Carnegie Mellon University

Submodular Optimization Under Uncertainty

Online, Dynamic and Streaming Algorithms

Roie Levin

Committee: Anupam Gupta, R. Ravi, David Woodruff, Chandra Chekuri, Seffi Naor

Intro

Classical Approximation Algorithms

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Classical Approximation Algorithms

Cost

Classical Approximation Algorithms

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Unrealistic to expect full/perfect information!

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Algorithms Under Uncertainty

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Interesting when movement is restricted…

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Thesis studies 3 restrictions:

Online — monotone solution

Dynamic — low movement

Streaming — low memory

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This talk

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This Talk: Submodular Cover [Wolsey 82]

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This talk:

f integer valued, all costs are 1.

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[Goyal+ 13][Loukides Gwadera 16][Zheng+ 17][Andreev+ 09][Lee+ 13][Lukovszki+ 18][Poularakis+ 17][Krause+ 08][Korsarz Nutov 15][Jorgensen+ 17][Chen+ 18][Beinhofer+ 13][Tzoumas+ 16][Tong+ 17][Liu+ 16][Mafuta Walingo 16][Yang+ 15][Rahimian Preciado 15][Izumi+ 10][Wu+ 15], etc…

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Porting submod cover to uncertain settings automatically ports all applications!

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Optimal in poly time, unless P=NP [Feige 98][Dinur Steurer 14].

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Sweet spot between generality and tractability!

Online/Dynamic Submodular Cover

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This talk: $f_i(\mathcal{N}) = f(\mathcal{N})$, all same.

My PhD Work

Online	Dynamic	Streaming
The Online Submodular Cover Problem [Gupta, L., SODA 20]	Fully-Dynamic Submodular Cover with Bounded Recourse [Gupta, L., FOCS 20]	Streaming Submodular Matching Meets the Primal Dual Method [L., Wajc, SODA 21]
Random Order Set Cover is as Easy as Offline [Gupta, Kehne, L., FOCS 21]		Robust Subspace Approximation in a Stream [L., Sevekari, Woodruff, NeurIPS 18]
Competitive Algorithms for Block-Aware Caching [Coester, Naor, L., Talmon, SPAA 22] New!		

… and Offline

Finding Skewed Subcubes Under a Distribution [Gopalan, L., Wieder ITCS 20]

$\in$ Thesis

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Talk Outline

- Intro
- Part I — Online/Dynamic Submodular Cover
- Part II — Application: Block-Aware Caching
- Part III — Random Order Online Set Cover
- Conclusion

Talk Outline

Intro

➡ Part I — Online/Dynamic Submodular Cover

Part II — Application: Block-Aware Caching

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Part I — Online/Dynamic Submodular Cover

with Anupam Gupta

Online Submodular Cover

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Decisions are irrevocable!!

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$$F = \sum_i f_i$$

Decisions are irrevocable!!

S can only grow over time…

Special Case: Online Set Cover

[Alon Awerbuch Azar Buchbinder Naor 03]

s_1 •

s_2 •

s_3 •

s_4 •

s_5 •

s_6 •

Special Case: Online Set Cover

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$$f_i(S) = \begin{cases} 1 & \text{if } v_i \text{ covered by } S \\ 0 & \text{otherwise} \end{cases}$$

Special Case: Online Set Cover

[Alon Awerbuch Azar Buchbinder Naor 03]

$$f_i(S) = \begin{cases} 1 & \text{if } v_i \text{ covered by } S \\ 0 & \text{otherwise} \end{cases}$$

$$F = \sum_i f_i = \text{\# elements covered}$$

Online Submodular Cover Results

$$F = \sum_i f_i$$

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Theorem [Gupta L. SODA20]:

*There is a **randomized poly time** algo for **Online Submod Cover** with expected competitive ratio:*

$$O(\log m \cdot \log F(\mathcal{N})).$$

Special Case: Online Set Cover

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Theorem (Online):

$O(\log m \cdot \log F(\mathcal{N}))$.

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Generalizes [Alon+ 03]

Fully-Dynamic Submodular Cover

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<u>Definition</u>: Recourse

$$\sum_t |S^t \triangle S^{t-1}|$$

Fully-Dynamic Submodular Cover

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Theorem [Gupta **L.** FOCS 20]:

*There is a **deterministic poly time** algorithm for **Fully-Dynamic Submodular** Cover with:*

(i) *competitive ratio* $O(\log F(\mathcal{N}))$.

(ii) *average recourse* $\tilde{O}(f(\mathcal{N}))$.

Special Case: Dynamic Set Cover

$$f_i(S) = \begin{cases} 1 & \text{if } v_i \text{ covered by } S \\ 0 & \text{otherwise} \end{cases}$$

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Special Case: Dynamic Set Cover

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Theorem (Dynamic):

(i) *competitive ratio* $O(\log F(\mathcal{N}))$.

