Online Covering

Secretaries, Prophets, and Universal Maps

FOCS 2021 + Forthcoming Work

Roie Levin

Anupam Gupta (CMU)

Gregory Kehne (Harvard)

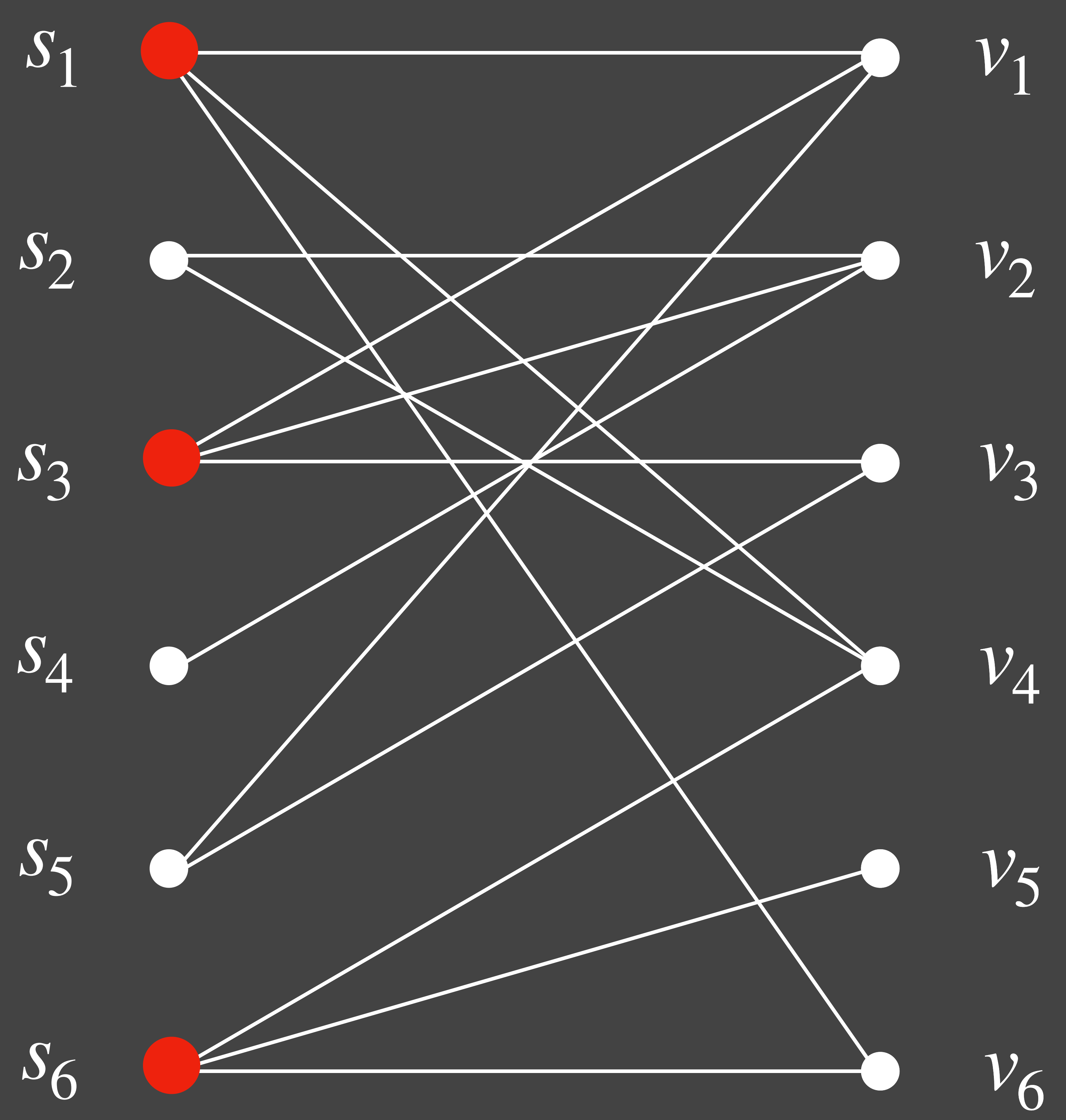
Set Cover

Apx: $\log n + 1$

[Johnson 74],[Lovasz 75], [Chvatal 79]

$\mathcal{S}$

m sets



$\mathcal{U}$

n elements

Online Set Cover

[Alon Awerbuch Azar Buchbinder Naor 03]

