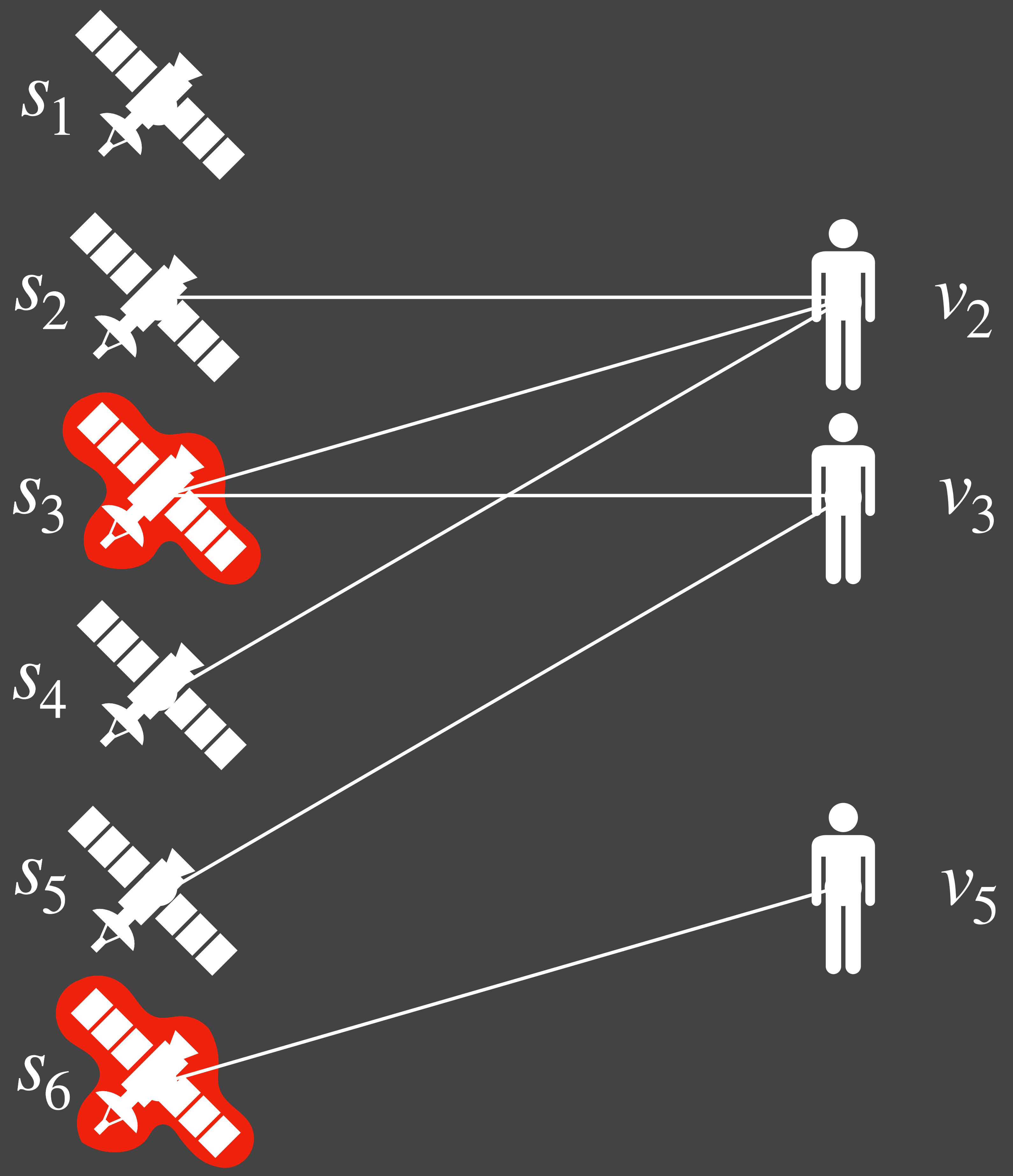

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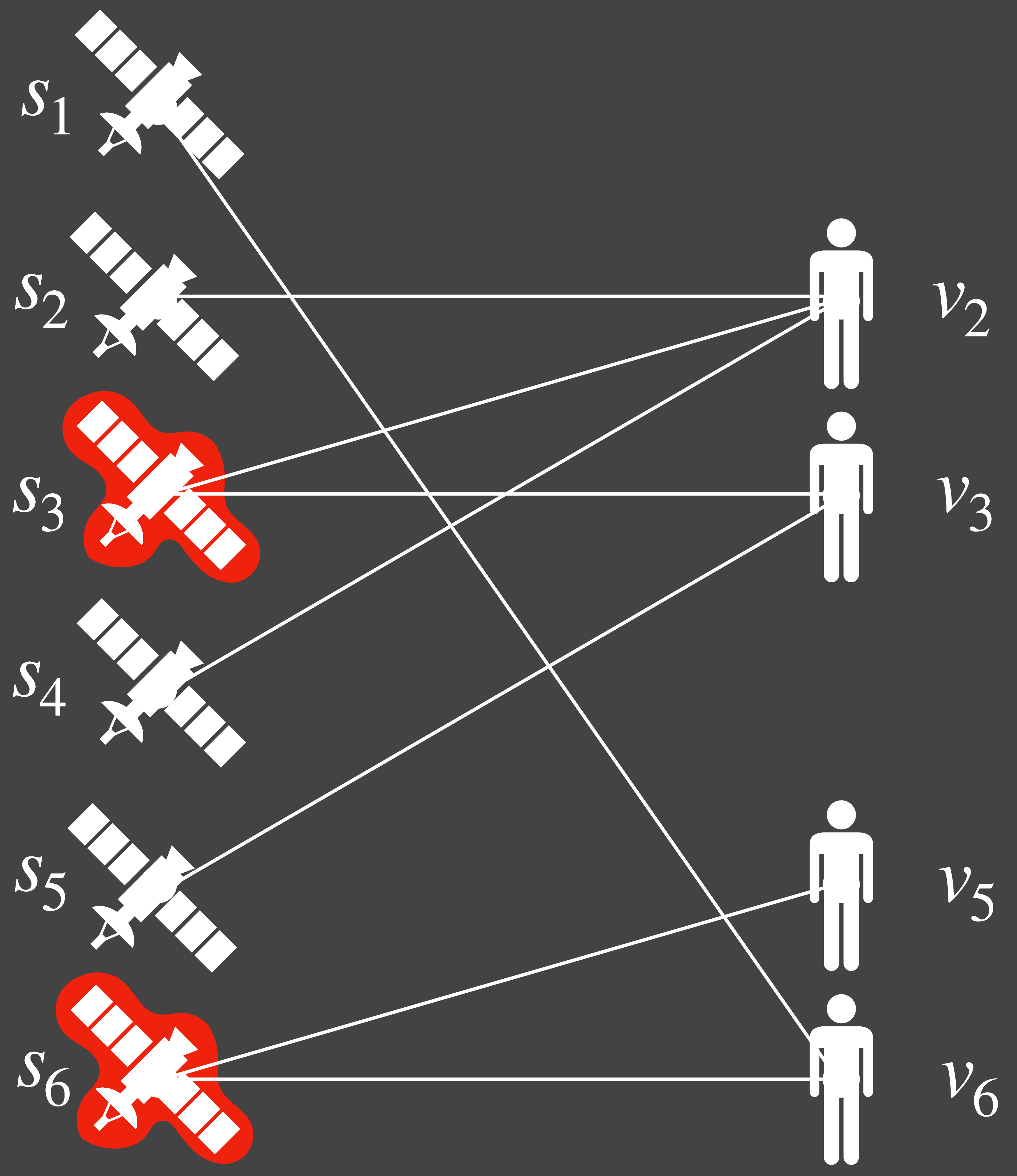

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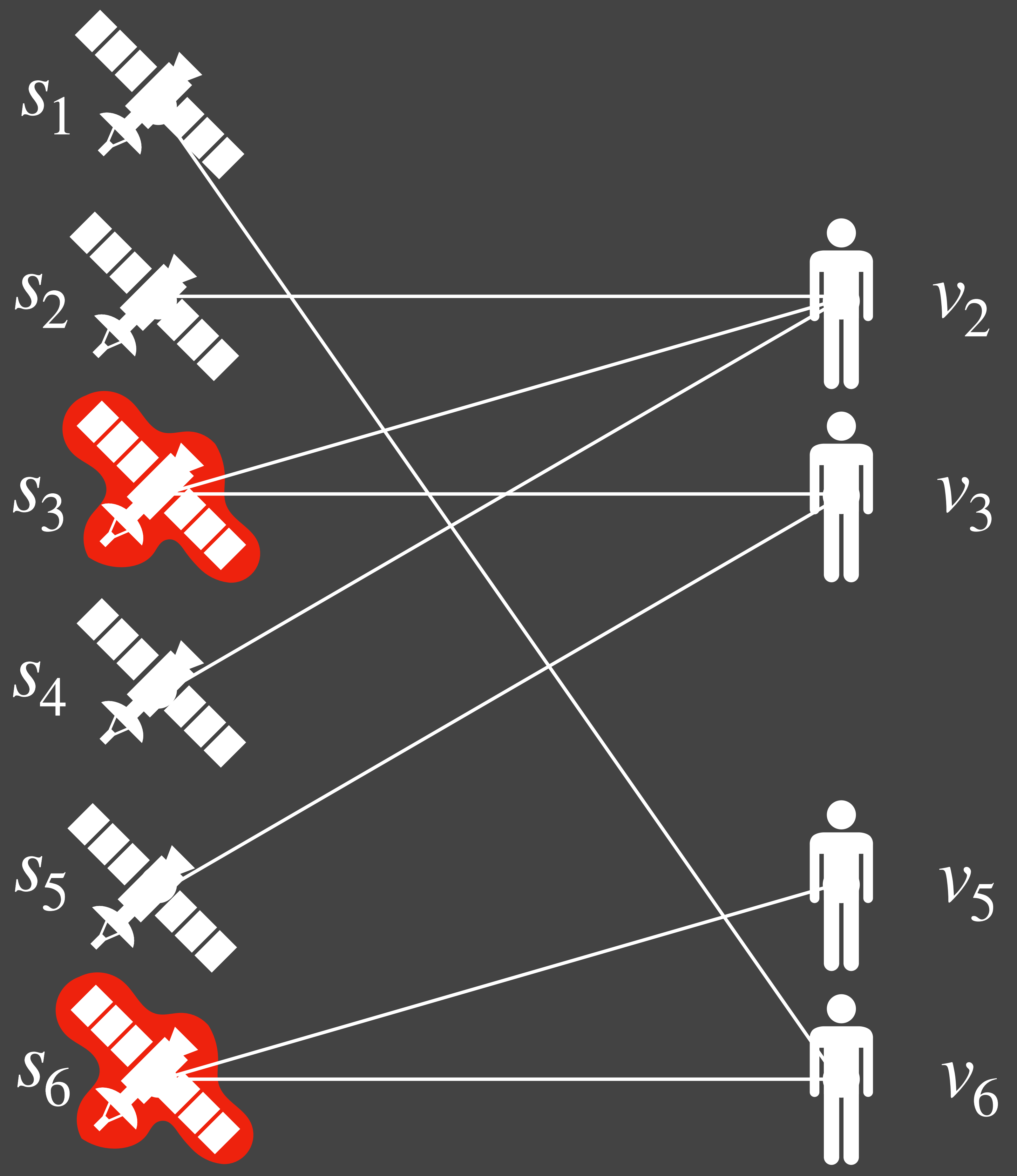

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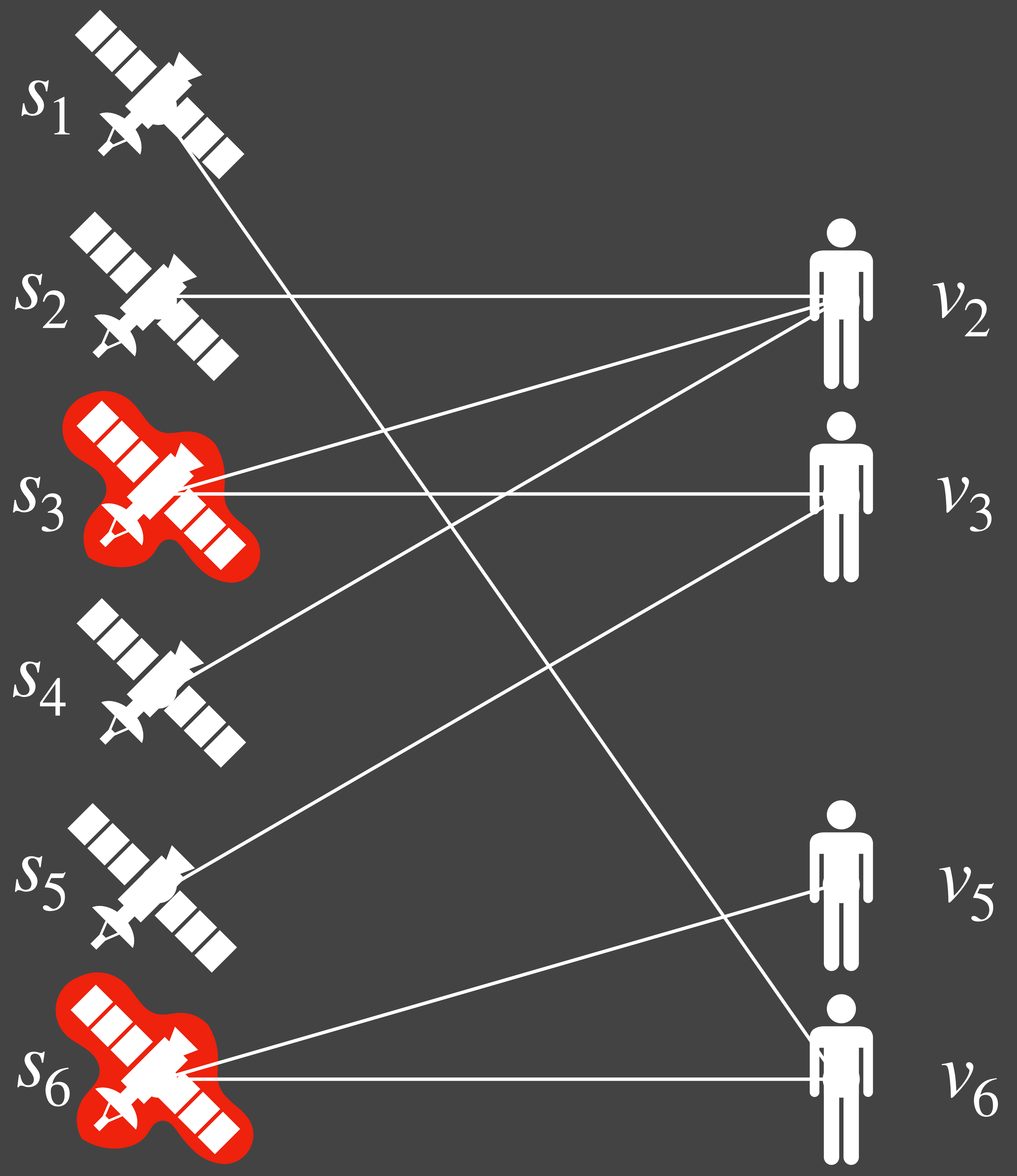

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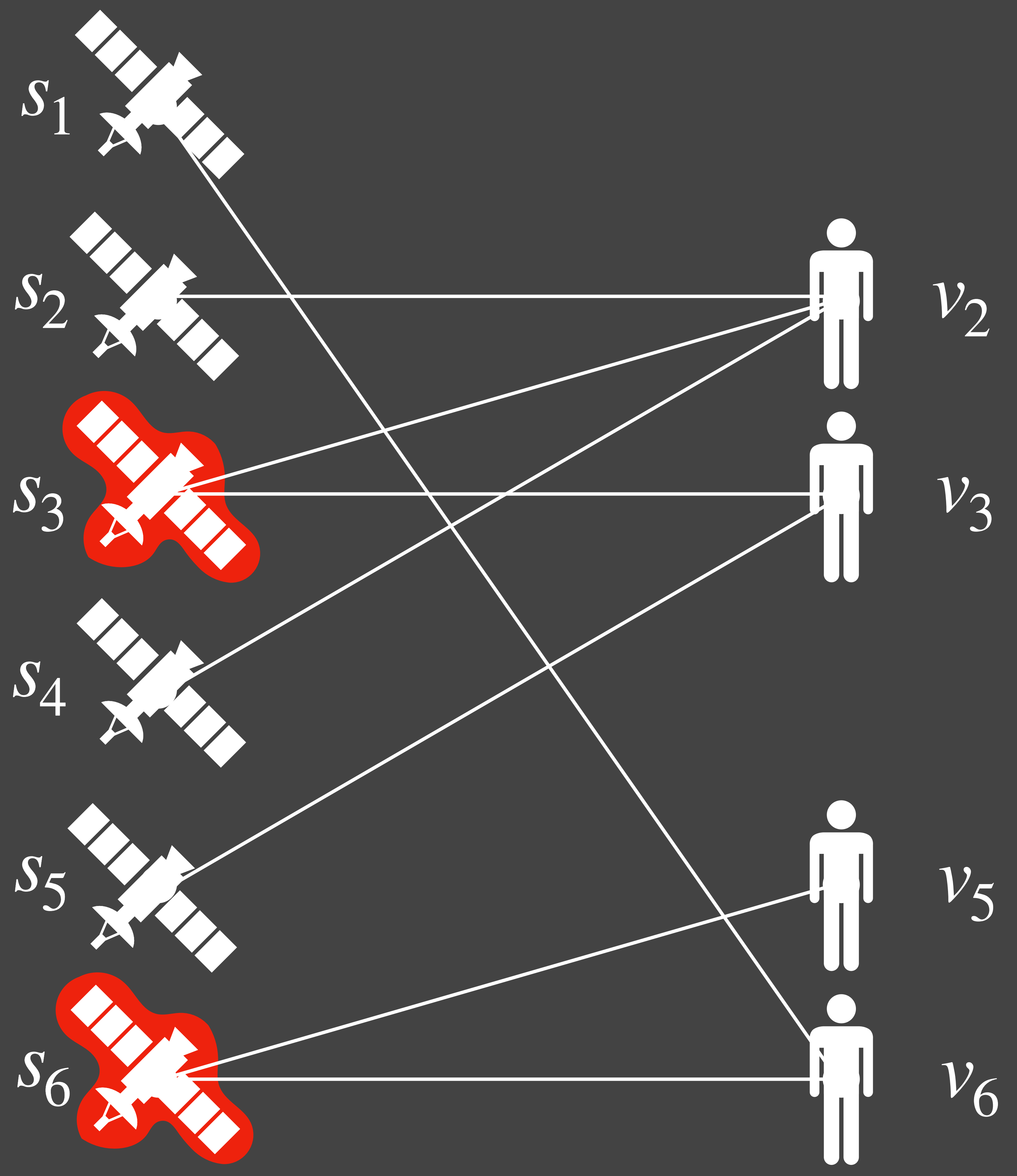

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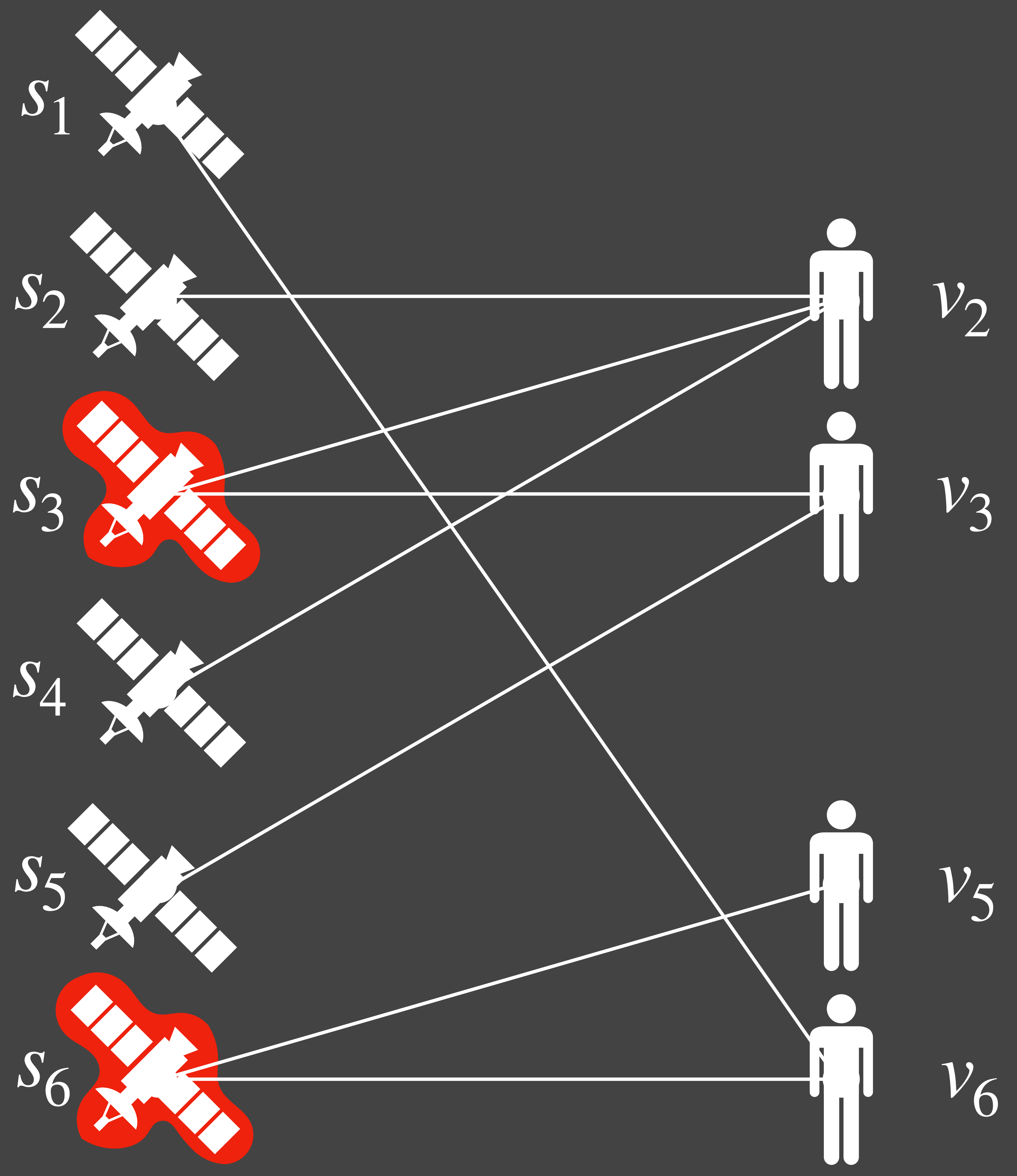

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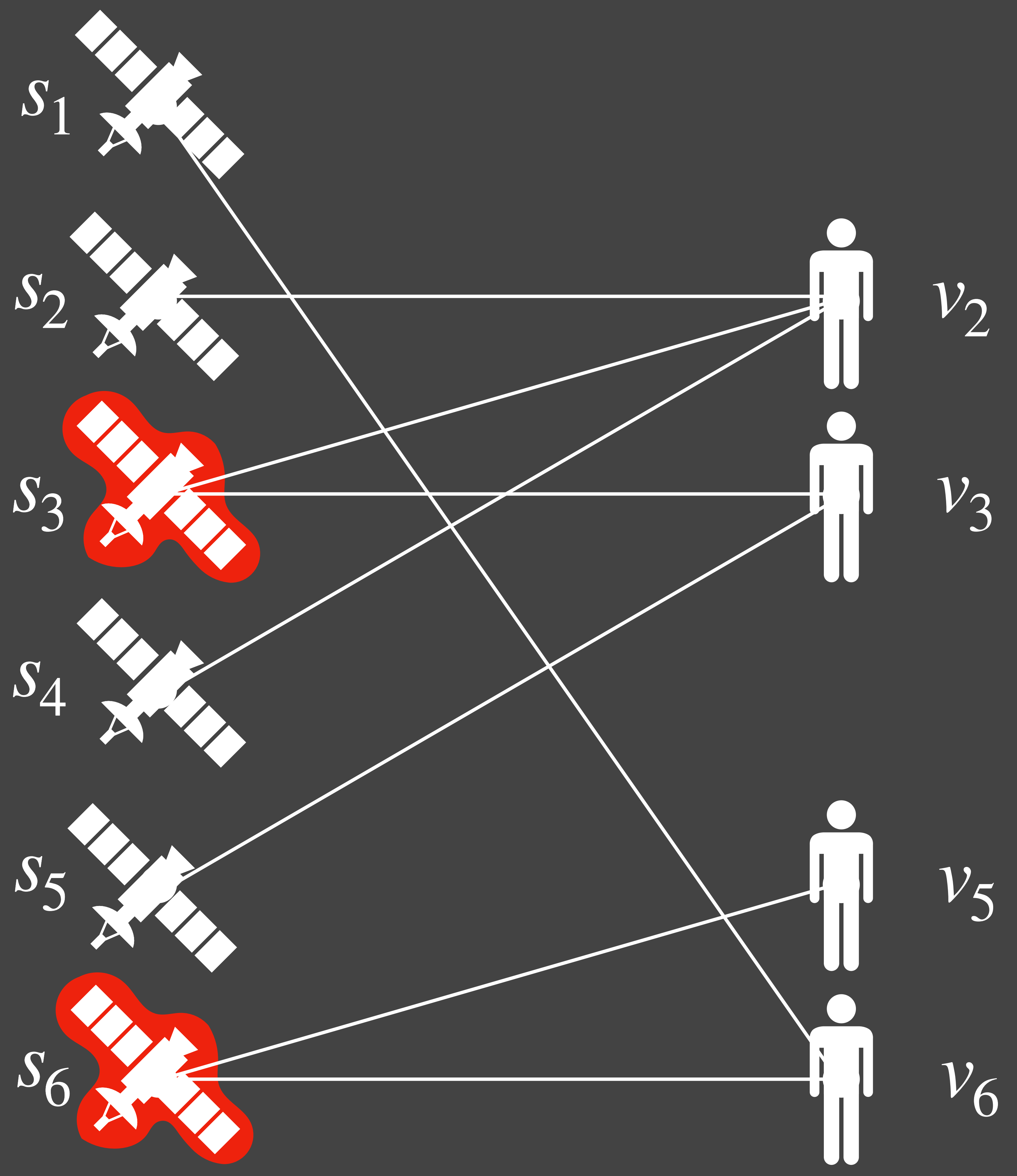

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Q: What is recourse/ approximation tradeoff?

Low Recourse Dynamic Algorithms

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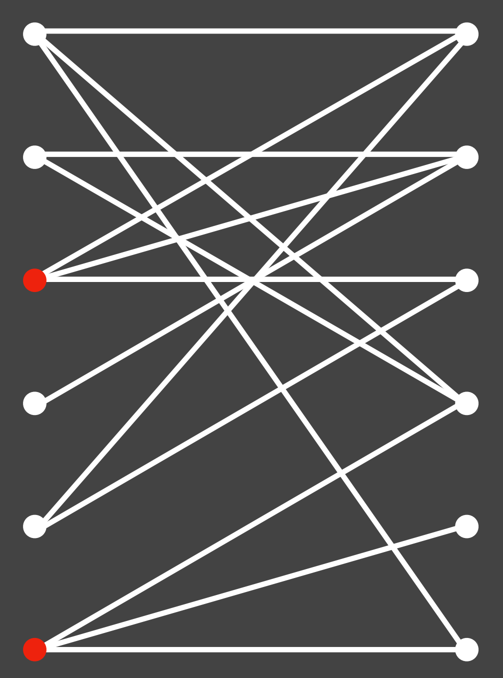

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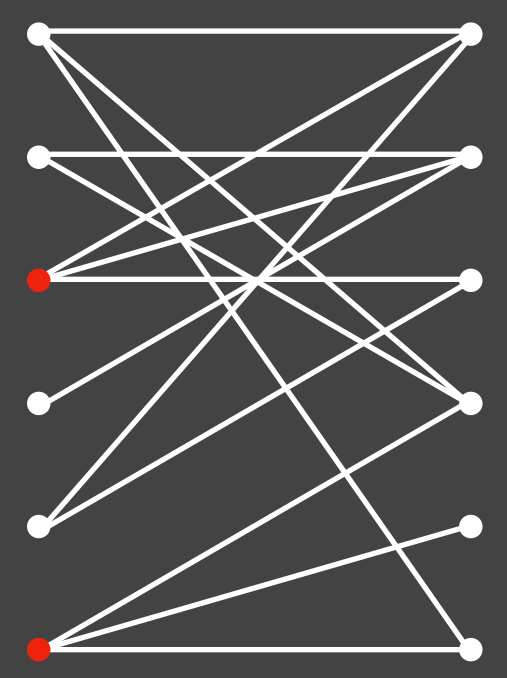

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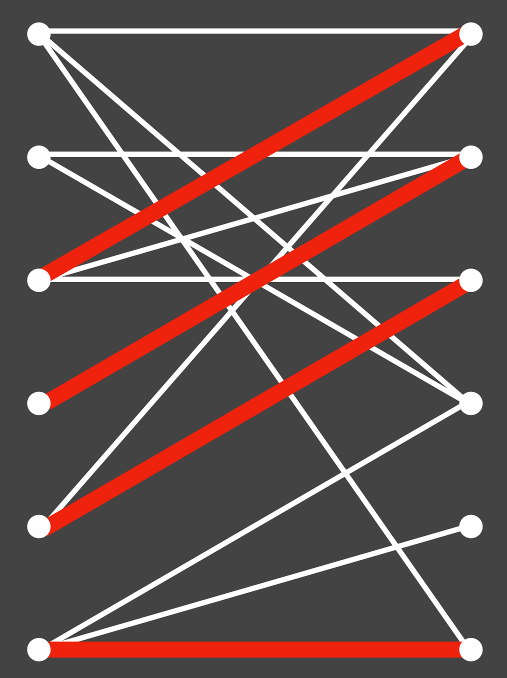

Matching

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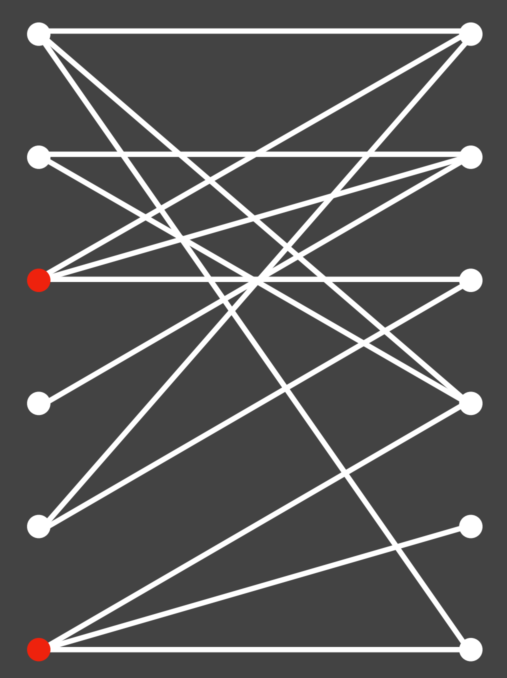

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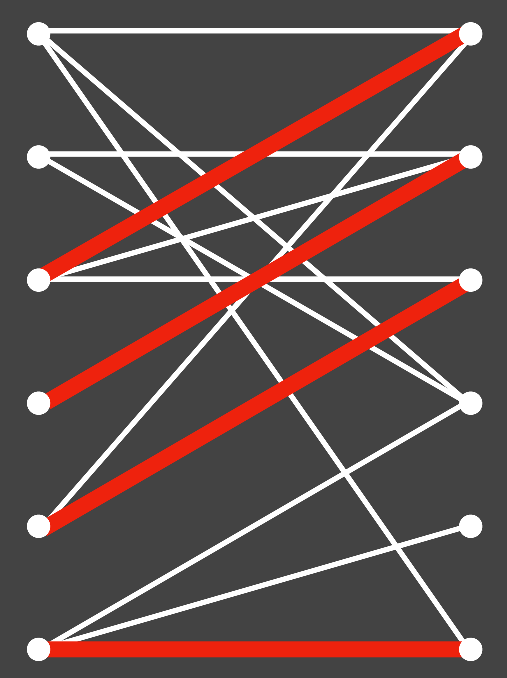

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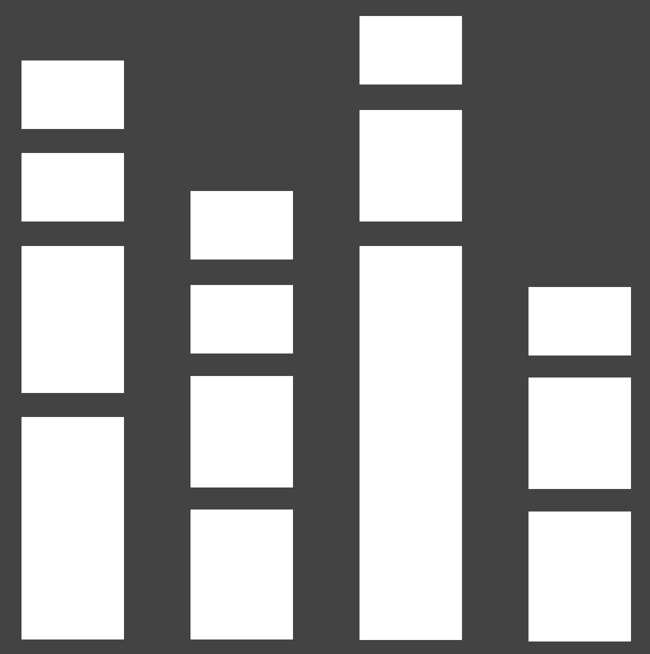

Load Balancing

Low Recourse Dynamic Algorithms

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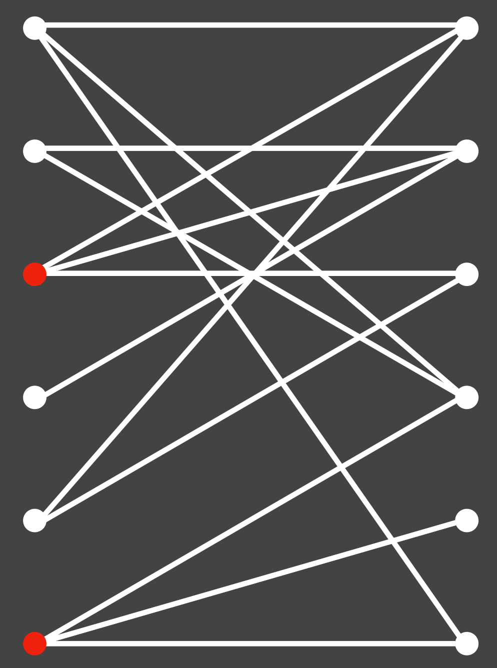

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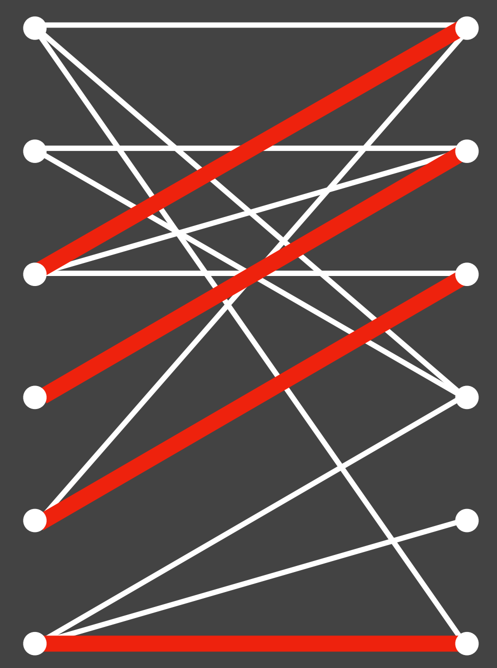

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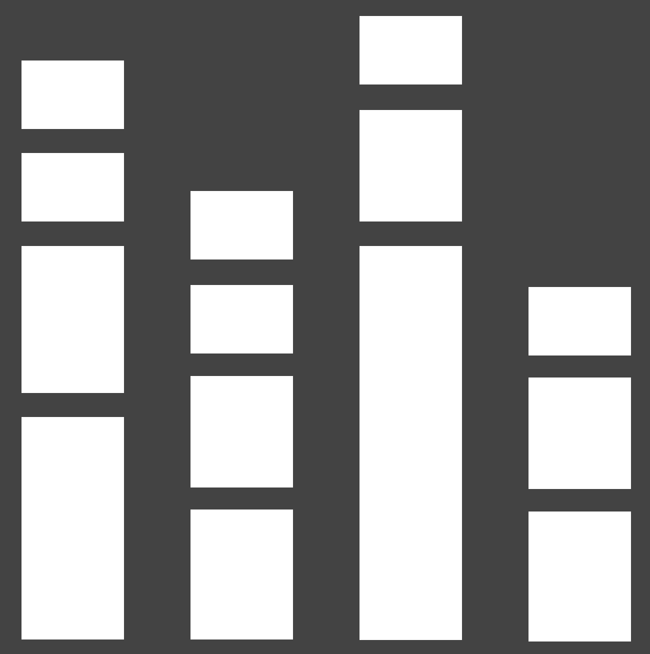

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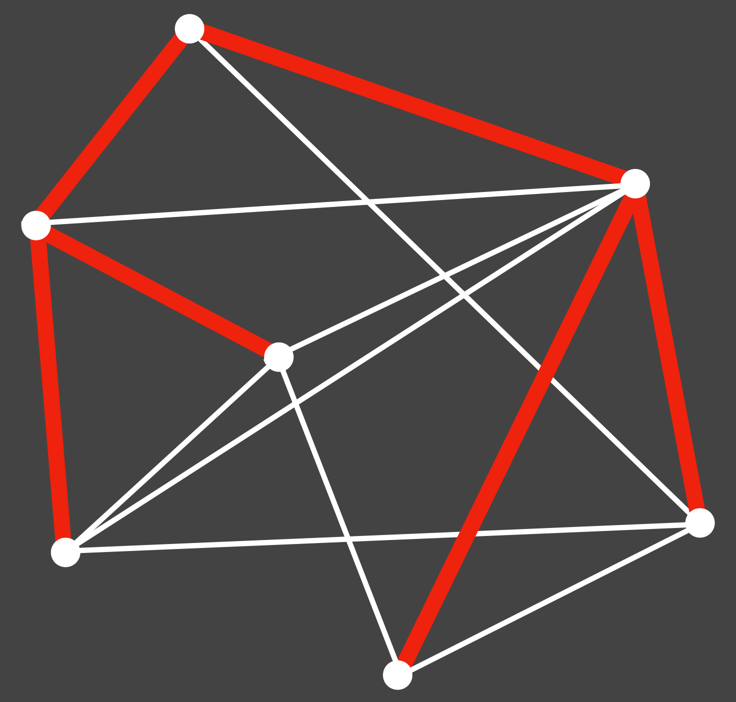

Minimum Spanning Tree

Low Recourse Dynamic Algorithms

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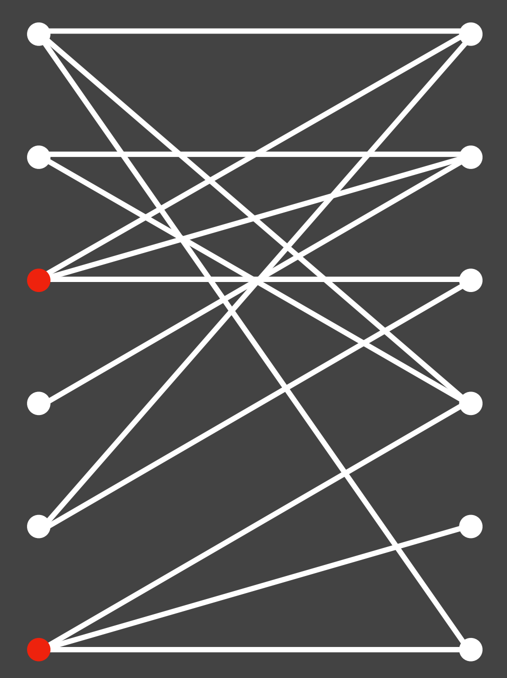
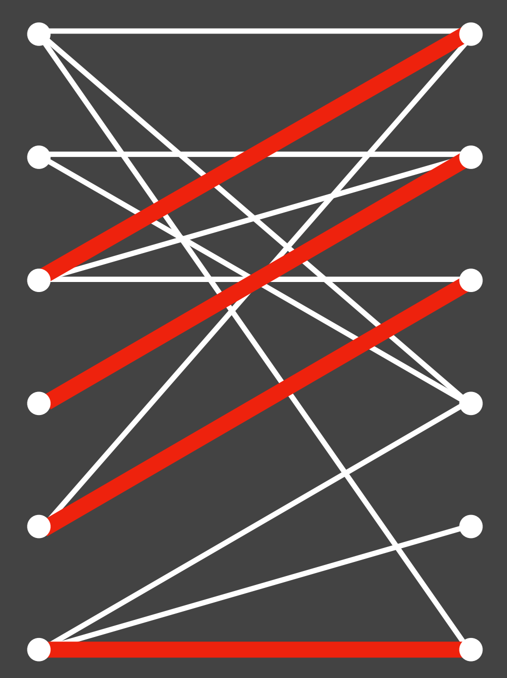


Set Cover

[Gupta Krishnaswamy Kumar Panigrahi 17] [Abboud+ 17] [Bhattacharya Henzinger Nanongkai 19] [Gupta **L.** 20] [Bhattacharya Henzinger Nanongkai Wu 21] [Assadi Solomon21]



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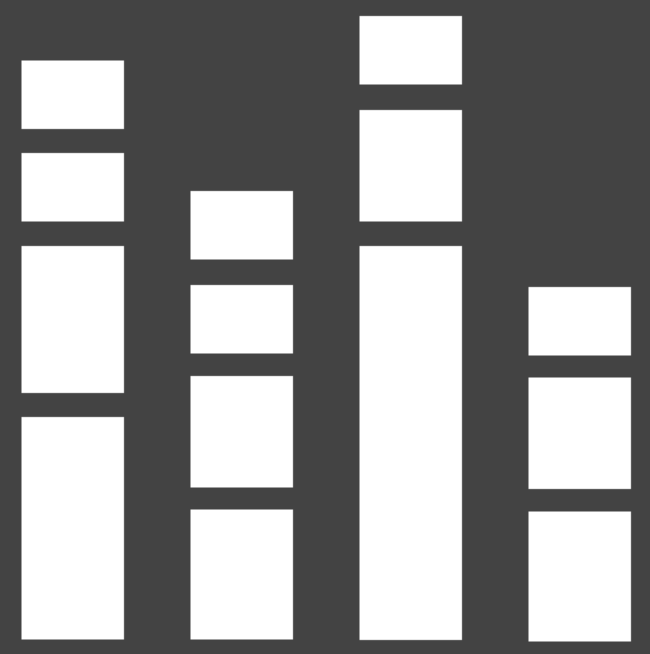
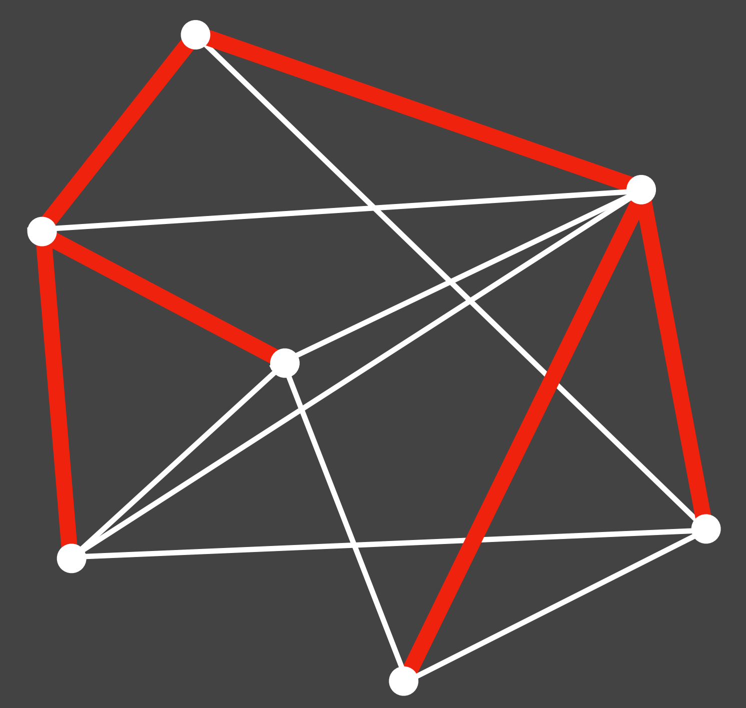
[Folklore] [Grove Kao Krishnan Vitter 95] [Chadhuri Daskalakis Kleinberg Lin 09] [Bosek Leniowski Sankowski Zych] [Bernstein Holm Rotenberg 18]



Load Balancing

[Awerbuch Azar Plotkin Warts 01] [Gupta Kumar Stein 14] [Krishnaswamy Li Suriyanarayana 23]



Minimum Spanning Tree

[Imase Waxman 91] [Gu Gupta Kumar 16] [Gupta Kumar 14] [Łącki+ 15] [Gupta **L.** 20]

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General recipe for designing low-recourse algorithms?