(ii) *average recourse* $O(f(\mathcal{N}))$.

Special Case: Dynamic Set Cover

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Generalizes [Gupta Kumar Krishnaswamy Panigrahi 17]

Recap so far

Part I: Online

- Inserts Only
- Decisions are **irrevocable**

Part II: Fully- Dynamic

- Inserts + Deletes
- Want minimum # edits, a.k.a. **recourse.**

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Modeling power of Submod Cover + robustness to uncertainty of Online/Dynamic algos.

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➡ Part I — Online/Dynamic Submodular Cover

Part II — Application: Block-Aware Caching

Part III — Random Order Online Set Cover

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Part II — Application: Block-Aware Caching

with Christian Coester, Seffi Naor, Ohad Talmon

Classic Caching

Cache of size k

n total pages

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Goal is to minimize number of evictions!

Block-Aware Caching [Beckmann+ 2021]

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n total pages, **in blocks of size β**

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n total pages, **in blocks of size β**

Goal is to minimize number of **blocks** evicted!

Results [Coester, Naor L., Talmon, SPAA 22]

New!

	Classic	Block-Aware
Offline	1	
Deterministic Online	k	
Randomized Online	O(log k)	

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New!

	Classic	Block-Aware
Offline	1	β
Deterministic Online	k	βk
Randomized Online	$O(\log k)$	$O(\beta \log k)$

Trivial!

Results [Coester, Naor L., Talmon, SPAA 22]

New!

	Classic	Block-Aware
Offline	1	O(log k)
Deterministic Online	k	k
Randomized Online	O(log k)	$O(\log^2 k)$

Our Result

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Our Result

Also show $\Omega(\beta)$ lower bound for randomized algorithms in **fetching cost model…**

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New!

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Offline	1	O(log k)
Deterministic Online	k	k
Randomized Online	O(log k)	$O(\log^2 k)$

Our Result

Also show $\Omega(\beta)$ lower bound for randomized algorithms in **fetching cost model…**

… separation of eviction/ fetching cost models!

What does this have to do with Submodular Cover?

$n = 9, \quad k = 4$

What does this have to do with Submodular Cover?

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Reduces overflow at time τ by ___.

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What does this have to do with Submodular Cover?

Reduces overflow at time τ by __1__.

$n = 9, \quad k = 4$

What does this have to do with Submodular Cover?

Reduces overflow at time τ by __2__.

$n = 9, \quad k = 4$

What does this have to do with Submodular Cover?

Reduces overflow at time τ by 3.

$n = 9, \quad k = 4$

What does this have to do with Submodular Cover?

Reduces overflow at time τ by 5.

$n = 9, \quad k = 4$

What does this have to do with Submodular Cover?

Reduces overflow at time τ by 5.

$f^\tau :=$ "reduction in overflow at time τ" is submodular!

$n = 9, \quad k = 4$

Formulation as Submodular Cover

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$$\min_{S} |S|$$

$$\forall \tau : \quad f^{\tau}(S) \geq n - k$$

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This is an instance of Online Submodular Cover!

Bounds from Part I <u>too weak</u>, depend on total time T.

We show our bounds via finer analysis… but reuse some ideas!

Talk Outline

Intro

Part I — Online/Dynamic Submodular Cover

➡ Part II — Application: Block-Aware Caching

Part III — Random Order Online Set Cover

Conclusion

Talk Outline

Intro

Part I — Online/Dynamic Submodular Cover

Part II — Application: Block-Aware Caching

➡ Part III — Random Order Online Set Cover

Conclusion

Part III — Random Order Online Set Cover

with Anupam Gupta and Gregory Kehne

Random Order (RO) Online Set Cover

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m sets

$\mathcal{U}$

n elements

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What is known?

Offline	log n + 1 [Johnson74],[Lovasz75], [Chvatal79]
Adversarial Online	O(log n log m) [Alon+03] [BuchbinderNaor09]
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Believable $o(\log n \log m)$ not possible...

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Theorem [Gupta Kehne **L.** 21]:

*There is a **randomized poly time** algorithm for RO **Online Set** Cover with competitive ratio* $O(\log mn)$.

New algorithm! We show how to learn distribution & solve at same time.

RO Set Cover

(Exponential Time Warmup)

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$k \approx |OPT|$

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$\Rightarrow R$ covers $\dfrac{|\mathscr{U}|}{4k}$ in expectation.

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RO Set Cover
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Case 1: (COVER)

$\mathscr{U}$ shrinks by $\left(1 - \frac{1}{4k}\right)$ in expectation.

Case 2: (LEARN)

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$|\mathscr{U}|$ initially n, $\Rightarrow$ $O(k \log n)$ COVER steps suffice.