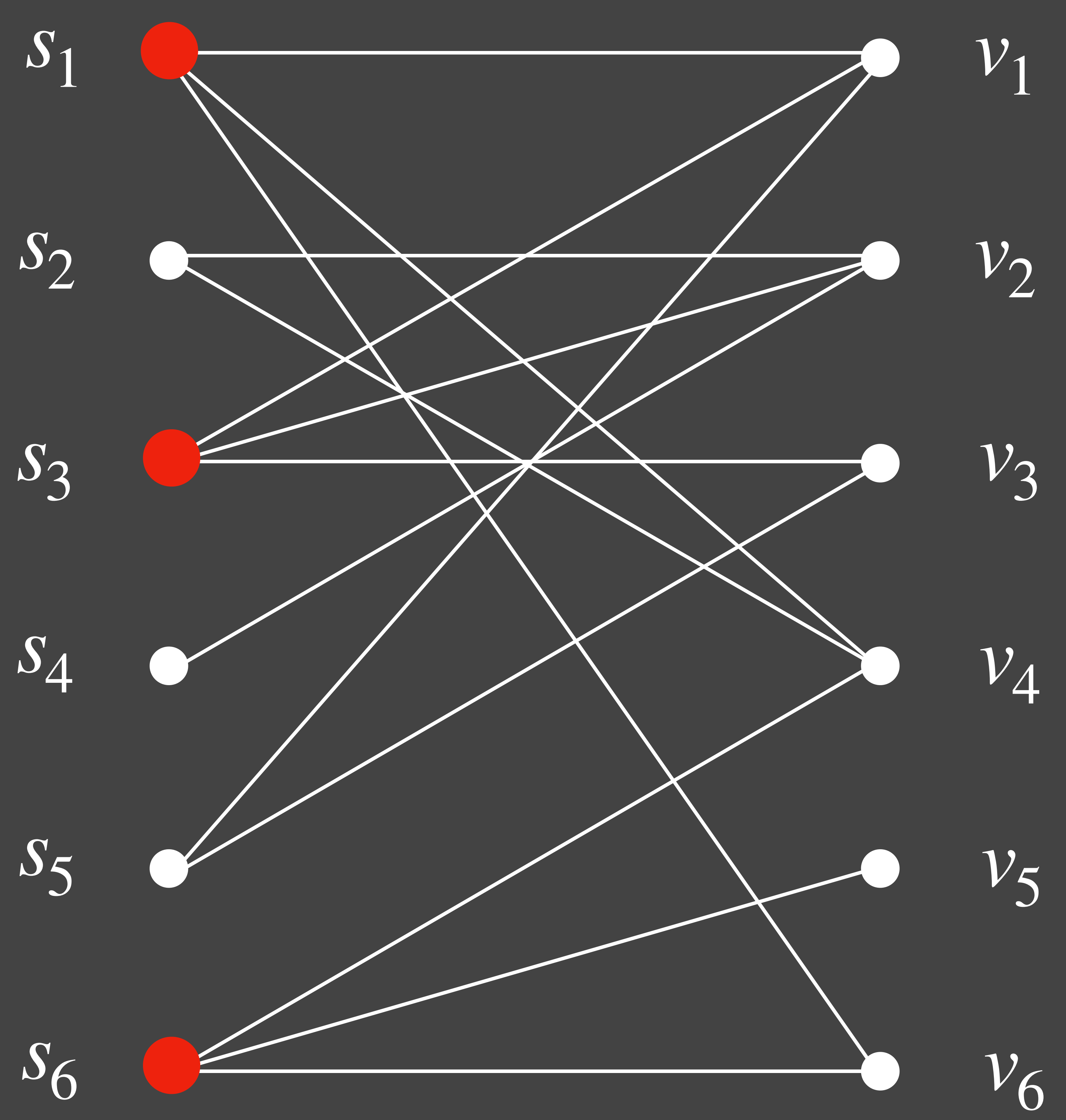
CR: $O(\log n \log m)$

[Alon+ 03]

[Buchbinder Naor 09]

$\mathcal{S}$

m sets



$\mathcal{U}$

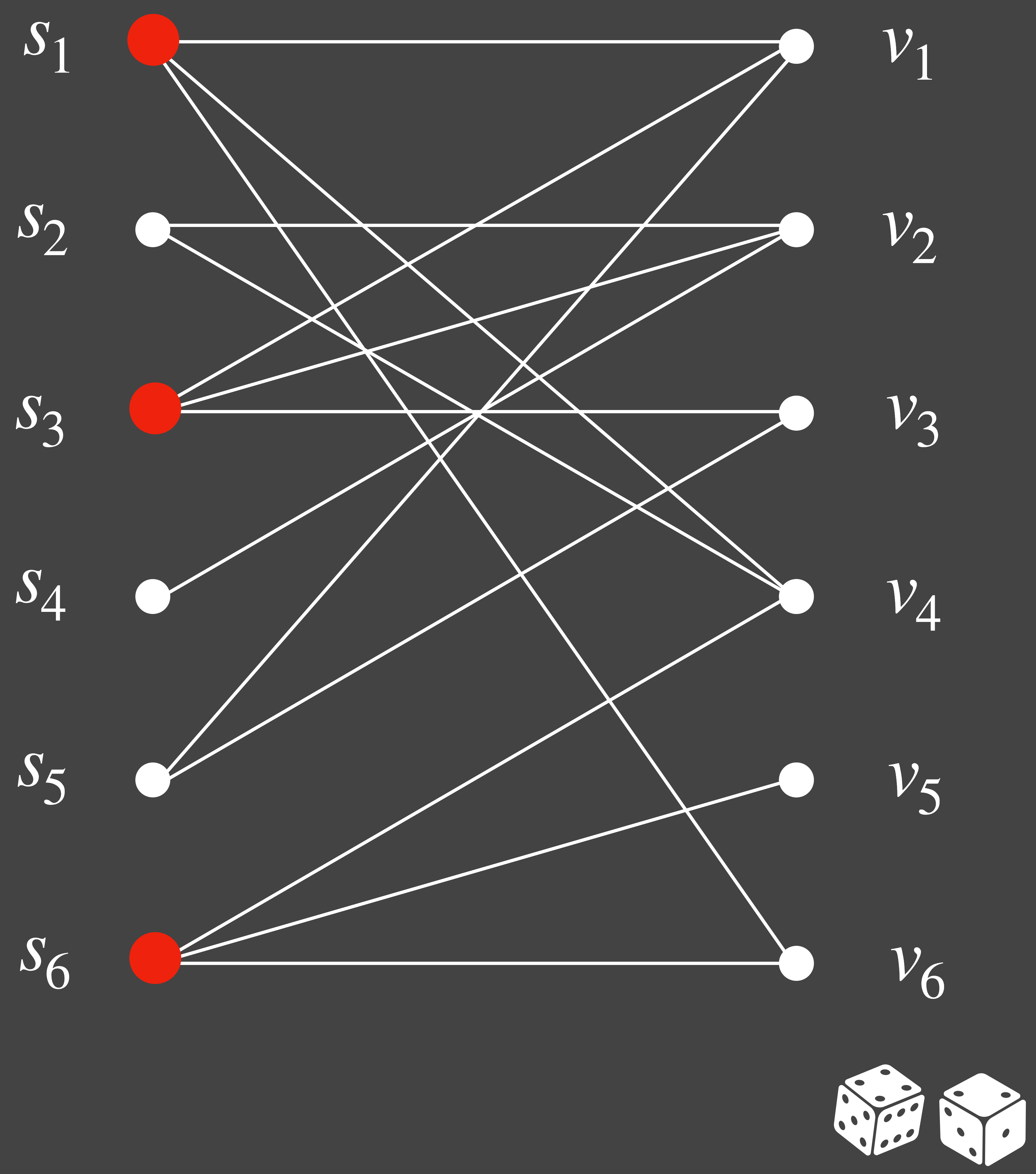
n elements

Q: What happens beyond the worst case?