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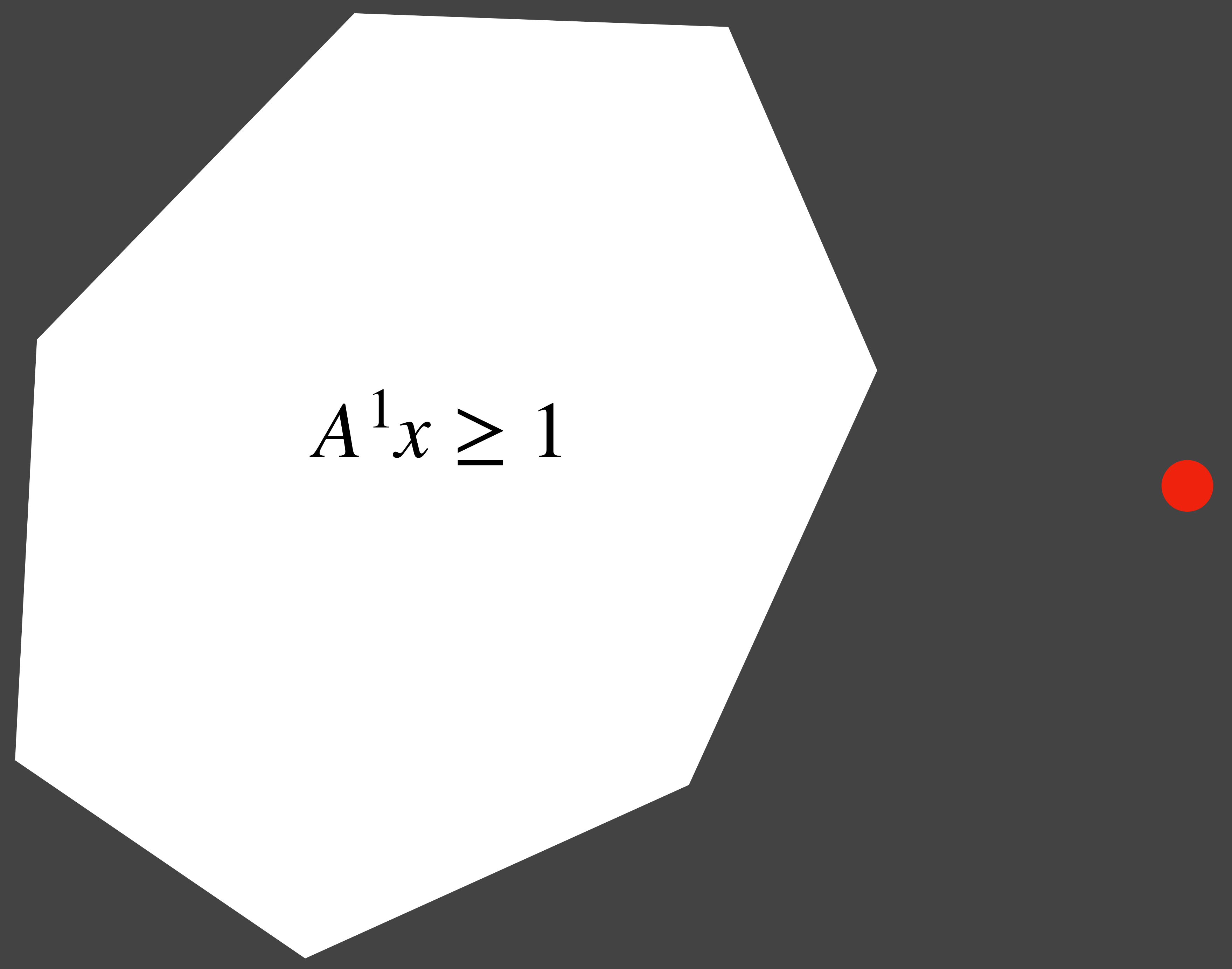

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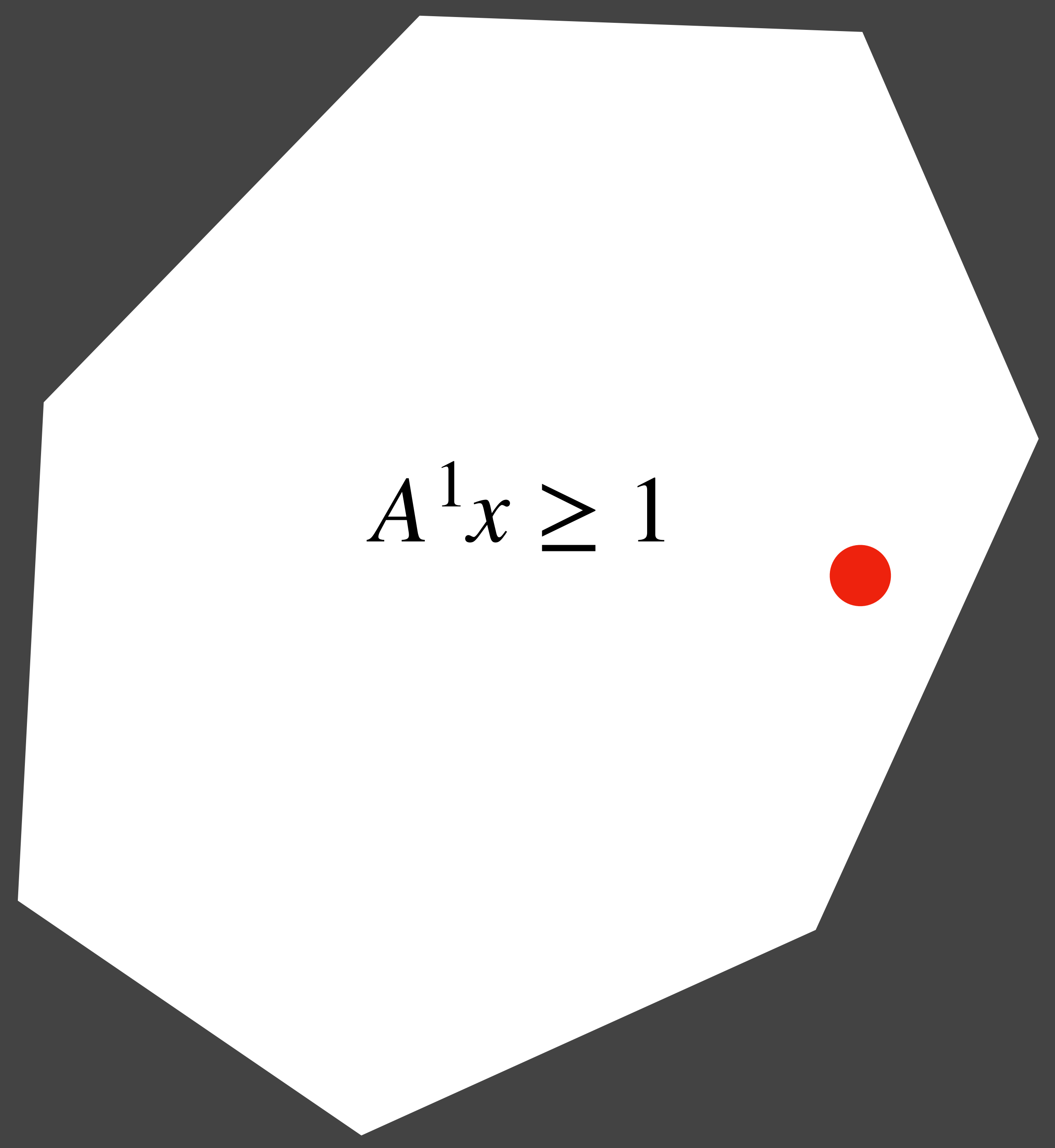

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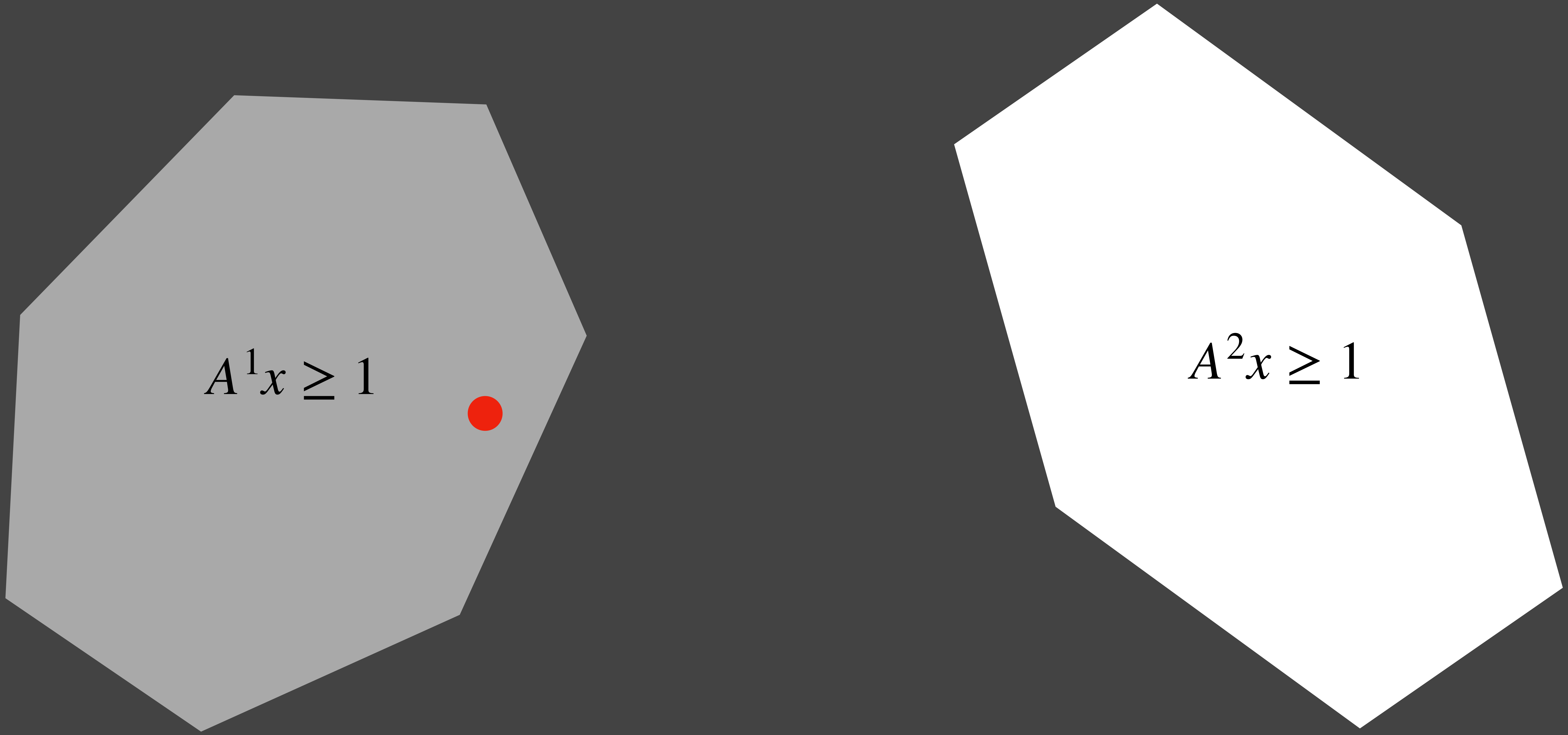

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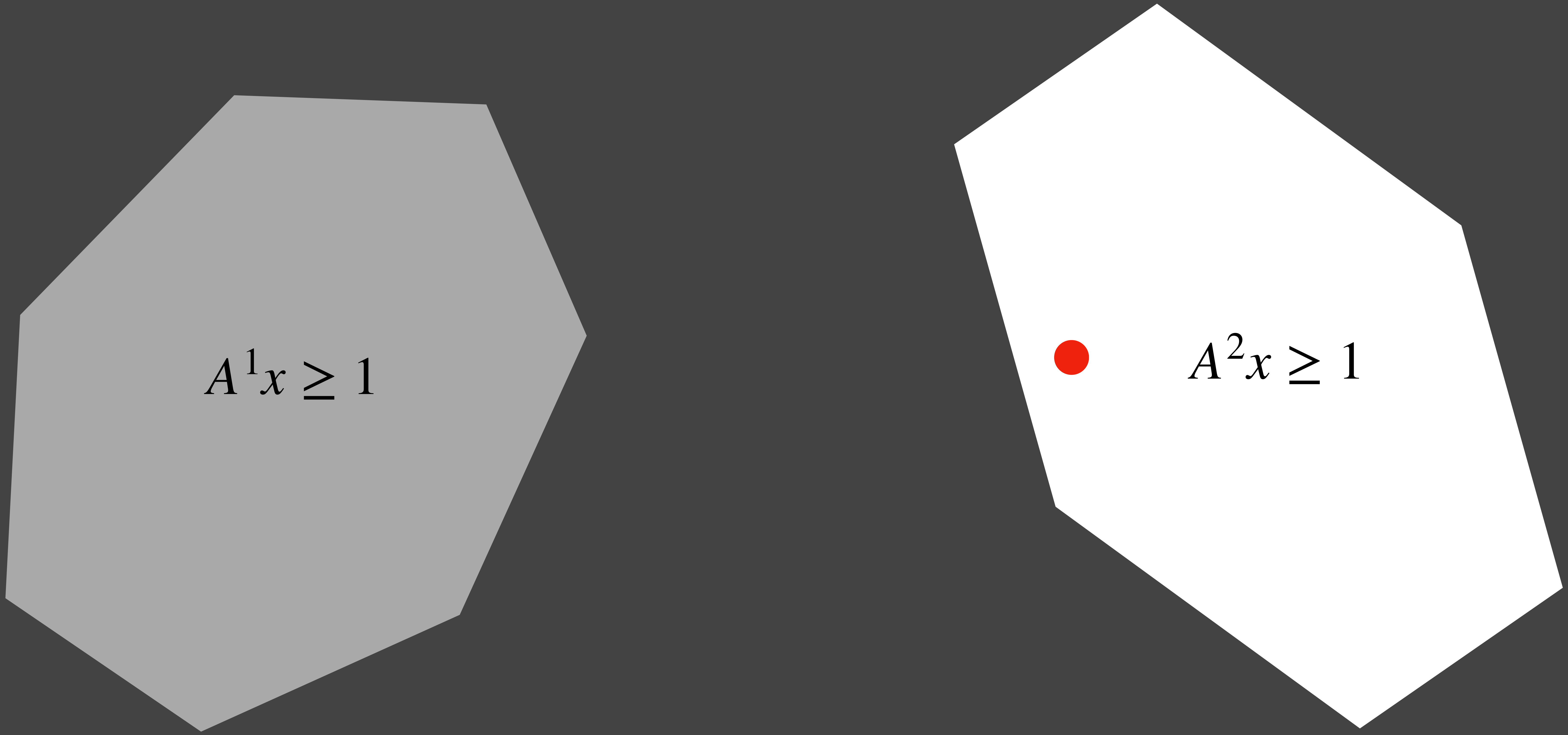

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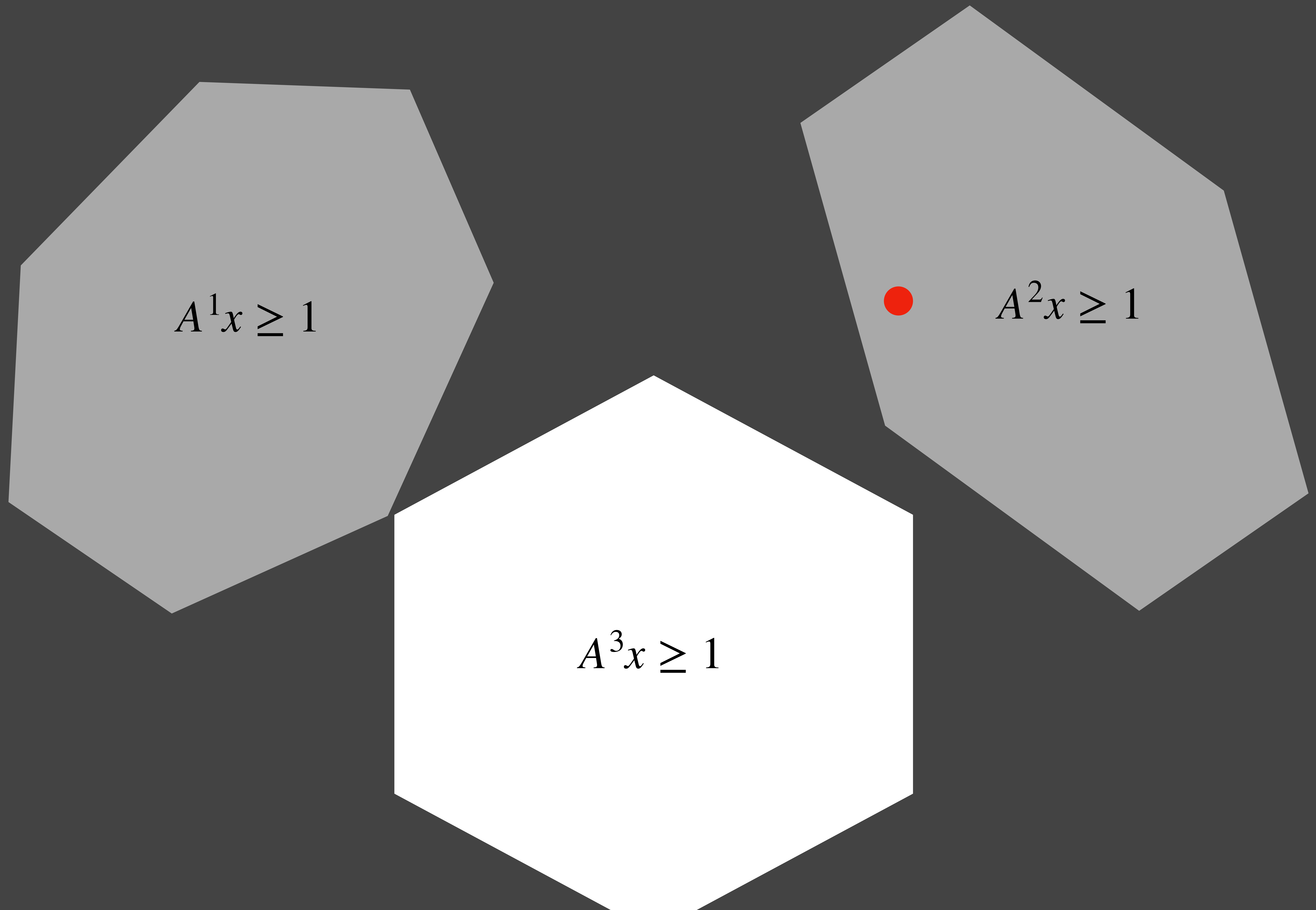

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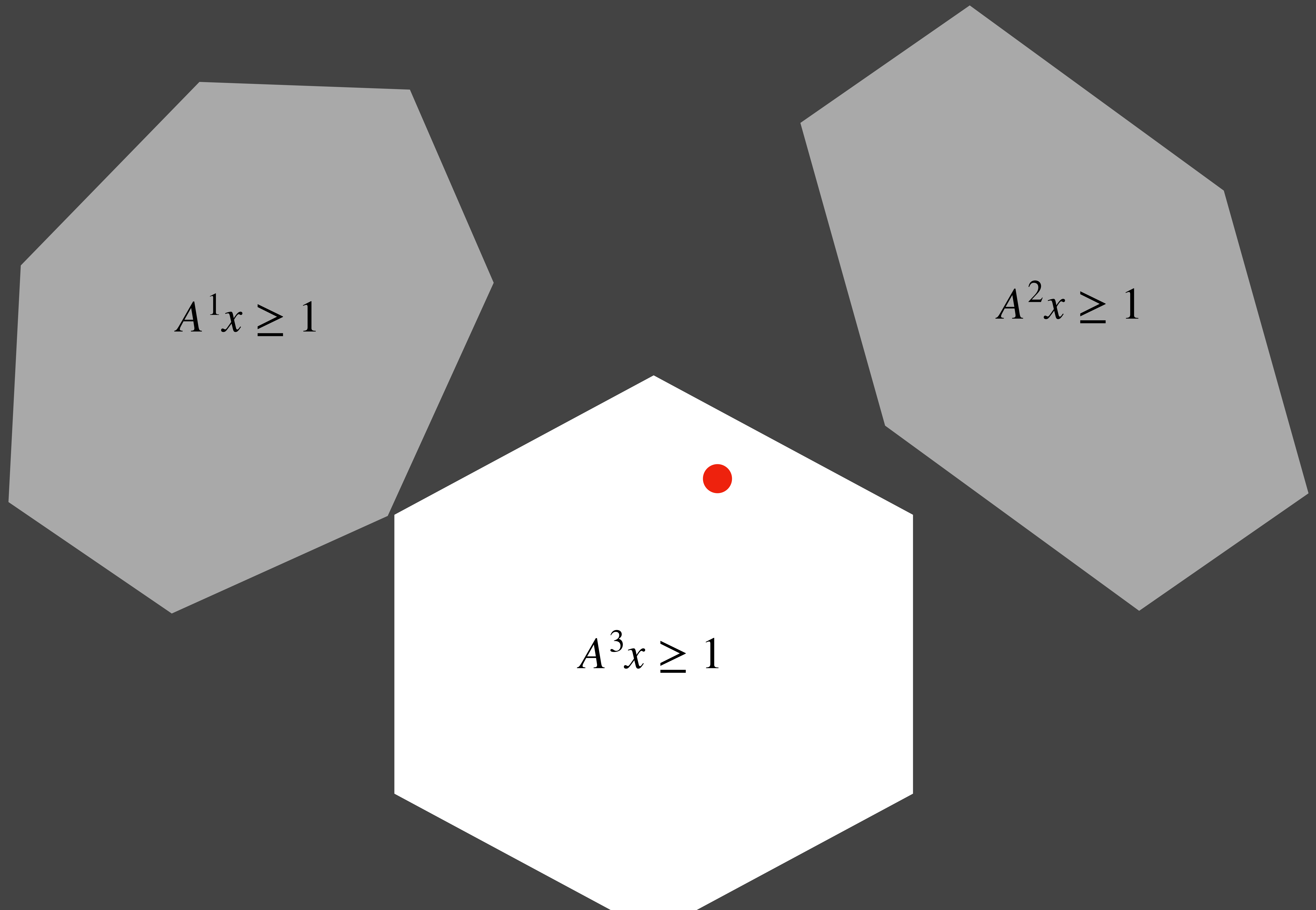

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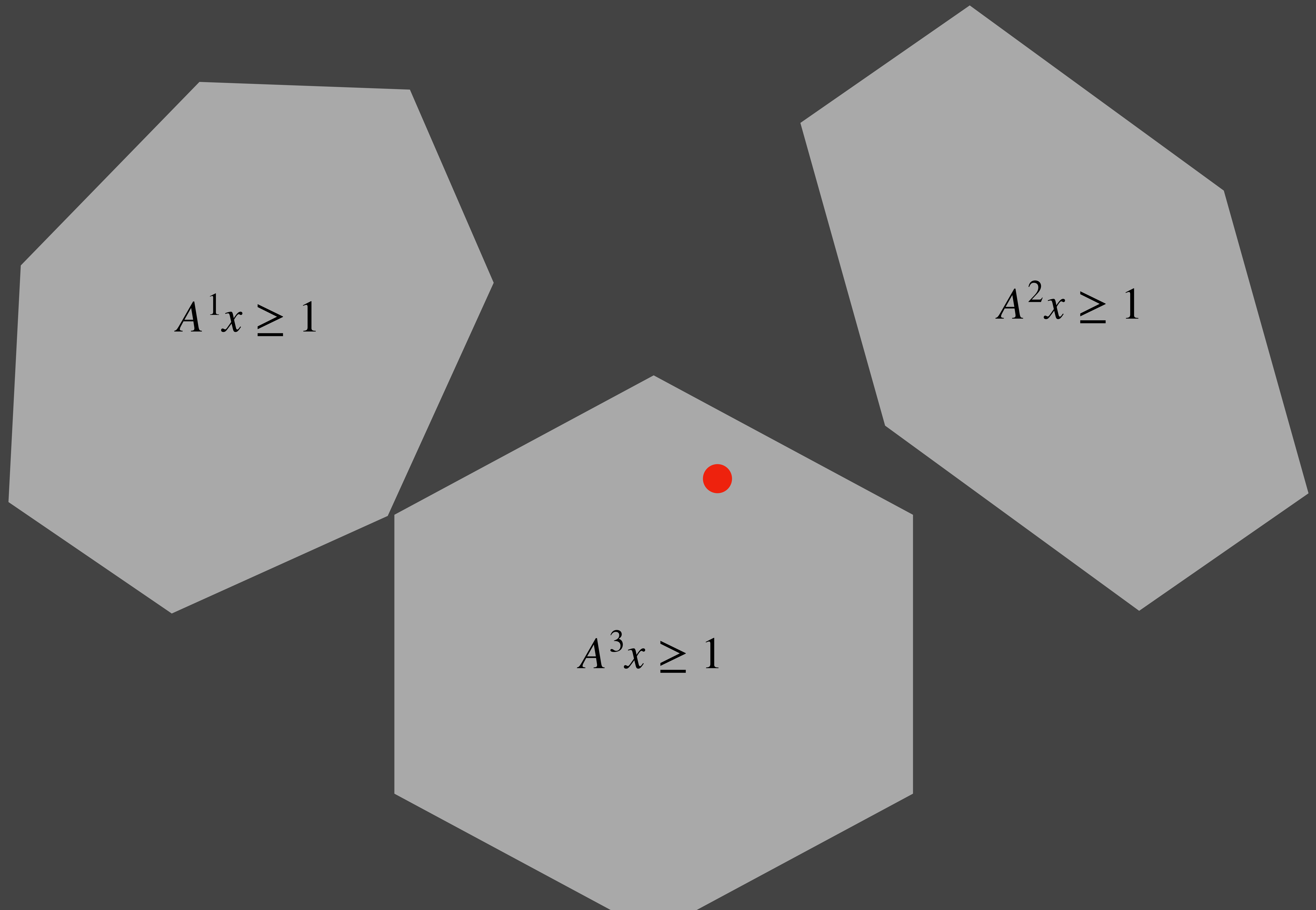

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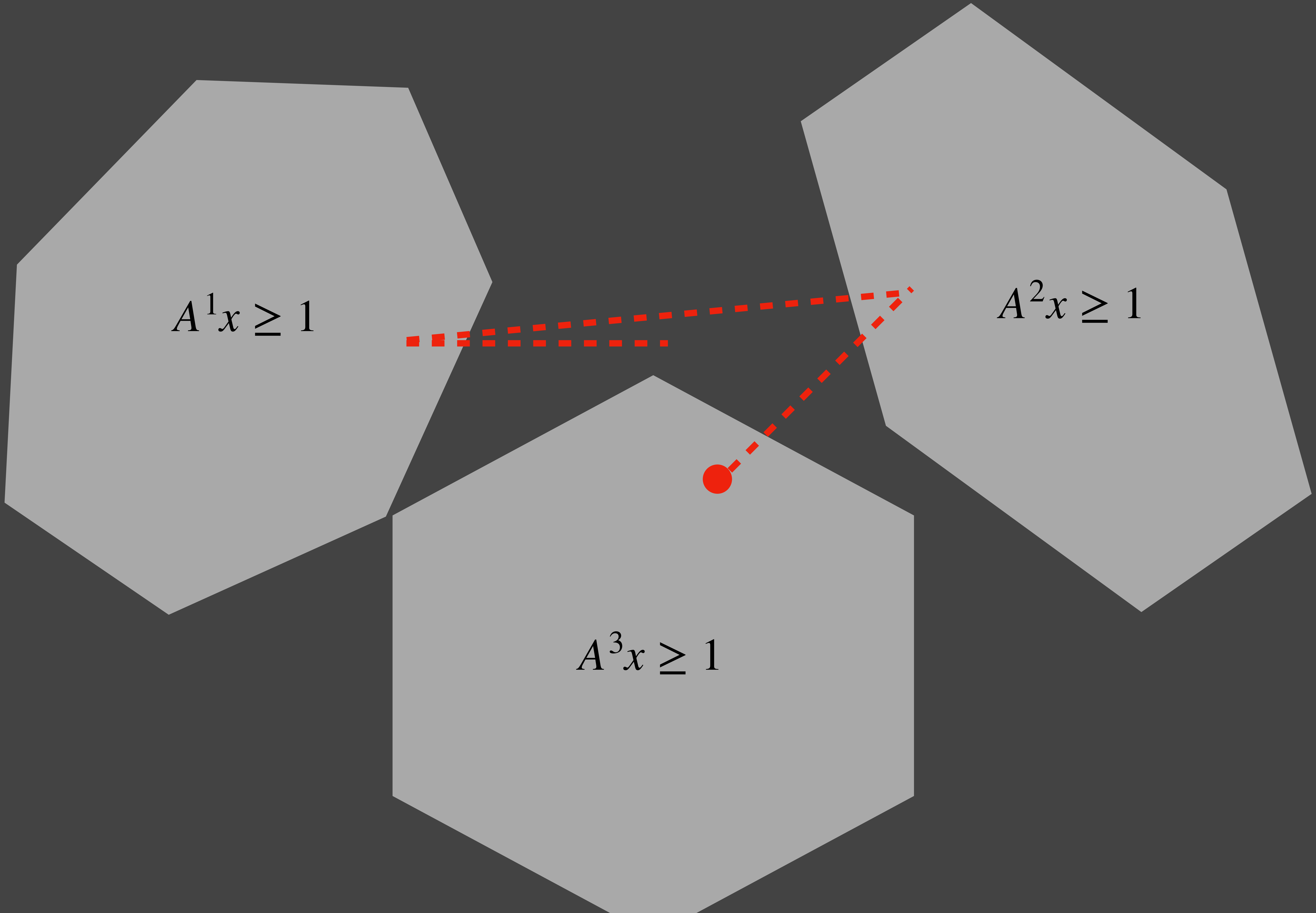
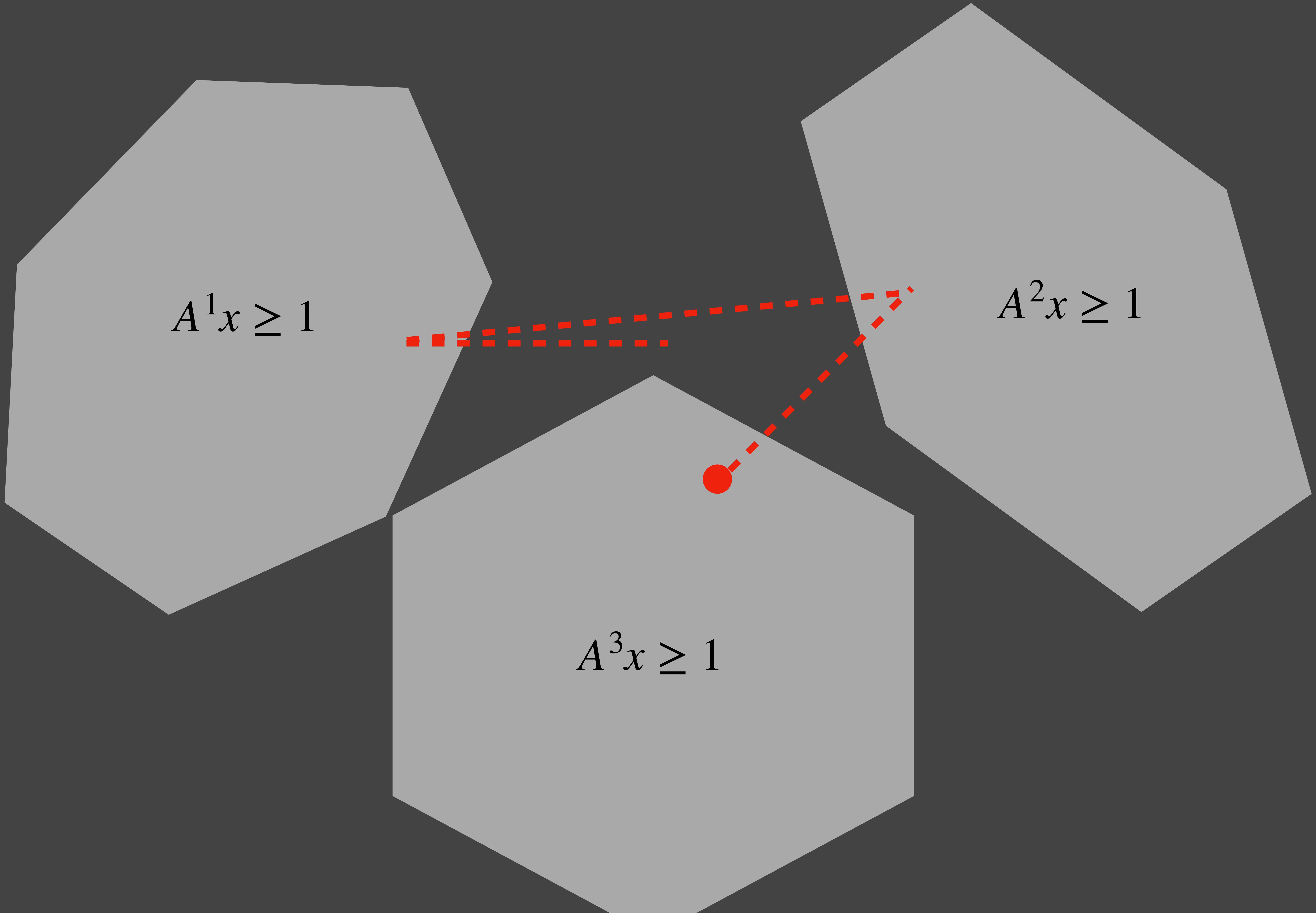


Goal: minimize distance traveled,

i.e. $\sum_t d(x^t, x^{t-1})$.

a.k.a. Convex Body Chasing!



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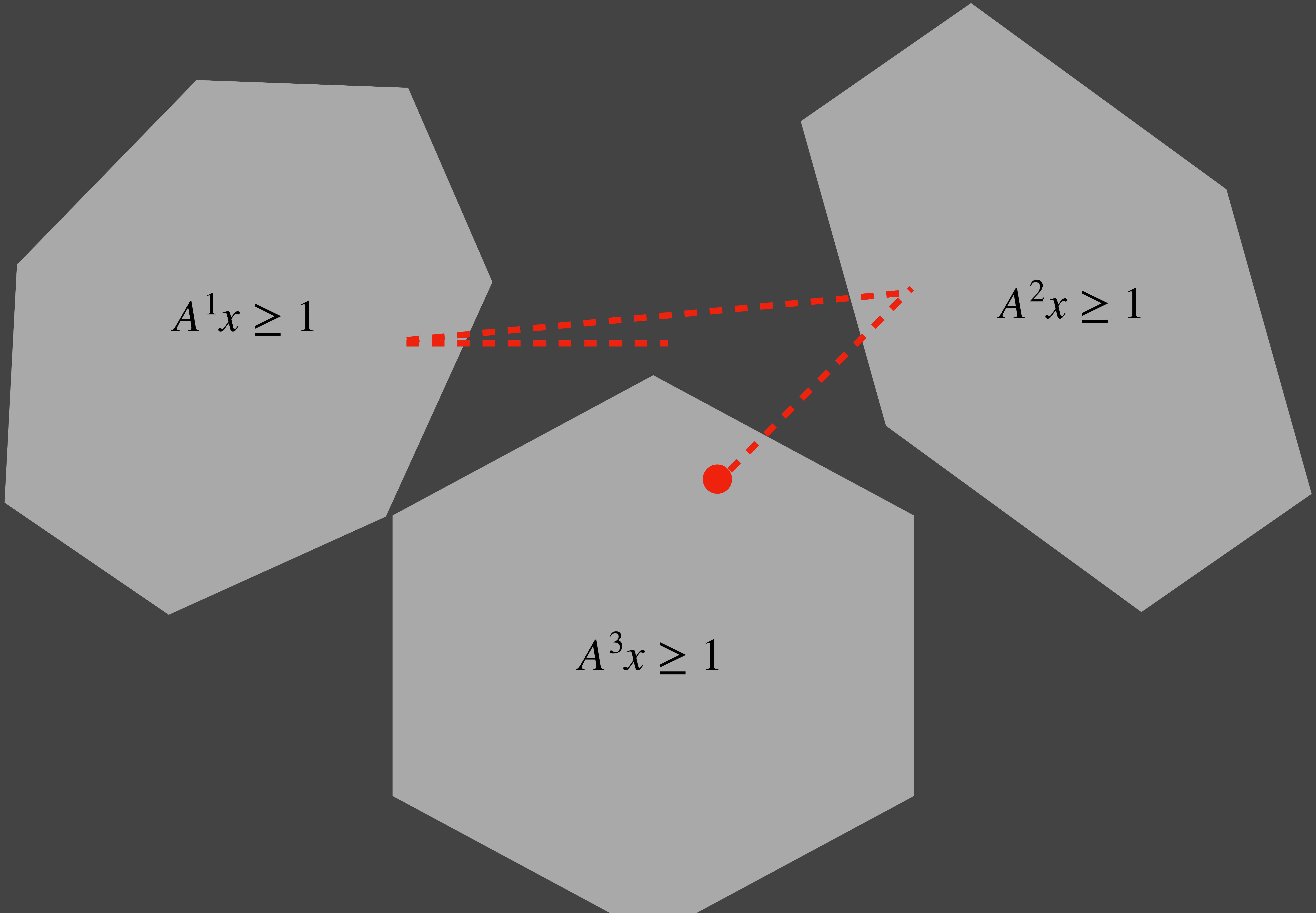

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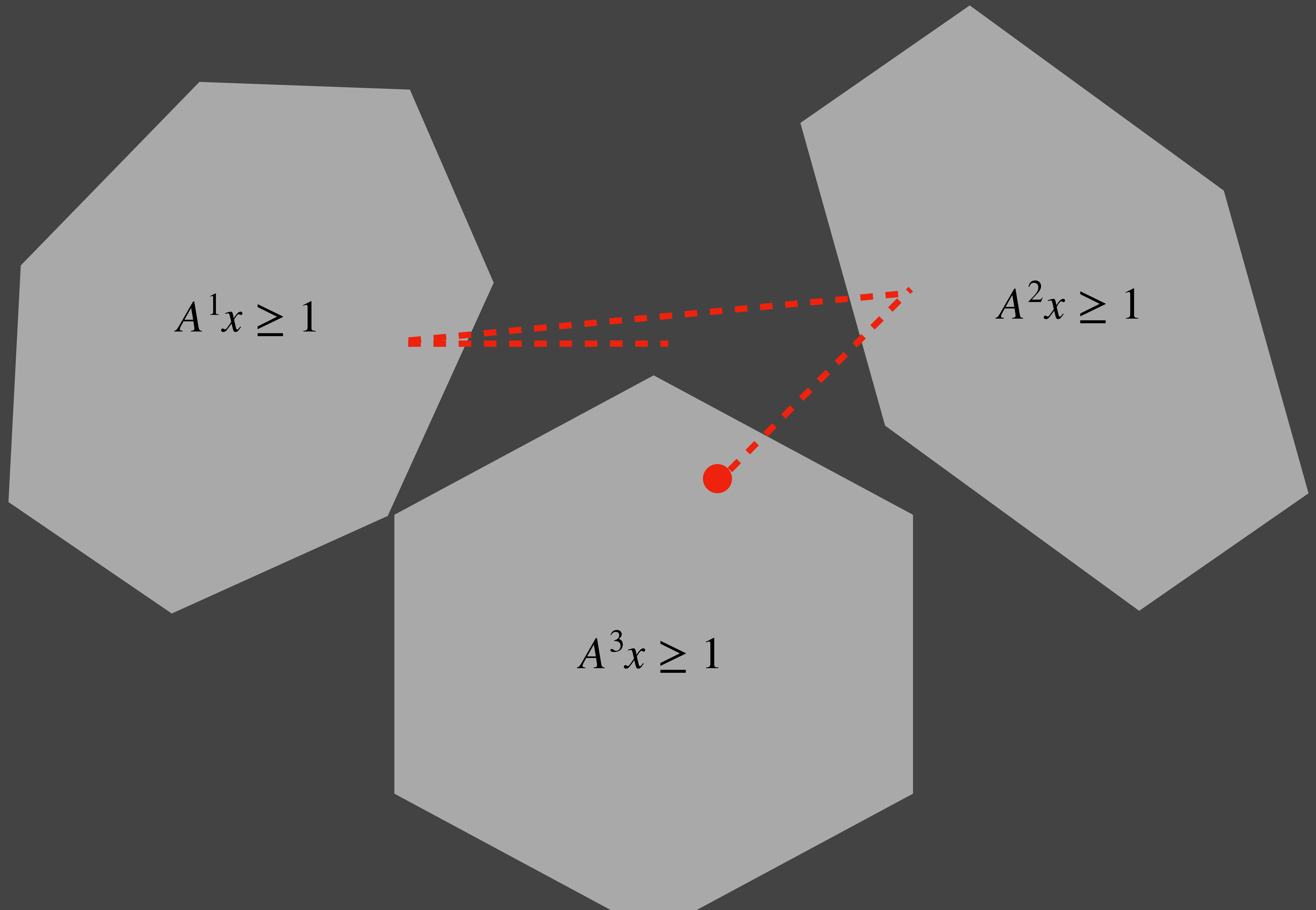

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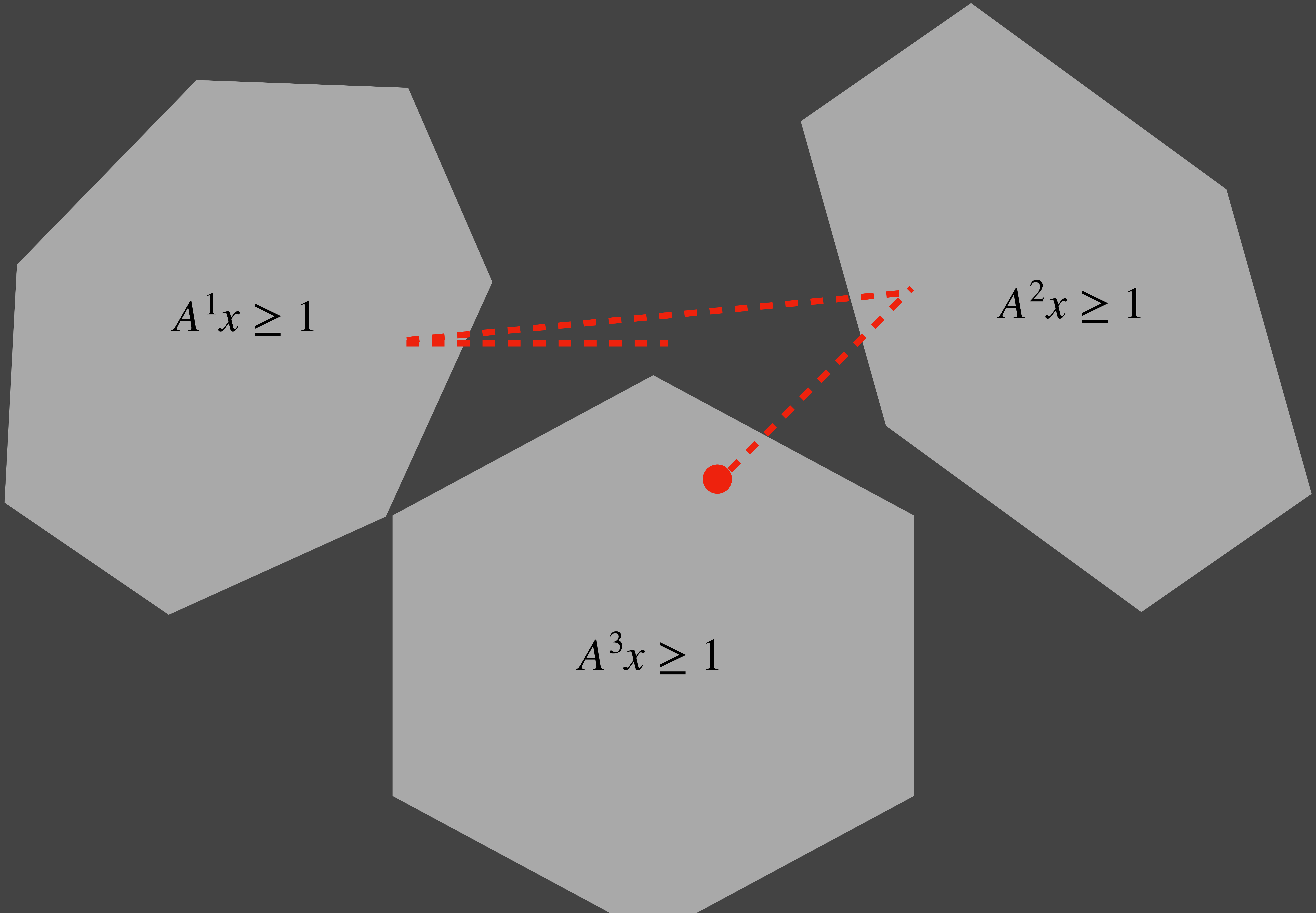

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Super cool! Co-best papers @ SODA 2020!