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But how to make polytime?

Can we reuse LEARN/COVER intuition?

LearnOrCover

(Unit cost)

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Init. $x \leftarrow 1/m$.

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@ time t, element v arrives:

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Renormalize $x \leftarrow x/\|x\|_1$.

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Idea! Measure convergence with potential function:

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Idea! Measure convergence with potential function:

$$\Phi(t) = c_1 \, \mathrm{KL}(x^* \,||\, x^t) + c_2 \log |\mathcal{U}^t|$$

$\mathcal{U}^t$:= uncovered elements @ time t

x^* := uniform distribution on OPT

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Claim 2: If v uncovered, then $E[\Delta\Phi] \leq -\frac{1}{k}$.

(Recall $k = |OPT|$)

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Bound $E_v[\Delta KL]$ over randomness of v.

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Bound $E_v[\Delta KL]$ over randomness of v. $\longleftarrow$ This is where we use RO!

Claim 2a: If v^t uncovered,

$$E_v[\Delta \mathrm{KL}] \leq (e-1) \cdot E_v\left[\sum_{S \ni v} x_S\right] - \frac{1}{k}.$$

Claim 2b: If v^t uncovered,

$$E_R[\Delta \log |\mathcal{U}^t|] \leq -E_v\left[\sum_{S \ni v} x_S\right].$$

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$$E[\Delta\Phi] = E_v[\Delta \mathrm{KL}] + (e-1) \cdot E_R[\Delta \log|\mathcal{U}^t|] \leq -\frac{1}{k}$$

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Since $\Phi(0) = O(\log(mn))$, expected total cost is $k \log(mn)$.

Claim 2a: If v^t uncovered,

$$E_v[\Delta \mathrm{KL}] \leq (e-1) \cdot E_v\left[\sum_{S \ni v} x_S\right] - \frac{1}{k}.$$

Proof:

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Claim 2a: If v^t uncovered,

$$E_v[\Delta \mathrm{KL}] \leq (e-1) \cdot E_v\left[\sum_{S \ni v} x_S\right] - \frac{1}{k}.$$

Proof:

$$\mathbf{KL}(x^* \,||\, x^t) - \mathbf{KL}(x^* \,||\, x^{t-1})$$

Claim 2b: If v^t uncovered,

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Proof:

$$\sum_S x_S^* \log\left(\frac{x_S^*}{x_S^t}\right) - \sum_S x_S^* \log\left(\frac{x_S^*}{x_S^{t-1}}\right)$$

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Use $\log(1+z) \leq z$, take expectation over v, ■.

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Proof:

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$$\log|\mathcal{U}^t| - \log|\mathcal{U}^{t-1}|$$

$$= \log\left(1 - \frac{|\mathcal{U}^{t-1}| - |\mathcal{U}^t|}{|\mathcal{U}^{t-1}|}\right)$$

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Claim 2b: If v^t uncovered,

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Proof:

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Claim 2b: If v^t uncovered,

$$E_R[\Delta \log|\mathcal{U}^t|] \leq -E_v\left[\sum_{S \ni v} x_S\right].$$

Proof:

$$\log|\mathcal{U}^t| - \log|\mathcal{U}^{t-1}|$$

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$$E_R[\Delta \log|\mathcal{U}^t|] \leq -E_v\left[\sum_{S \ni v} x_S\right].$$

Proof:

$$\log|\mathcal{U}^t| - \log|\mathcal{U}^{t-1}|$$

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Take expectation over R.

$$E_R[\Delta \log|\mathcal{U}^t|] \leq -\frac{1}{|\mathcal{U}^{t-1}|} \sum_R x_R \sum_{v \in \mathcal{U}^{t-1}} \mathbb{1}\{R \ni v\}$$

$$= -\frac{1}{|\mathcal{U}^{t-1}|} \sum_{v \in \mathcal{U}^{t-1}} \sum_{R \ni v} x_R. \quad ■$$

Extensions & Lower bounds

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Recall, in Part I [Gupta L. 20], we show $O(\log m \log(n \cdot f(\mathcal{N})))$ for adversarial order.

Online Set Cover With-a-Sample

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Online set cover, but random 1/2 of elements known upfront (see [Kaplan Naori Raz 21]).

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S_2 •

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Theorem:

*There is a **randomized poly time** algorithm for **Online Set** Cover With-a-Sample with competitive ratio* $O(\log(mn))$.

Reduction to LearnOrCover!

S_1 •

S_2 •

S_3 •

S_4 •

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Part I — Online/Dynamic Submodular Cover

Part II — Application: Block-Aware Caching

➡ Part III — Random Order Online Set Cover

Conclusion

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My Amazing Collaborators (so far!)

My Family

Thanks!