Relaxation 1: Random Order (RO)

$\mathcal{S}$

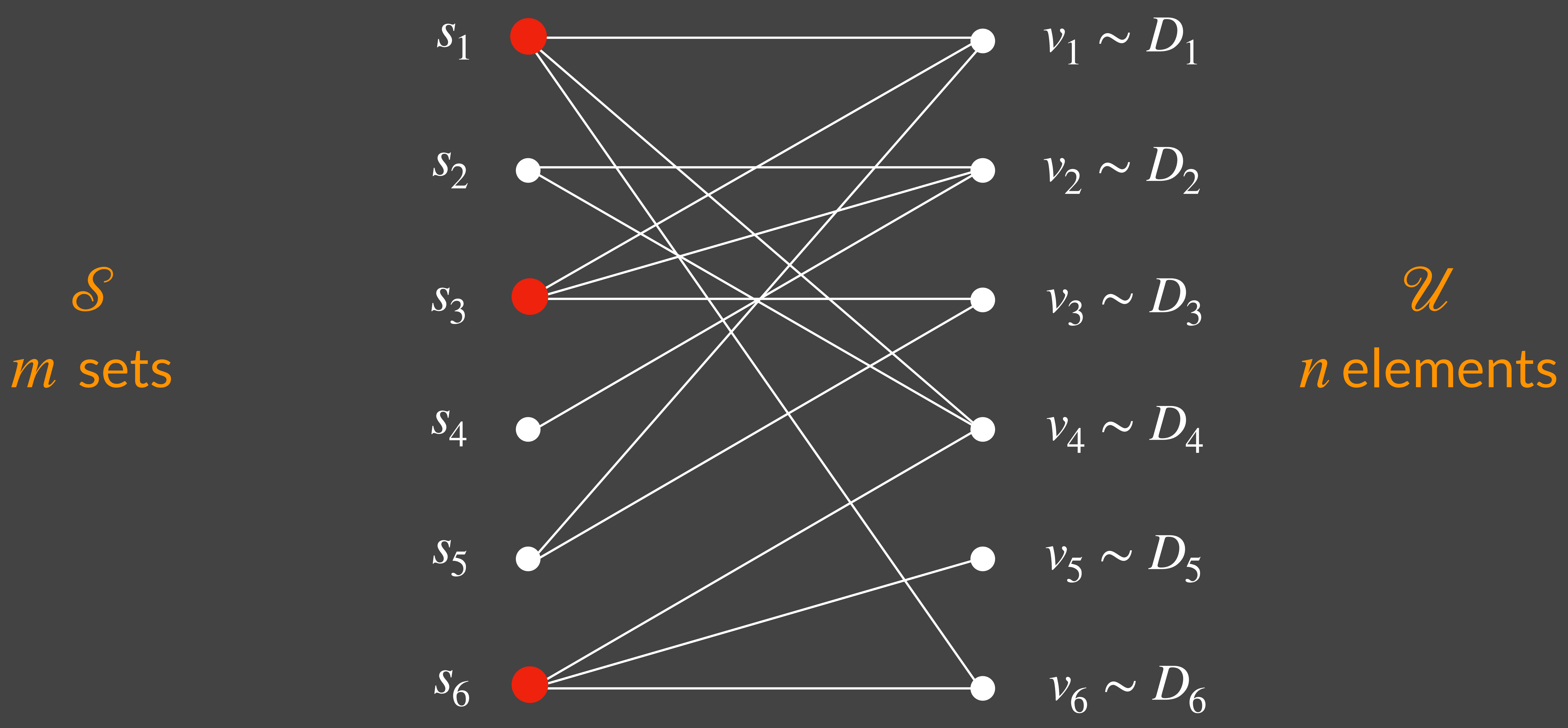
m sets



$\mathcal{U}$

n elements

Relaxation 2: Random Instance



The Landscape

$m =$ # sets
$n =$ # elements

Instance

Arrival Order	Random	Adversarial
Random	O(log(m [support size])) [Gupta Grandoni Leonardi Miettinen Sankowski Singh 08]	Secretary: O(log mn) Our work
Adversarial	Prophet: New! O(log mn) Our work	O(log n log m) [Alon+ 03] [Buchbinder Naor 09]

Theorem [Gupta Kehne **L.** FOCS 21]:

There is a poly time algorithm for ***secretary*** *Covering IPs with competitive ratio* $O(\log mn)$.

Theorem [Gupta Kehne **L.** 22]:

There is a poly time algorithm for ***prophet*** *Covering IPs with competitive ratio* $O(\log mn)$.

Bonus Properties!

1. Only need single sample from each D_i!
2. Universal! Gives sample complexity bound $O(n)$.

Talk Outline

- Intro
- Secretary
  - LearnOrCover in Exponential Time
  - LearnOrCover in Poly Time
- (Single Sample) Prophet
- Conclusion & Extensions

Set Cover via Random Rounding

2 Stage algorithm!

(I) Solve LP.

$$\min \sum_S x_S$$

$$\forall v \in \mathcal{U} : \quad \sum_{S \ni v} x_S \geq 1$$

$$\forall S \in \mathcal{S} : \quad x_S \geq 0$$

This is relaxation, so $c(x) \leq c(\text{OPT})$.

(II) Round.

Buy S with probability x_S.

Expected cost is $c(x)$!

Can show $\forall v \in \mathcal{U}$, covered with constant prob.

Repeat $O(\log n)$ times, union bound.

Expected Cost: $O(\log n) \cdot \text{OPT}$

How [Alon+ 03] works

Same 2 Stages!

(I) Solve LP Online.