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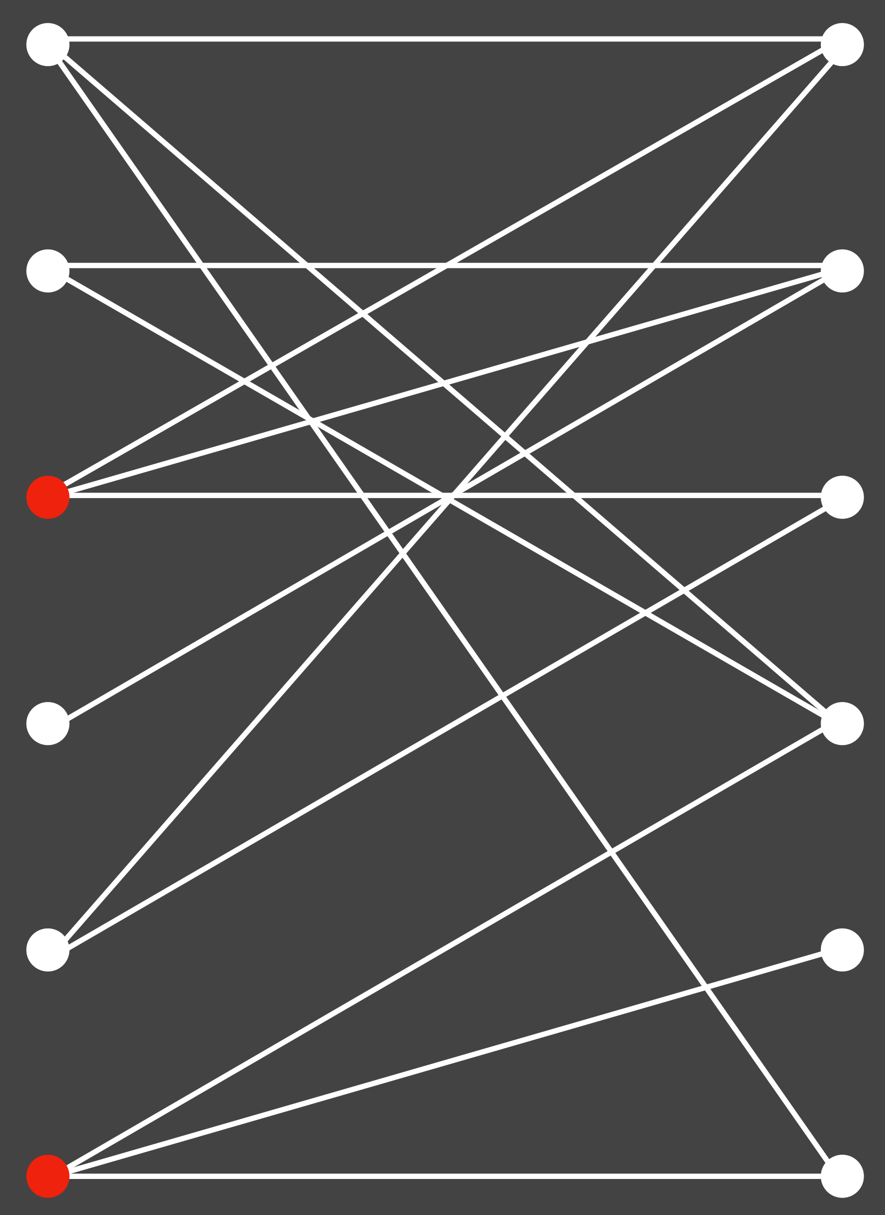
e.g. Set Cover.



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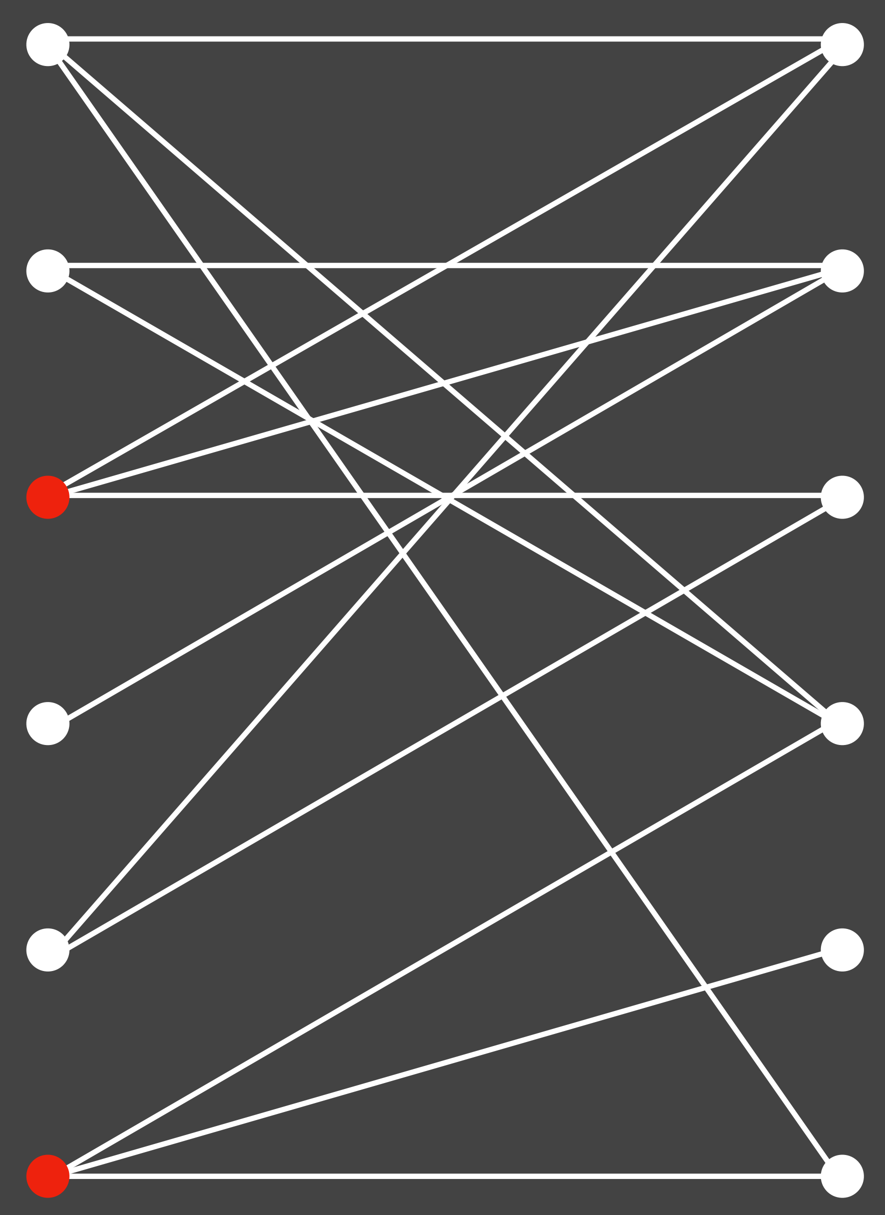

$\Omega(\sqrt{n})$ lower bound [Bubeck Klartag Lee Li Sellke 20] for all l_p metrics, $p \geq 1$.

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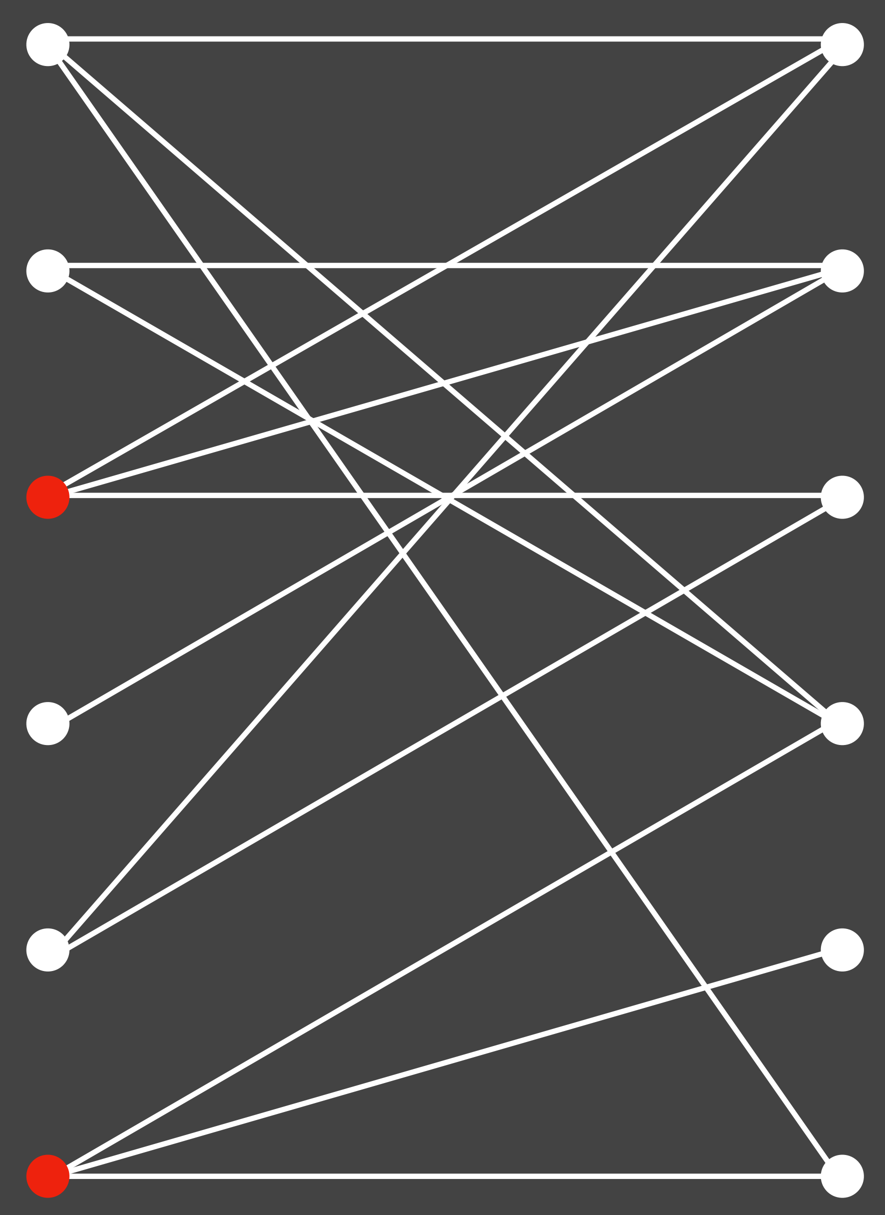

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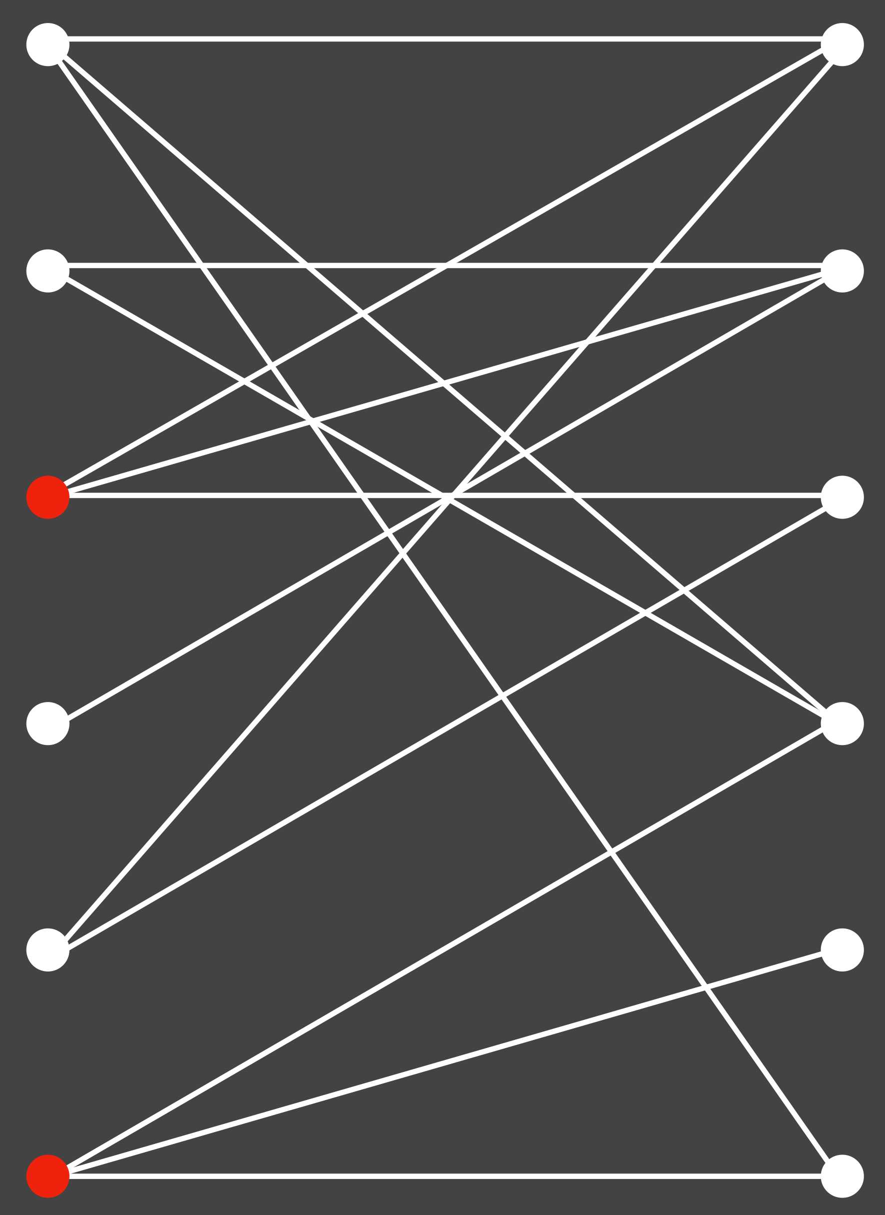

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Q: Expressive AND tractable special families of Convex Body Chasing?

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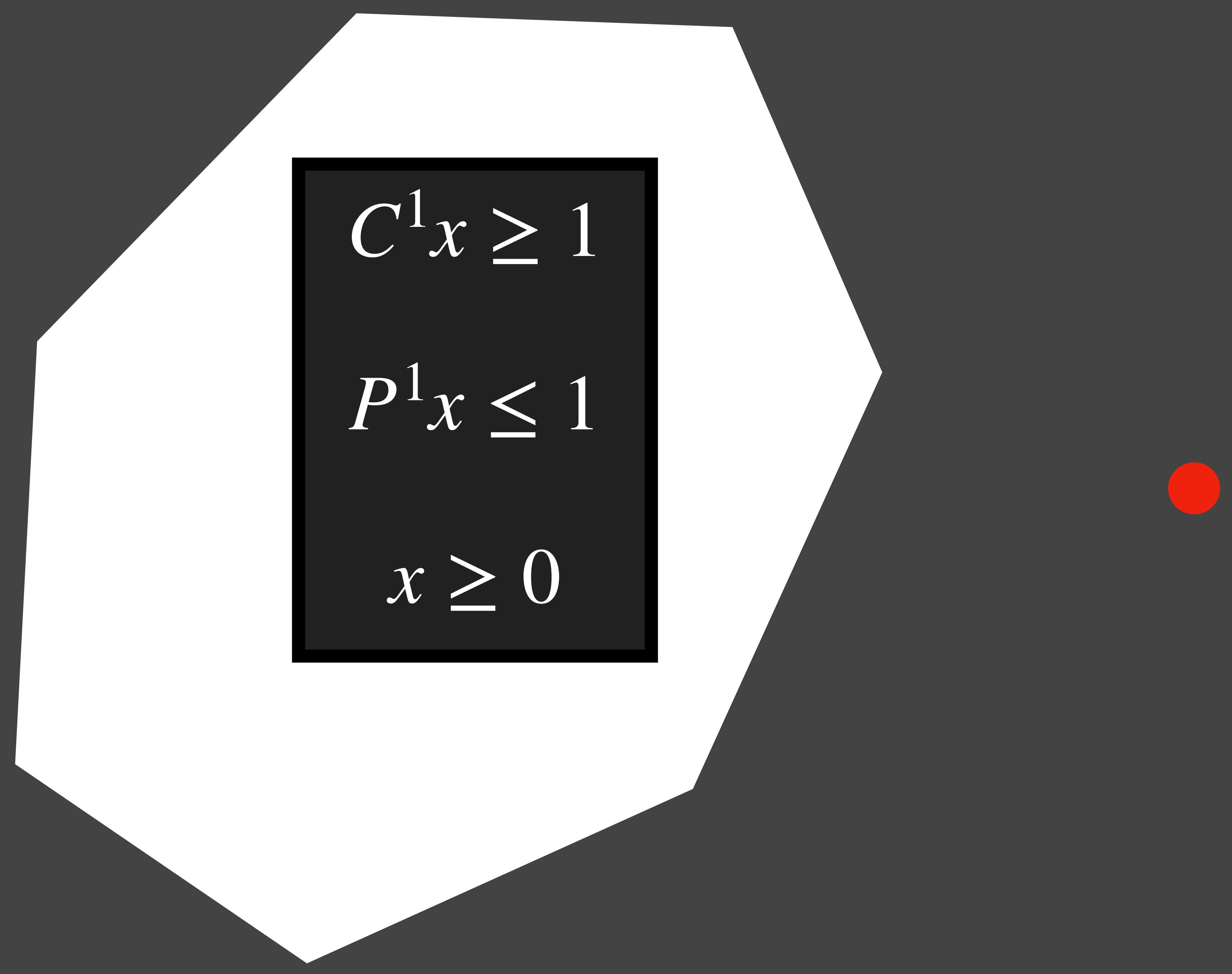

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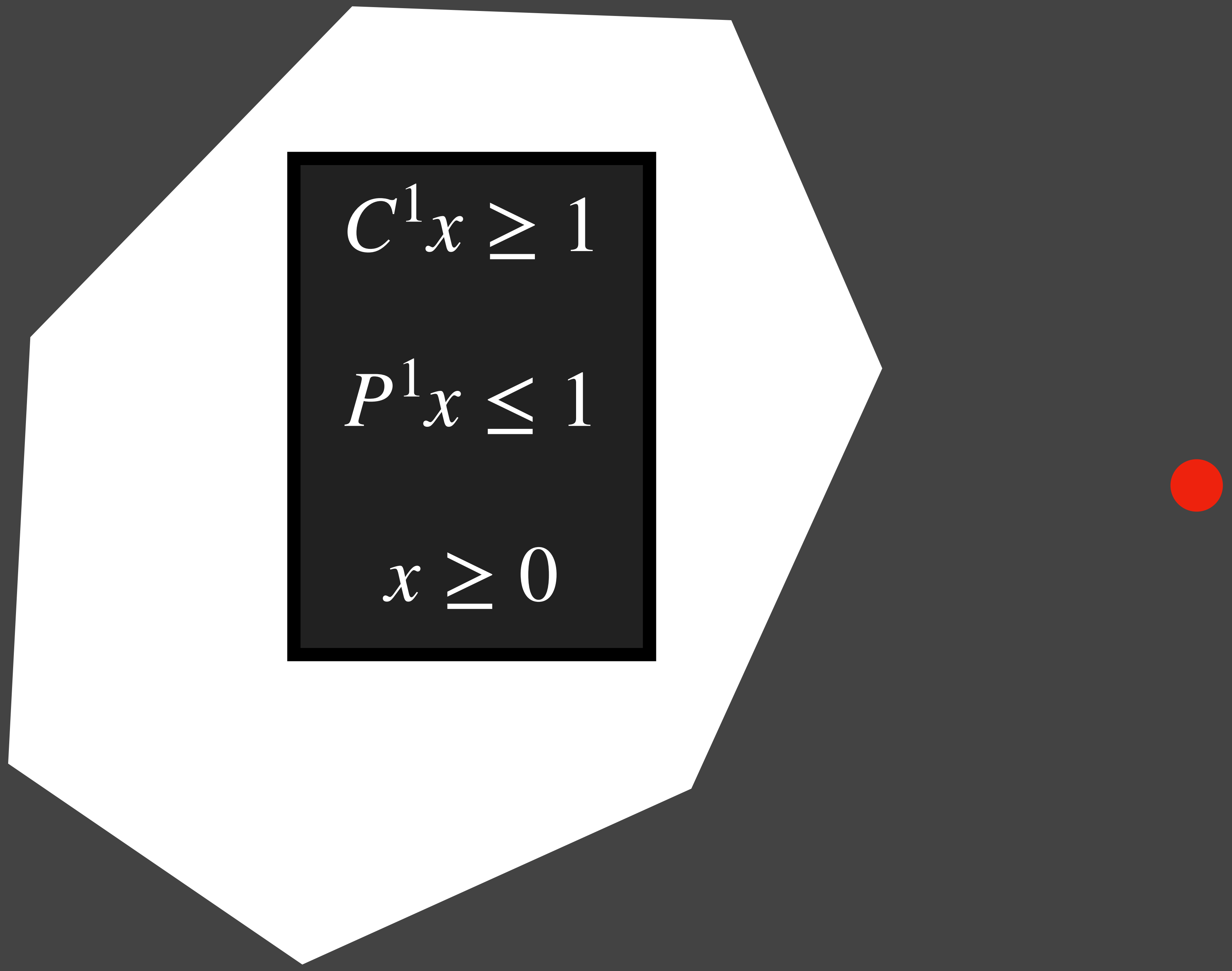

C^t, P^t have nonnegative entries

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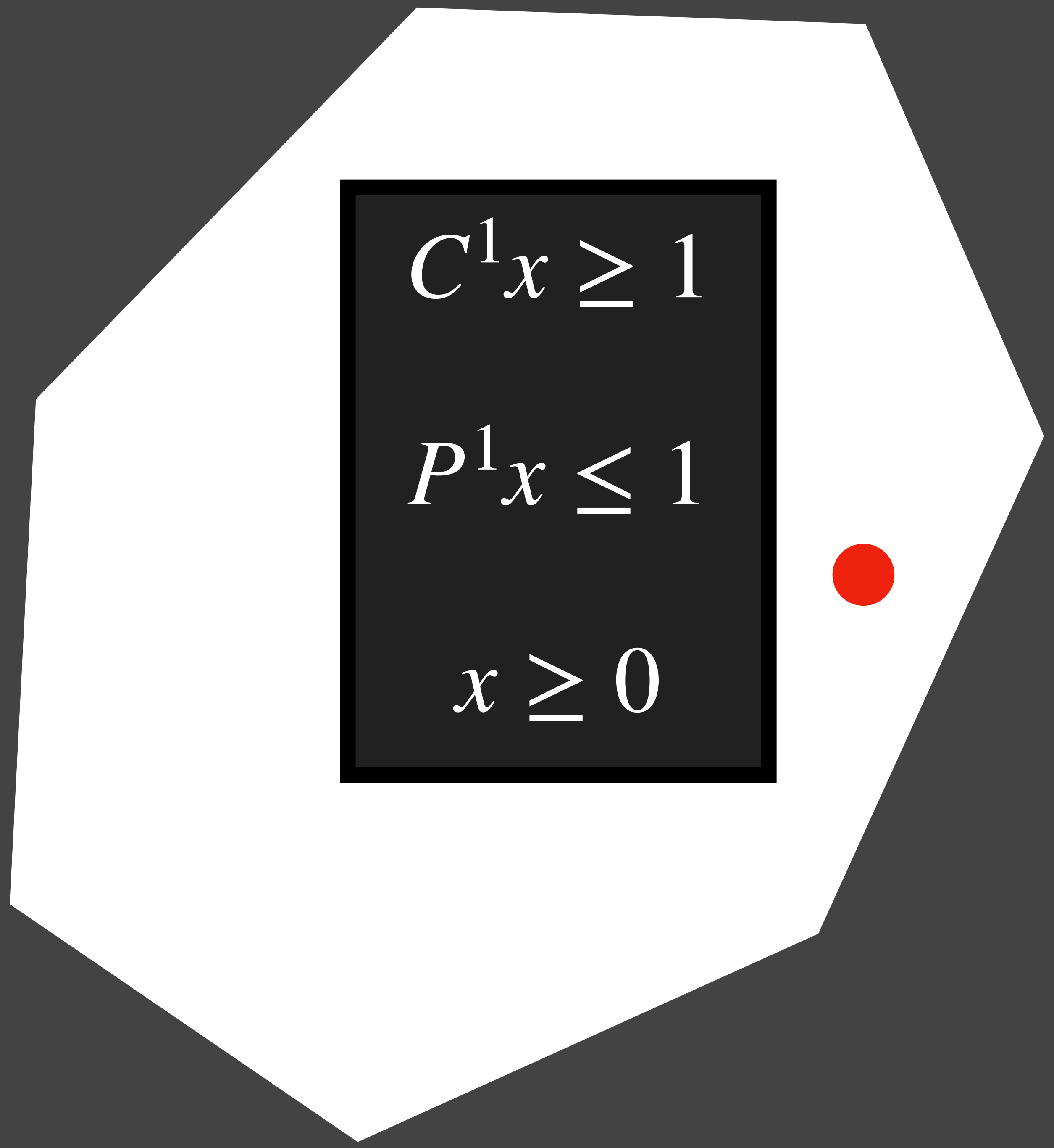

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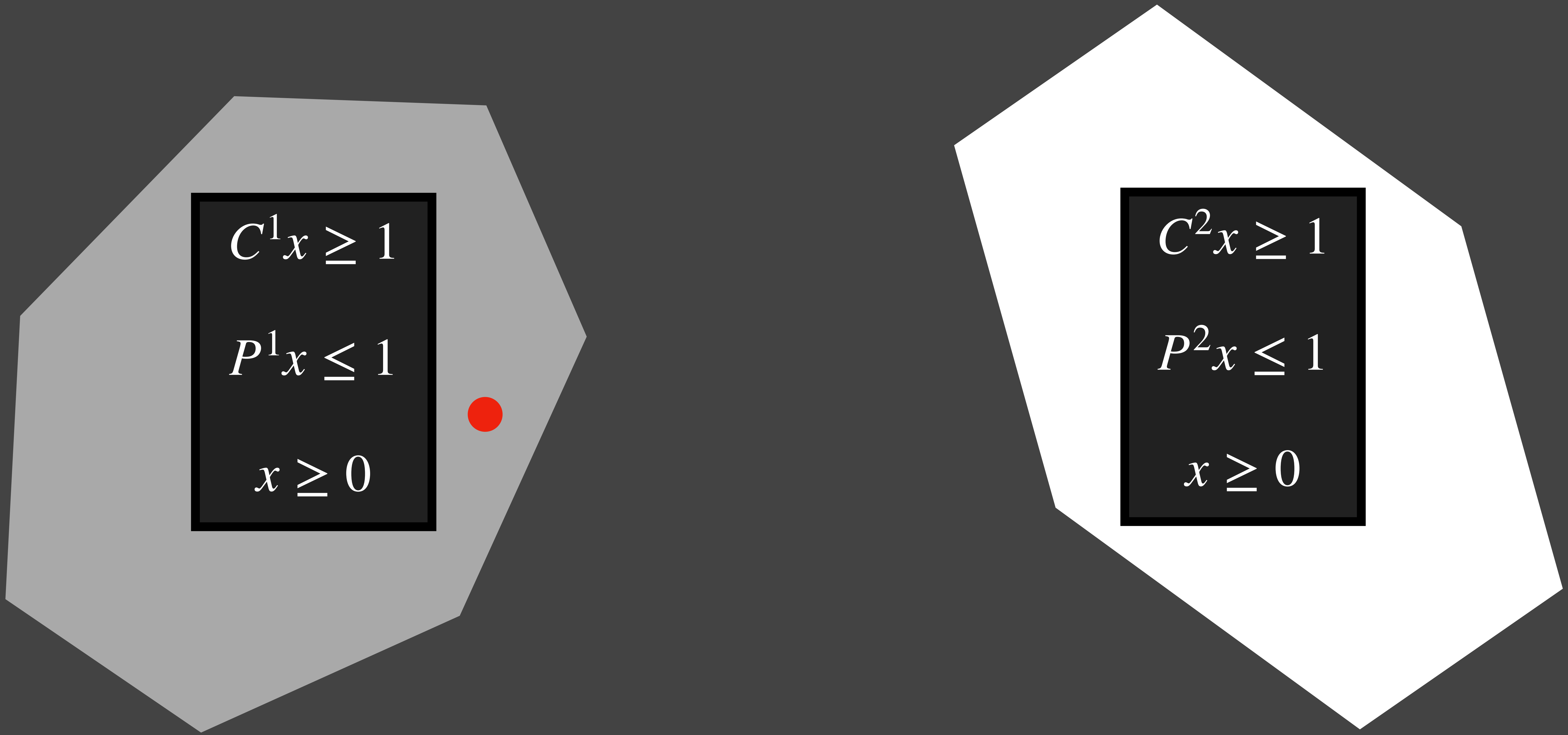

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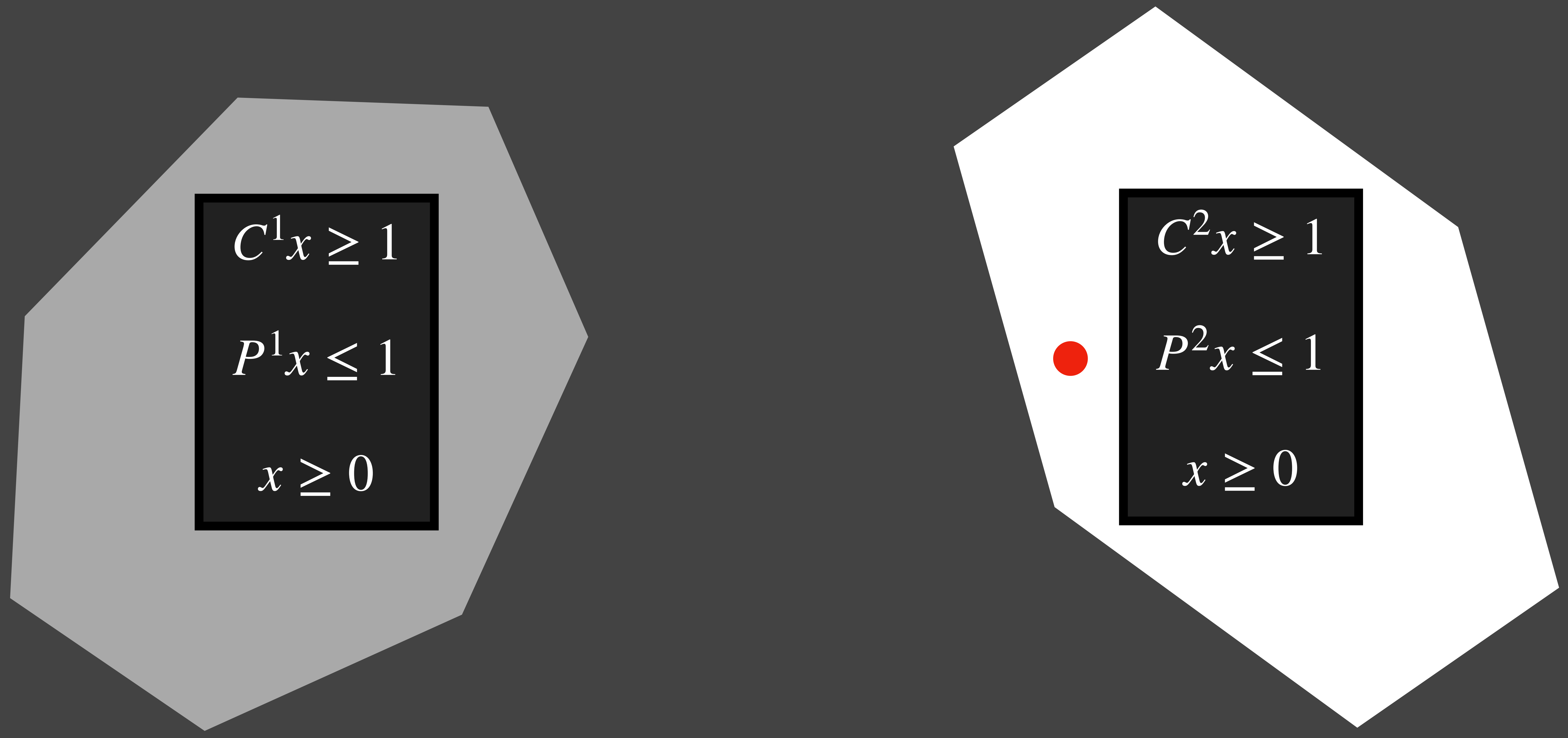

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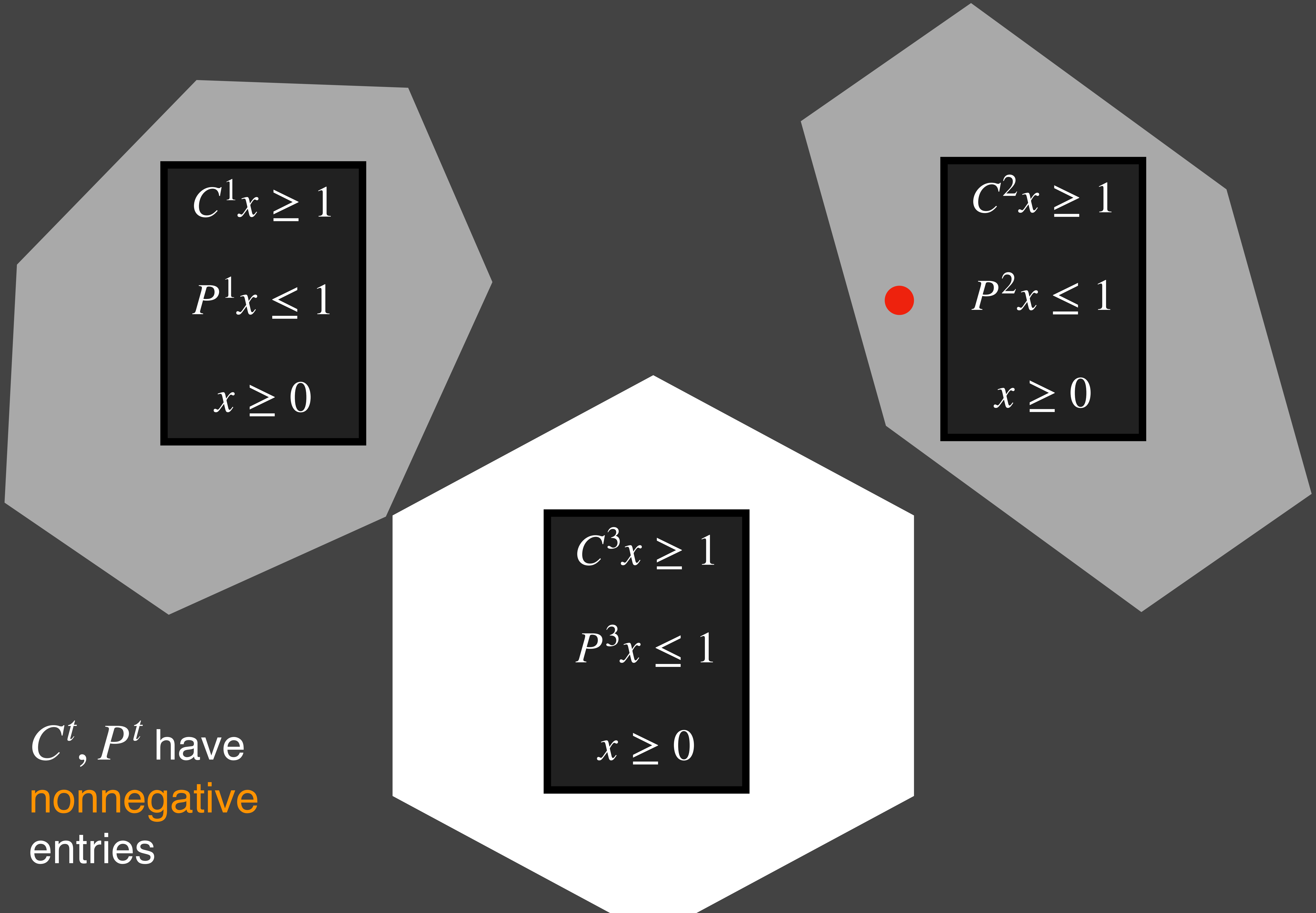

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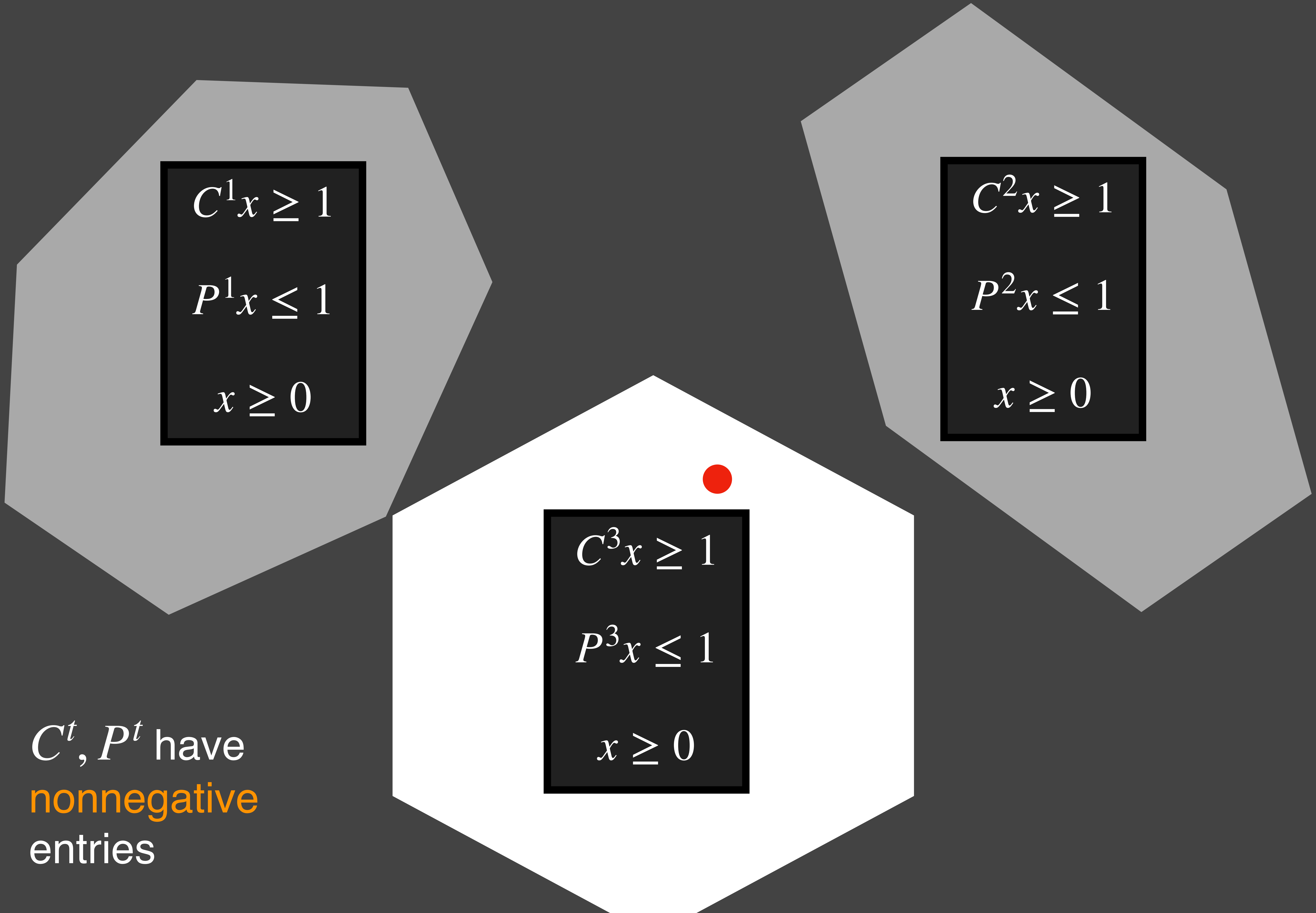

