Vectorisation example : Atomic decay via Spontaneous Emission

Worked example : JC Hamiltonian

- Recall :

$$H = \frac{\hbar\omega_1}{2}\hat{\sigma}_z + \hbar\omega_2\left(a^{\dagger}a + \frac{1}{2}\right) + \hbar G\left(a\sigma_{eg} + a^{\dagger}\sigma_{ge}\right)$$

$$|\psi(0)\rangle = |e,0\rangle \longrightarrow \rho(0) = |e,0\rangle\langle e,0|$$

- At the beginning of the project, we worked out :

$$\hat{H} = \hbar \begin{array}{c|cc} & \langle g,n+1| & \langle e,n| \\ \hline |g,n+1\rangle & \omega_2\left(n+\frac{5}{2}\right) - \frac{\omega_1}{2} & G\sqrt{n+1} \\ |e,n\rangle & \cancel{\hbar} G\sqrt{n+1} & \cancel{\hbar}\omega_2\left(n+\frac{1}{2}\right) + \frac{\omega_1}{2} \end{array}$$

- Simplifications : $\omega_1 = \omega_2 = \omega$, $n = 0$ and $\hbar = 1$:

$$\hat{H} = \begin{bmatrix} \omega & G \\ G & \omega \end{bmatrix}$$

- For our jump superoperator, $\hat{L}$, we choose the spontaneous emission :

$$\hat{L} = \gamma|g\rangle\langle e| = \gamma|g\rangle \otimes \mathbb{1}\langle e| \otimes \mathbb{1}^{\dagger}$$

- Lastly, we say that :

$$|\dot{\rho}(t)\rangle\rangle = \hat{\mathcal{L}}|\rho(t)\rangle\rangle \longrightarrow \begin{bmatrix} \dot{\rho}_{g1,g1}(t) \\ \dot{\rho}_{g1,e0}(t) \\ \dot{\rho}_{e0,g1}(t) \\ \dot{\rho}_{e0,e0}(t) \end{bmatrix} = \begin{bmatrix} \\ \hat{\mathcal{L}} \\ \\ \end{bmatrix} \begin{bmatrix} \rho_{g1,g1}(t) \\ \rho_{g1,e0}(t) \\ \rho_{e0,g1}(t) \\ \rho_{e0,e0}(t) \end{bmatrix}$$

where :

$$\hat{\mathcal{L}} = -i\left(\mathbb{1}\otimes H - H^{T}\otimes\mathbb{1}\right) + \cancel{\hbar}\left[\left(L^{*}\otimes L - \frac{1}{2}\left(\mathbb{1}\otimes L^{\dagger}L\right) - \frac{1}{2}\left(L^{\dagger}L\right)^{T}\otimes\mathbb{1}\right]$$

· So, we need to calculate the components of $\hat{\mathcal{L}}$ in order to get a full matrix representation.

<u>Calculation.</u>

$$\cdot\ \mathbb{1}\otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \omega & a \\ a & \omega \end{pmatrix} = \begin{bmatrix} \omega & a & 0 & 0 \\ a & \omega & 0 & 0 \\ 0 & 0 & \omega & a \\ 0 & 0 & a & \omega \end{bmatrix}$$

$$\cdot\ H^T \otimes \mathbb{1} = \begin{pmatrix} \omega & a \\ a & \omega \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} \omega & 0 & a & 0 \\ 0 & \omega & 0 & a \\ a & 0 & \omega & 0 \\ 0 & a & 0 & \omega \end{bmatrix}$$

$\therefore$

$$-i(\mathbb{1}\otimes H - H^T \otimes \mathbb{1}) = -i \begin{bmatrix} 0 & a & -a & 0 \\ a & 0 & 0 & -a \\ -a & 0 & 0 & a \\ 0 & -a & a & 0 \end{bmatrix}$$

· For L, let's use matrix representation, being careful of basis.

$$L = \gamma \begin{array}{c|c} \begin{matrix} \langle g| \otimes \mathbb{1}^\dagger & \langle e| \otimes \mathbb{1} \end{matrix} & \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{matrix} |g\rangle \\ |e\rangle \end{matrix} \end{array}$$

$\Rightarrow$ note that, as shown previously, L action on QHO subsys is identity $\mathbb{1}$.

$$L = \gamma \cdot \begin{array}{cc} \begin{matrix} \langle g,1| & \langle e,0| \end{matrix} & \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{matrix} |g,1\rangle \\ |e,0\rangle \end{matrix} \end{array}$$

· Term by term calculation:

$$L^* \otimes L = |\gamma|^2 \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- $\mathbb{1}\otimes L^{\dagger}L$: $L^{\dagger}L = \begin{bmatrix}0&0\\ \phi&0\end{bmatrix}\cdot\begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&1\end{bmatrix}|\gamma|^2$

$$\Rightarrow \mathbb{1}\otimes\begin{bmatrix}0&0\\0&1\end{bmatrix}|\gamma|^2 = |\gamma|^2\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

- $(L^{\dagger}L)^{T}\otimes\mathbb{1} = |\gamma|^2\begin{bmatrix}0&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix} = |\gamma|^2\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&\cancel{0}\,1&0\\0&0&0&1\end{bmatrix}$

- Now, sum the jump-operator terms:

$$\hat{\mathcal{L}}_L = L^{*}\otimes L - \frac{1}{2}\mathbb{1}\otimes L^{\dagger}L - \frac{1}{2}(L^{\dagger}L)^{T}\otimes\mathbb{1}$$

$$= |\gamma|^2\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} - |\gamma|^2\begin{bmatrix}0&0&0&0\\0&\frac{1}{2}&0&0\\0&0&0&0\\0&0&0&\frac{1}{2}\end{bmatrix} - |\gamma|^2\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&\frac{1}{2}&0\\0&0&0&\frac{1}{2}\end{bmatrix}$$

$$= |\gamma|^2\begin{bmatrix}0&0&0&1\\0&-\frac{1}{2}&0&0\\0&0&-\frac{1}{2}&0\\0&0&0&-1\end{bmatrix}$$

- Finally add $\hat{\mathcal{L}}_H + \hat{\mathcal{L}}_L = \hat{\mathcal{L}}$

$$\hat{\mathcal{L}}|\rho\rangle\rangle = \begin{bmatrix}0&-i\Omega&i\Omega&|\gamma|^2\\-i\Omega&-\frac{1}{2}|\gamma|^2&0&i\Omega\\i\Omega&0&-\frac{1}{2}|\gamma|^2&-i\Omega\\0&i\Omega&-i\Omega&-|\gamma|^2\end{bmatrix}\begin{bmatrix}\rho_{g1,g1}(t)\\ \rho_{g1,e0}(t)\\ \rho_{e0,g1}(t)\\ \rho_{e0,e0}(t)\end{bmatrix} = \begin{bmatrix}\dot\rho_{g1,g1}(t)\\ \dot\rho_{g1,e0}(t)\\ \dot\rho_{e0,g1}(t)\\ \dot\rho_{e0,e0}(t)\end{bmatrix}$$

- We now have four differential equations to solve, with initial condition $\rho(0) = |e,0\rangle\langle e,0|$.