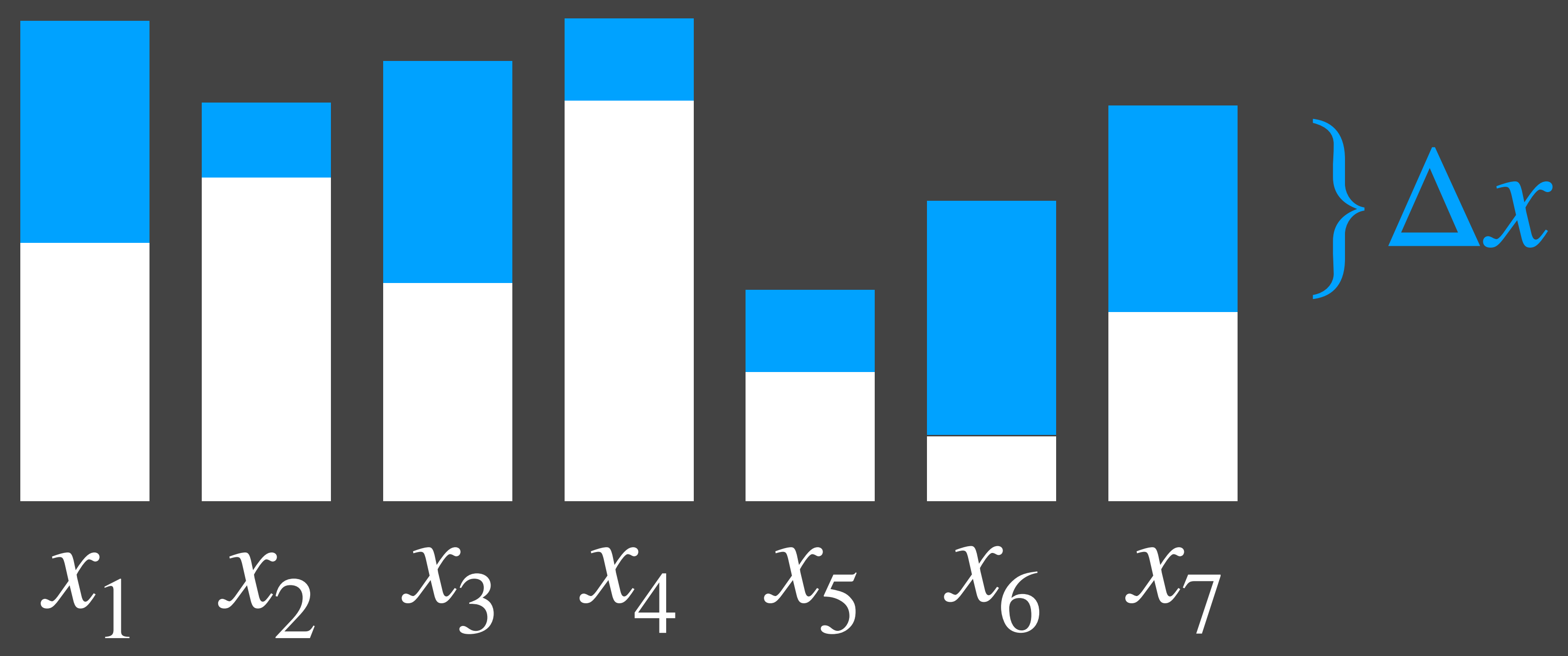
$$\min \sum_S x_S$$

$$\forall v \in \mathcal{U}: \quad \sum_{S \ni v} x_S \geq 1$$

$$\forall S \in \mathcal{S}: \quad x_S \geq 0$$

Can guarantee x is $O(\log m)$-apx, and only increases ***monotonically.***

(II) Round Online.



Take S with prob. $\propto \Delta x_S$.

Suffices to analyze ***offline*** rounding.
Repeat $\log n$ times, union bound.

Expected Cost: $O(\log n \log m) \cdot \mathrm{OPT}$

Online LP Solver of [Alon+ 03]

(P)

$$\min \sum_{S} x_S$$

$$\forall v \in \mathcal{U} : \quad \sum_{S \ni v} x_S \geq 1$$

$$\forall S \in \mathcal{S} : \quad x_S \geq 0$$

(D)

$$\max \sum_{v} y_v$$

$$\forall S \in \mathcal{S} : \quad \sum_{v \in S} y_v \leq 1$$

$$\forall v \in \mathcal{U} : \quad y_v \geq 0$$

Init $x \leftarrow 1/m$.

While v (fractionally) uncovered:

- $\times 2$ to x_S for all $S \ni v$.
- $+1$ to y_v.

Claim 1: x feasible for (**P**).

Claim 2: $c(x) \leq c(y)$

Claim 3: $y/\log m$ feasible for (**D**).

Neither stage of [Alon+ 03] can be improved!

Independent rounding loses $\Omega(\log n)$.

Theorem [Gupta Kehne **L.**]: $\Omega(\log m)$ for fractional algos in RO.

Theorem [Gupta Kehne **L.**]: algo of [Alon+ 03] gets $\Omega(\log m \log n)$ in RO.

New algorithm needed!

We maintain coarse solution x, neither feasible nor monotone,
but round x anyway…

Talk Outline

Intro

Secretary

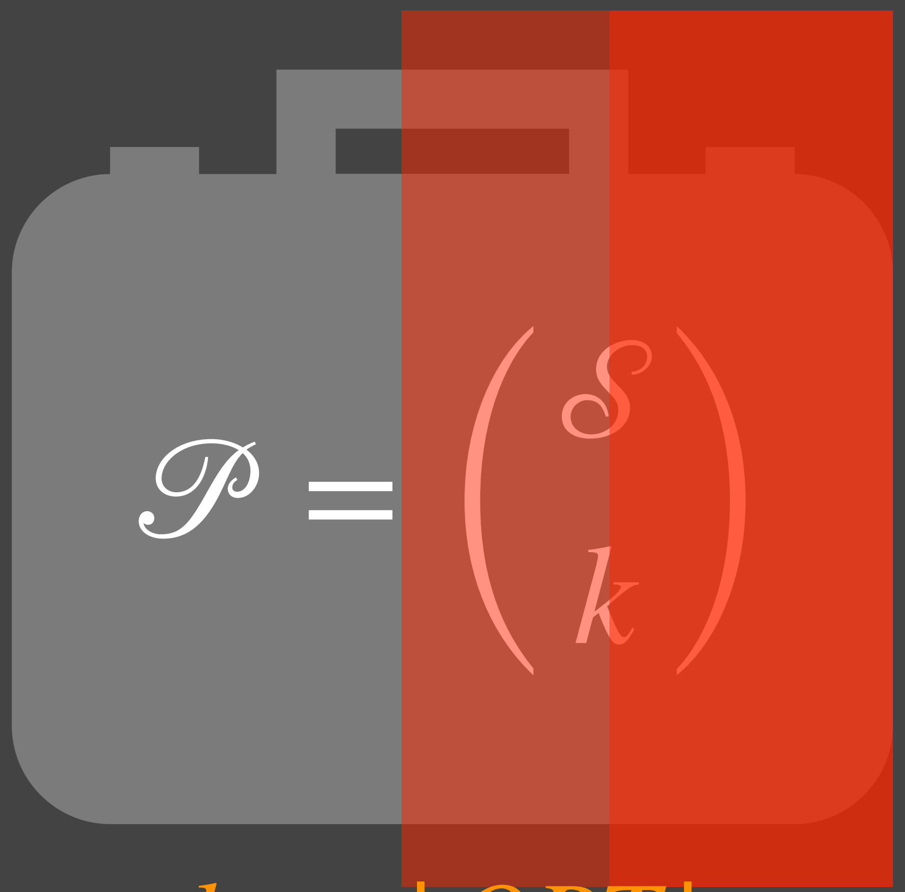
LearnOrCover in Exponential Time

LearnOrCover in Poly Time

(Single Sample) Prophet

Conclusion & Extensions

LearnOrCover
(Unit cost, exp time warmup)



$k := |OPT|$

@ time t, element v arrives:

If v covered, do nothing.

Else:

(I) choose $T \sim \mathscr{P}$, buy random $R \sim T$.

(II) "Prune" $T \not\ni v$ from $\mathscr{P}$.