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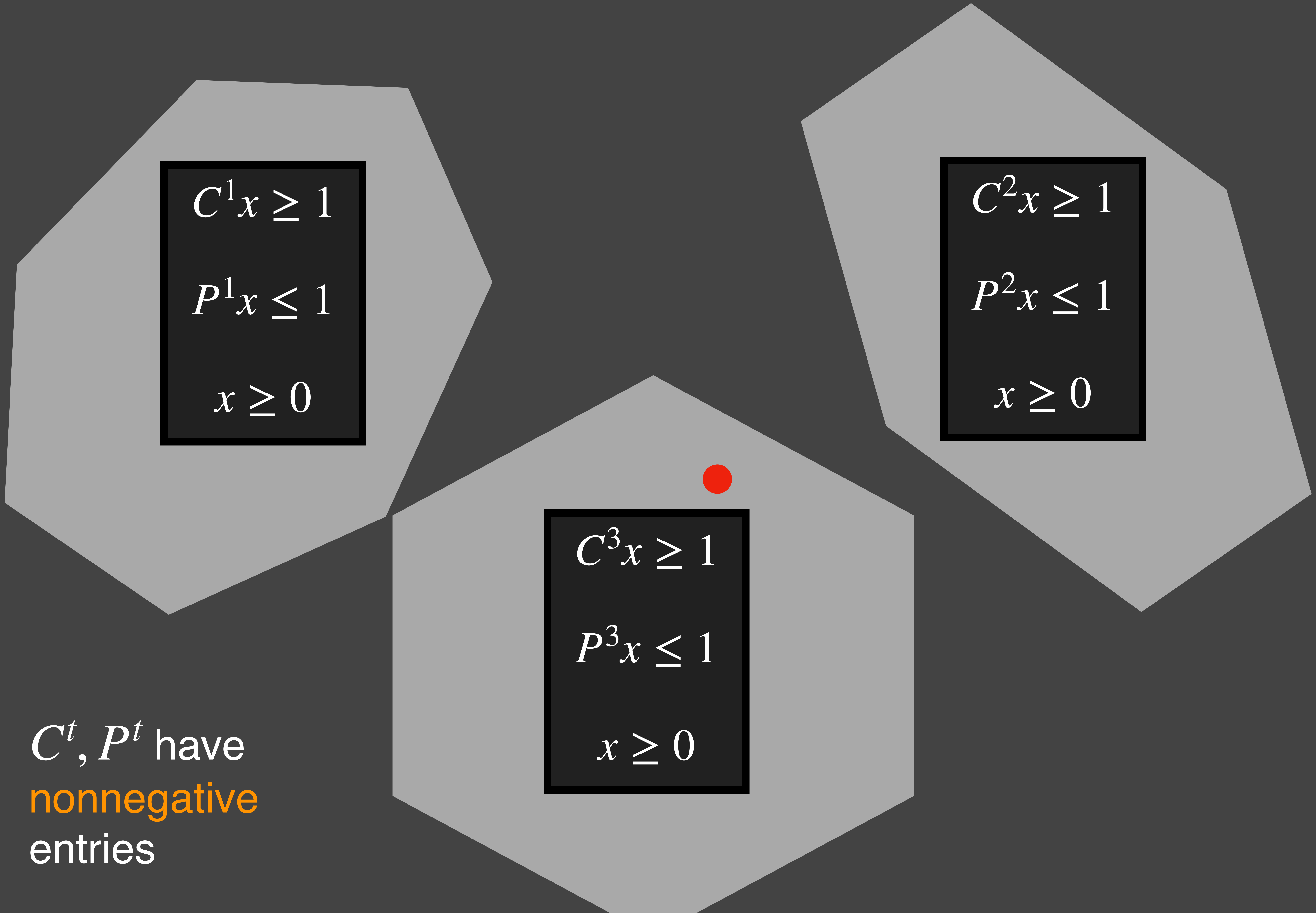

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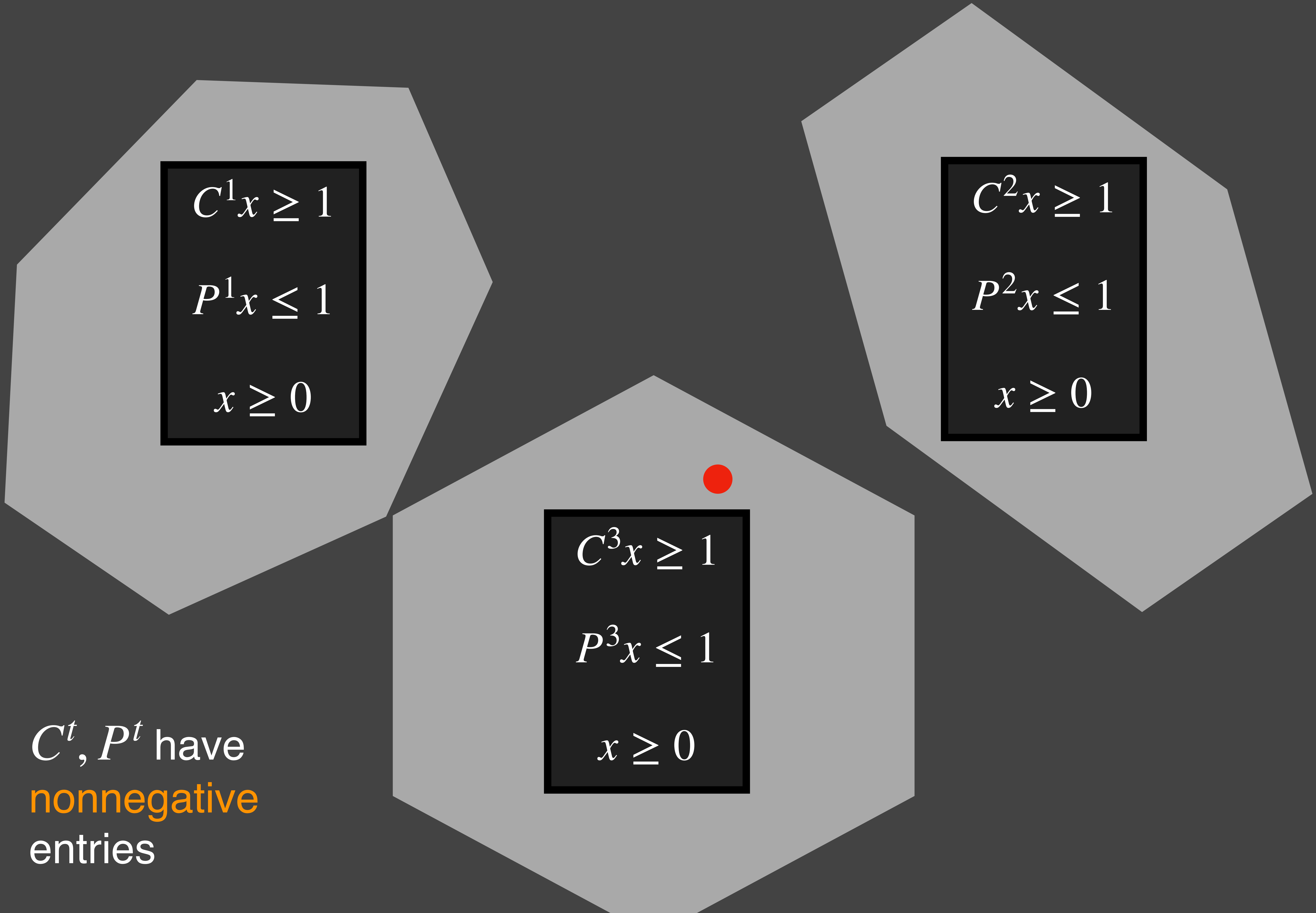

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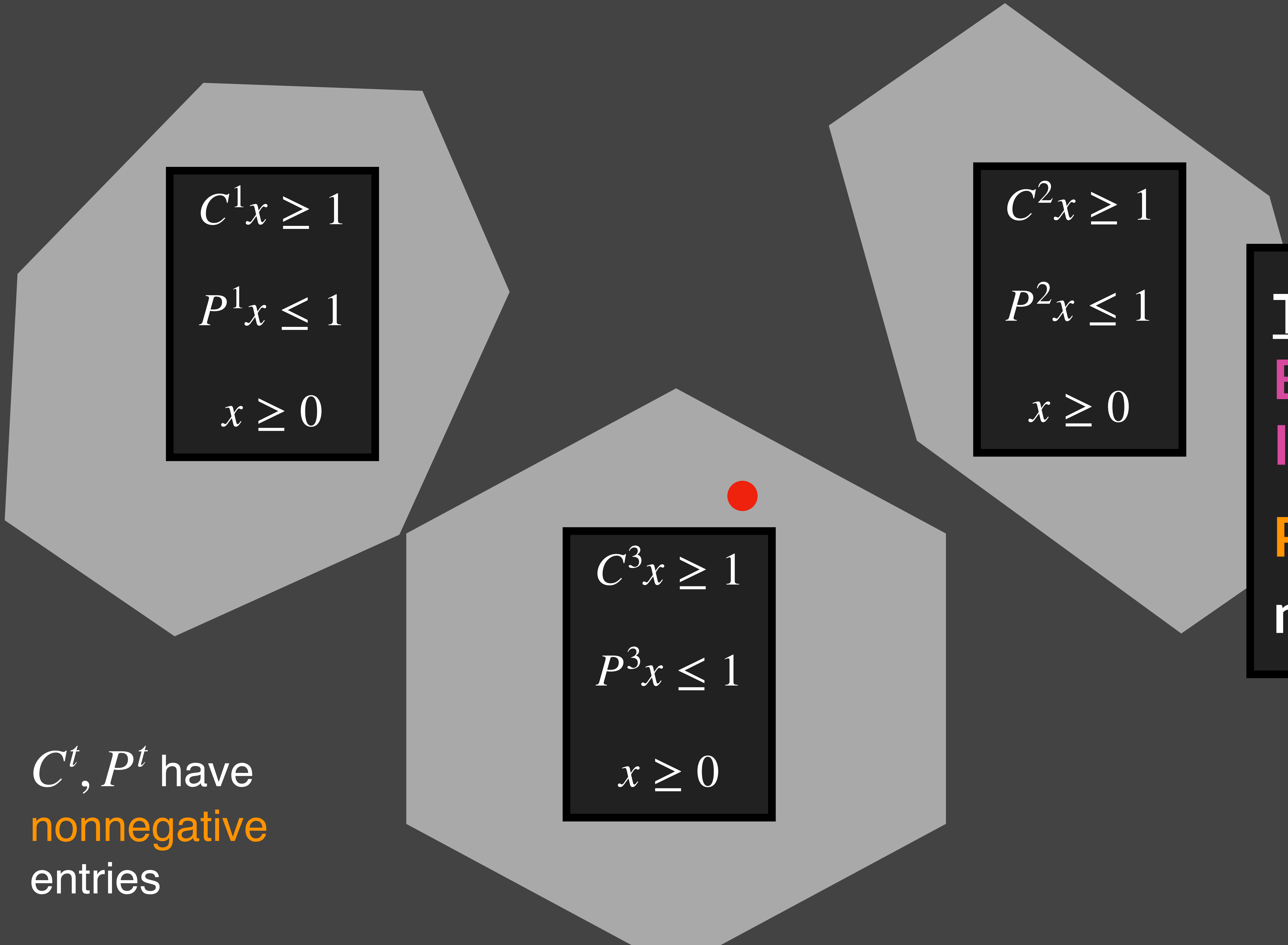

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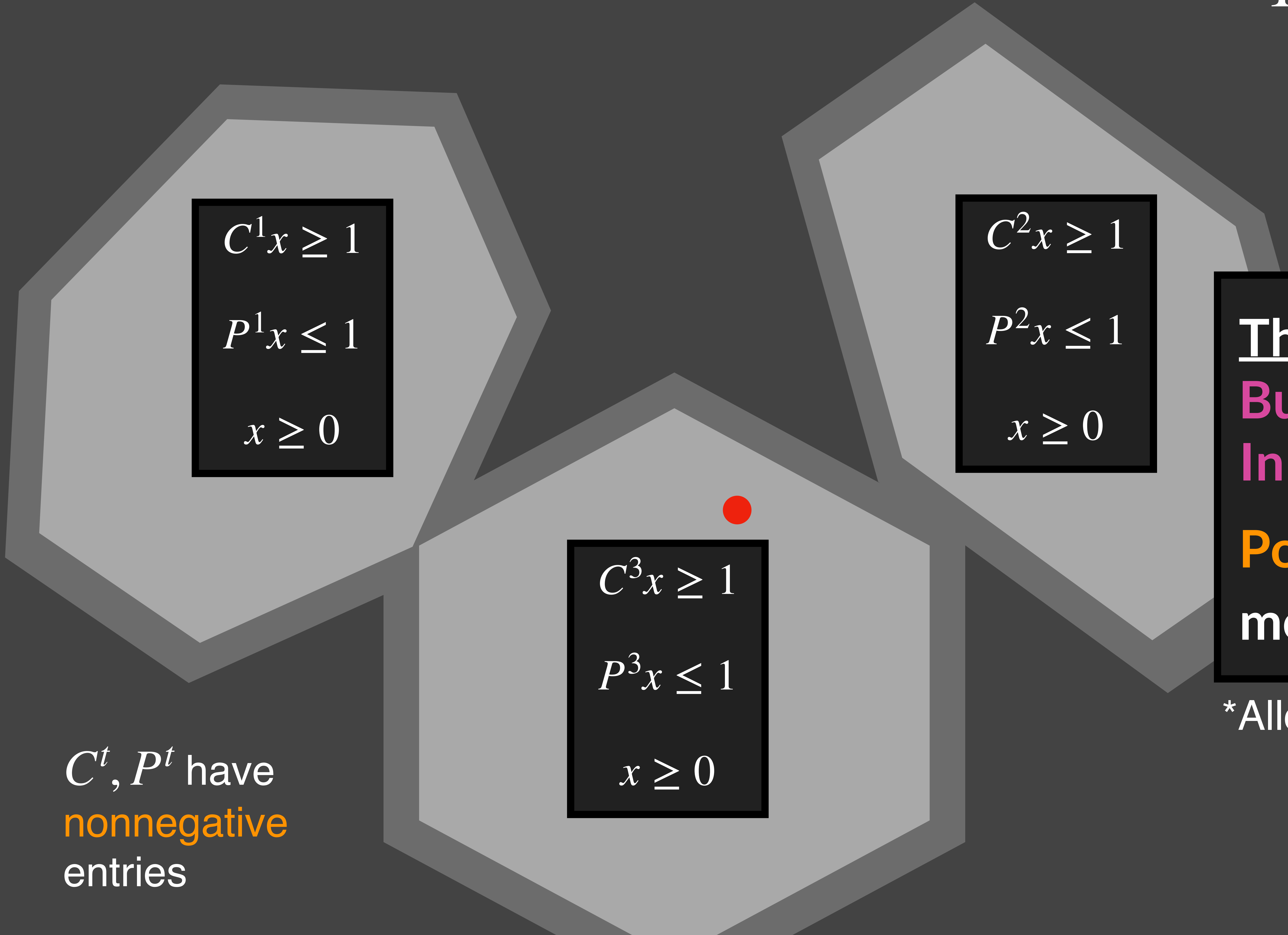

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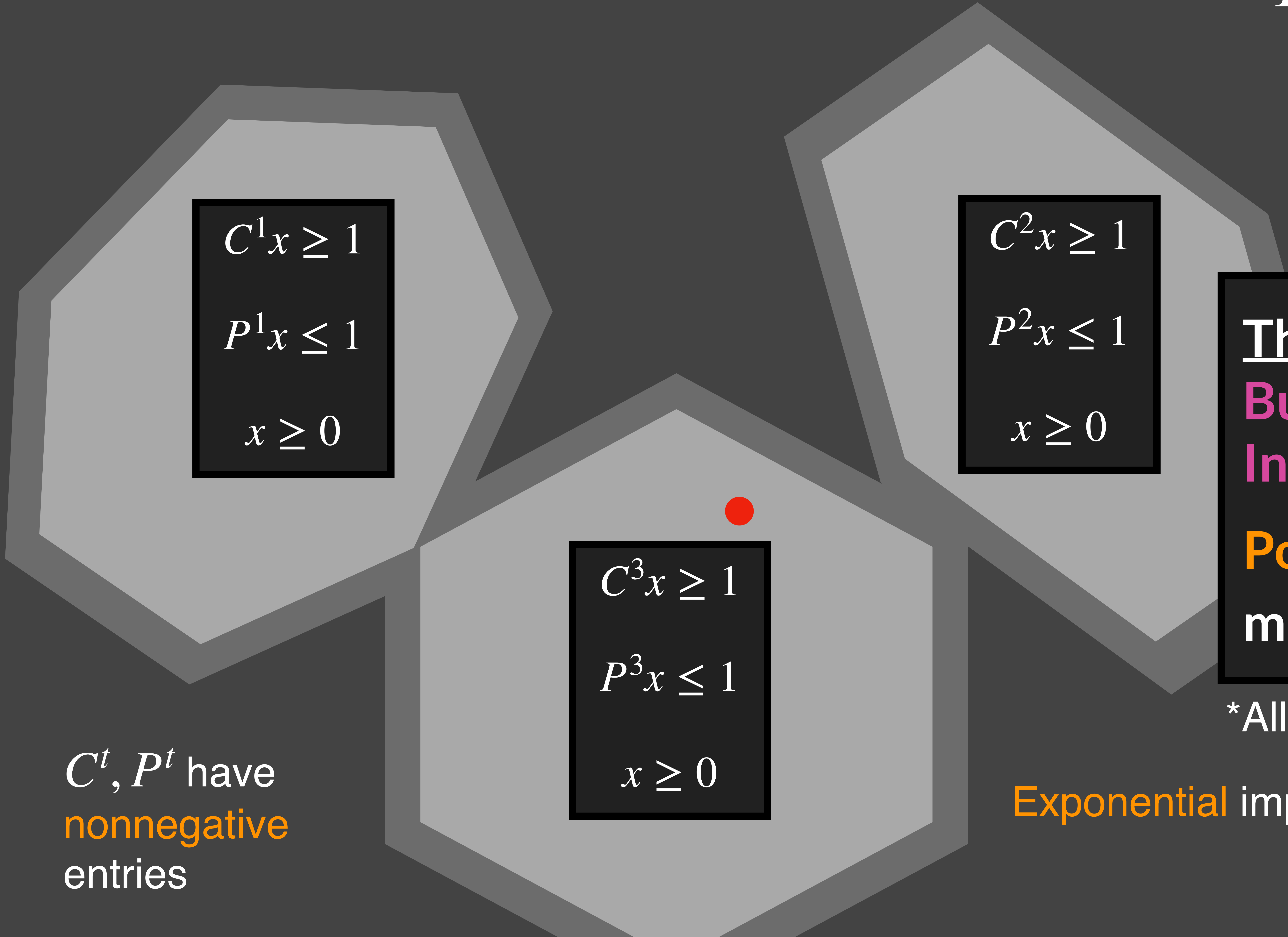

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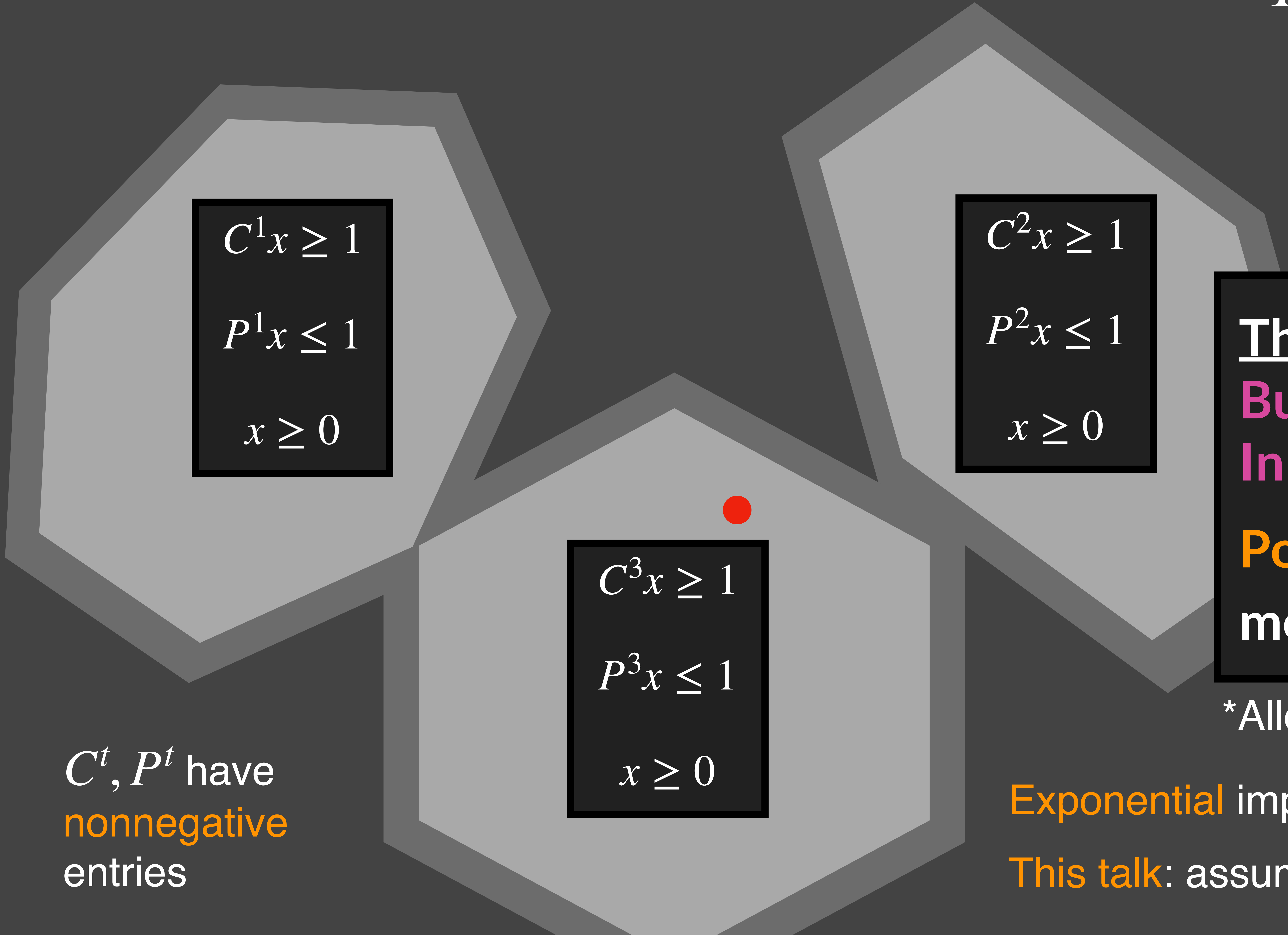

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This talk: assume C^t, P^t are 0/1 matrices.

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Insert only.

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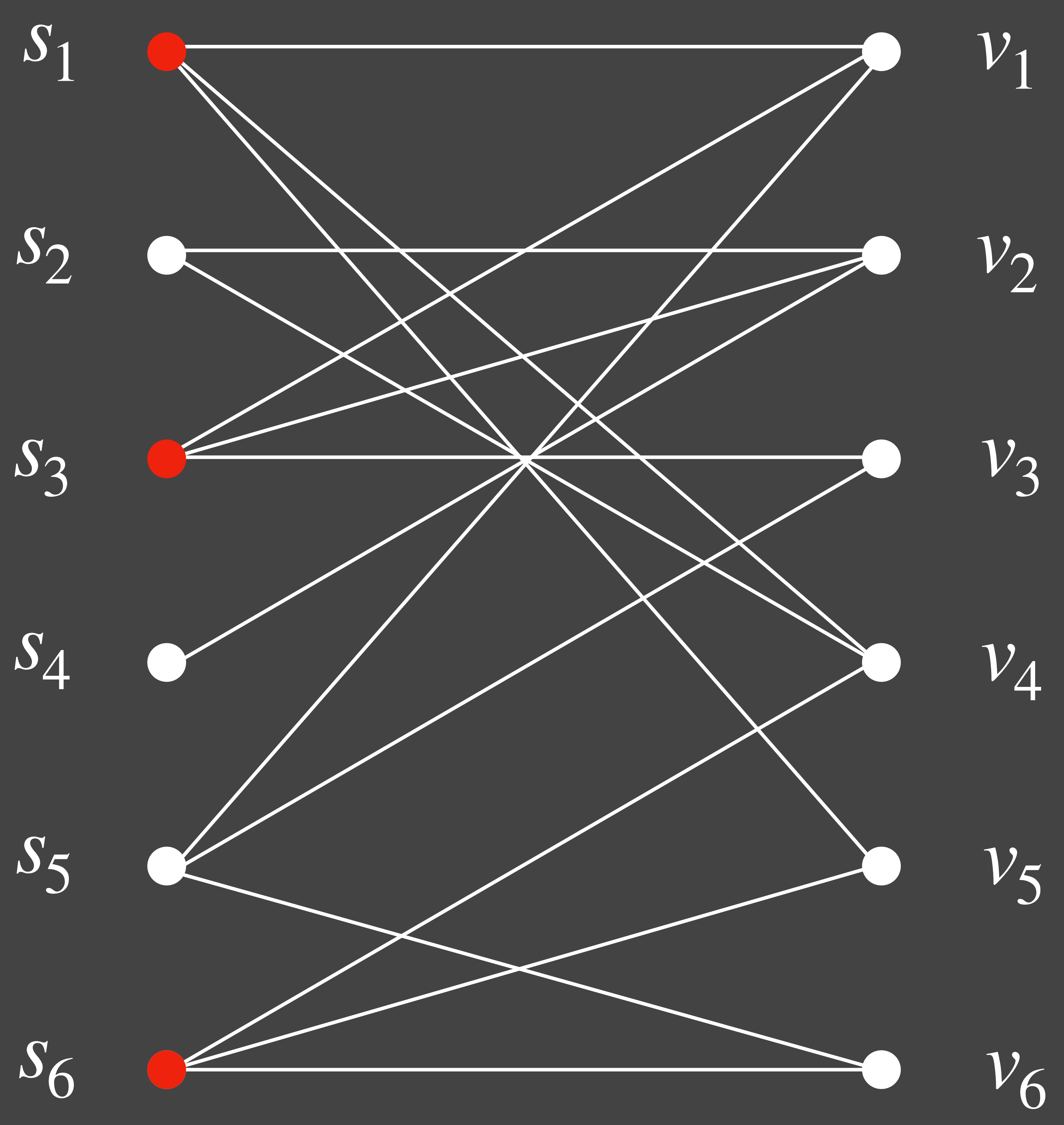

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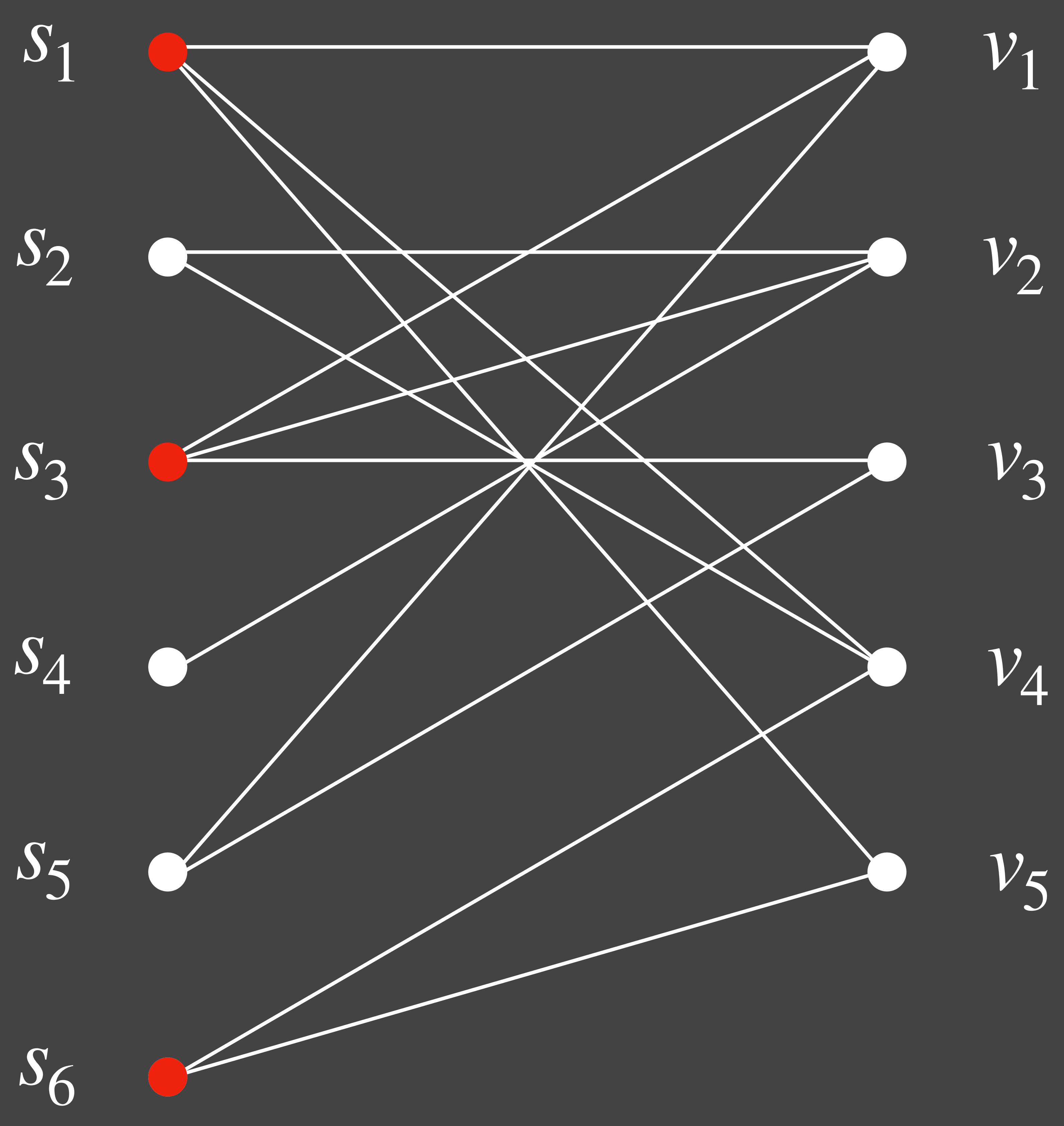

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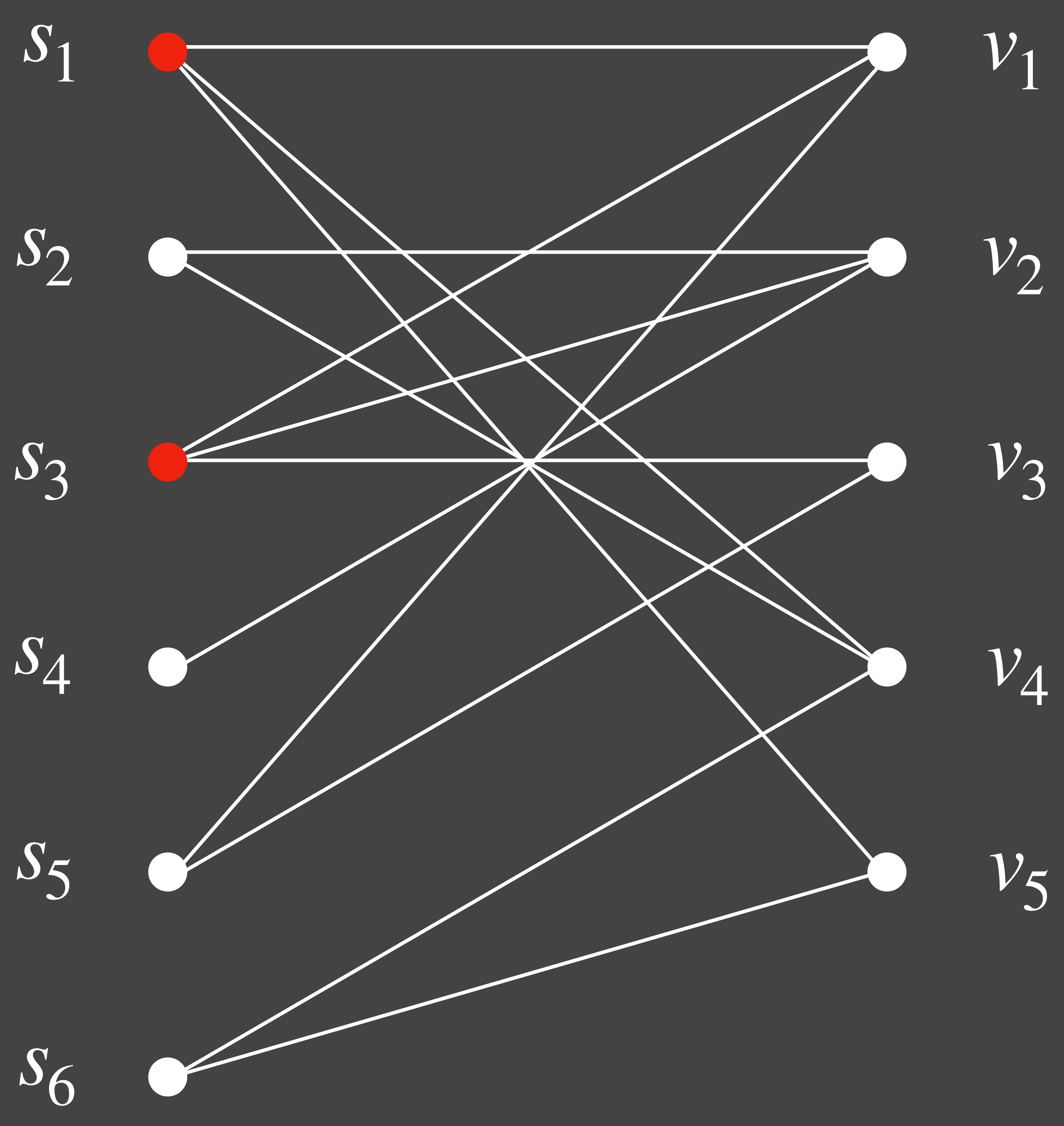

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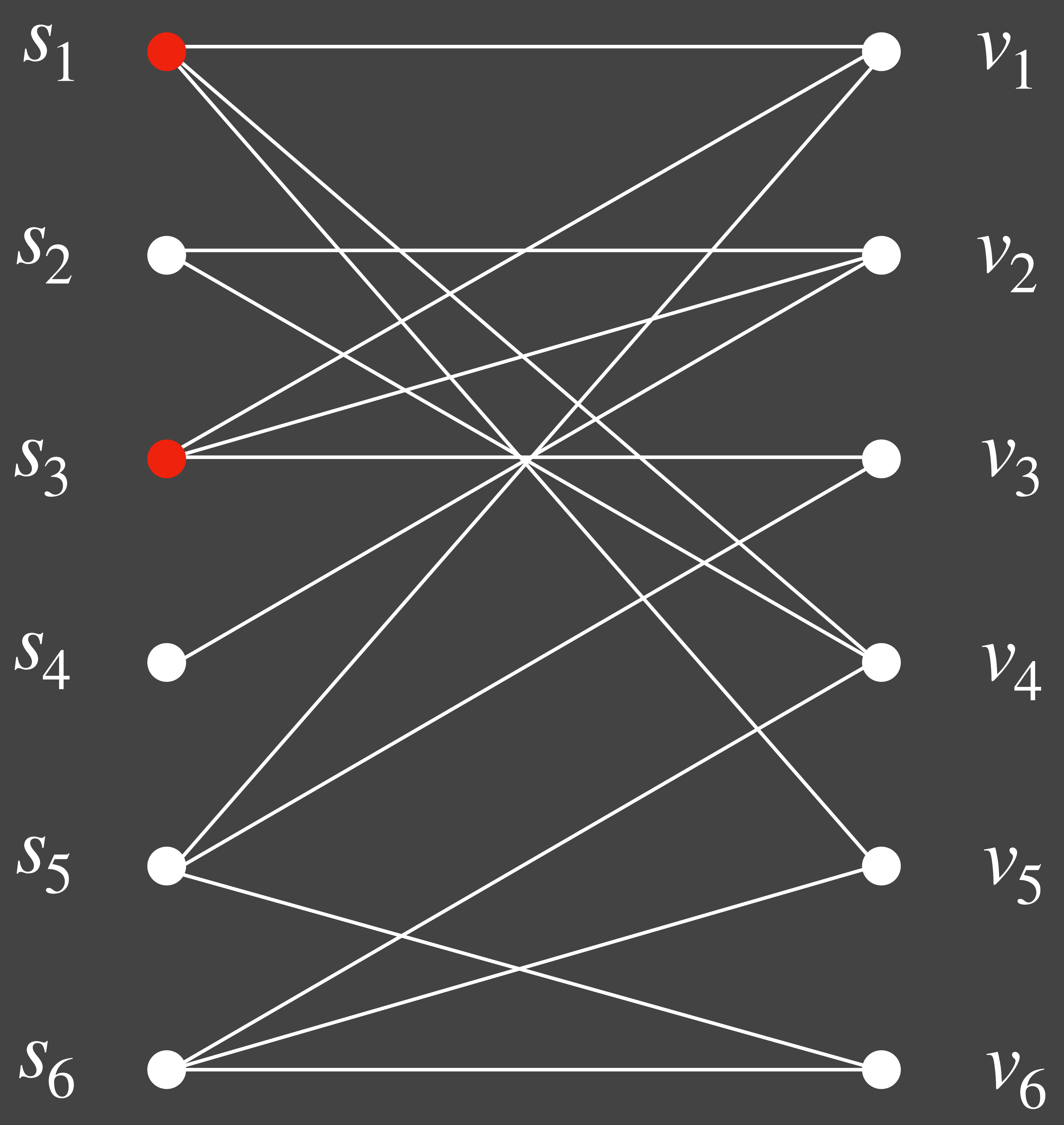

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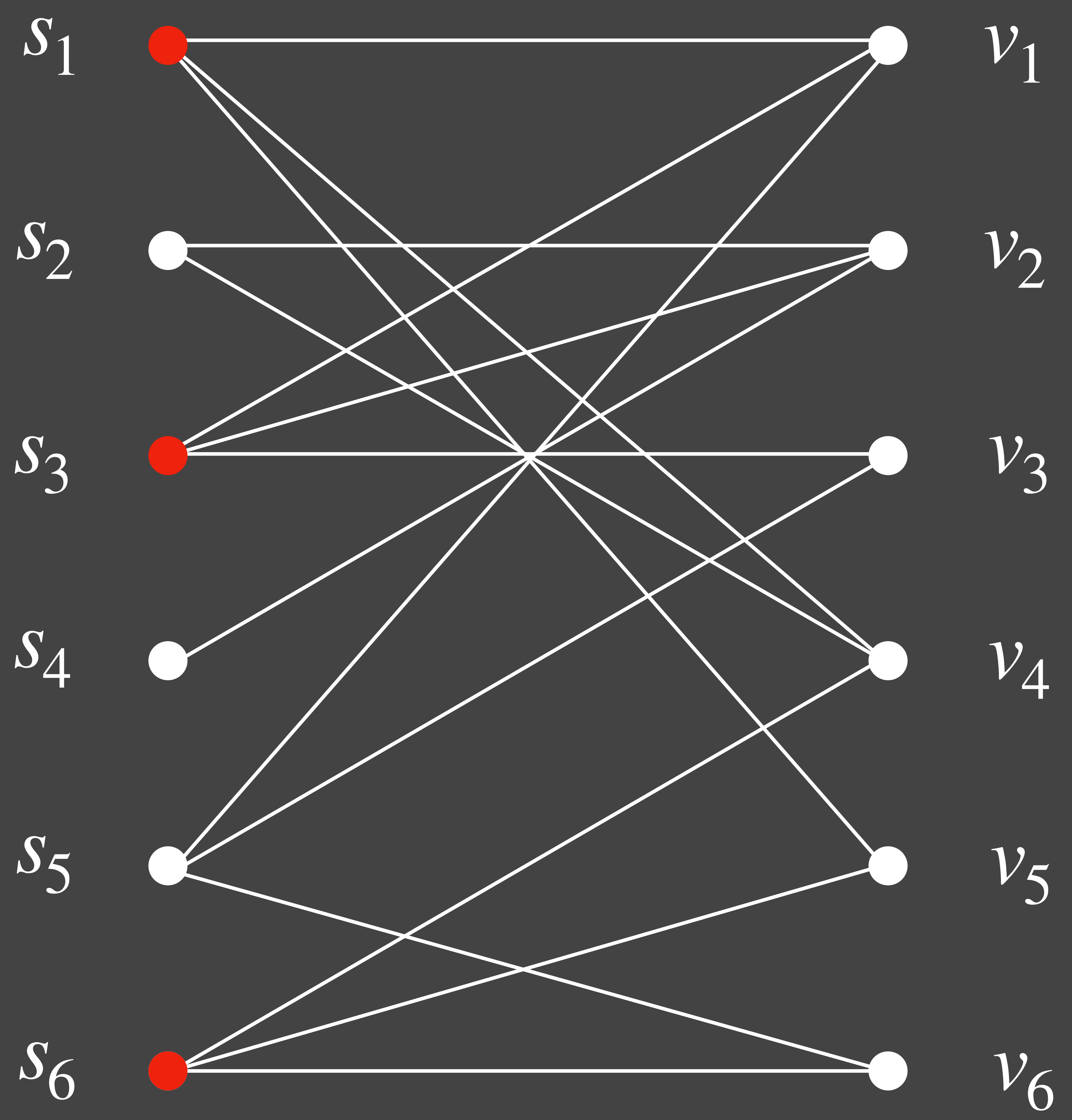

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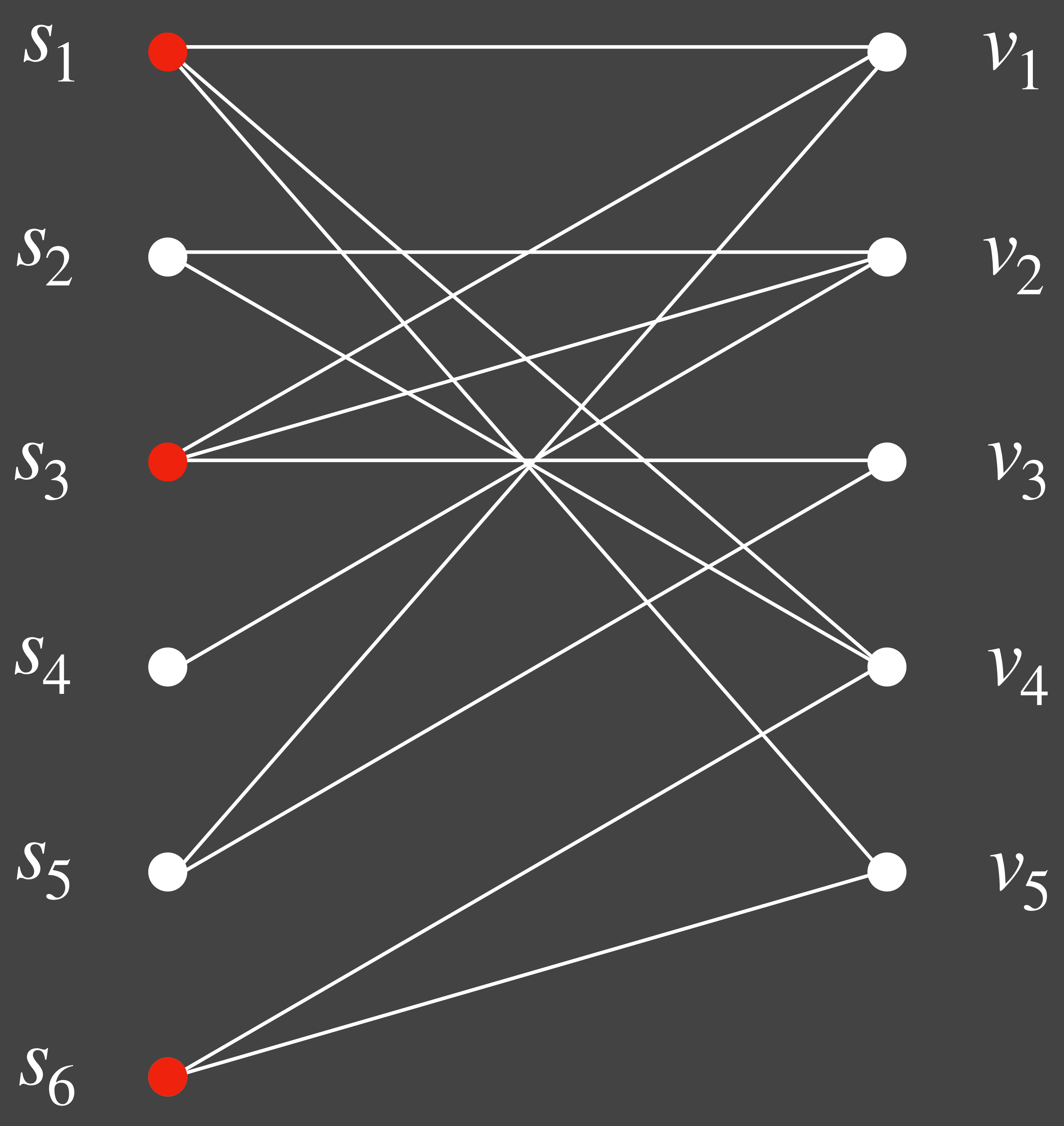