Buy arbitrary set to cover v.

Case 1: $\geq 1/2$ **of** $T \in \mathscr{P}$ **cover** $\geq 1/2$ **of** $\mathscr{U}$.

R covers $\dfrac{|\mathscr{U}|}{4k}$ in expectation.

$\mathscr{U}$ shrinks by $\left(1 - \dfrac{1}{4k}\right)$ in expectation.

Case 2: $> 1/2$ **of** $T \in \mathscr{P}$ **cover** $< 1/2$ **of** $\mathscr{U}$.

$\geq 1/2$ of $T \in \mathscr{P}$ pruned w.p. $1/2$.

$\mathscr{P}$ shrinks by $3/4$ in expectation.

RO Set Cover

(Exponential Time Warmup)

Case 1: (COVER)

$\mathscr{U}$ shrinks by $\left(1 - \frac{1}{4k}\right)$ in expectation.

Case 2: (LEARN)

$\mathscr{P}$ shrinks by $3/4$ in expectation.

$|\mathscr{U}|$ initially n, $\Rightarrow$ $O(k \log n)$ COVER steps suffice.

$|\mathscr{P}|$ initially $\binom{m}{k} \approx m^k$, $\Rightarrow$ $O(k \log m)$ LEARN steps suffice.

$\Rightarrow O(k \log mn)$ steps suffice.

But how to make polytime?

Can we reuse LEARN/COVER intuition?

Talk Outline

Intro

Secretary

➡ LearnOrCover in Exponential Time

➡ LearnOrCover in Poly Time

(Single Sample) Prophet

Conclusion & Extensions

LearnOrCover
(Unit cost)

Init. $x \leftarrow 1/m$.

@ time t, element v arrives:

If v covered, do nothing.

Else:

(I) Buy random $R \sim x$.

(II) $\forall S \ni v$, set $x_S \leftarrow e \cdot x_S$.

Renormalize $x \leftarrow x/\|x\|_1$.

Buy arbitrary set to cover v.

$\sum_S x_S^* \log \frac{x_S^*}{x_S^t}$

Idea! Measure convergence with potential function:

$$\boxed{\Phi(t) = c_1 \, \mathrm{KL}(x^* \,||\, x^t) + c_2 \log |\mathscr{U}^t|}$$

$\mathscr{U}^t$:= uncovered elements @ time t

x^* := uniform distribution on OPT

Claim 1: $\Phi(0) = O(\log mn)$, and $\Phi(t) \geq 0$.

Claim 2: If v uncovered, then $E[\Delta\Phi] \leq -\frac{1}{k}$.

(Recall $k = |OPT|$)

Bound $E_R[\Delta \log |\mathscr{U}^t|]$ over randomness of R.

Bound $E_v[\Delta KL]$ over randomness of v. $\longleftarrow$ This is where we use RO!

Claim 2a: If v^t uncovered,

$$E_v[\Delta \mathrm{KL}] \le (e-1) \cdot E_v\left[\sum_{S \ni v} x_S\right] - \frac{1}{k}.$$

Claim 2b: If v^t uncovered,

$$E_R[\Delta \log |\mathcal{U}^t|] \le -E_v\left[\sum_{S \ni v} x_S\right].$$

$$E[\Delta\Phi] = E_v[\Delta \mathrm{KL}] + (e-1) \cdot E_R[\Delta \log |\mathcal{U}^t|] \le -\frac{1}{k}$$

Since $\Phi(0) = O(\log(mn))$, expected total cost is $k \log(mn)$.

Claim 2a: If v^t uncovered,

$$E_v[\Delta \mathrm{KL}] \le (e-1)\cdot E_v\left[\sum_{S\ni v} x_S\right] - \frac{1}{k}.$$

Proof:

$$\mathrm{KL}(x^* \| x^t) - \mathrm{KL}(x^* \| x^{t-1}) \quad \sum_S x^*_S \log\left(\frac{x^*_S}{x^t_S}\right) - \sum_S x^*_S \log\left(\frac{x^*_S}{x^{t-1}_S}\right)$$

$$= \sum_S x^*_S \log\left(\frac{x^{t-1}_S}{x^t_S}\right)$$

$$= \sum_S \underset{=1}{\cancel{x^*_S}} \log \|x\|_1 - \sum_{S\ni v} x^*_S \underset{=1}{\cancel{\log e}}$$

$$= \log\left(\underbrace{\sum_S x^{t-1}_S}_{=1} + \sum_{S\ni v}(e-1)\cdot x^{t-1}_S\right) - \underbrace{\sum_{S\ni v} x^*_S}_{\ge 1/k}$$

$$\le \log\left(1 + \sum_{S\ni v}(e-1)\cdot x_S\right) - \frac{1}{k}.$$

Use $\log(1+z) \le z$, take expectation over v, ■.

Claim 2b: If v^t uncovered,

$$E_R[\Delta \log |\mathcal{U}^t|] \le -E_v\left[\sum_{S\ni v} x_S\right].$$

Proof:

$$\log|\mathcal{U}^t| - \log|\mathcal{U}^{t-1}|$$

$$= \log\left(1 - \frac{|\mathcal{U}^{t-1}| - |\mathcal{U}^t|}{|\mathcal{U}^{t-1}|}\right)$$

Use $\log(1-z) \le -z$.

$$\le -\frac{1}{|\mathcal{U}^{t-1}|}\sum_{v\in\mathcal{U}^{t-1}} \mathbb{1}\{R\ni v\}.$$

Take expectation over R.