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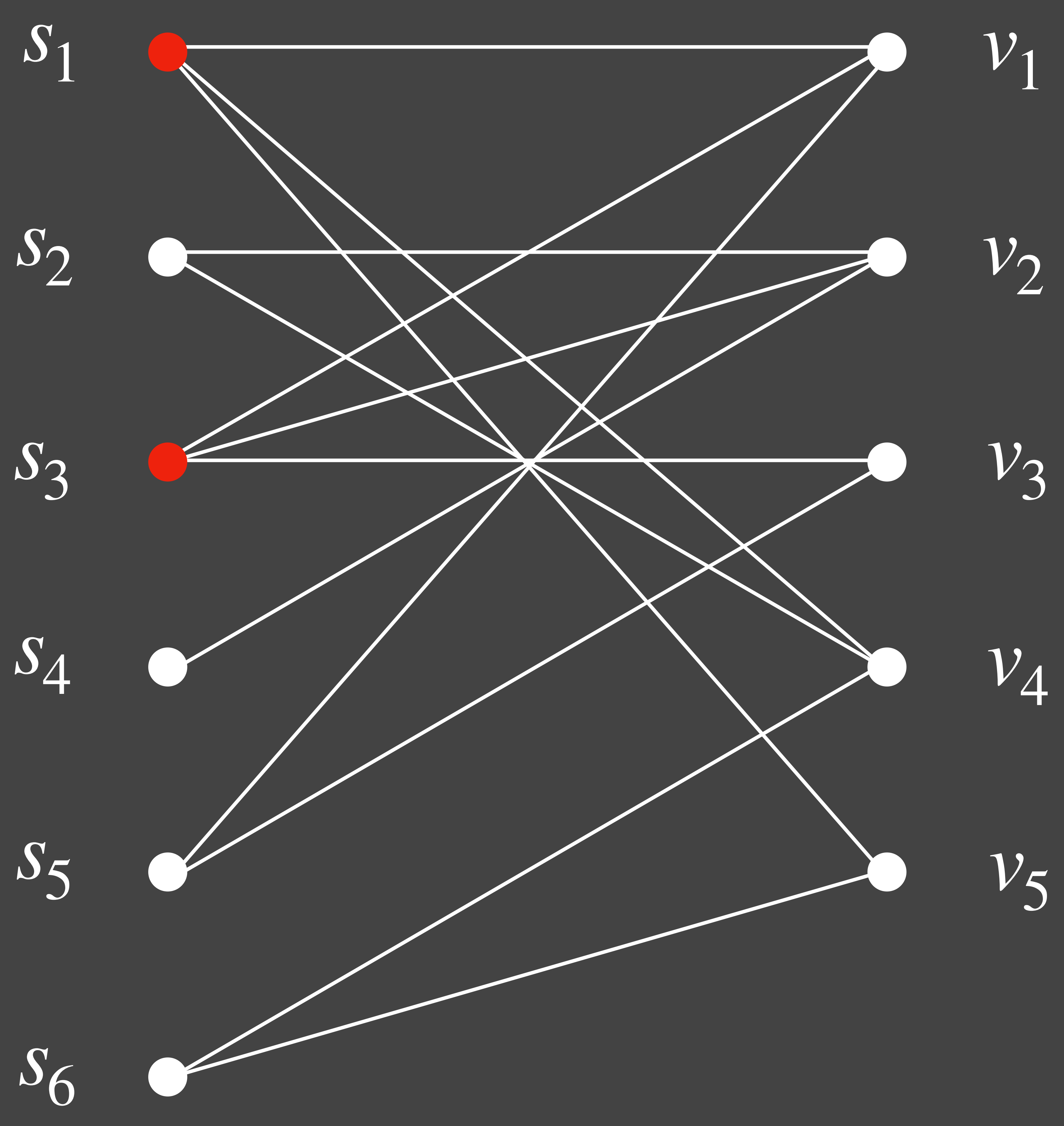

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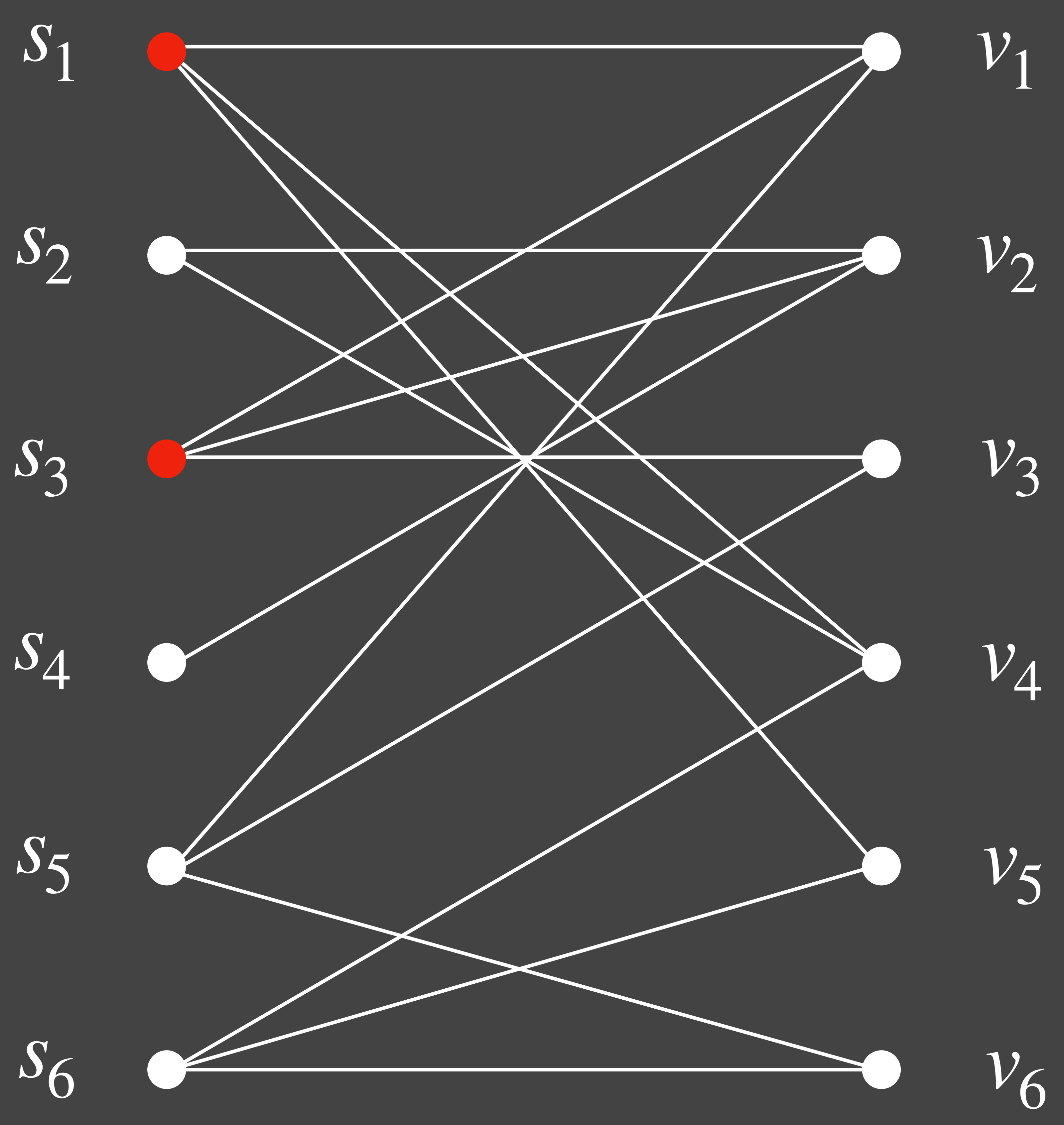

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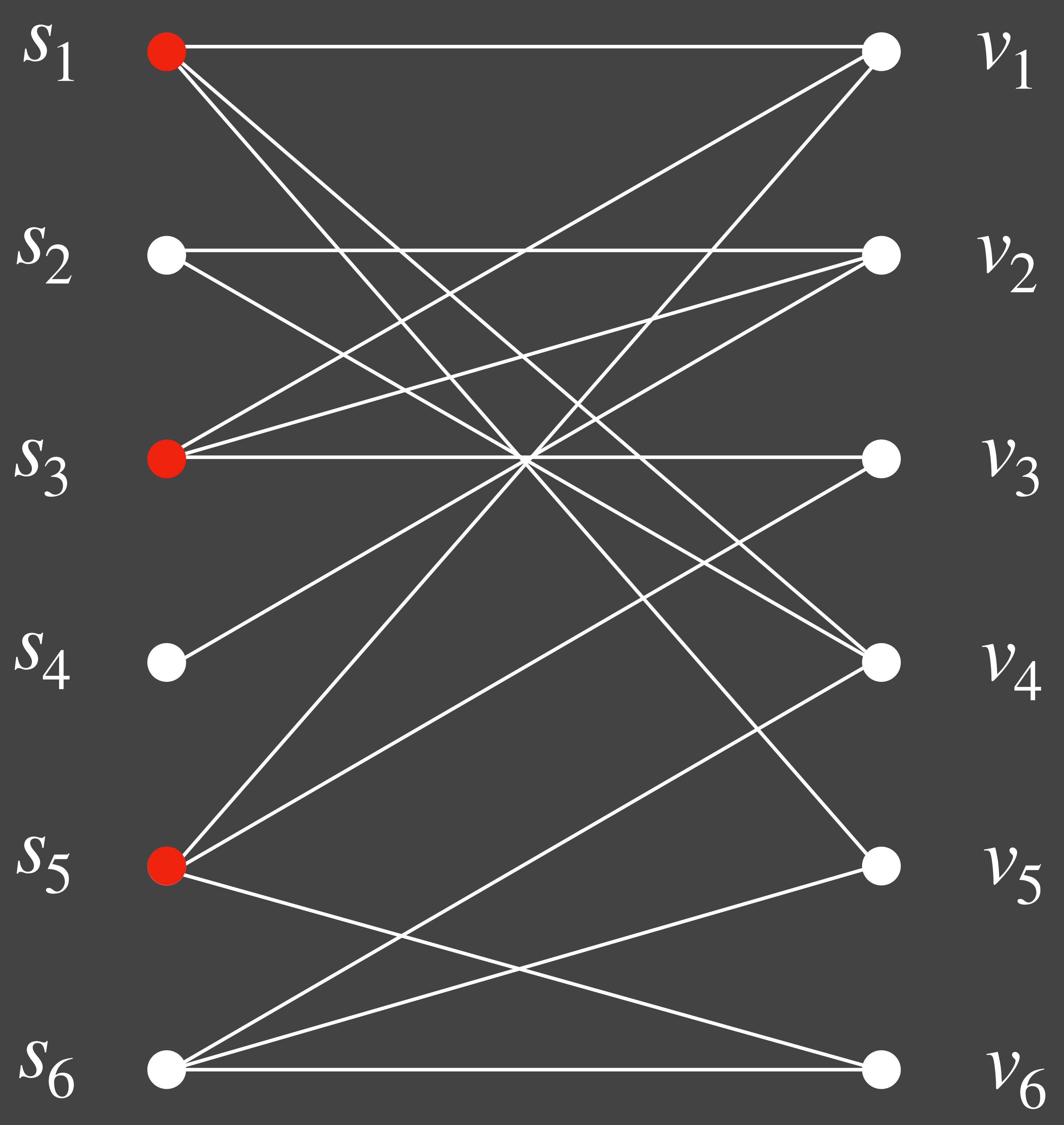

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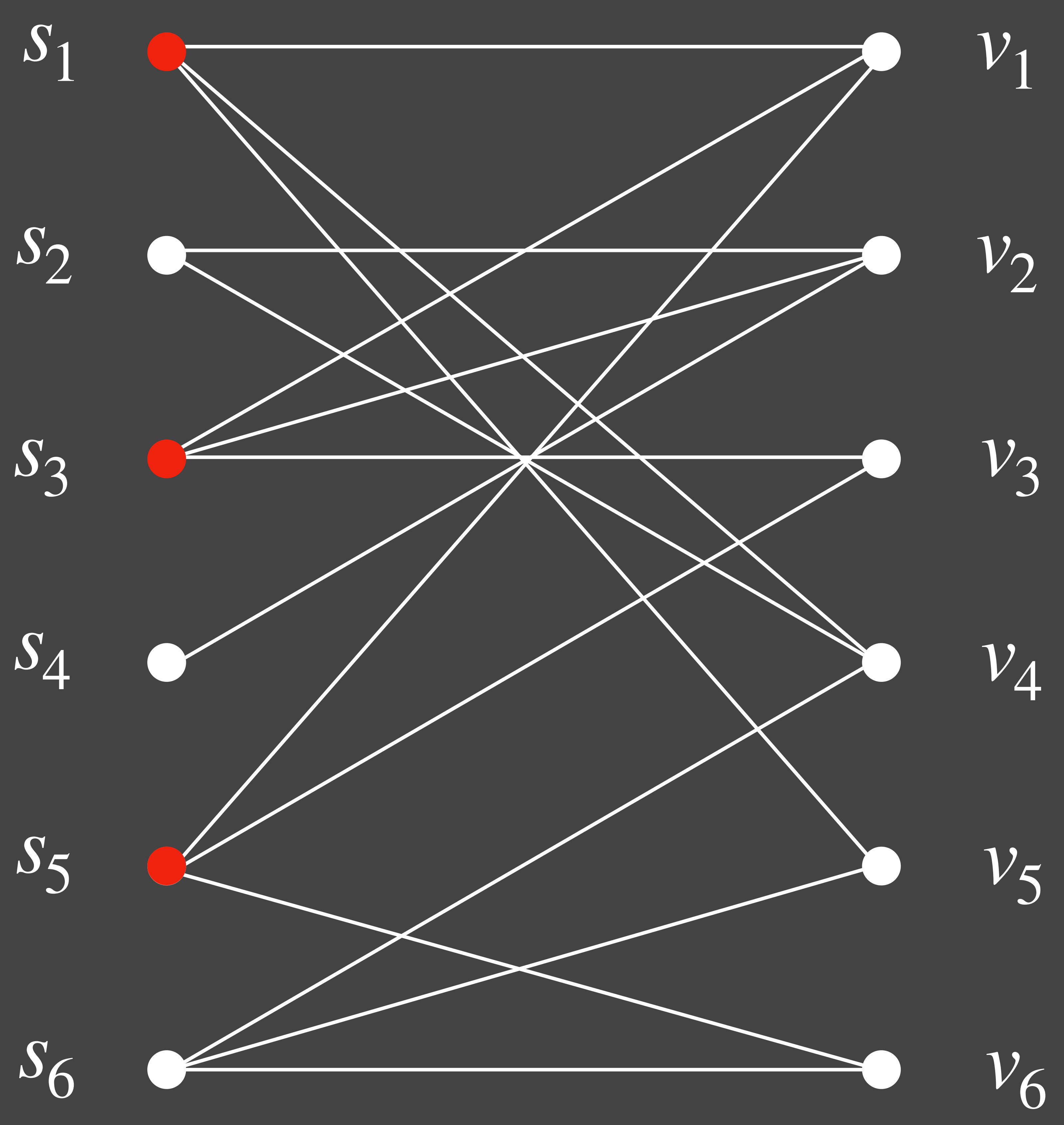

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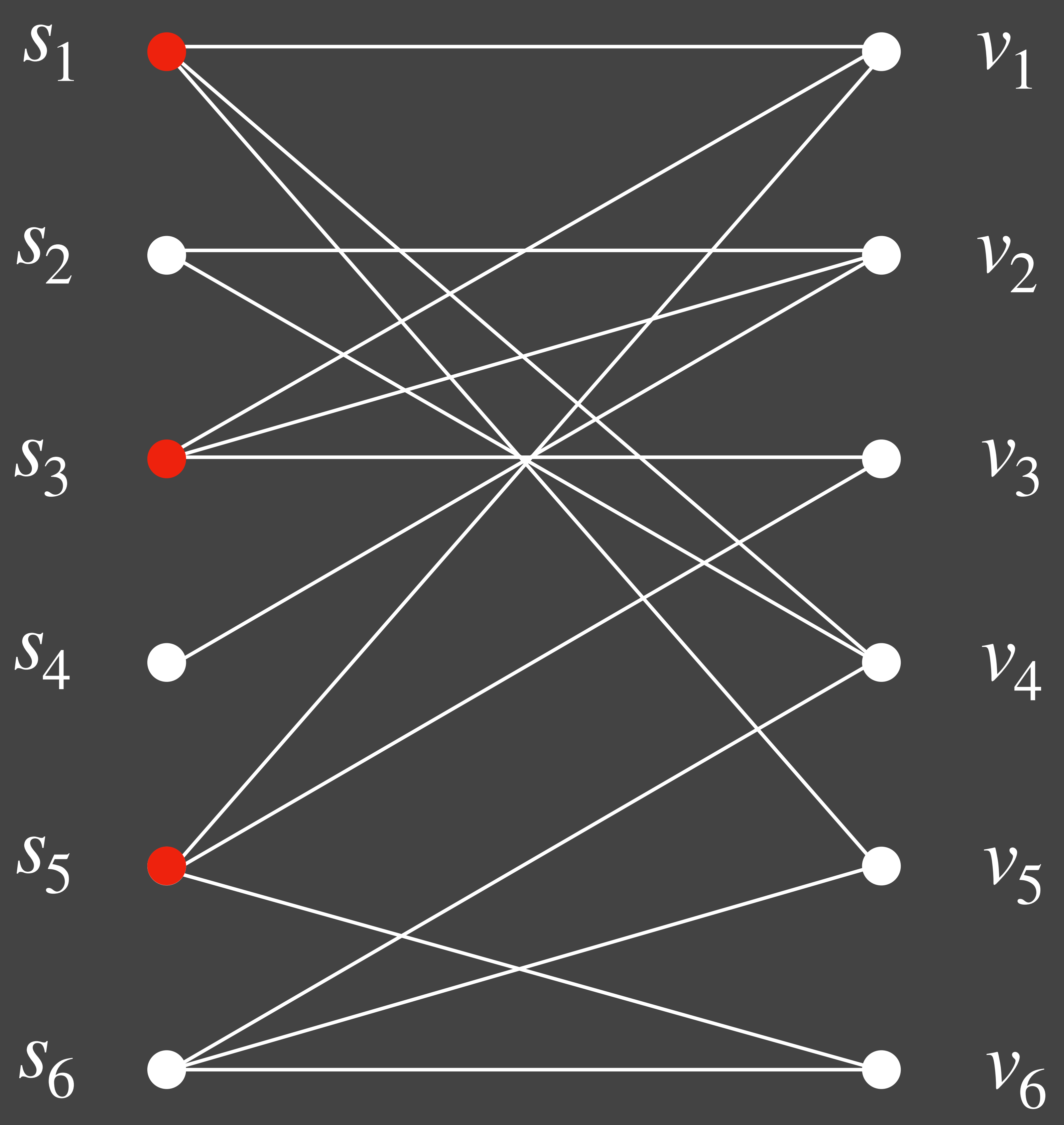

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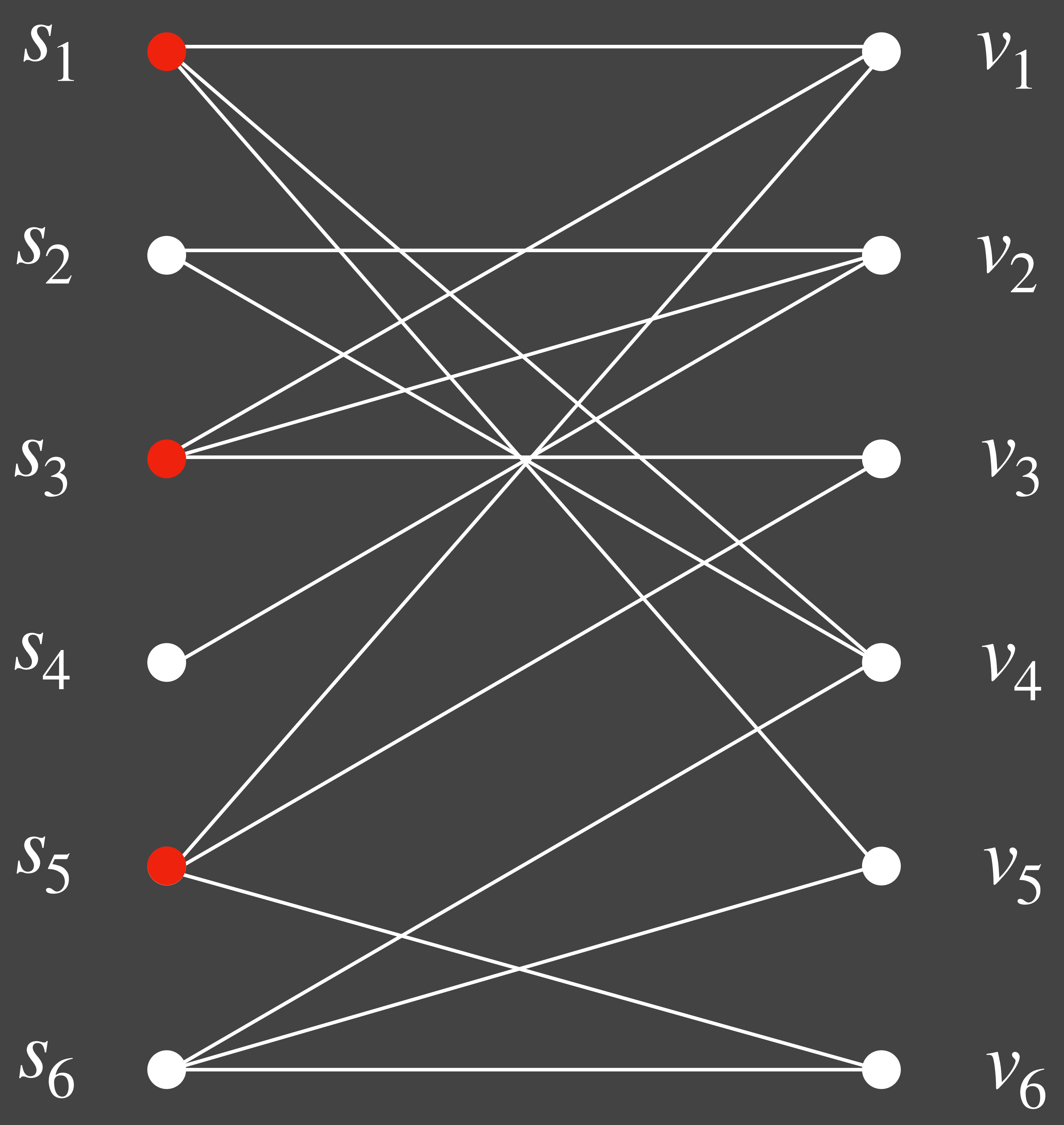

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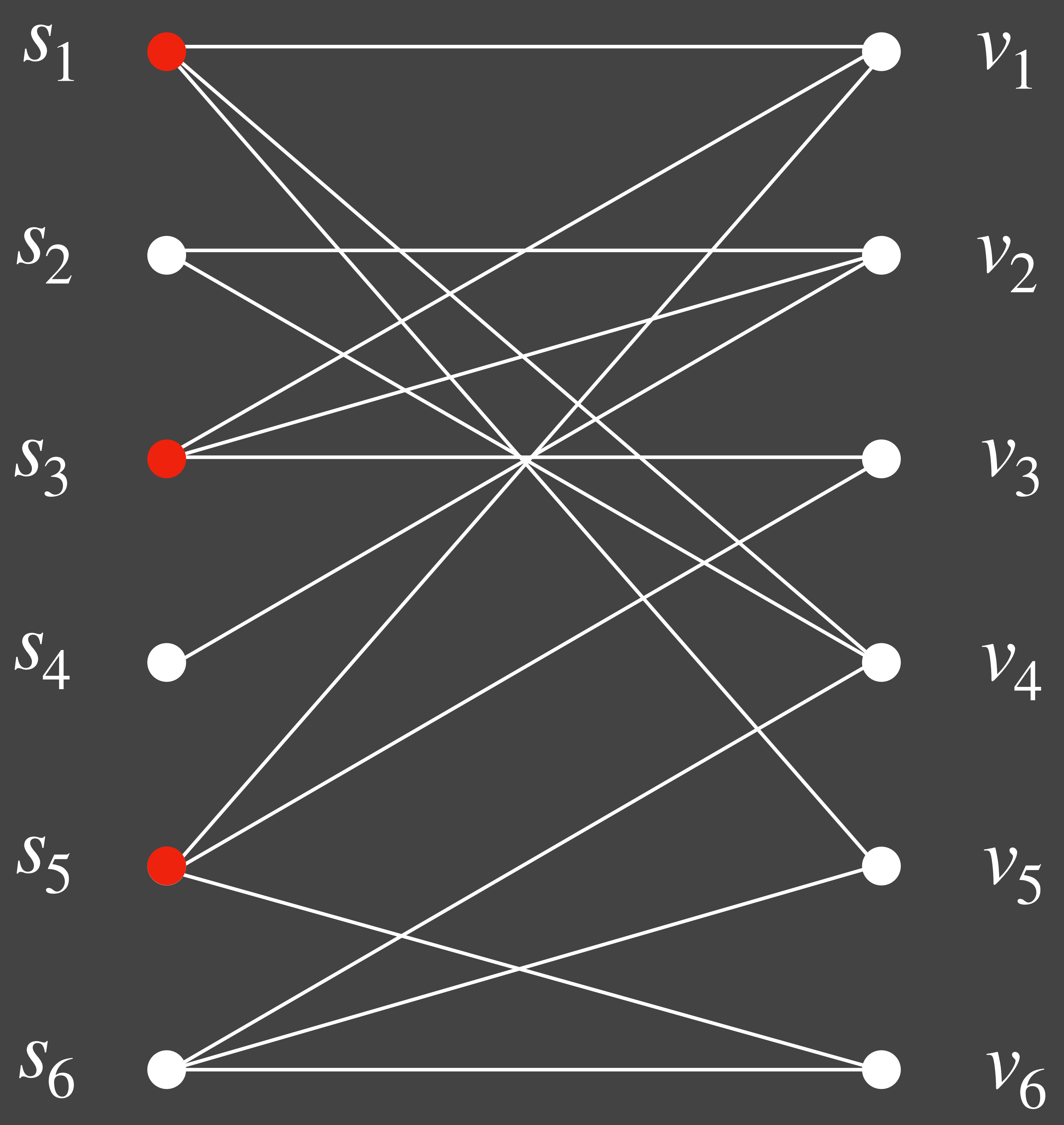

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Competitive recourse is the answer!

The Fractional Algorithm

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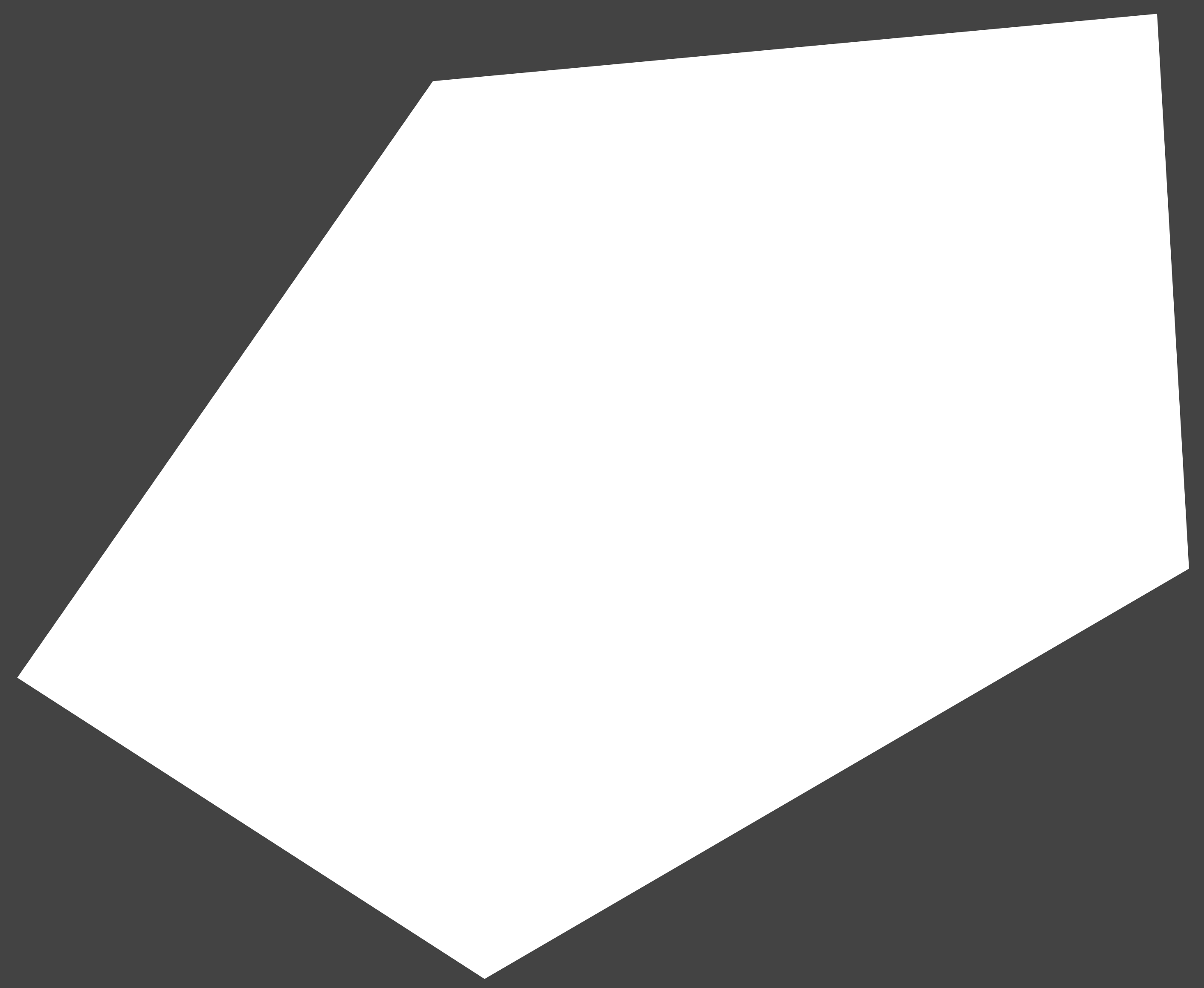

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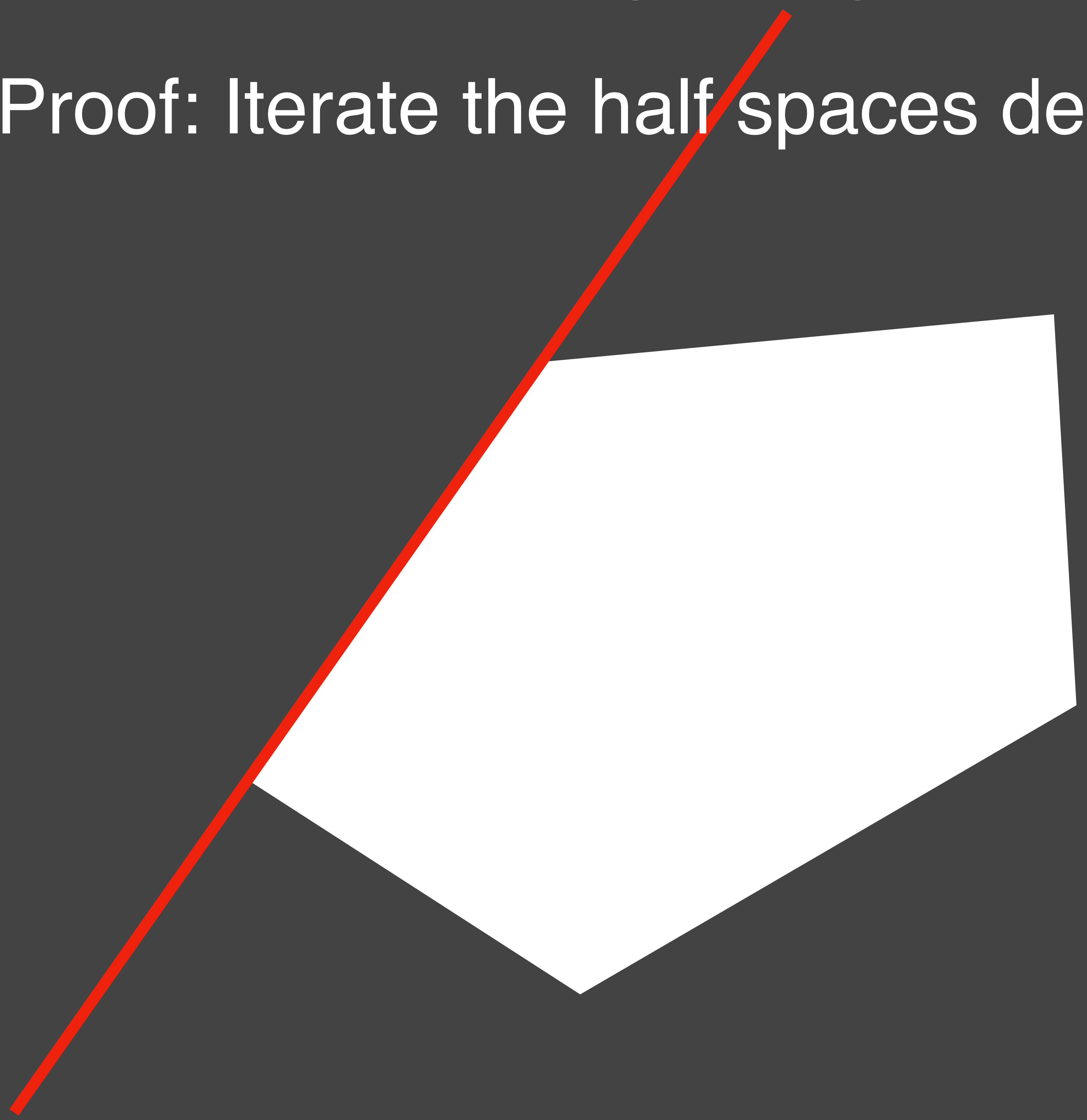

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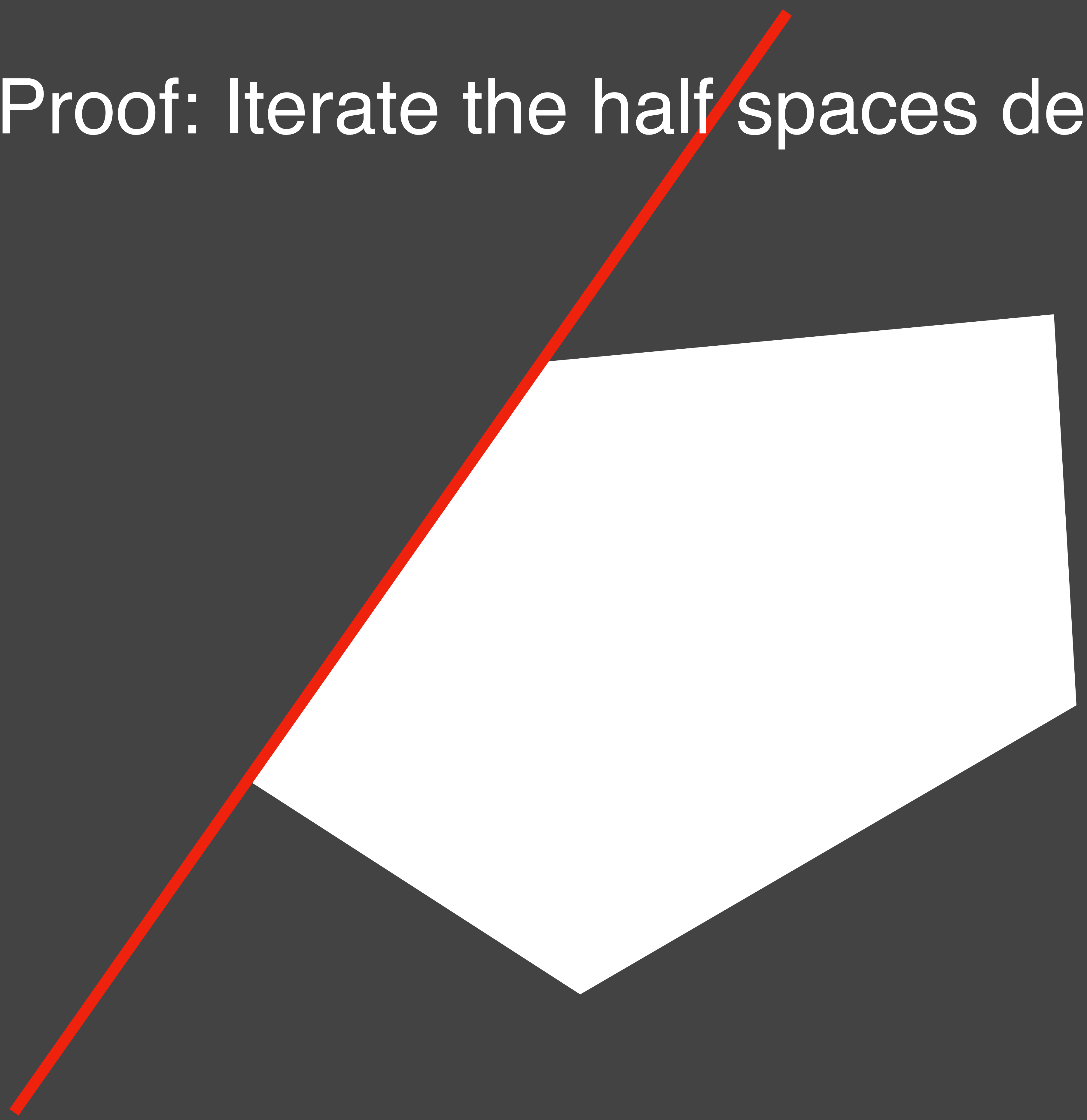

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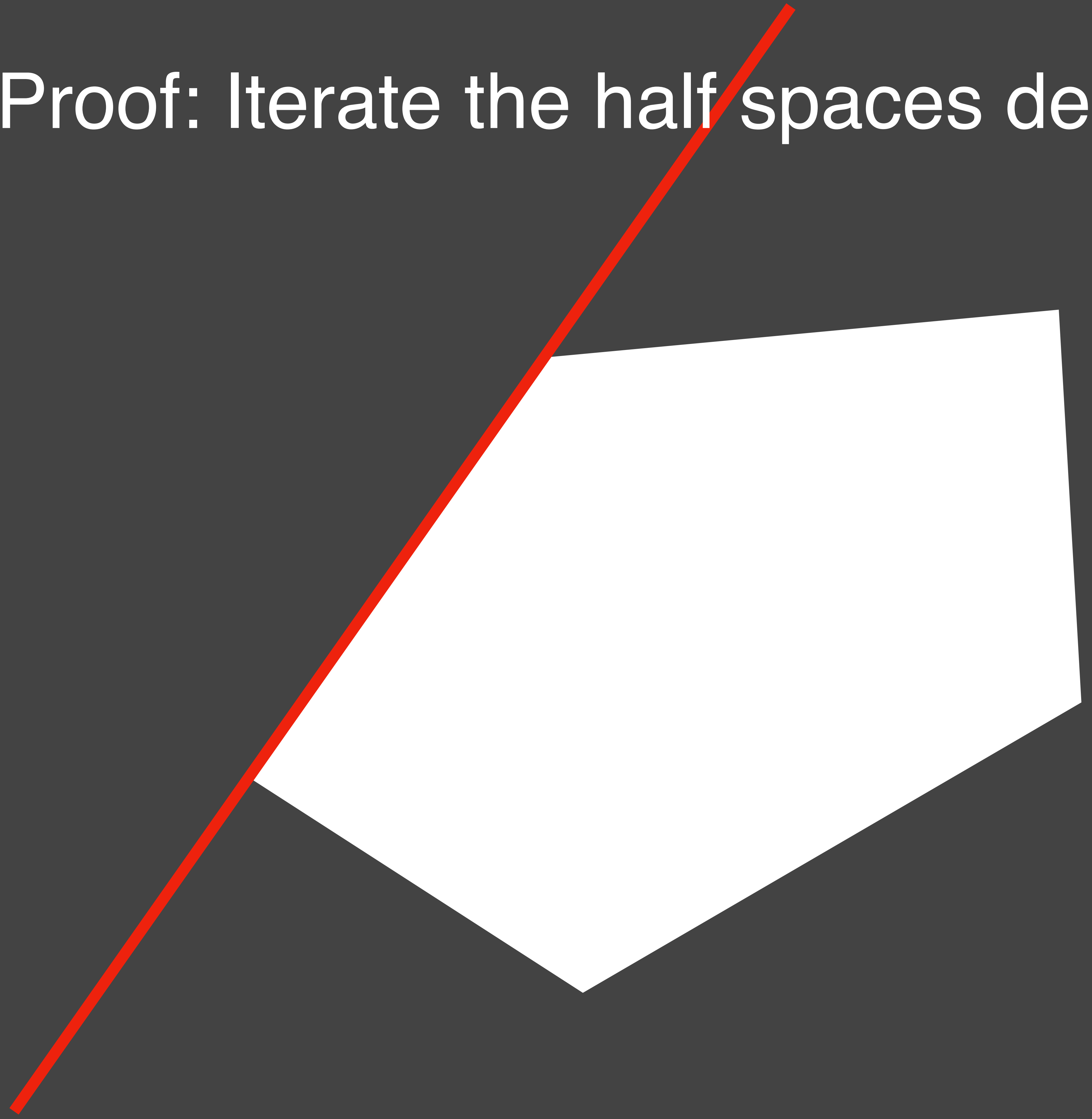