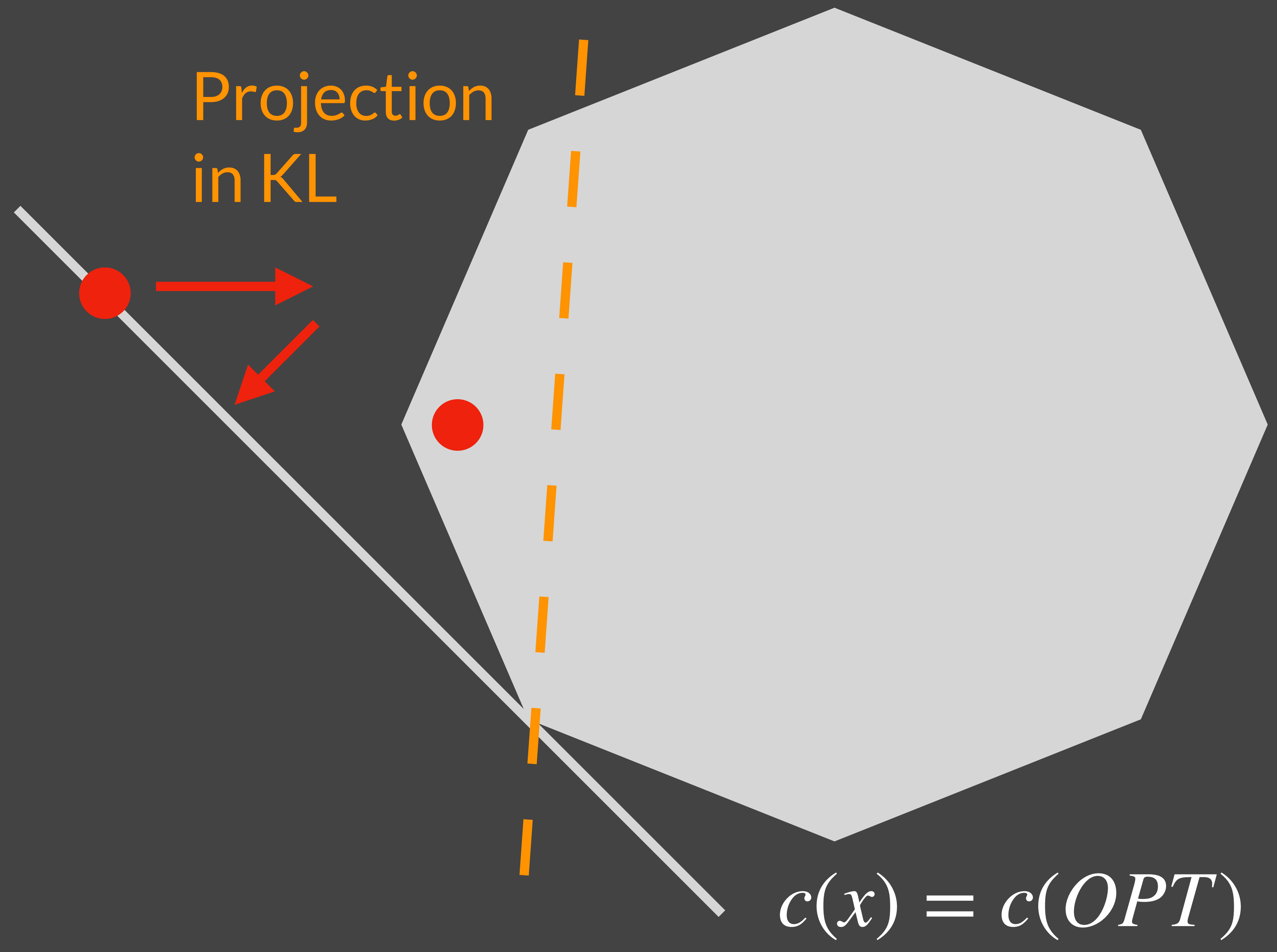
$$E_R[\Delta \log|\mathcal{U}^t|] \le -\frac{1}{|\mathcal{U}^{t-1}|}\sum_R x_R \sum_{v\in\mathcal{U}^{t-1}} \mathbb{1}\{R\ni v\}$$

$$= -\frac{1}{|\mathcal{U}^{t-1}|}\sum_{v\in\mathcal{U}^{t-1}}\sum_{R\ni v} x_R. \quad ■$$

LearnOrCover

(Some philosophy)

Perspective 1:



[Alon+ 03]
LearnOrCover
[Buchbinder Gupta Molinaro Naor 19]

Perspective 2:

Define

$$f(x) := \sum_{v} \max\left(0, 1 - \sum_{S \ni v} x_S\right)$$

(Goal is to minimize f in smallest # of steps)

$$\nabla f|_S(x) = \text{\# uncovered elements in } S$$
$$\propto E[\mathbb{1}\{v \in S \mid v \text{ uncovered}\}]$$