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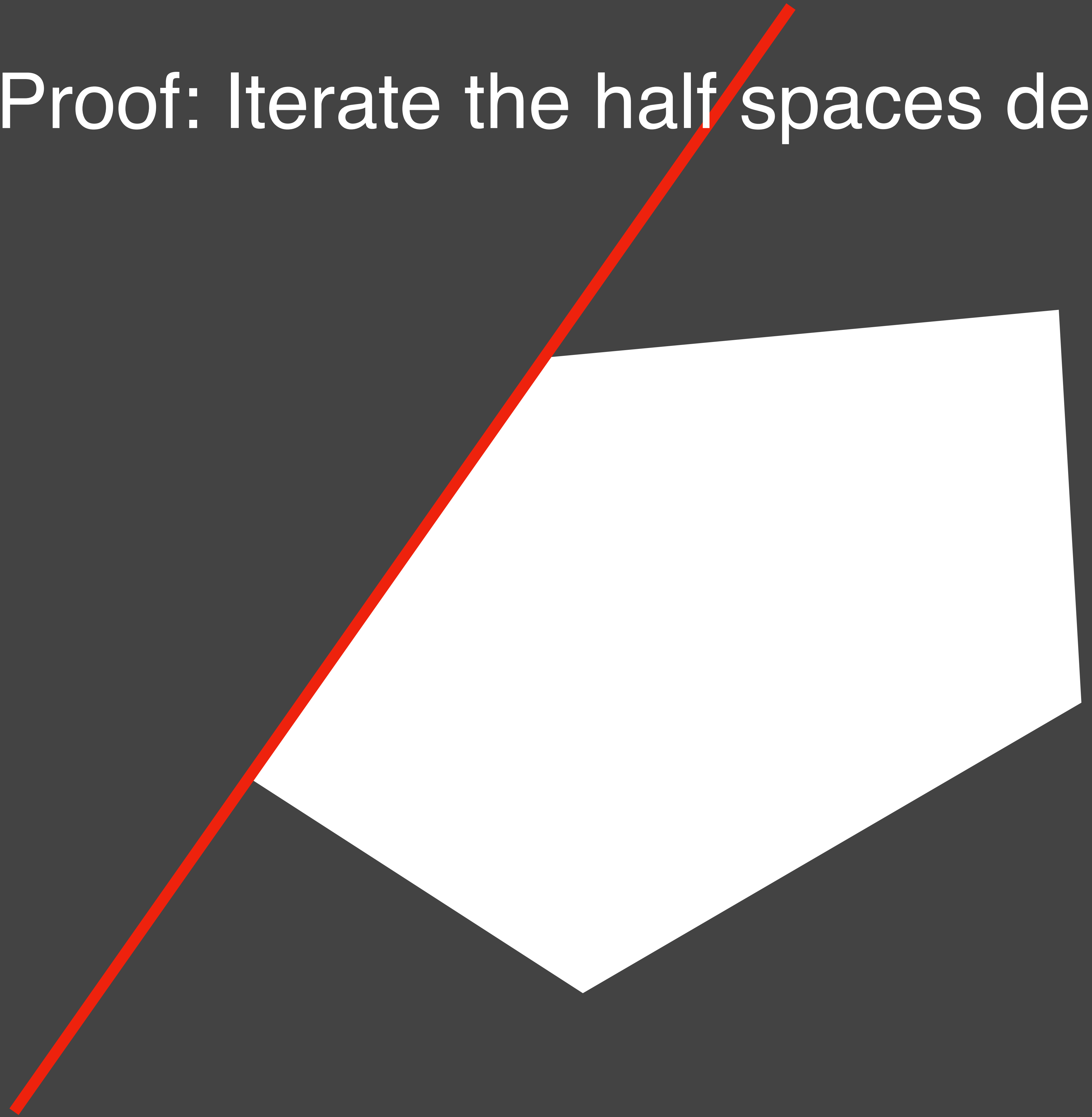

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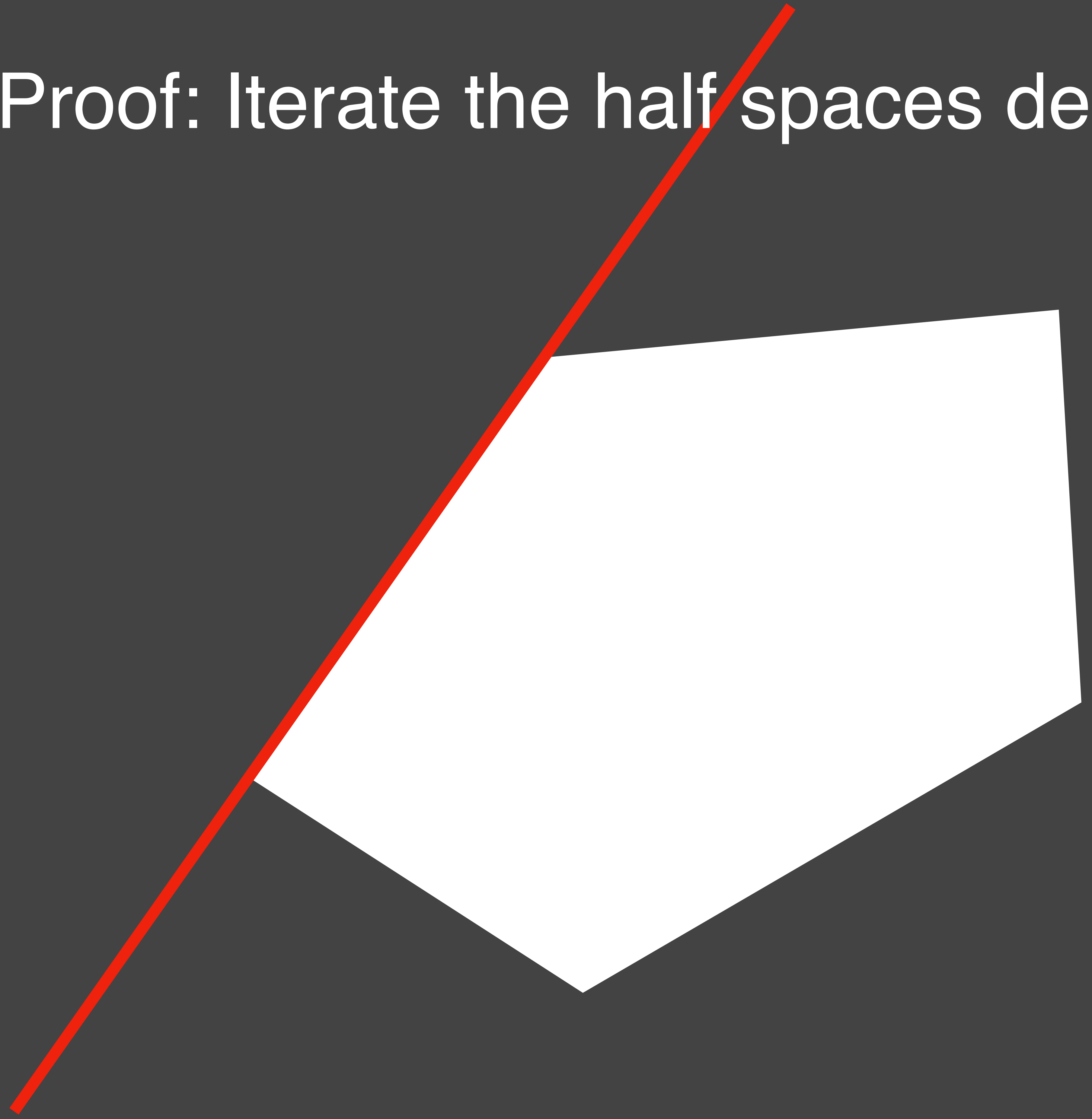

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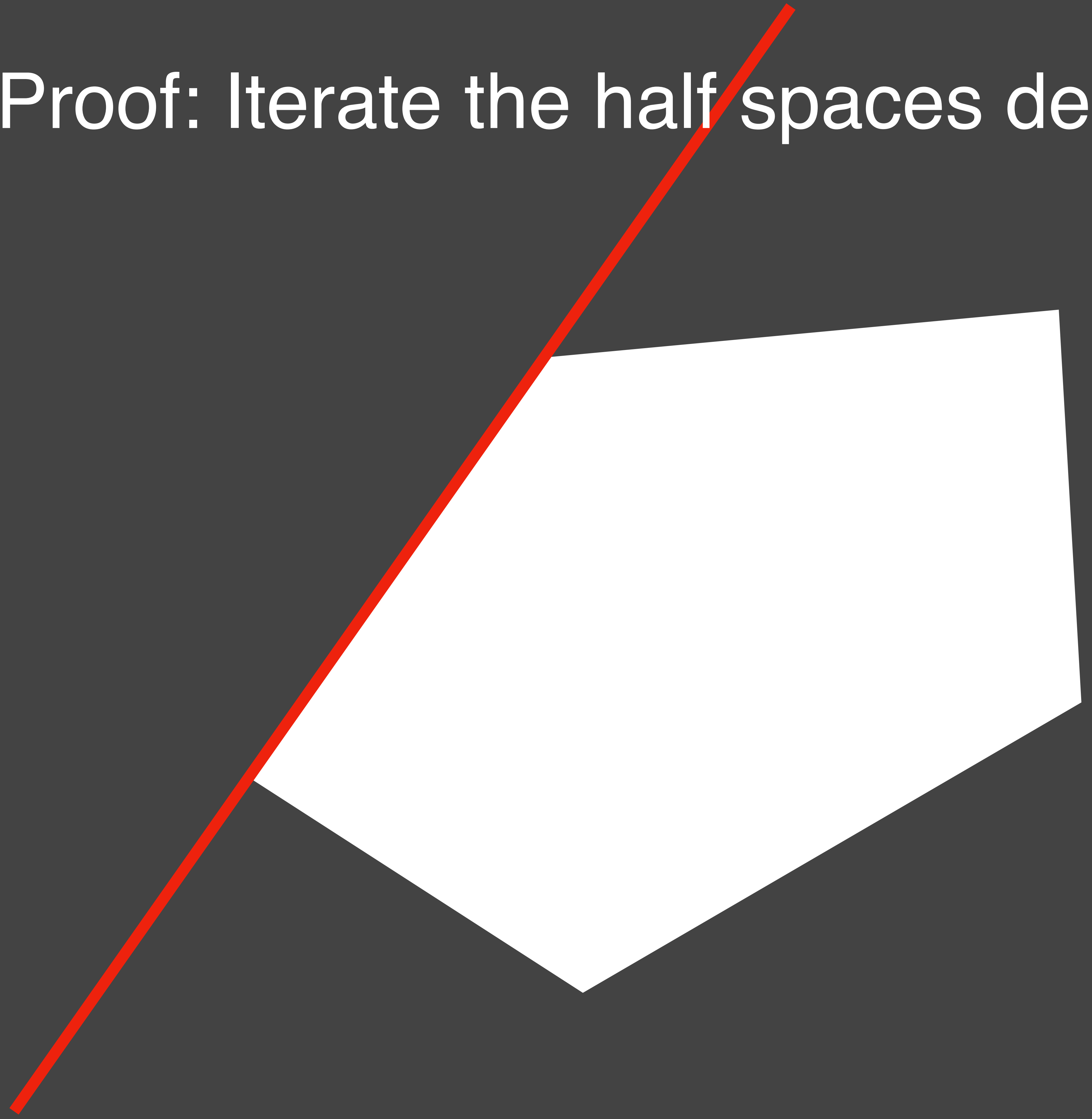

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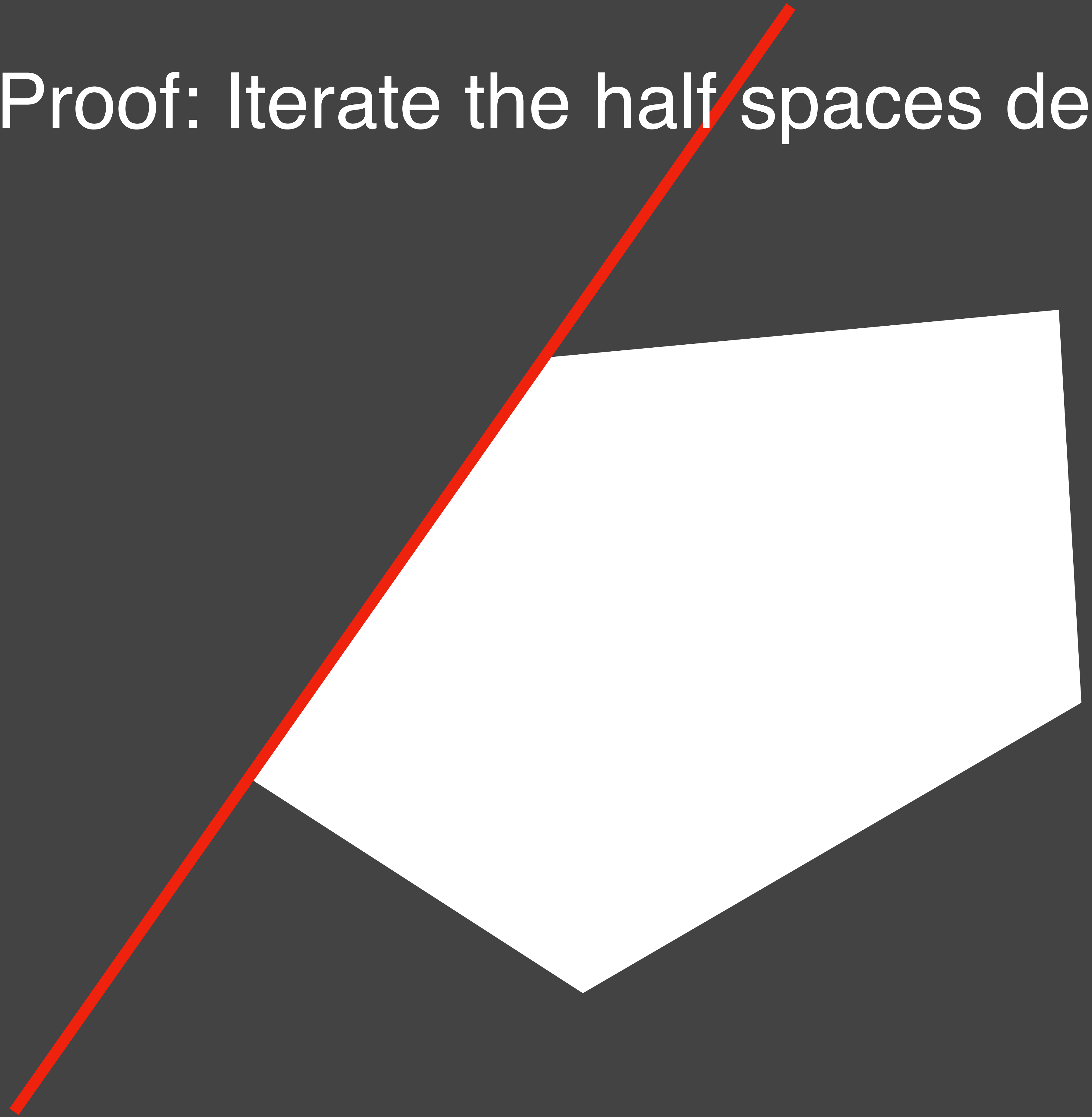

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I.e. all coefficients are positive, variables on same side of $\leq$.

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$$\left(\mathsf{KL}(a\,||\,b) = \sum_i a_i \log \frac{a_i}{b_i} - a_i + b_i\right)$$

When covering constraint arrives

$$x^t = \boxed{\begin{array}{c} \min_z \ \mathsf{KL}(z+\delta\,||\,x^{t-1}+\delta) \quad \text{s.t.} \\ \langle c^t, z\rangle \geq 1 \qquad (y^t) \\ x \geq 0 \end{array}}$$

(for some small δ)

By KKT:

$$\log \frac{x^t + \delta}{x^{t-1} + \delta} = c_i^t y^t$$

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Multiplicative weights update (almost)!

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We fit a dual to ALG's solution! How?

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Set $r = \log\left(\dfrac{1+4n/\epsilon}{1+4n\cdot x^{t-1}/\epsilon}\right)$.

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By weak duality:

Theorem [BBLS]:

Positive Body Chasing with movement $O(\log(n/\epsilon)/\epsilon) \cdot$ **OPT.**

Relating ALG and Dual Objective

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Lemma 2a: $\text{ALG} \leq \left(1 + \frac{\epsilon}{4}\right)\sum_t y^t.$

Lemma 2b: $0 \leq \left(1 + \frac{\epsilon}{4}\right)\sum_t y^t - (1+\epsilon)\sum_t z^t.$

Relating ALG and Dual Objective

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$$\underline{\textbf{Lemma 2a}}\text{:}\quad \text{ALG} \le \left(1 + \frac{\epsilon}{4}\right)\sum_t y^t.$$

Linear combination gives Lemma 2.

$$\underline{\textbf{Lemma 2b}}\text{:}\quad 0 \le \left(1 + \frac{\epsilon}{4}\right)\sum_t y^t - (1+\epsilon)\sum_t z^t.$$

Slack for this argument needs resource augmentation, i.e. violate packing by ϵ.

Lemma 2a Proof

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Lemma 2a Proof

Lemma 2a: ALG $\leq \left(1 + \frac{\epsilon}{4}\right) \sum_t y^t.$

$$\sum_i (x_i^t - x_i^{t-1})_+ \leq \sum_{i:\, x_i^t > x_i^{t-1}} (x_i^t + \delta) \ \log\left(\frac{x_i^t + \delta}{x_i^{t-1} + \delta}\right)$$

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$$\leq \sum_{t \in C} \sum_{i:\, c_i^t > 0} (x_i^t + \delta) \log \frac{x_i^t + \delta}{x_i^{t-1} + \delta} + \sum_{t \in P} \sum_{i:\, p_i^t > 0} x_i^t \log \frac{x_i^t}{x_i^{t-1}}$$

$$\leq \sum_{t \in C} y^t \sum_{i:\, c_i^t > 0} (x_i^t + \delta) - \sum_{t \in P} z^t \sum_{i:\, p_i^t > 0} x_i^t$$

$$\leq \left(1 + \frac{\epsilon}{4}\right) \sum_{t \in C} y^t - (1+\epsilon) \sum_{t \in P} z^t. \quad \blacksquare$$

KKT conditions:

$$\log \frac{x^t + \delta}{x^{t-1} + \delta} = c_i^t y^t$$

$$\log \frac{x^t}{x^{t-1}} = -p_i^t z^t$$

Set $\delta := \frac{\epsilon}{4n}$.

$$\mathrm{KL}(a \,||\, b) = \sum_i a_i \log \frac{a_i}{b_i} - a_i + b_i$$

Comparison to Online Covering

Online Covering [Buchbinder Naor 09]

Our KL Projection Algorithm

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$$\min_z \ \mathsf{KL}(z+\delta \,||\, x^{t-1}+\delta) \quad \text{s.t.}$$

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Was a barrier to prior work.

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[AGGT 20] & [Sellke 20]

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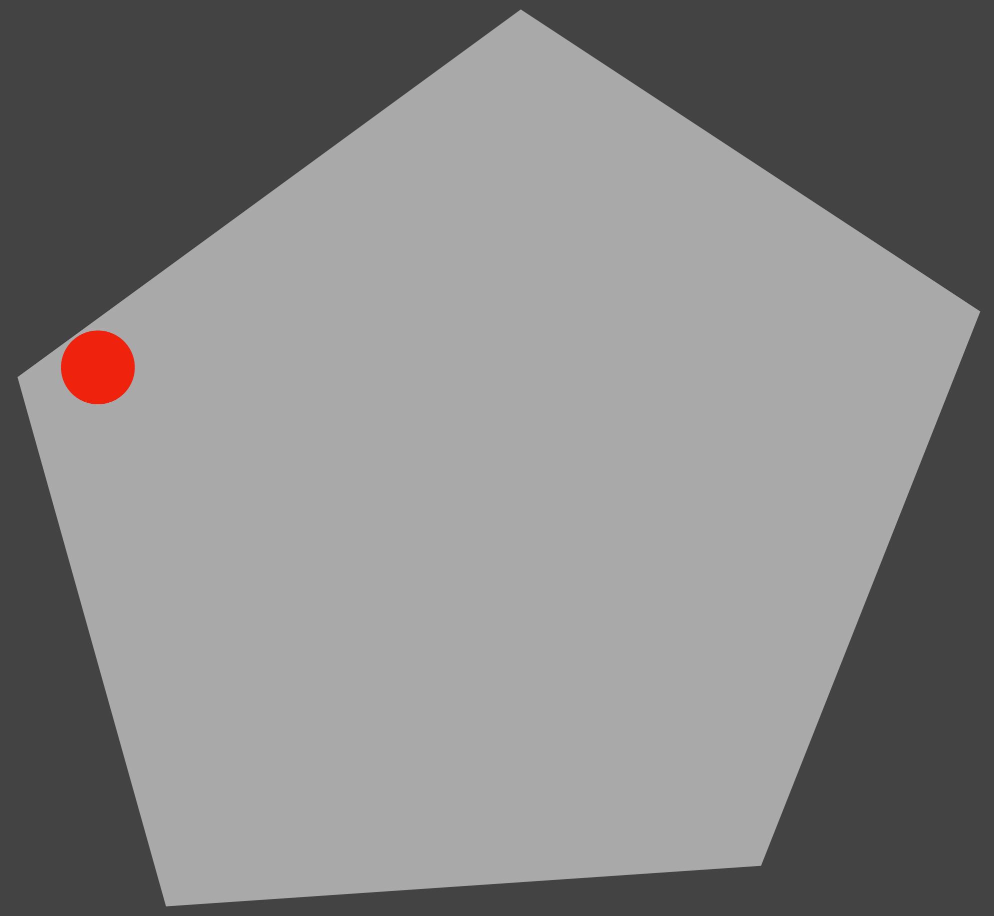

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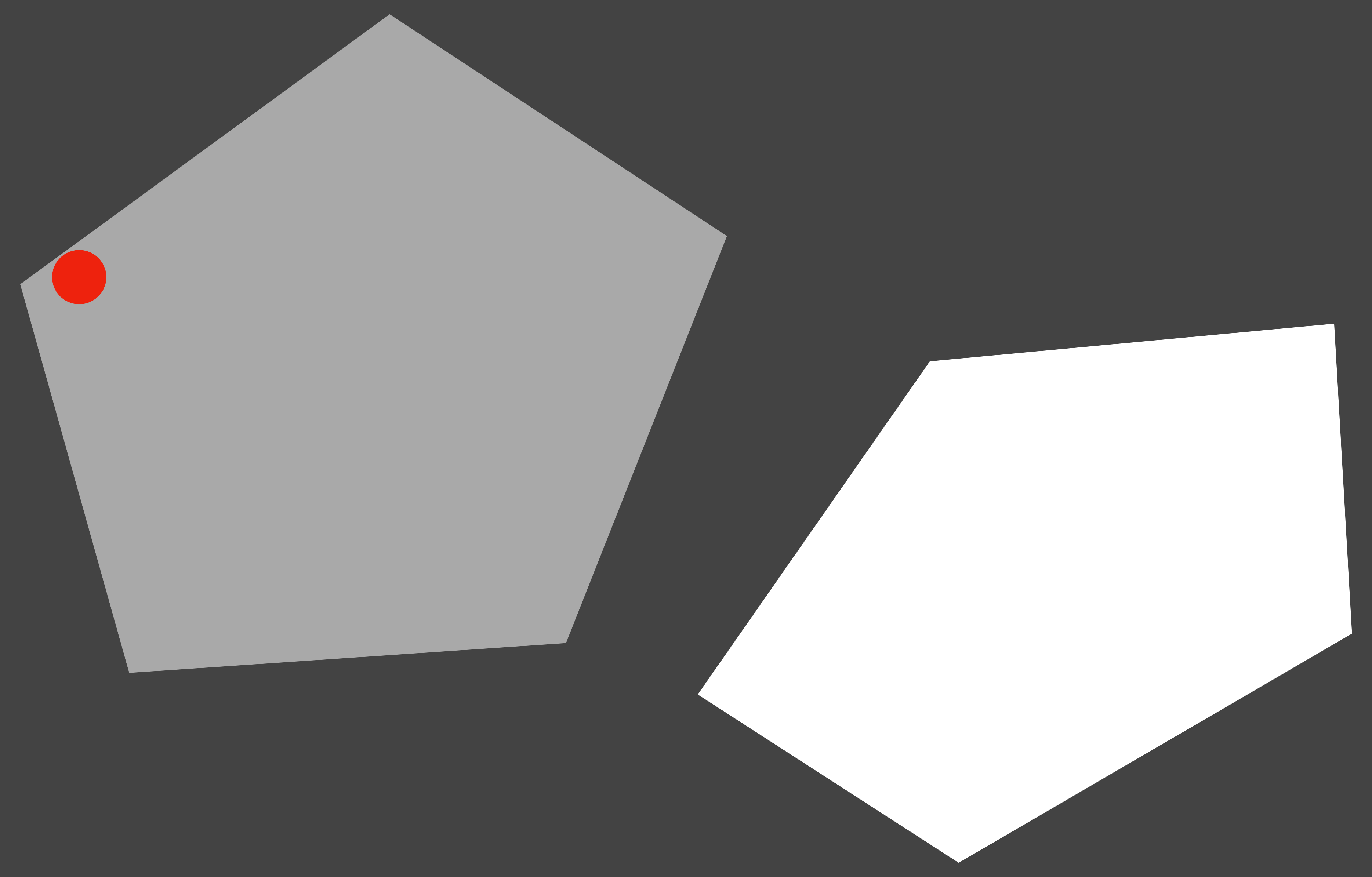

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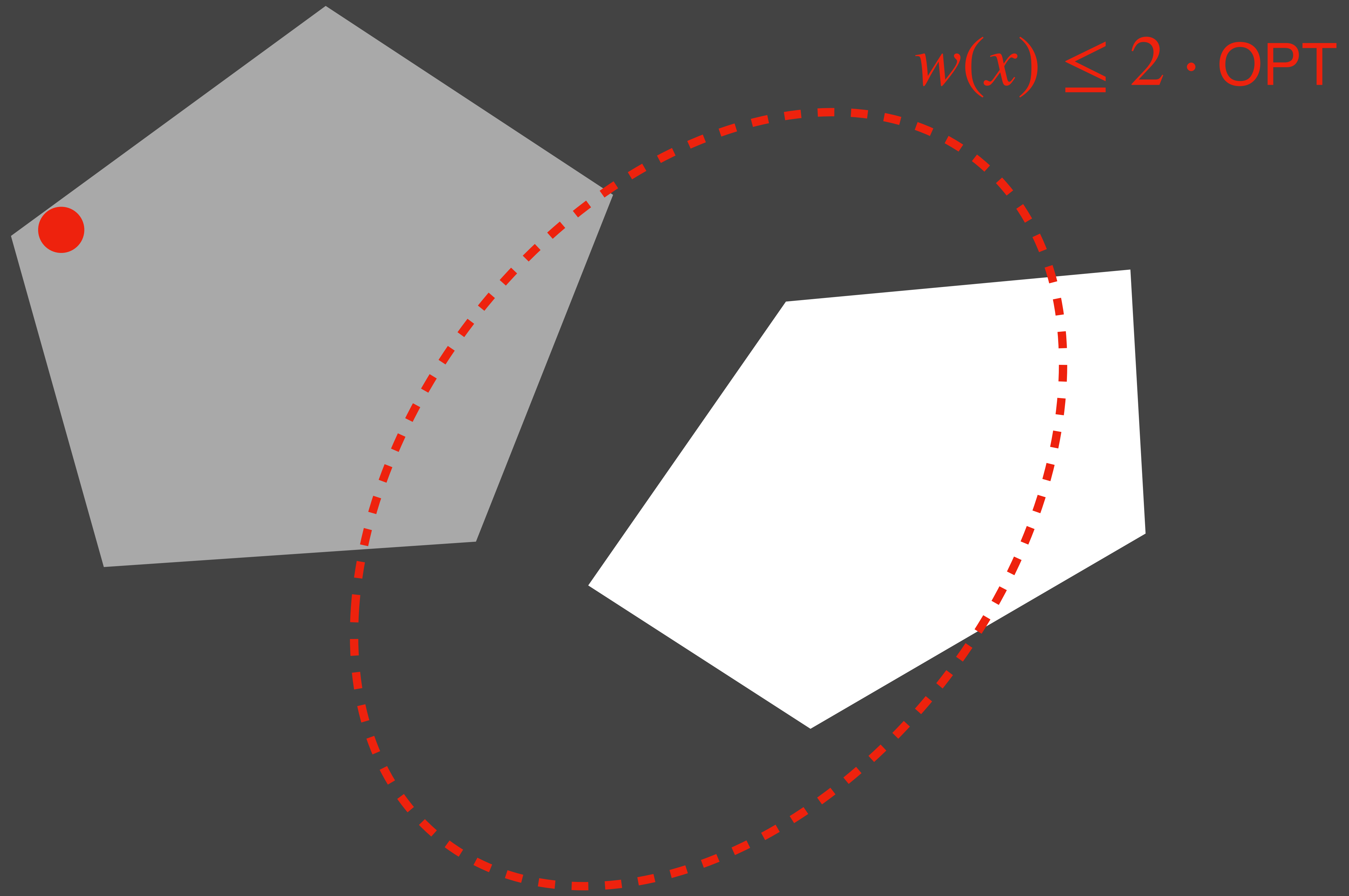

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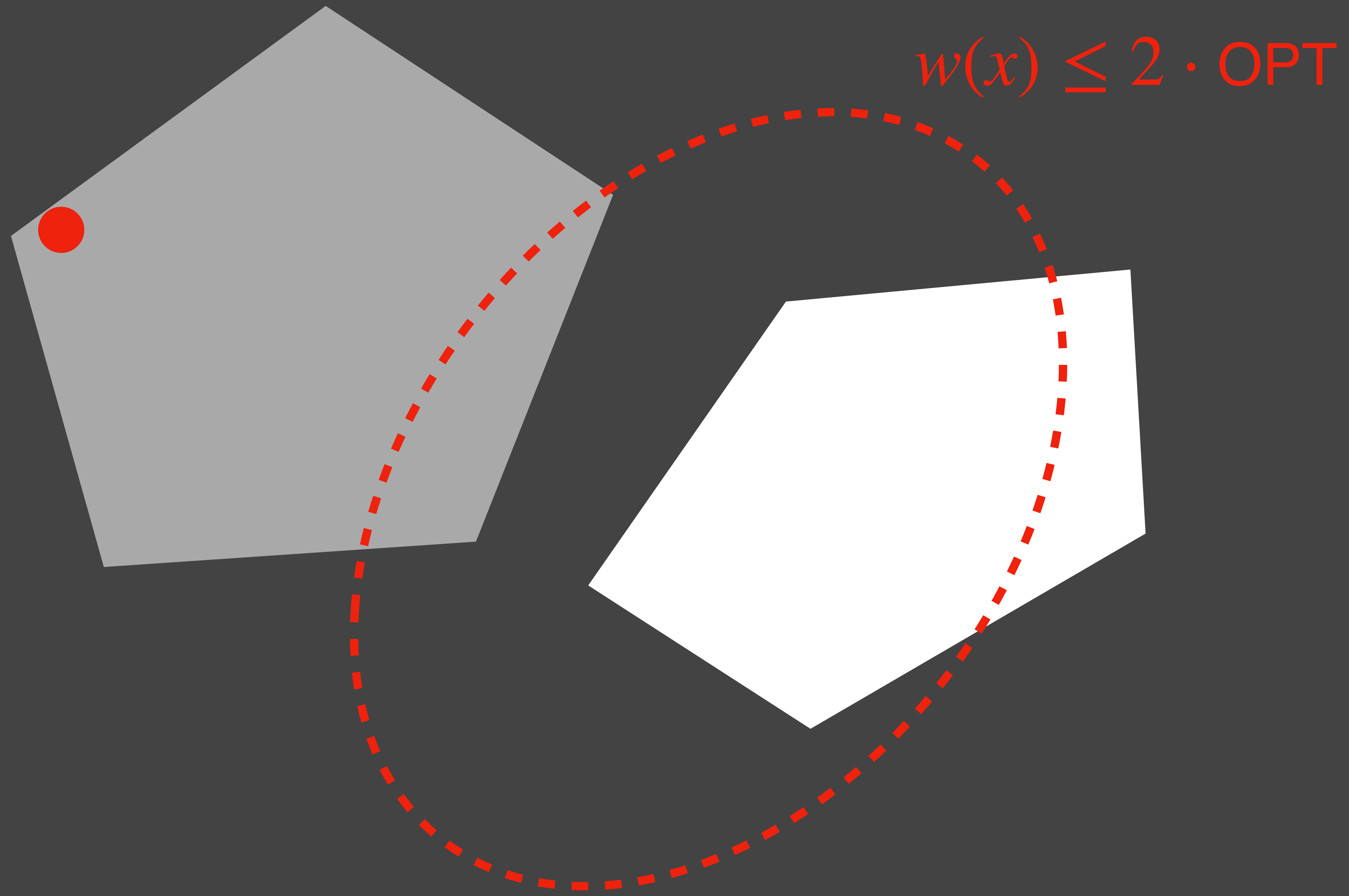

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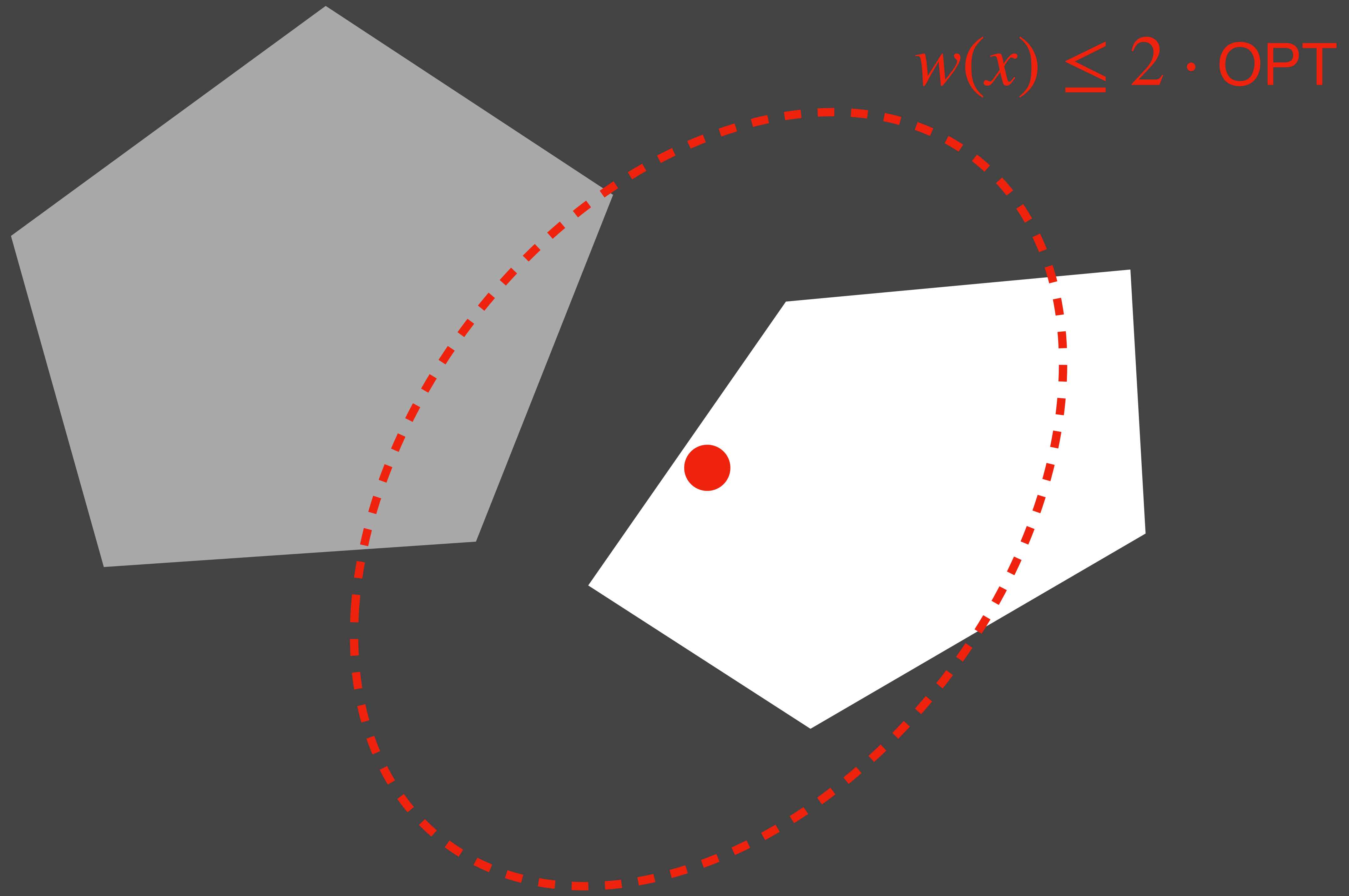

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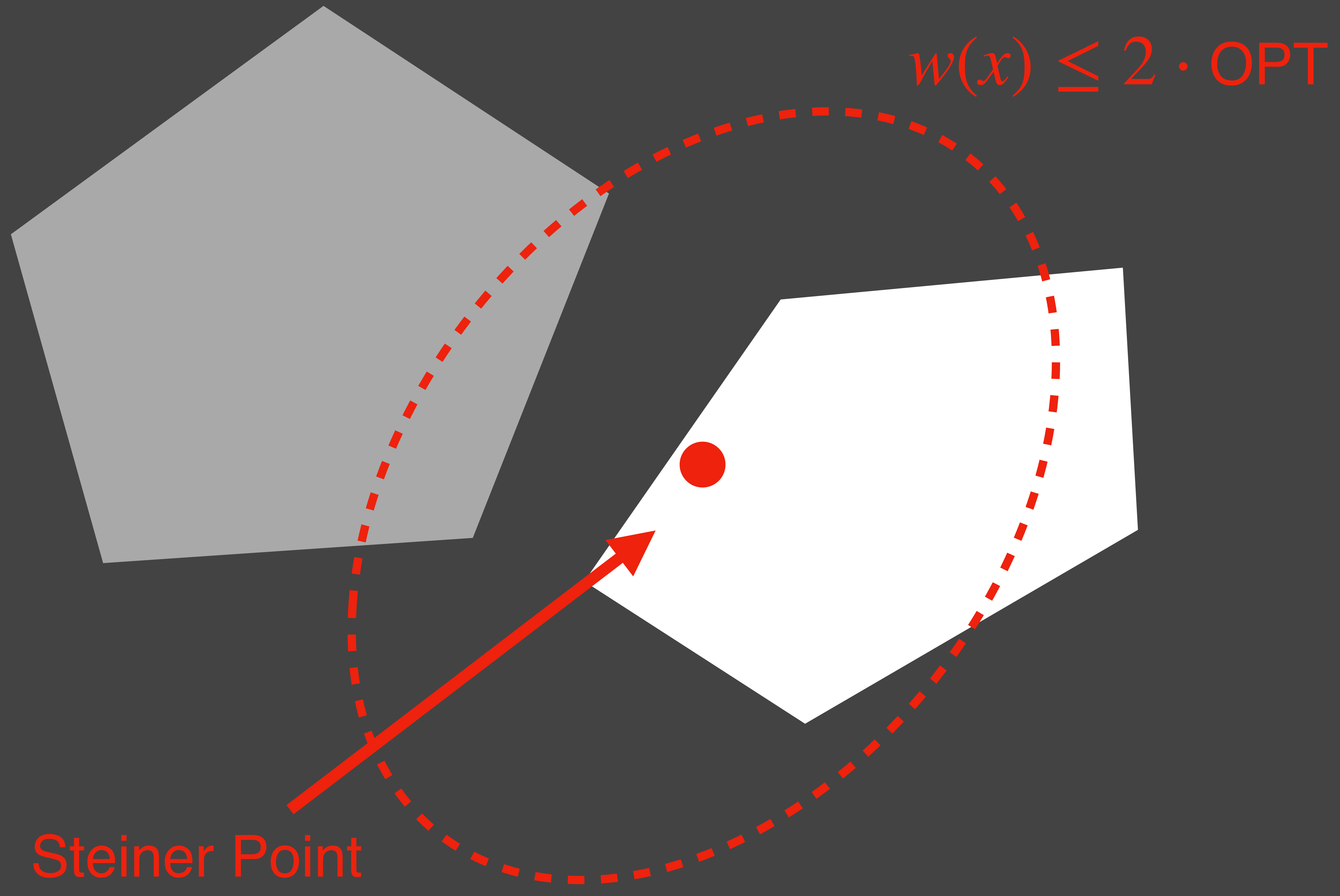

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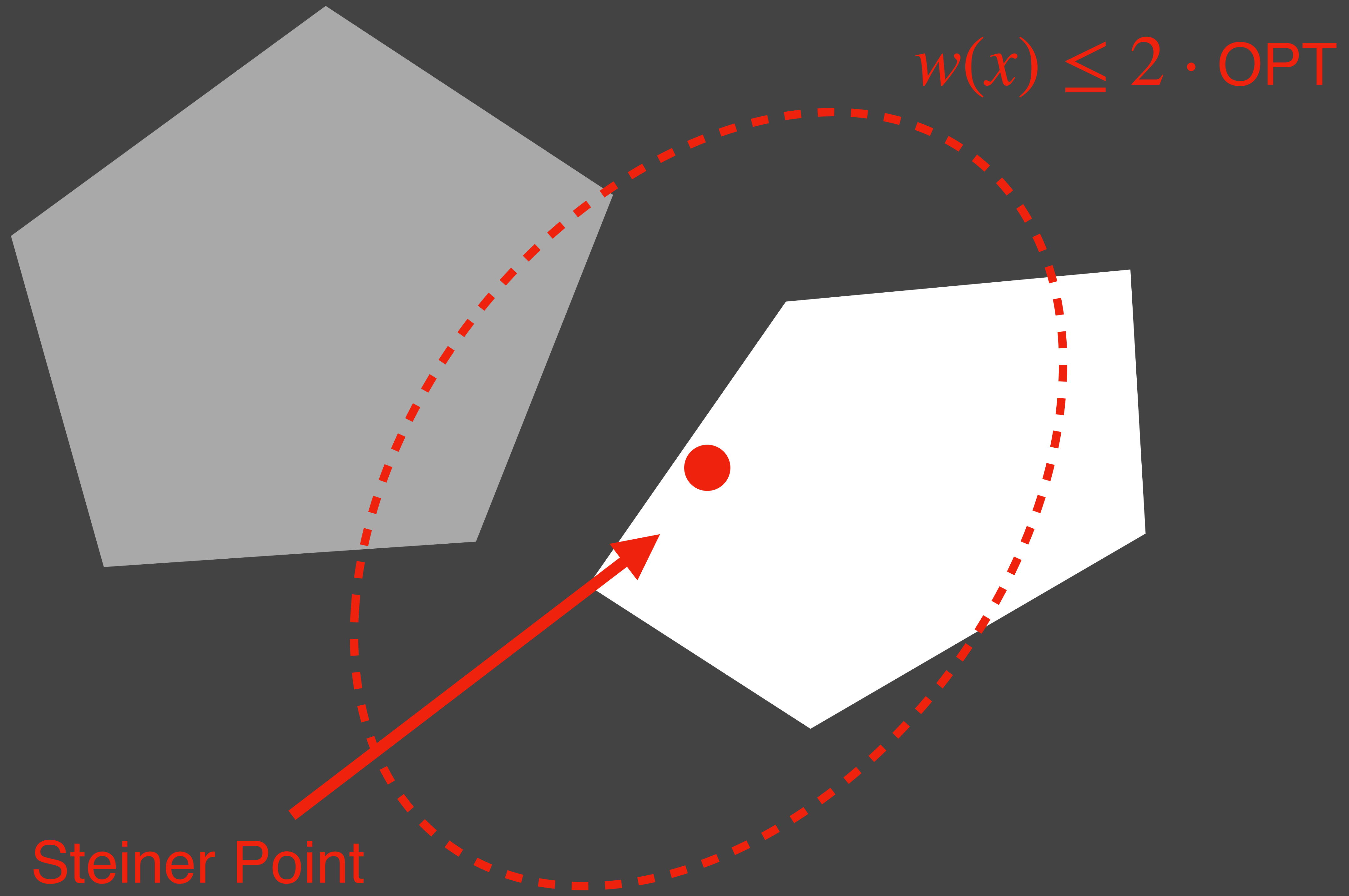