RO reveals stochastic gradient…

Talk Outline

Intro

Secretary

LearnOrCover in Exponential Time

➔ LearnOrCover in Poly Time

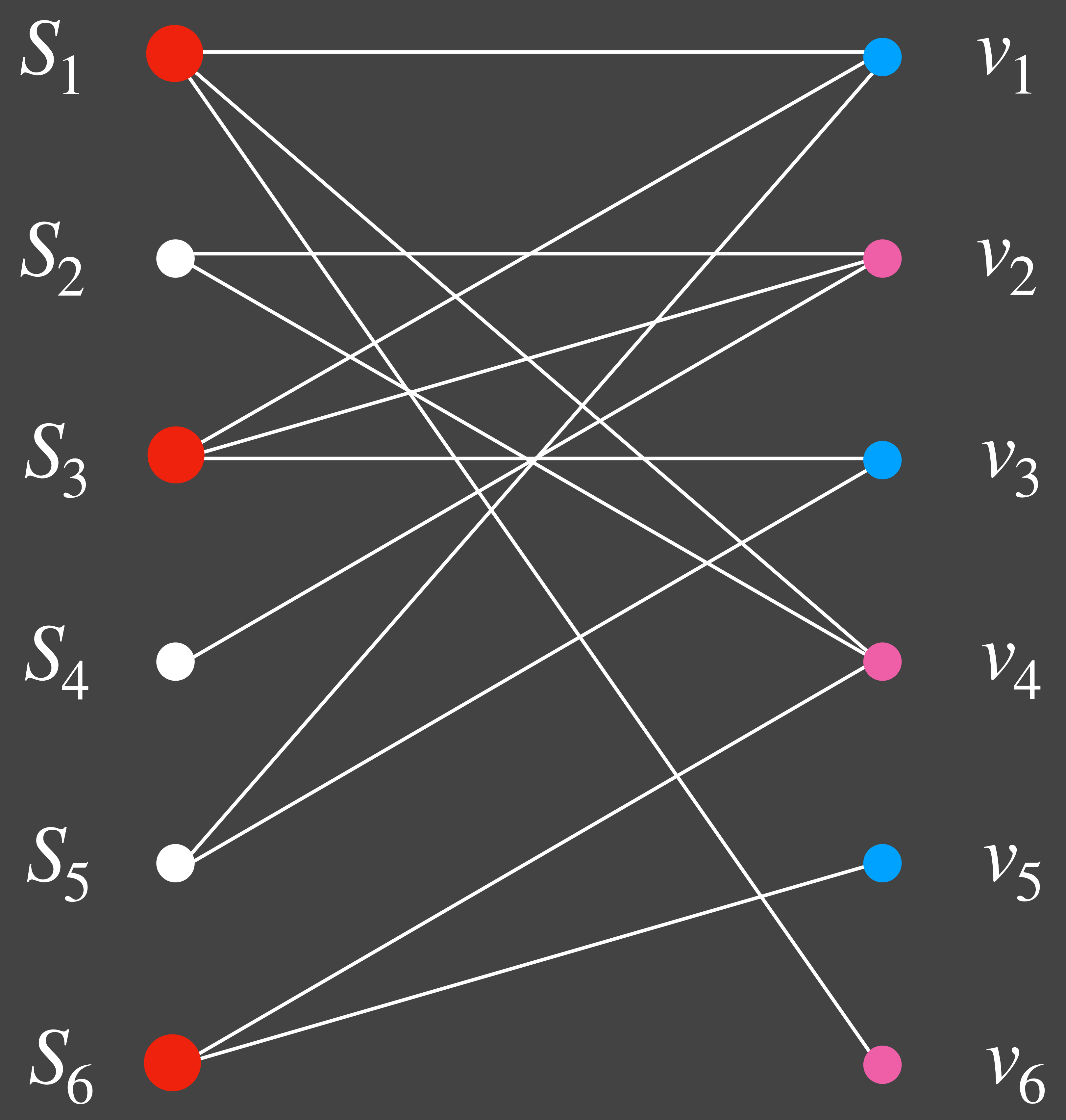
➔ (Single Sample) Prophet

Conclusion & Extensions

Special Case: the With-a-Sample model

Online set cover, but random 1/2 of elements known upfront (see [Kaplan Naori Raz 21]).

Remaining fraction revealed in adversarial order.

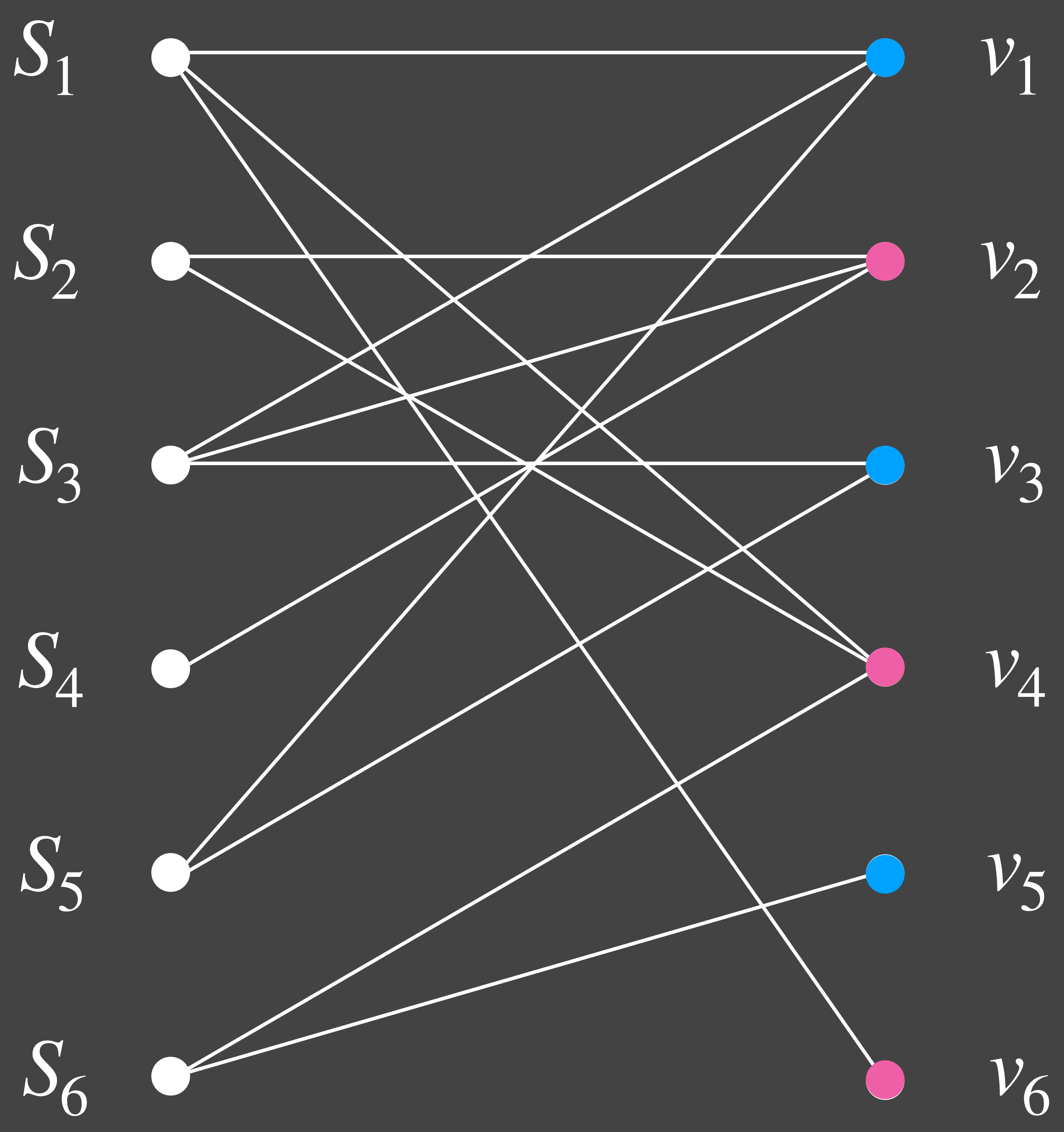


Theorem:

*There is a poly time algorithm for Online **Set** Cover **With-a-Sample** with competitive ratio* $O(\log(mn))$.

Reduction to LearnOrCover!

Idea:
1. Run LearnOrCover on samples.
2. Buy arbitrary sets for remaining elements.



Pretend colored pink (sampled)/blue (adversarial) on arrival.

@ time t:

If v^t pink, feed to LearnOrCover.

If v^t blue, buy arbitrary set to cover.

Recall LearnOrCover proof template:

Claim 1: $\Phi(0) = O(\log mn)$, and $\Phi(t) \geq 0$.