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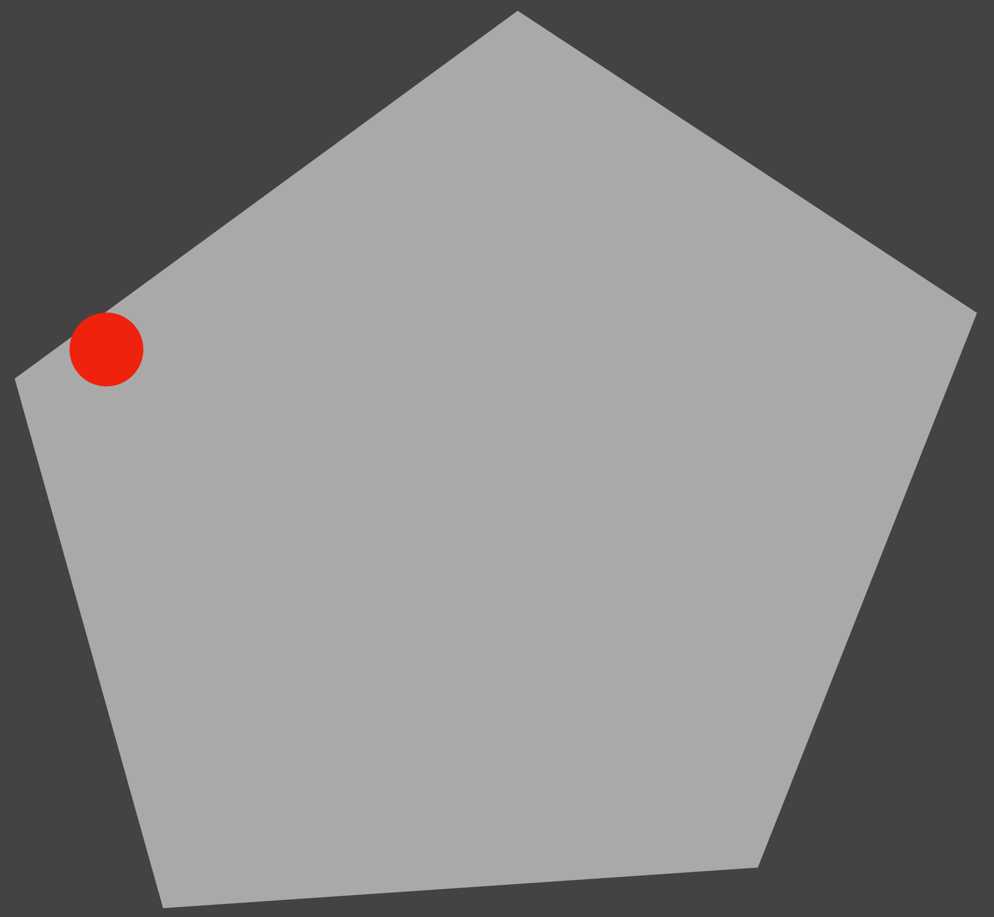

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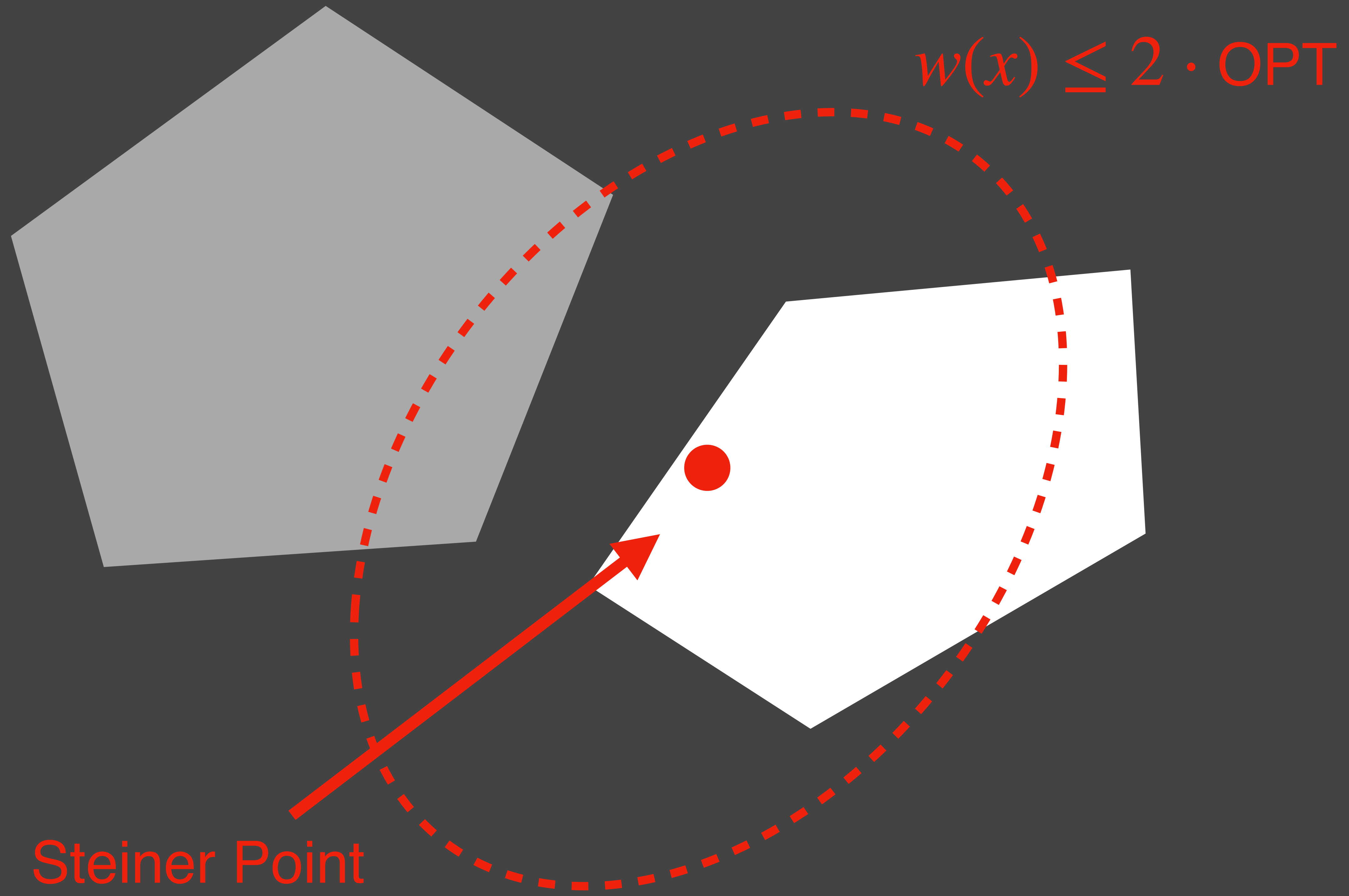

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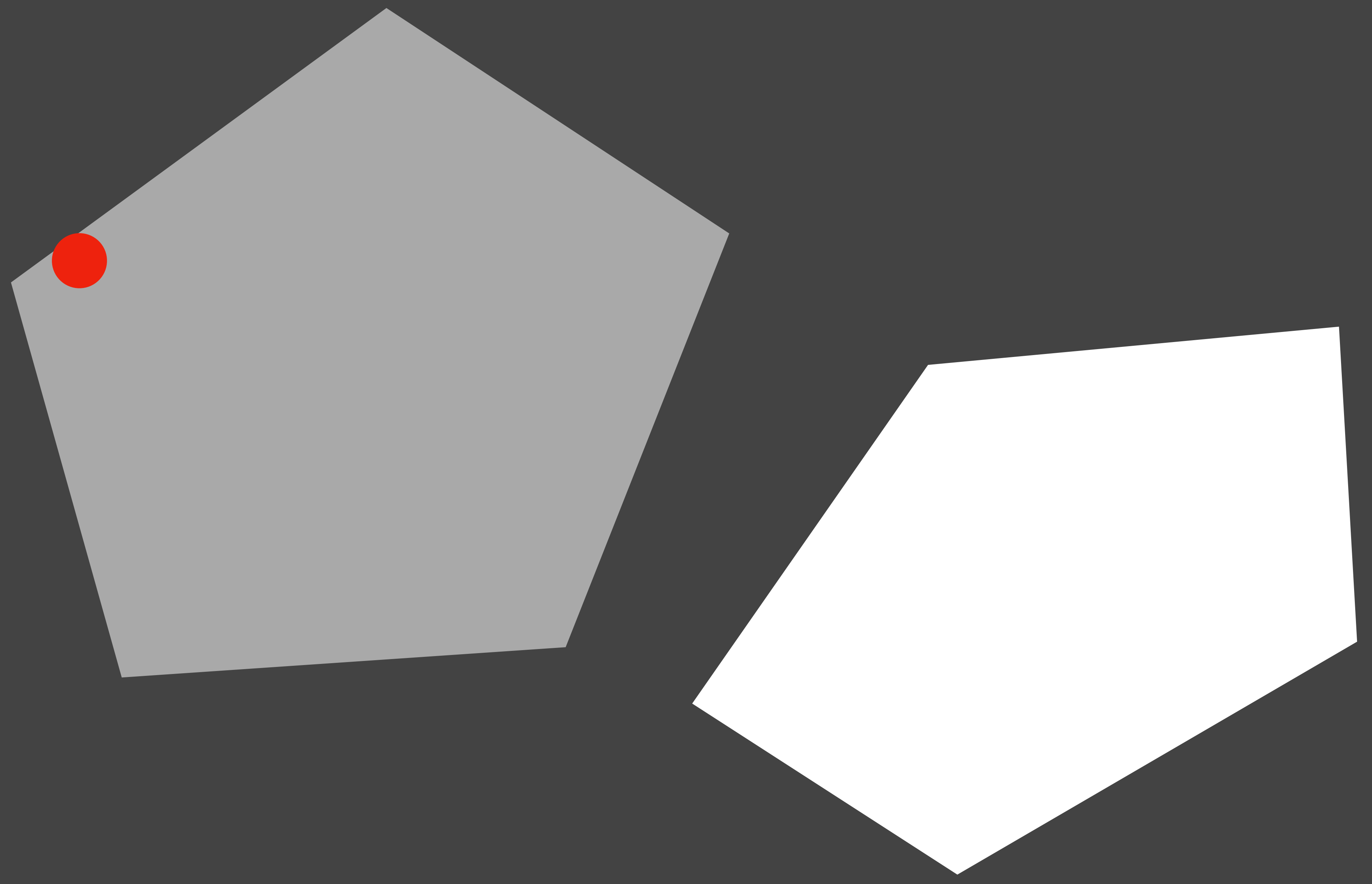

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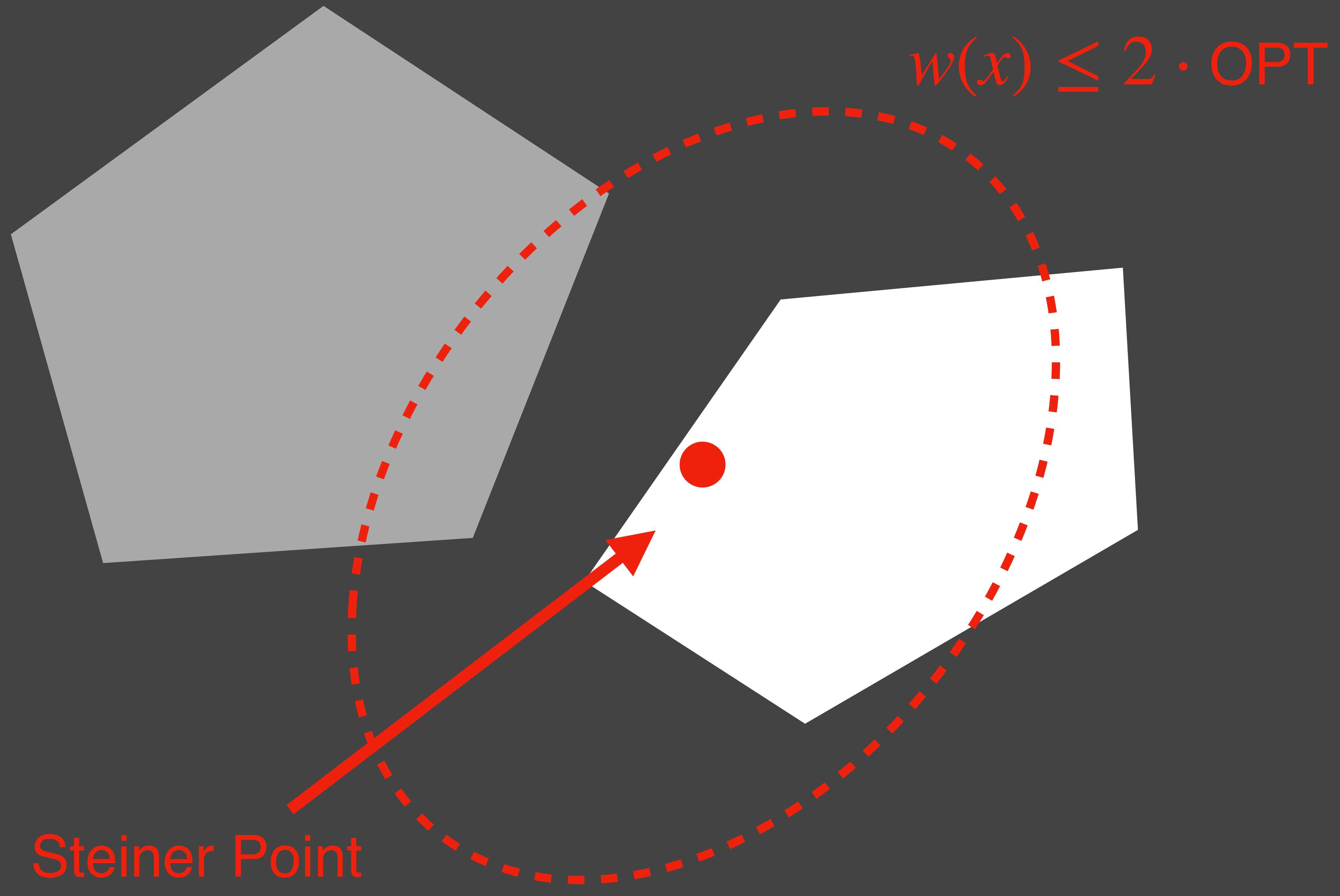

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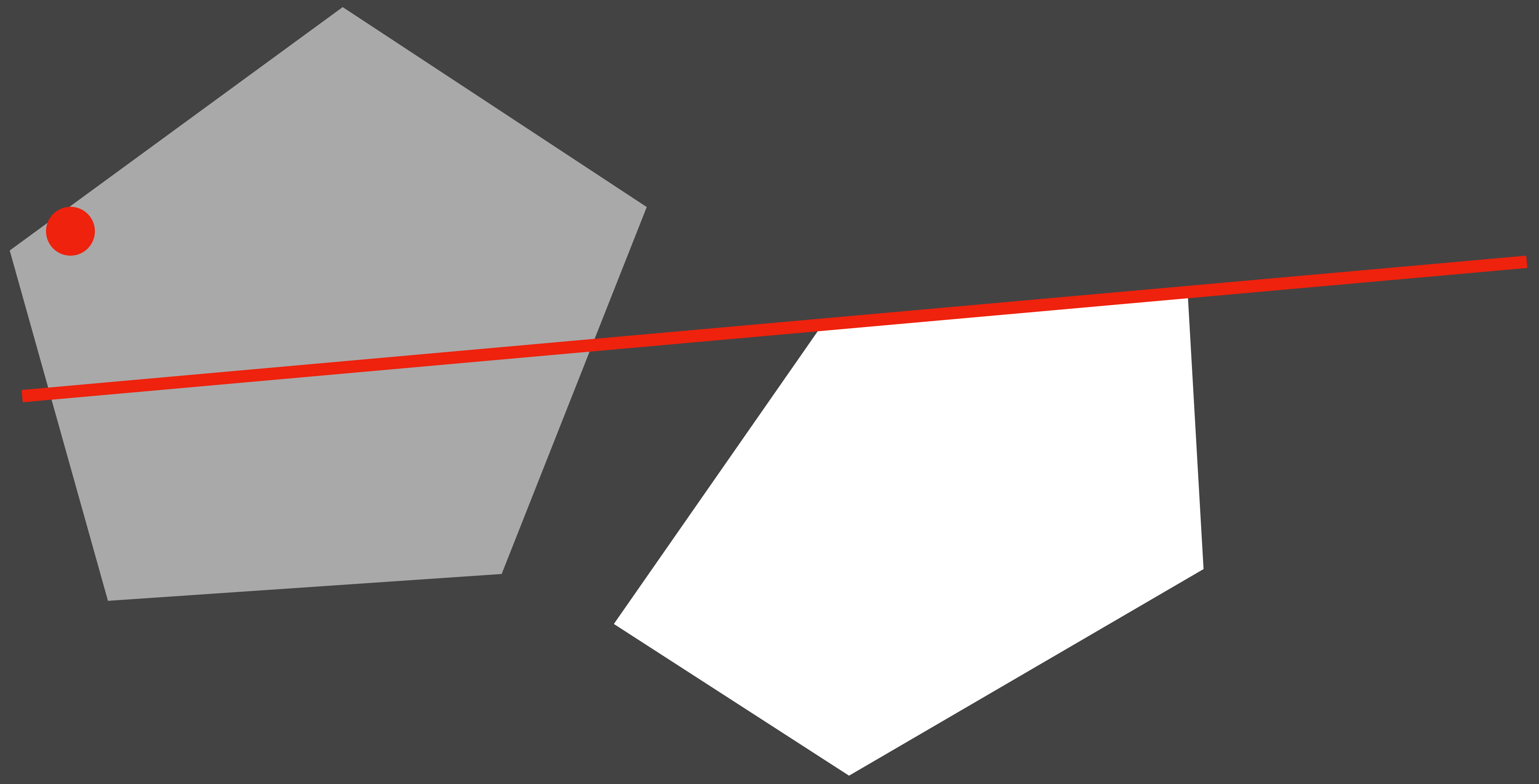

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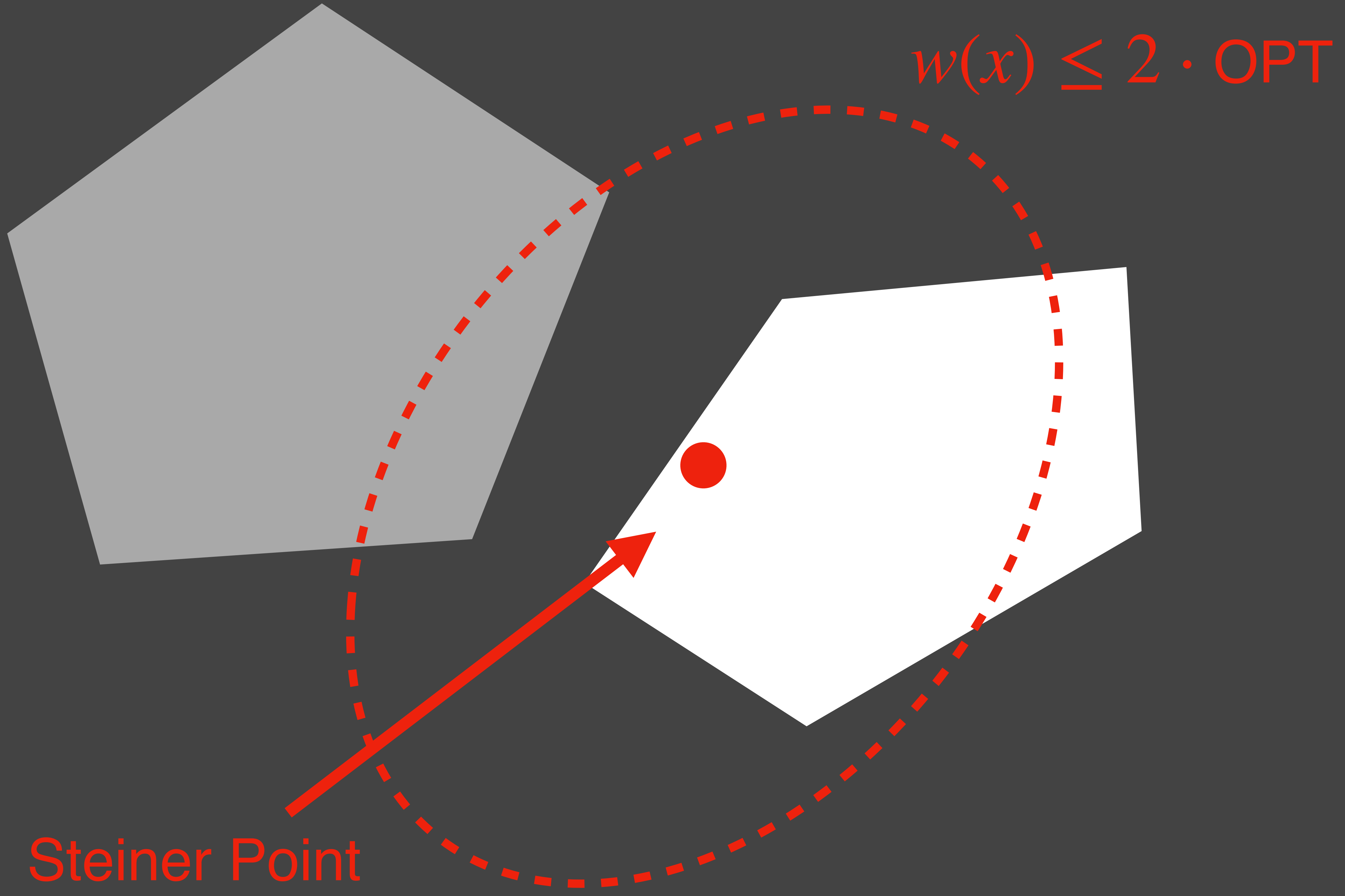

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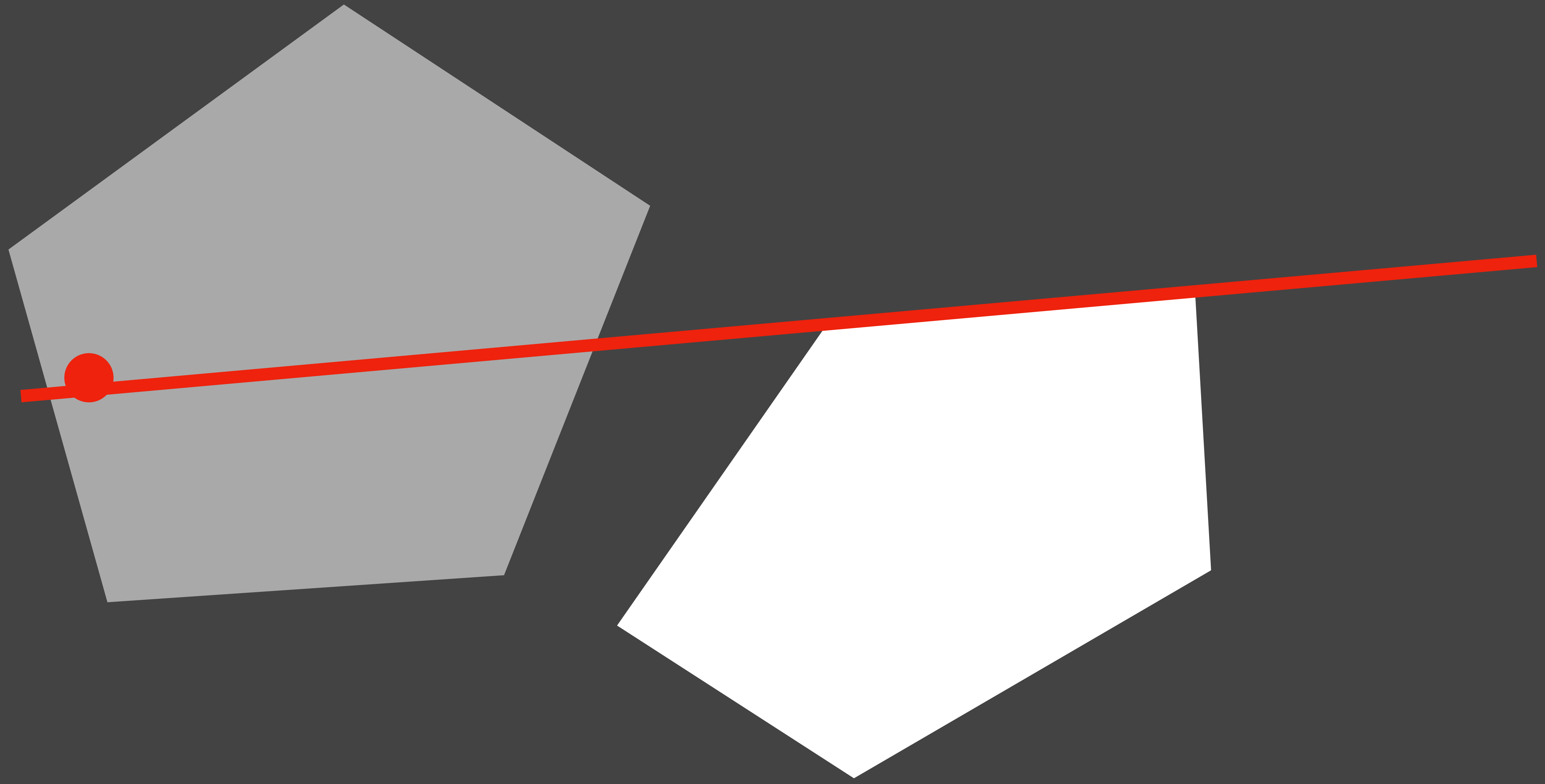

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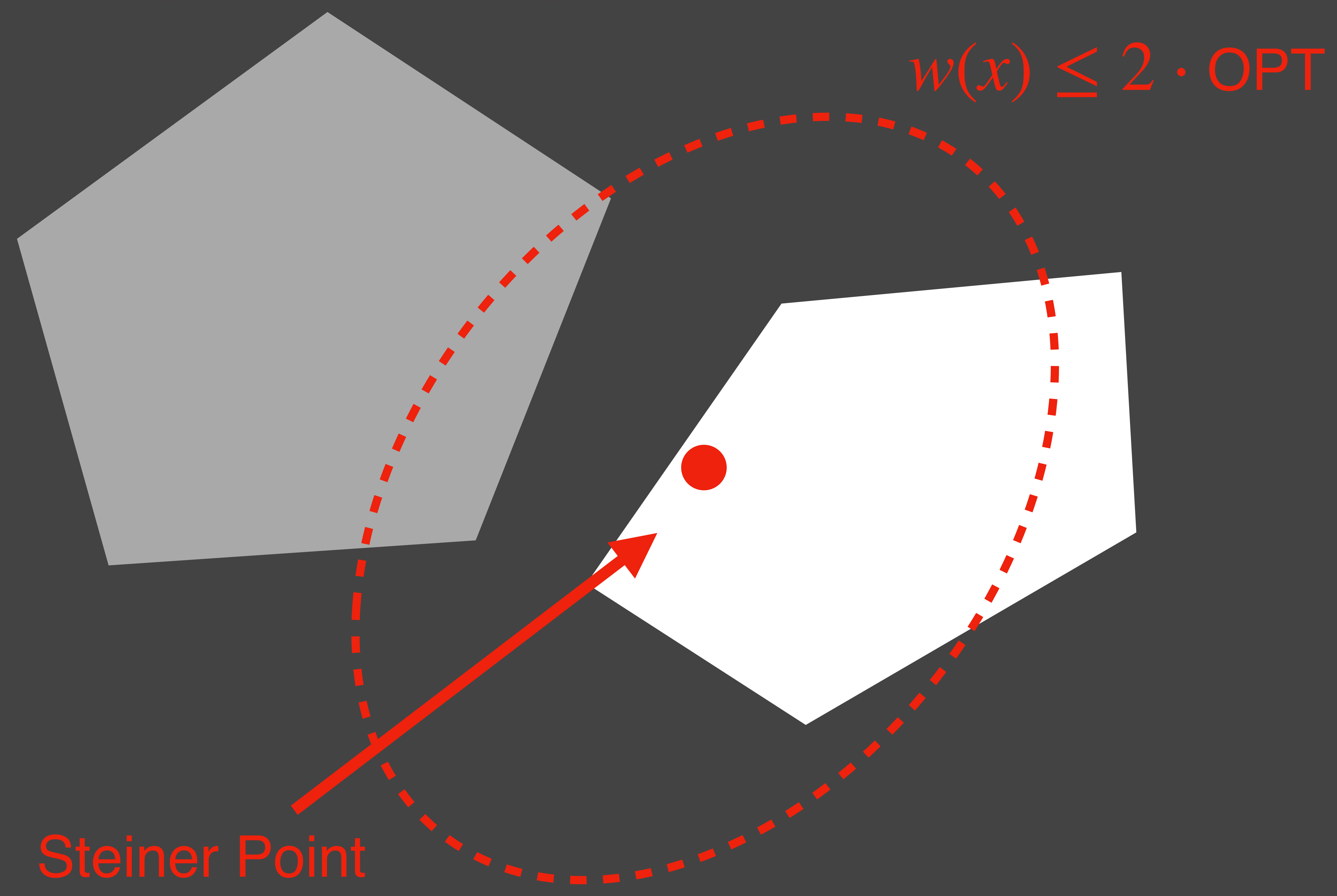

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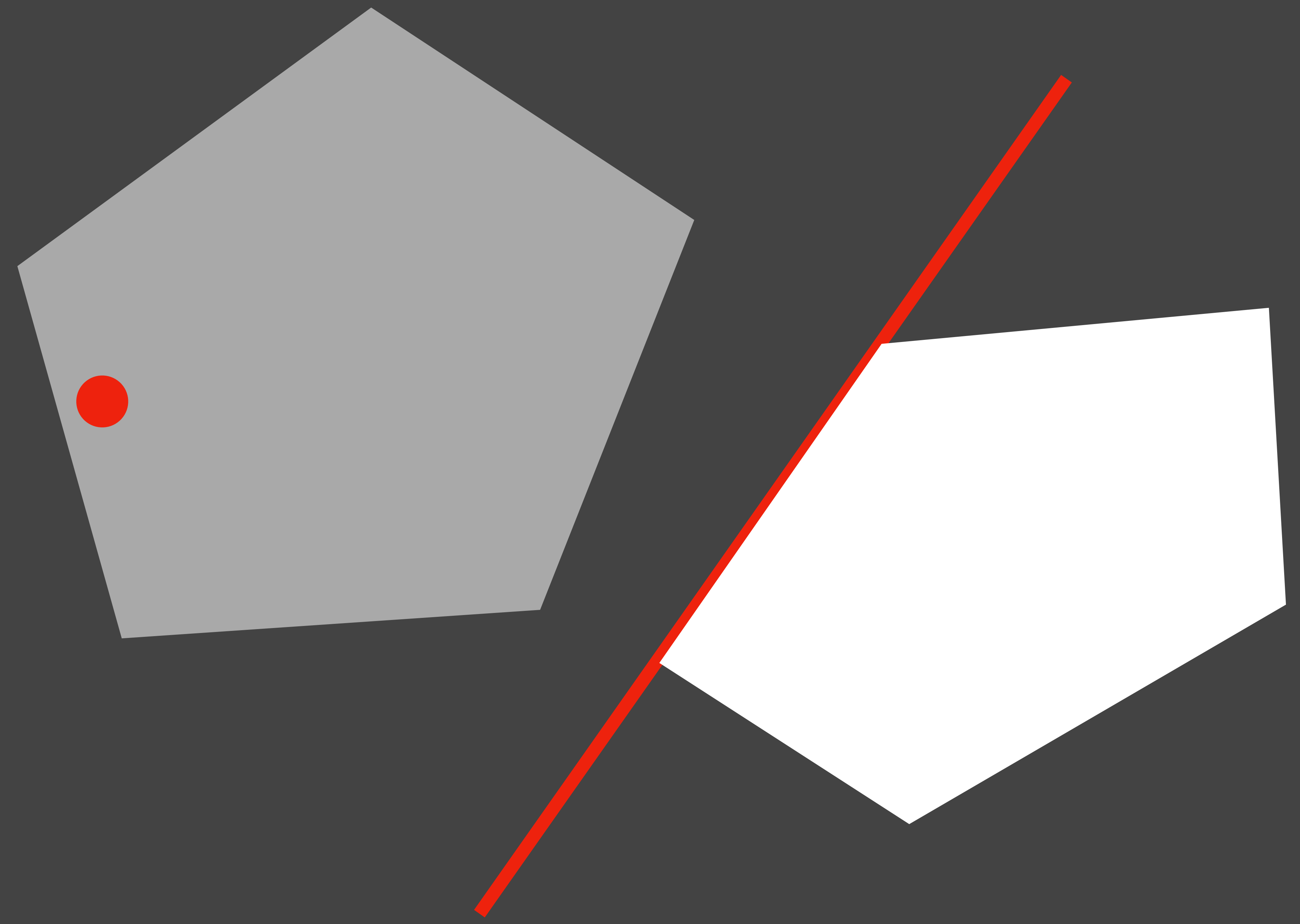

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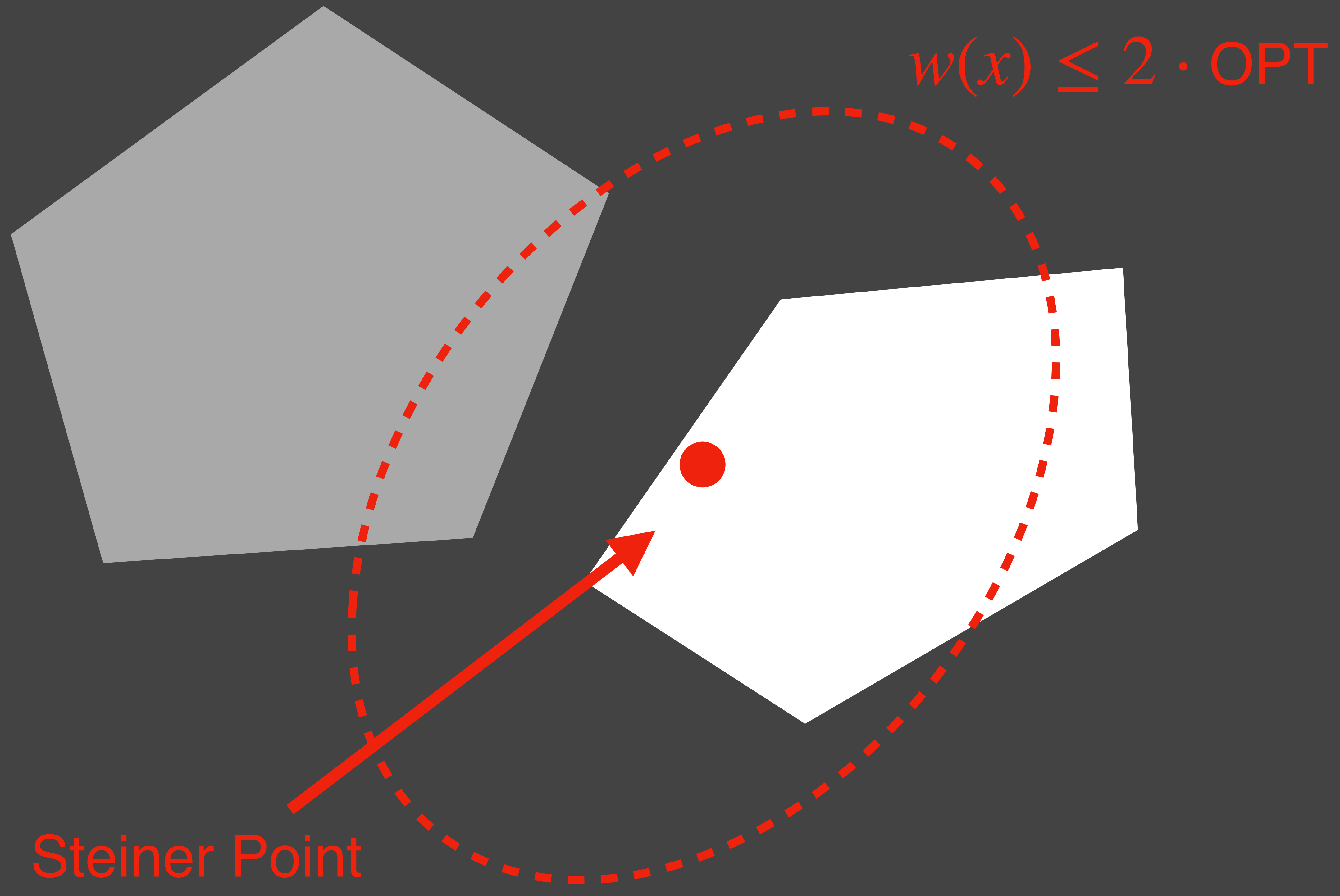

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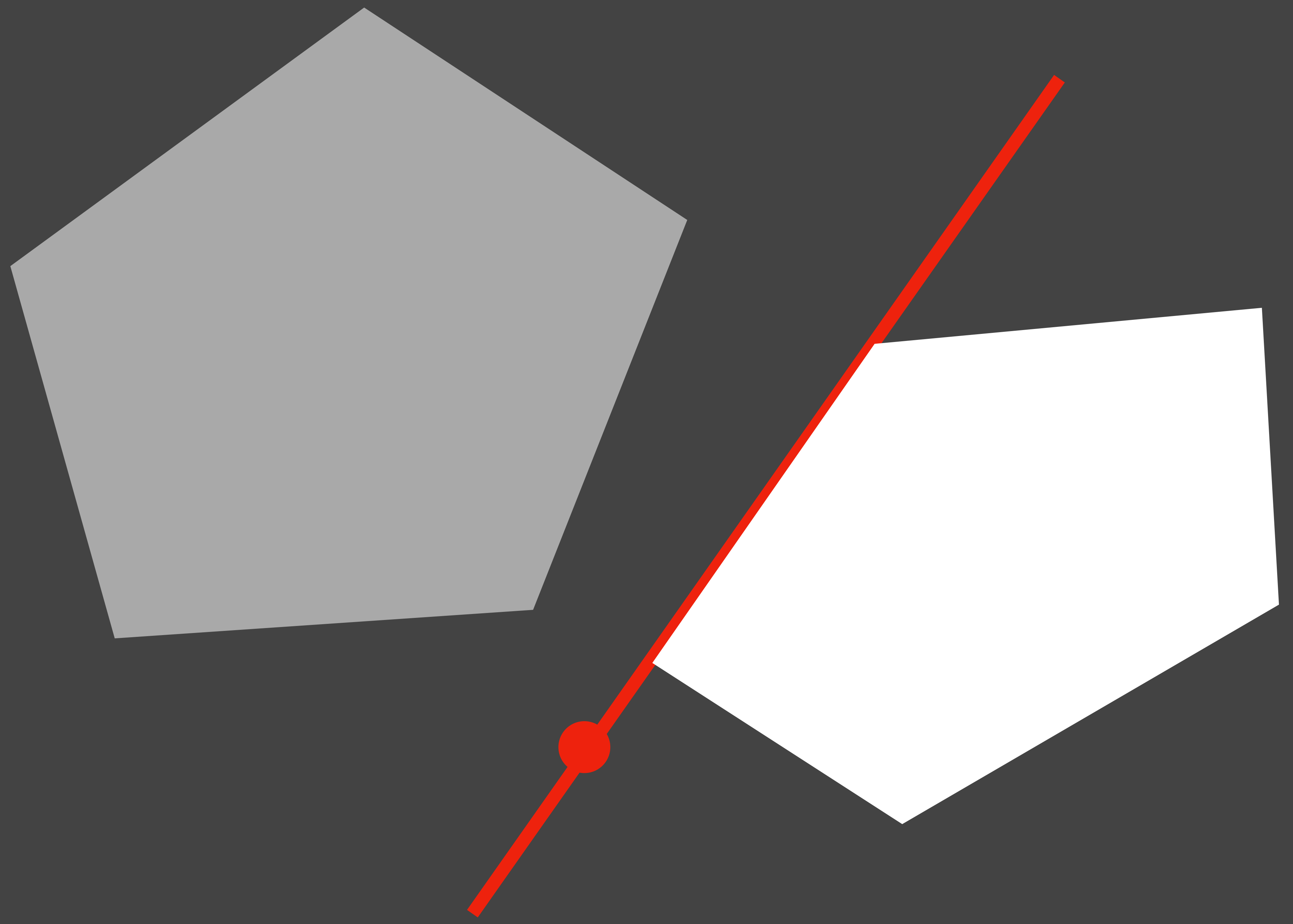

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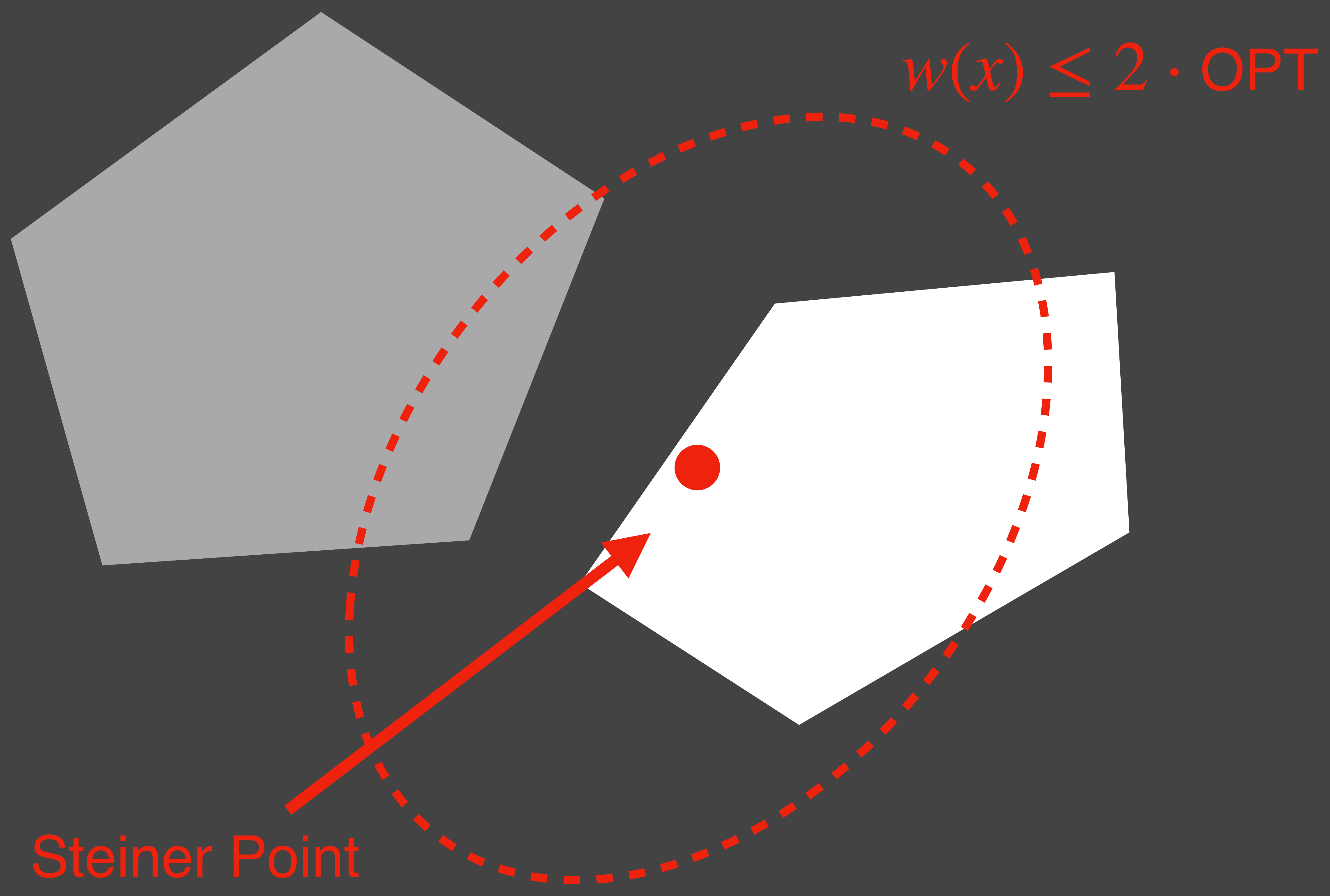

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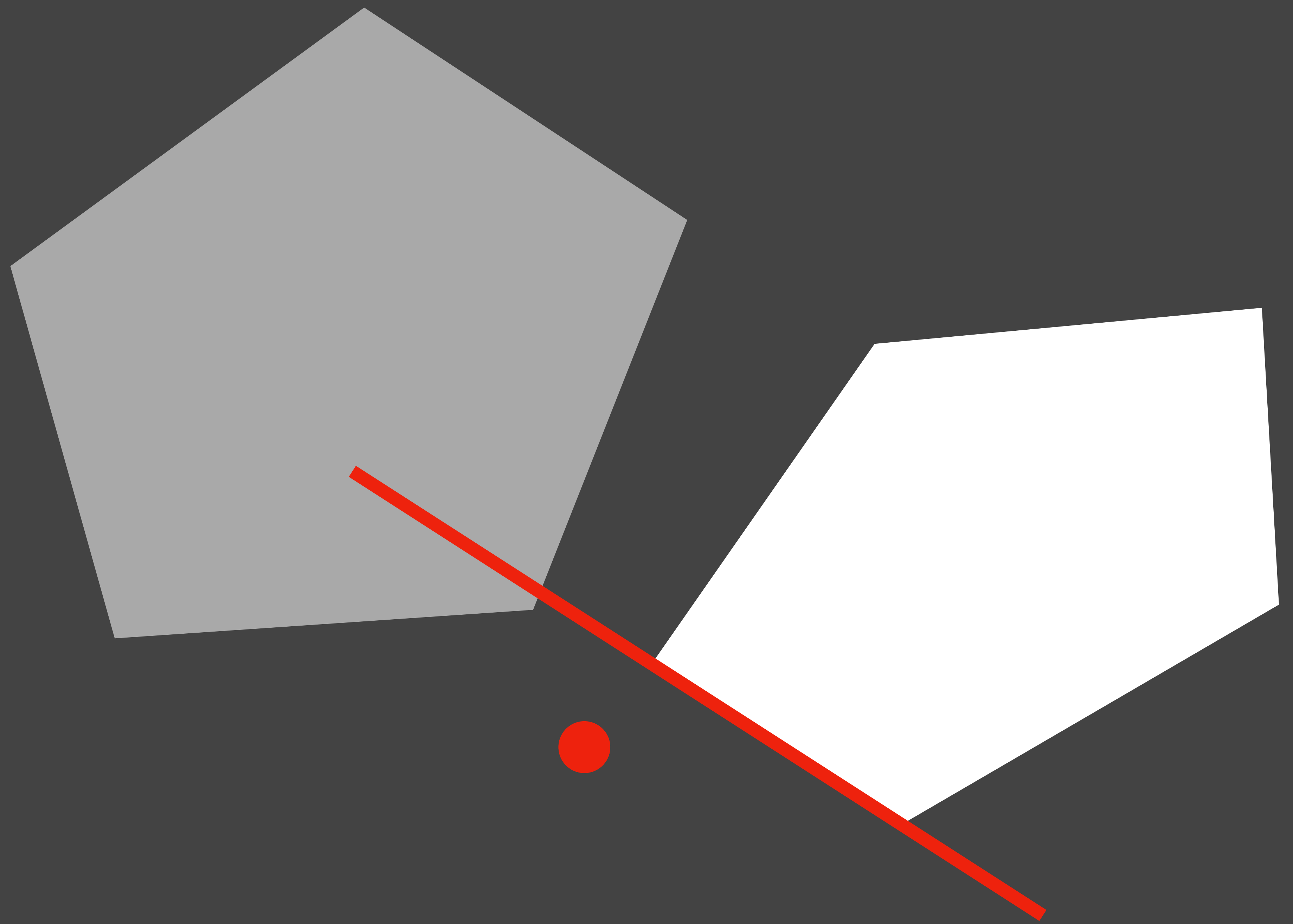

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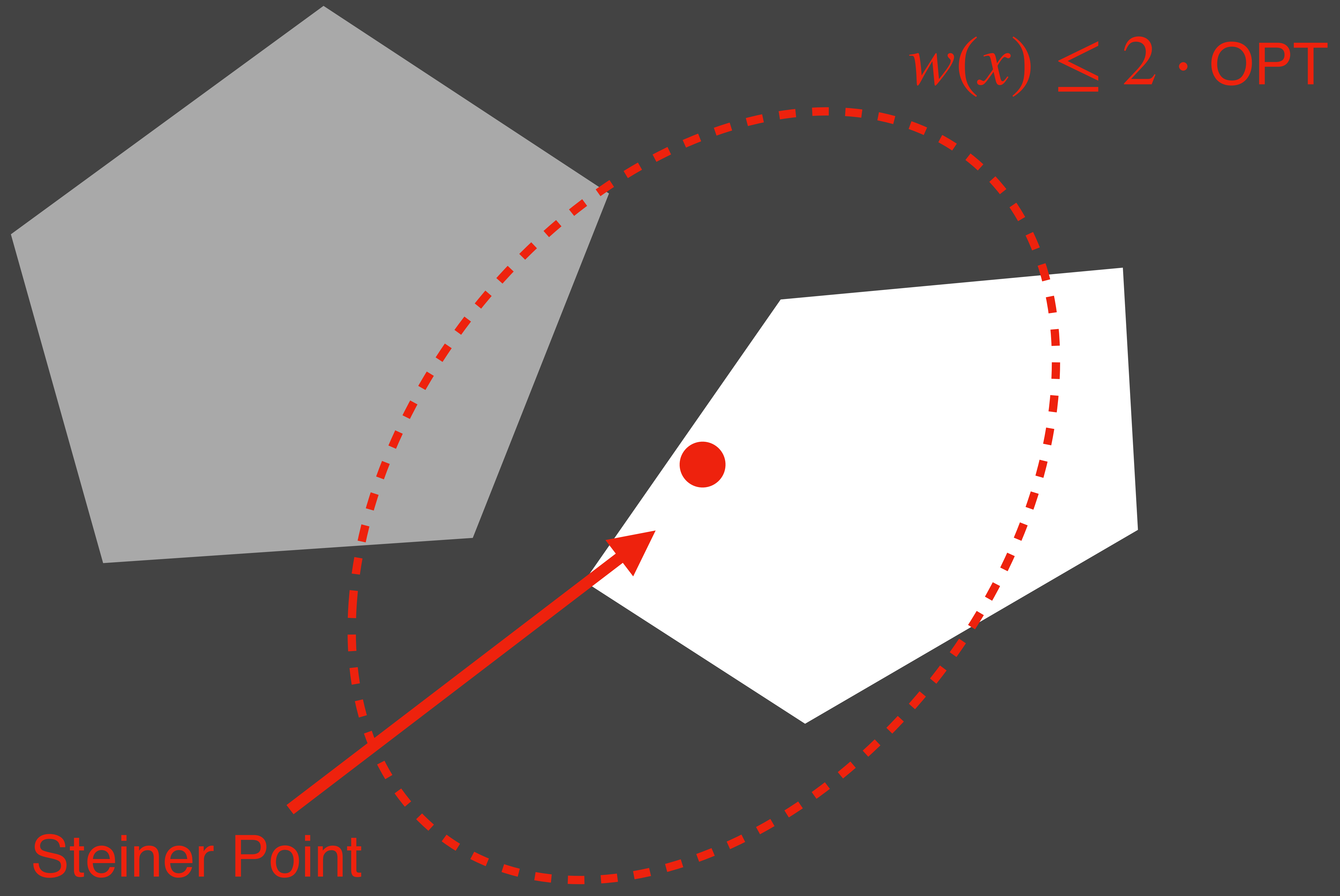

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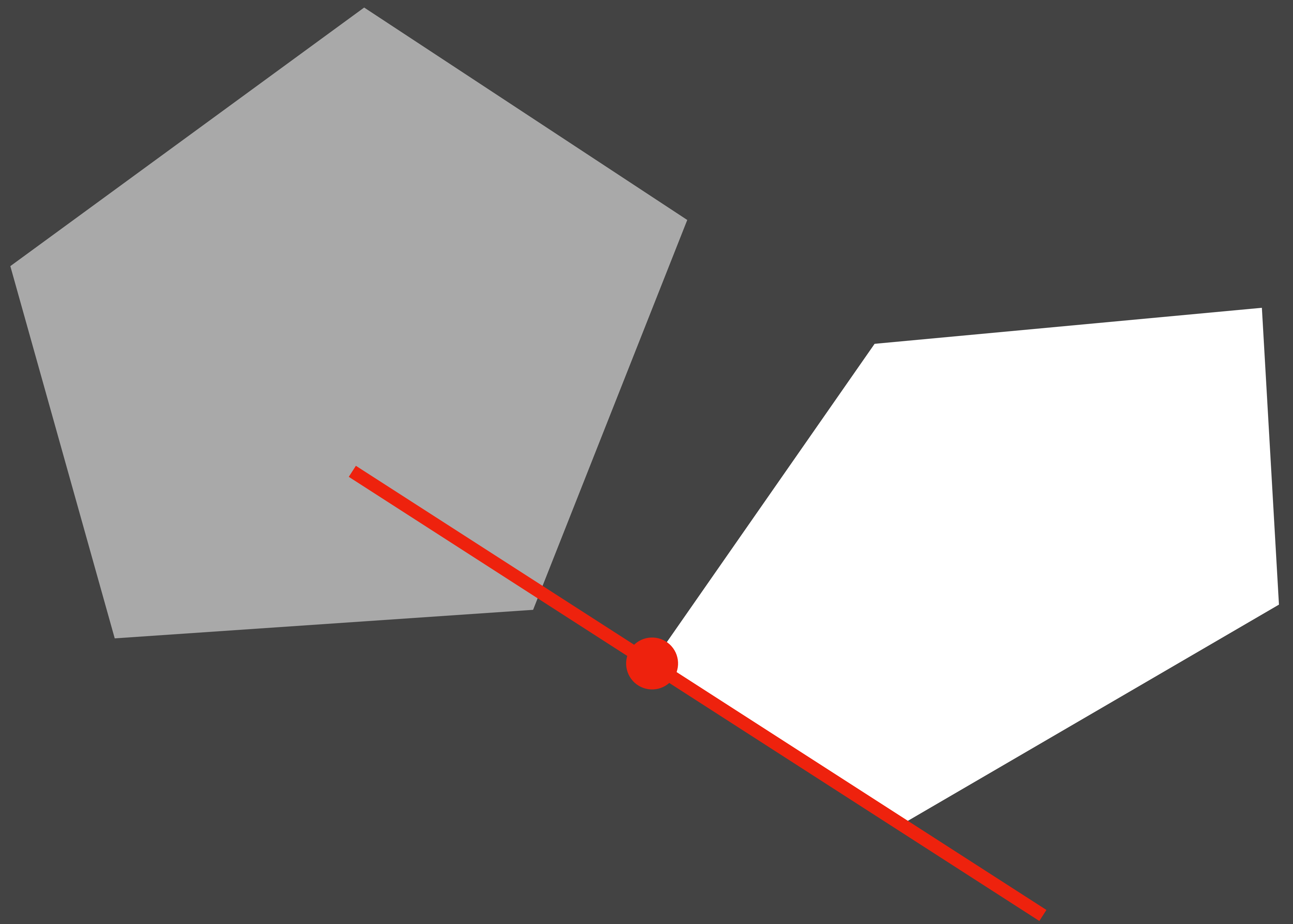

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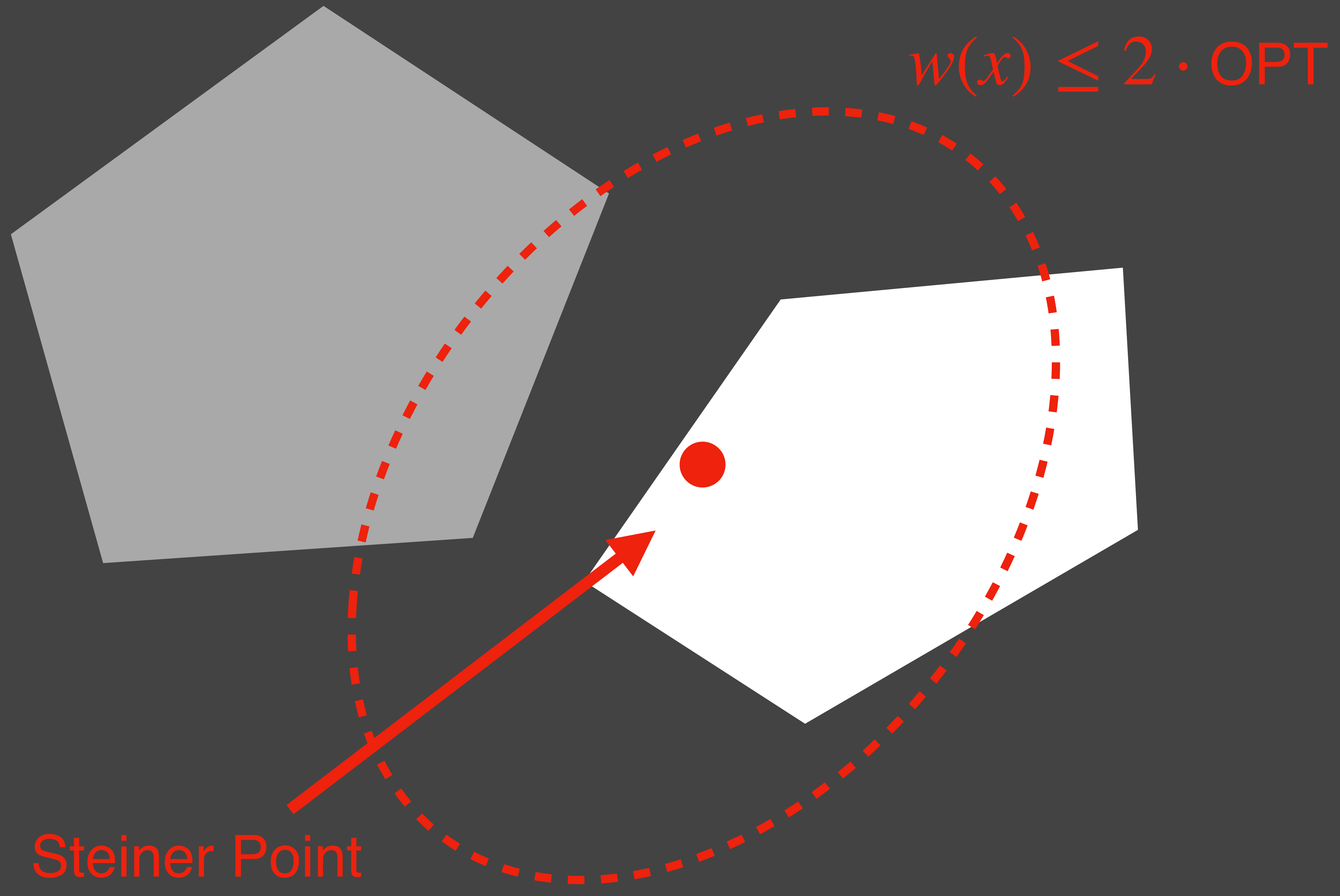

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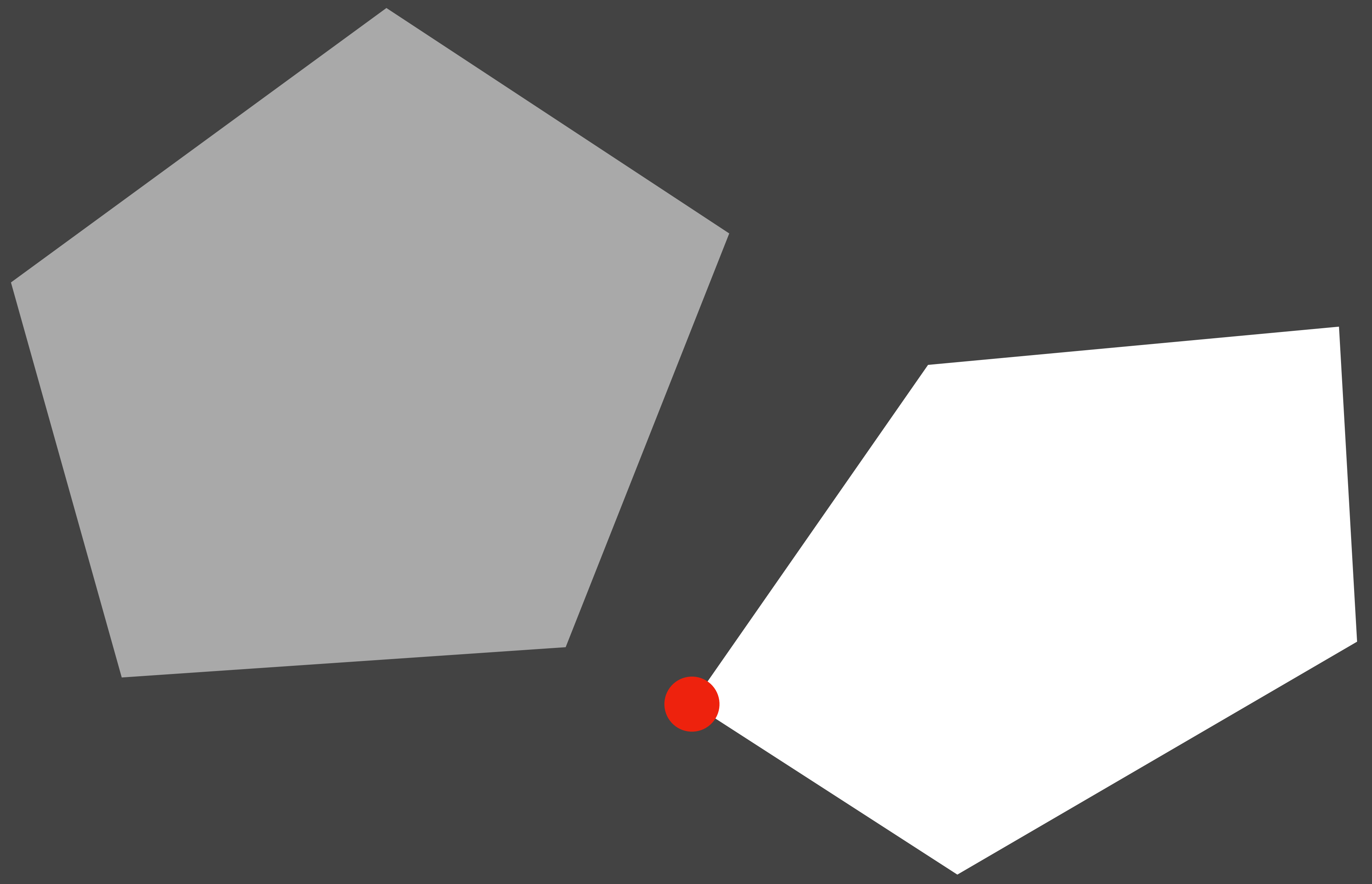

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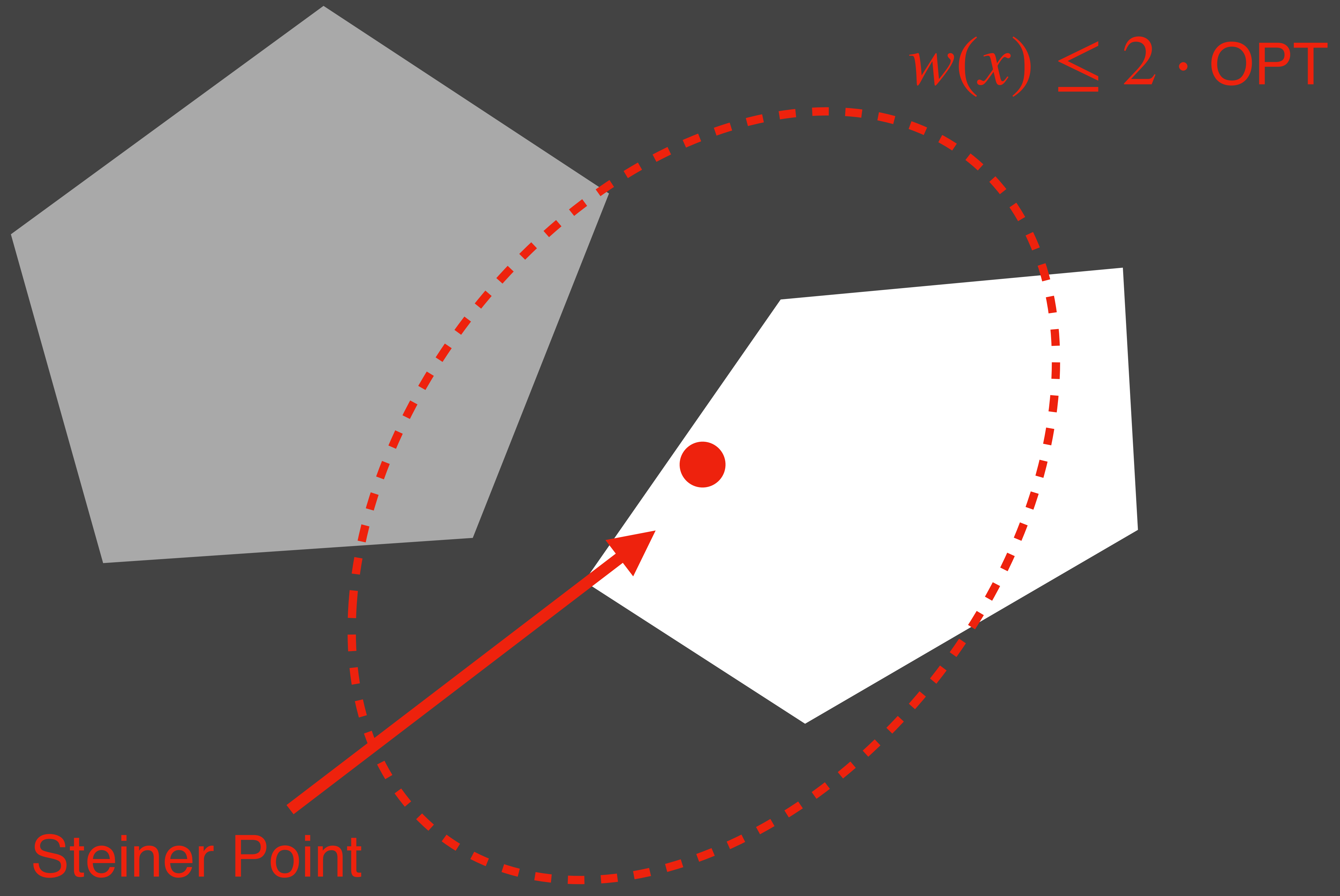

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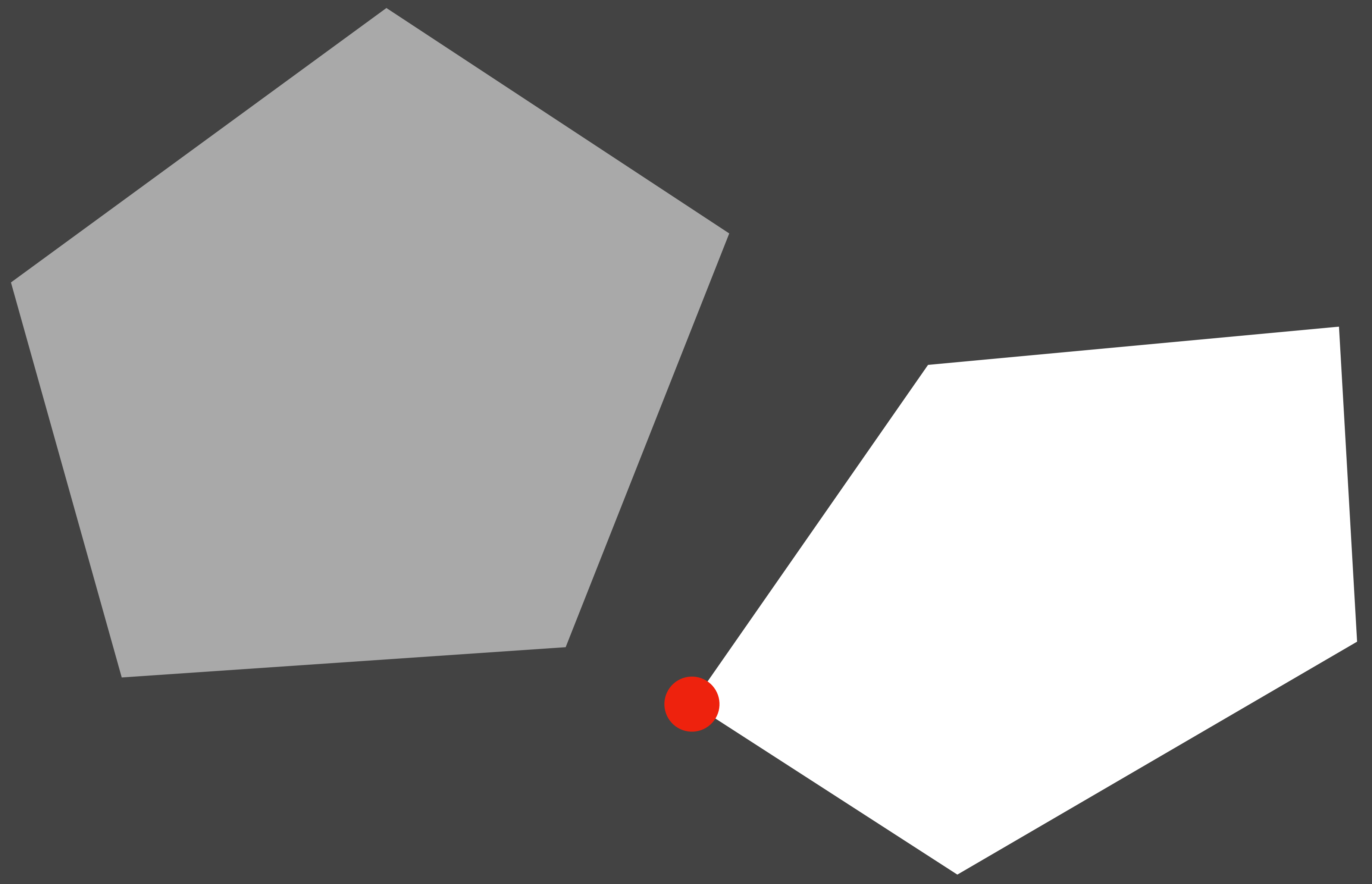

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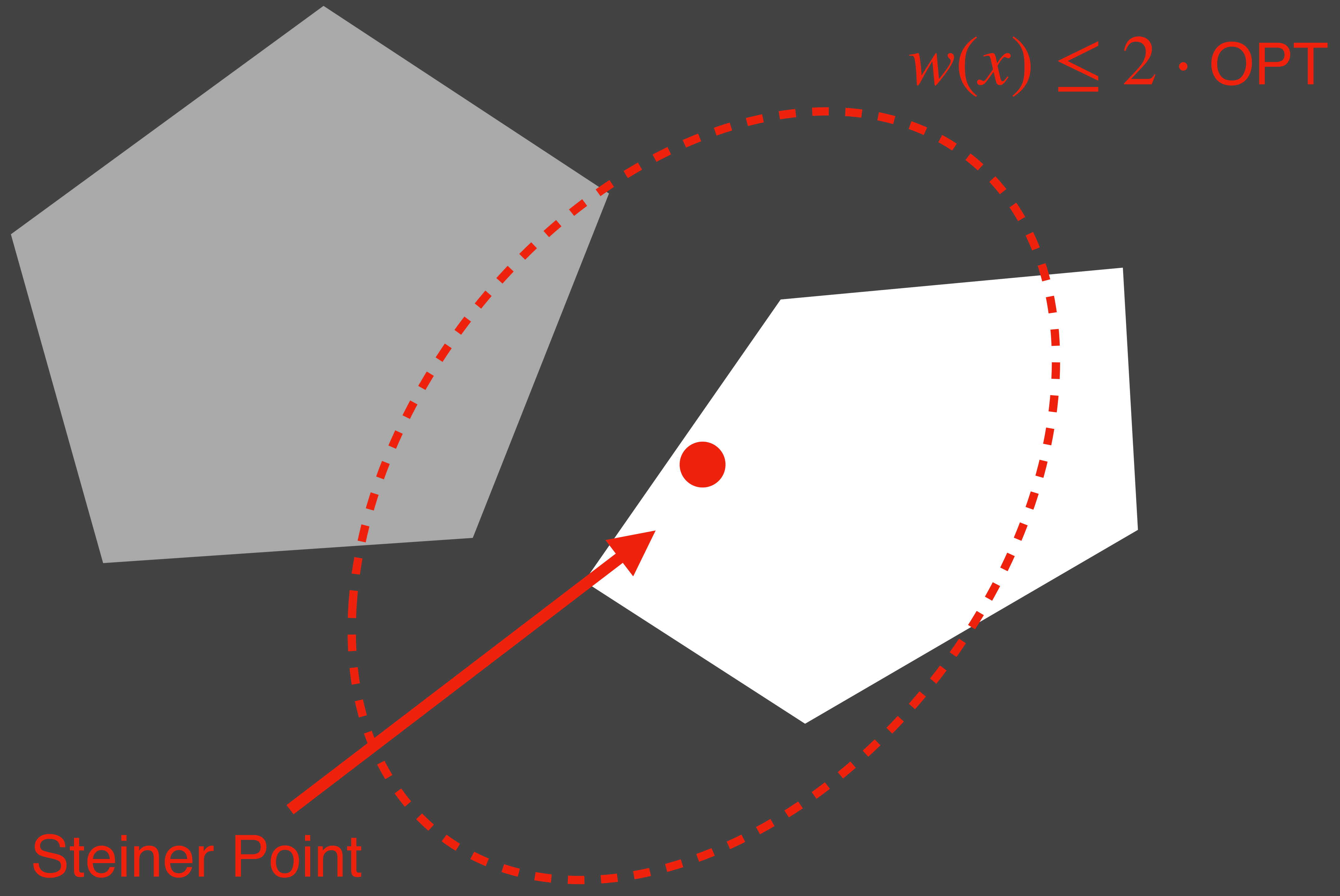

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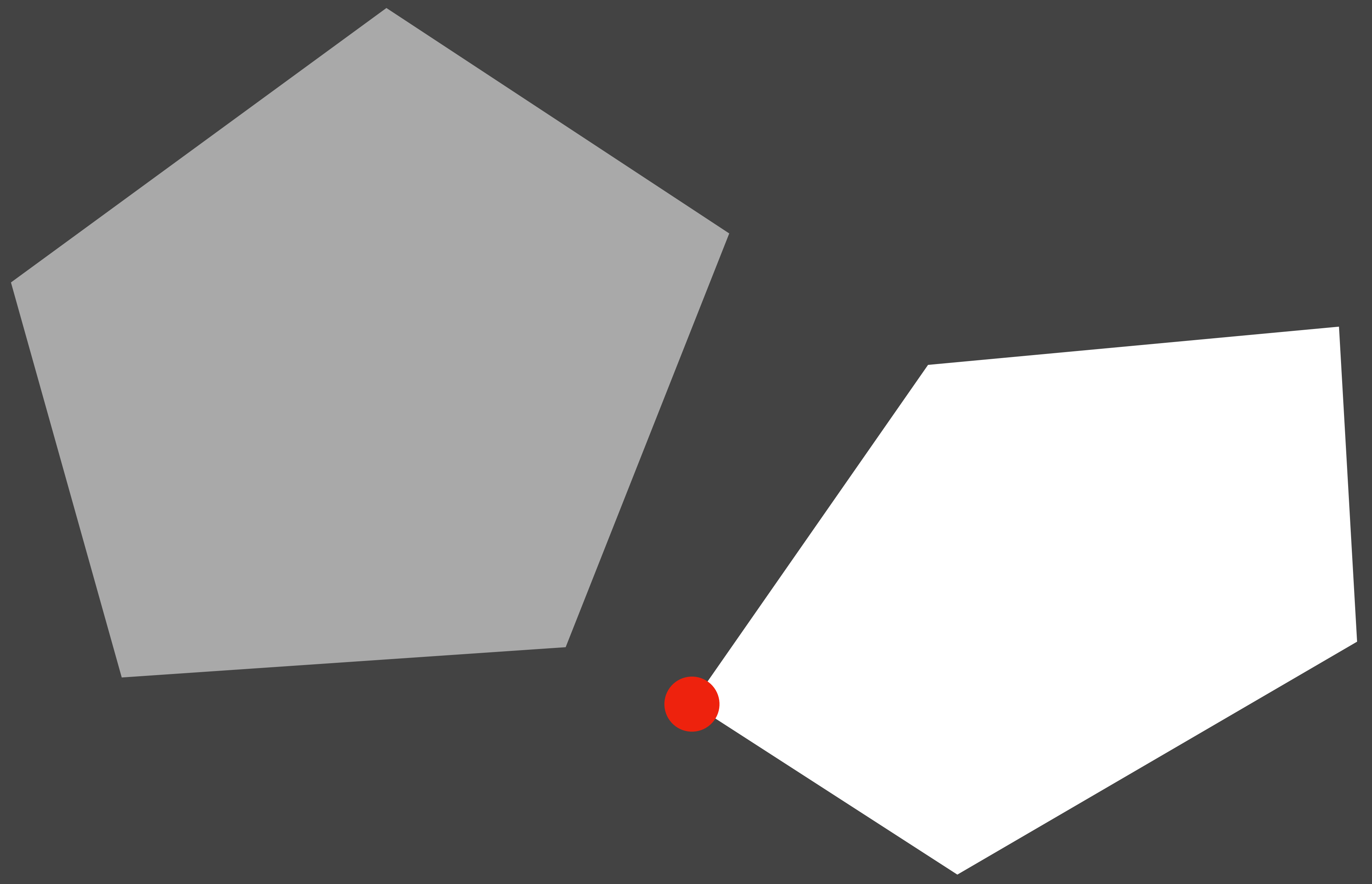

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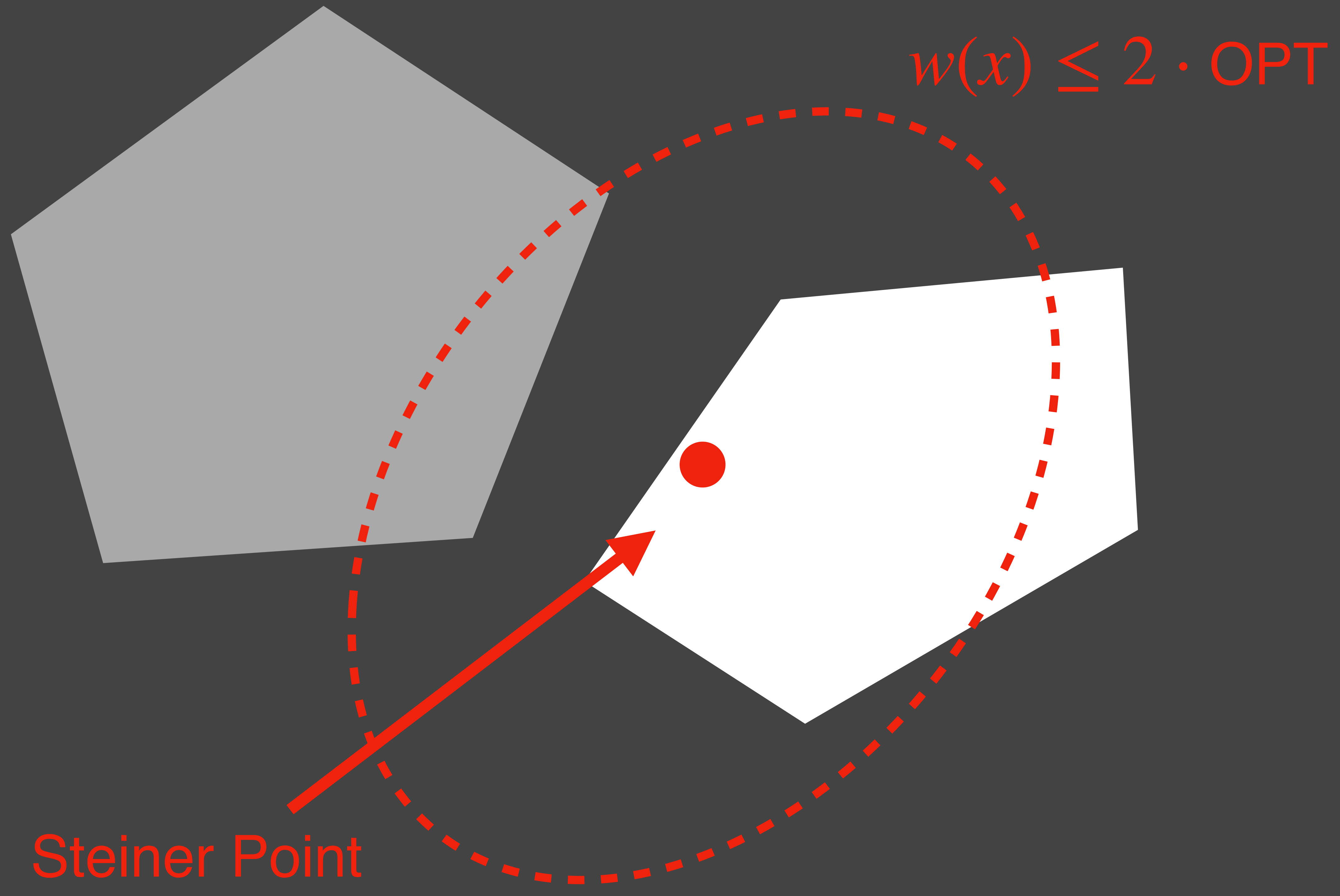

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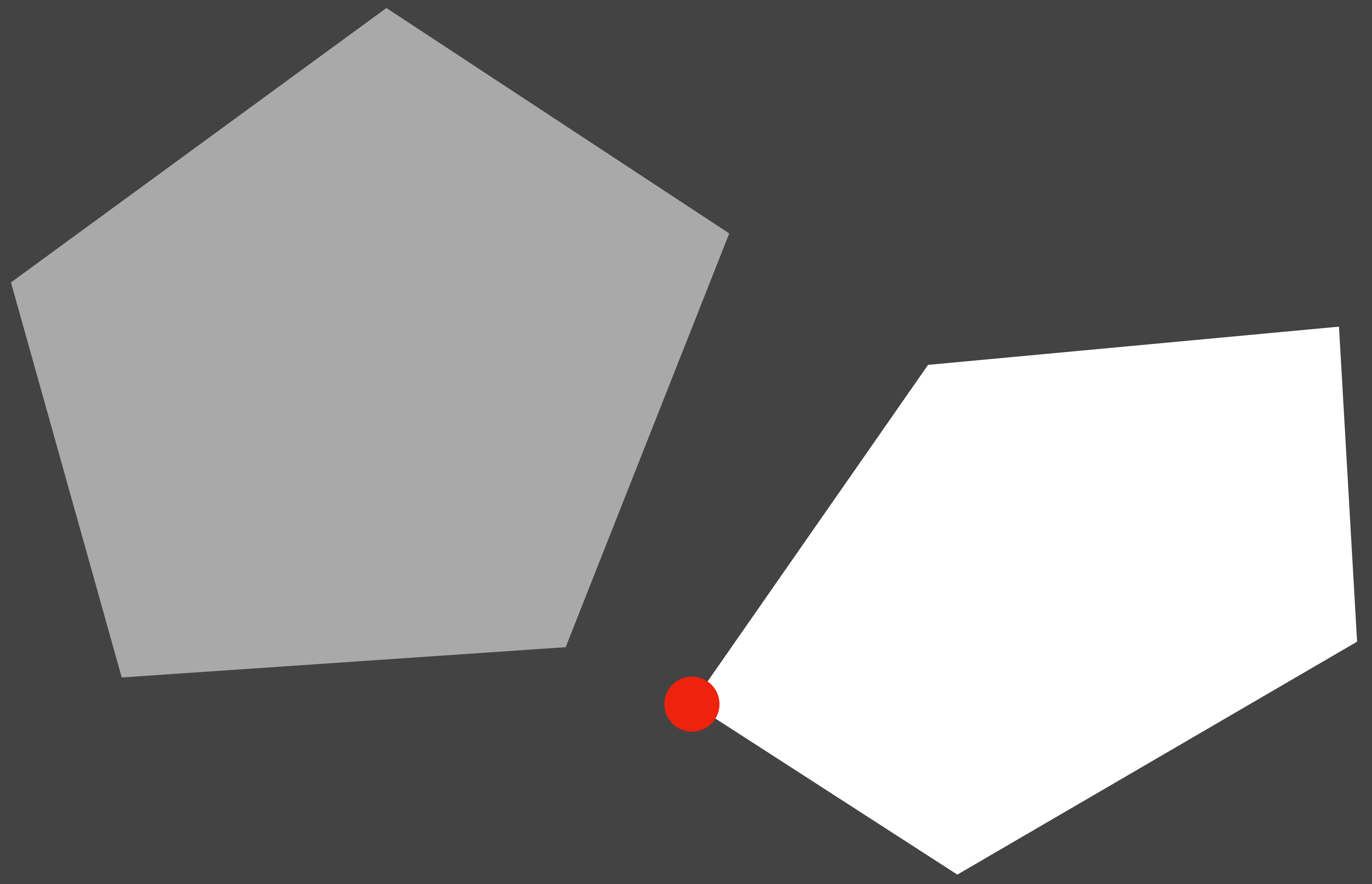

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Removing resource augmentation seems to need memory.

Improved Guarantee

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Beyond 0/1 matrices:

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Theorem [BBLS]:

Positive Body Chasing with movement $O(\log(n\Delta/\epsilon)/\epsilon) \cdot$ OPT.

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To overcome, we go back in time and modify old duals. (ALG still online!)

Rounding

Sample Application 1 : Set Cover

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$$K_t = \left\{ x \;\middle|\; \sum_{S \ni e} x_S \geq 1 \ \ \forall e \in U^t, \ \sum_{S} x_S \leq \beta \cdot \mathsf{OPT}^t \right\}$$

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$\mathrm{OPT}^t :=$ cost of best cover @ time t.

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(1) Approx $O(\log n) \cdot \beta \cdot \mathrm{OPT}^t$.

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Cost of solution is $O(\log n) \cdot \beta \cdot \mathsf{OPT}^t$, feasible w.h.p. @ every time t.

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$$K_t = \left\{ x \;\middle|\; \sum_{S \ni e} x_S \geq 1 \quad \forall e \in U^t, \; \sum_S x_S \leq \beta \cdot \mathsf{OPT}^t \right\}$$

Theorem [BBLS]:

Dynamic Set Cover with:

(1) Approx $O(f) \cdot \beta \cdot \mathsf{OPT}^t$.

(1) Recourse $O(f \log f) \cdot \mathsf{OPT}_{\mathsf{recourse}}(\beta)$.

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Matching recourse $= O(\epsilon^{-1}) \cdot$ [# edges updates to H] $= O(\epsilon^{-1}) \cdot [O(\log n) \cdot \mathrm{OPT}_{\text{recourse}}(\beta)]$.

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E.g. Can we capture clustering problems?

Remove resource augmentation?

Thanks!