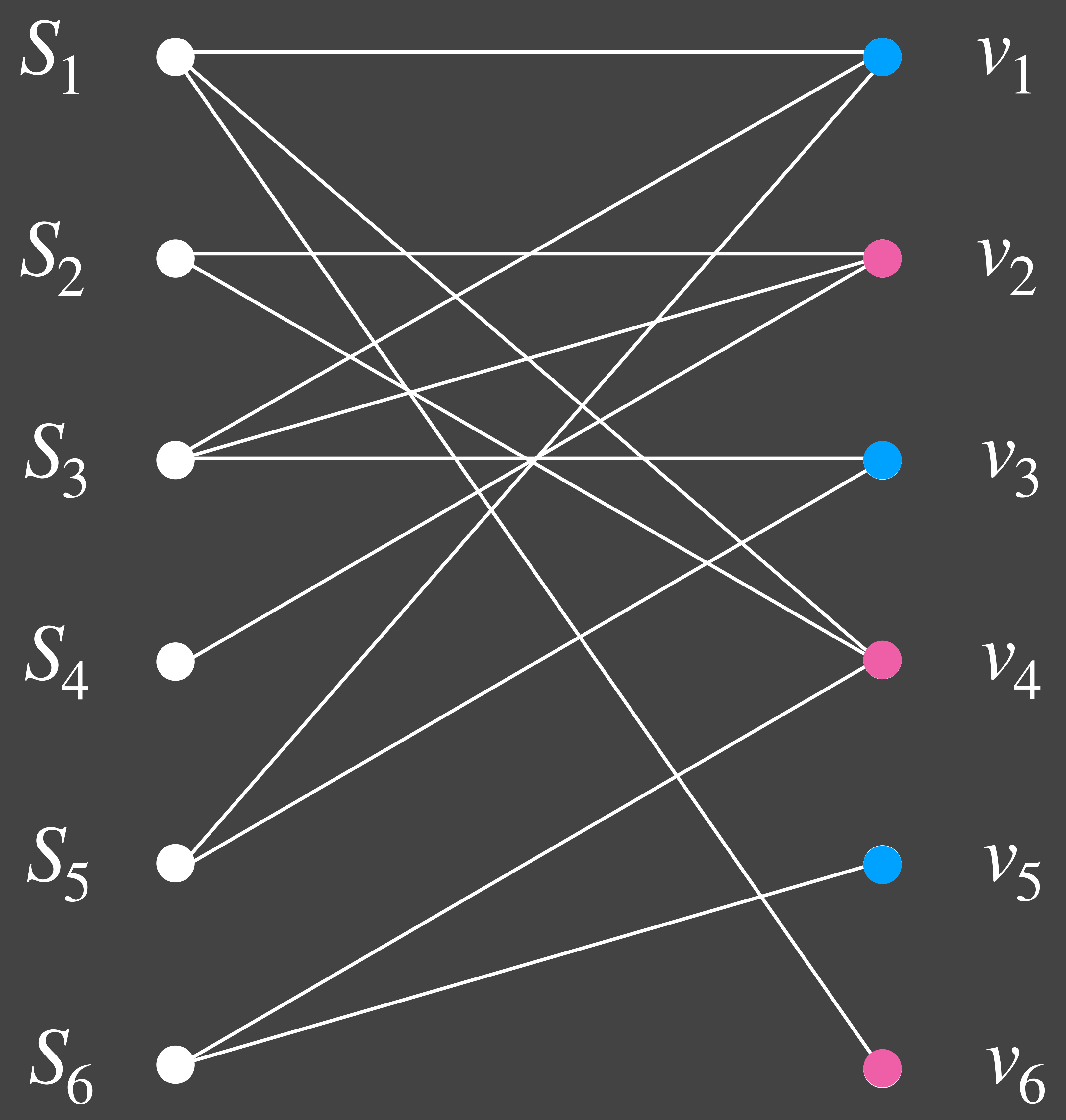
Claim 2: If v^t uncovered, then $E[\Delta\Phi] \leq -\Omega\left(\frac{1}{k}\right)$.

Φ only deceases during pink steps (so with prob. 1/2), but still $E[\Delta\Phi] \leq -\Omega\left(\frac{1}{k}\right)$.

Universality

<u>Idea</u>: Reduction!

1. Run LearnOrCover on samples.
2. Buy arbitrary sets for remaining elements.



Can build map $f : \mathscr{U} \to \mathscr{S}$ ***<u>before</u>*** we see any actual elements.

When $u \in \mathscr{U}$ arrives, commit to buying $f(u)$!

Our result shows only need $O(n)$ samples to build this map.

Talk Outline

Intro

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LearnOrCover in Exponential Time

LearnOrCover in Poly Time

➔ (Single Sample) Prophet

➔ Conclusion & Extensions

Conclusion

Theorem: $O(\log mn)$*-comp. algo for RO Covering IPs.*

Theorem: $O(\log mn)$*-comp. algo for Prophet Covering IPs.*

+ Single-Sample!
+ Universal!

Theorem: Same results *for Non-metric facility location.*

Theorem: $\Omega(\log m \log f(\mathcal{N}))$ for RO submodular cover.

Open Questions:

Does the LearnOrCover idea lend itself to other problems?
Harder covering problems? Covering IPs w/ box constraints?
Unified theory? Reinterpret old RO results as LearnOrCover?

Thanks!

Backup Slides

Online Covering IPs

$$\min \quad c^\intercal x$$

$$a_1^\intercal x \geq 1$$
$$a_2^\intercal x \geq 1$$
$$a_3^\intercal x \geq 1$$
$$a_4^\intercal x \geq 1$$
$$a_5^\intercal x \geq 1$$

$$x \in \mathbb{Z}_{\geq 0}^m$$

Goal: Maintain feasible solution x that is *monotonically* increasing.

Set Cover is the special case where constraint matrix A is 0/1.

LearnOrCover for non-unit costs

Main issue: # uncovered elements not good proxy for cost.

(Assuming WLOG $c(OPT) = 1$)

κ_v := cost of cheapest set covering v

Main Idea: tune learning & sampling rates as a function of κ_v.

LearnOrCover

Init. $x_S \leftarrow 1/(c_S \cdot m)$.

@ time t, element v arrives:

If v covered, do nothing.

Else:

(I) Buy every set R w.p. $\kappa_v x_R$.

(II) $\forall S \ni v$, set $x_S \leftarrow e^{\kappa_v / c_S} \cdot x_S$.

Renormalize $x \leftarrow x / \langle c, x \rangle$.

Buy cheapest set to cover v.

Claim 1: $\Phi(0) = c(\mathrm{OPT}) \cdot O(\log mn)$, and $\Phi(t) \geq 0$.

Claim 2: $E[\Delta\Phi] = -\Omega(\kappa_v)$.

Claim 3: $E[\Delta \mathrm{cost(ALG)}] = O(\kappa_v)$.

$$\Rightarrow \quad E[\Delta\Phi + \Delta \mathrm{cost(ALG)}] = 0.$$

$$E[\mathrm{cost(ALG)}] \leq \Phi(0).$$