Quantum Harmonic Oscillator and 2LS Composite System: Eigenvalues and Eigenstates

- Here, we will follow the steps below to determine the eigenvalues and states of the composite system of a Quantum Harmonic Oscillator and 2 Level System (2LS).

    1. Define Hamiltonians for 2LS, QHO and the coupled interaction. Combine into total Hamiltonian.

    2. Combine all into a matrix using expectation values.

    3. Calculate eigenvalues of matrix.

    4. Calculate eigenstates.

    5. Diagonalize the matrix.

1. Hamiltonian Definitions

- The total Hamiltonian will take the form:

$$\hat{H} = \hat{H}_{2LS} + \hat{H}_0 + \hat{H}_{int}$$

⇒ Let's index for ease, setting 2LS → 1, 0 → 2.

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12}$$

- $\hat{H}_1 = \hat{H}_{2LS} = \frac{\hbar \omega_1}{2} \hat{\sigma}_z$

where $\hbar = \frac{h}{2\pi}$

ω_1 is the frequency, such that it influences energy of the 2LS.

$\hat{\sigma}_z$ is the Pauli z-matrix, $\hat{\sigma}_z = |1\rangle\langle 1| - |0\rangle\langle 0|$

- $\hat{H}_2 = \hbar\omega_2 \left(a^\dagger a + \frac{1}{2}\right)$ where ω_2 is the angular frequency of the QHO and influences energy of that system.

  $a^\dagger$ is the raising operator;

  $a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$

  a is the lowering operator

  $a|n\rangle = \sqrt{n}\,|n-1\rangle$

- Note that $\hat{H}_2$ is written sometimes as $\hbar\omega_2 a^\dagger a$, and omits the $1/2$. This is sometimes done to simplify calculations. However, we will use the full version as above.

- $\hat{H}_3 = \hbar G (a\sigma_{10} + a^\dagger \sigma_{01})$ where G is the measures coupling of system.

  $\sigma_{10} = |1\rangle\langle 0|$

  $\sigma_{01} = |0\rangle\langle 1|$

- $\hat{H} = \frac{\hbar\omega_1}{2}\hat{\sigma}_z + \hbar\omega_2\left(a^\dagger a + \frac{1}{2}\right) + \hbar G (a\sigma_{10} + a^\dagger \sigma_{01})$ //

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2. Matrix Representation of a Hamiltonian

- The total composite system $|\psi\rangle$ will be a tensor product of the 2LS state $|\psi_1\rangle$ and the QHO state $|\psi_2\rangle$:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

- For a 2LS state, the basis states are $|g\rangle$ and $|e\rangle$ (denoting ground and excited states respectively).

  - Let us relabel $\hat{\sigma}_z, \hat{\sigma}_{01}, \hat{\sigma}_{10}$ to match the 2LS basis state labels:

$$\begin{aligned} \hat{\sigma}_z &= |e\rangle\langle e| - |g\rangle\langle g| \\ \hat{\sigma}_{01} = \hat{\sigma}_- &= |g\rangle\langle e| \\ \hat{\sigma}_{10} = \hat{\sigma}_+ &= |e\rangle\langle g| \end{aligned}$$

- For a QHO state, the basis states are $\{|n\rangle\}$, where $n = 0, 1, 2, \ldots$

- Thus, our total joint state / composite system is:

$$|\psi\rangle = \alpha|g, n+1\rangle + \beta|e, n\rangle$$

2.1 Matrix element calculations

- Let's start off with the diagonals.

$$\Rightarrow \langle g, n+1|\hat{H}|g, n+1\rangle = \frac{\hbar\omega_1}{2}\langle g|\hat{\sigma}_z|g\rangle\langle n+1|\mathbb{1}|n+1\rangle$$

$$+ \hbar\omega_2\left(\langle n+1|a^\dagger a|n+1\rangle\langle g|\mathbb{1}|g\rangle + \tfrac{1}{2}\right) + 0$$

$$= -\frac{\hbar\omega_1}{2} + \hbar\omega_2\left(n + \tfrac{3}{2}\right)$$

$$\Rightarrow \langle \cancel{g}e, n|\hat{H}|e, n\rangle = \frac{\hbar\omega_1}{2}\langle e|\hat{\sigma}_z|e\rangle\langle n|\mathbb{1}|n\rangle$$

$$+ \hbar\omega_2\left(\langle n|a^\dagger a|n\rangle\langle e|\mathbb{1}|e\rangle + \tfrac{1}{2}\right) + 0$$

$$= \frac{\hbar\omega_1}{2} + \hbar\omega_2\left(n + \tfrac{1}{2}\right)$$

$\Rightarrow$ Now for the off-diagonals:

$$\Rightarrow \langle g, n+1|\hat{H}|e, n\rangle = 0 + 0 + \hbar G\big(\langle n+1|a|n\rangle\langle g|\hat{\sigma}_+|e\rangle \overset{0}{=}$$
$$+ \langle n+1|a^\dagger|n\rangle\langle g|\hat{\sigma}_-|e\rangle\big)$$
$$= \hbar G(\sqrt{n+1})$$

$$\Rightarrow \langle e, n|\hat{H}|g, n+1\rangle = 0 + 0 + \hbar G\big(\langle n|a|n+1\rangle\langle e|\hat{\sigma}_+|g\rangle$$
$$+ \langle n|a^\dagger|n+1\rangle\underbrace{\langle e|\hat{\sigma}_-|g\rangle}_{=0}\big)$$
$$= \hbar G(\sqrt{n+1})$$

- Thus, our matrix built on $|g, n+1\rangle$, and $|e, n\rangle$ is:

$$\hat{H} = \begin{pmatrix} \hbar\omega_2(n+\frac{3}{2}) - \frac{\hbar\omega_1}{2} & \hbar G\sqrt{n+1} \\ \hbar G\sqrt{n+1} & \hbar\omega_2(n+\frac{1}{2}) + \frac{\hbar\omega_1}{2} \end{pmatrix} //$$

3. Eigenvalues of Matrix

- Solve $(H - \lambda I)\underline{x} = 0$

$$\det(H - \lambda I) = 0$$

$$\left(\hbar\omega_2\left(n+\frac{3}{2}\right) - \frac{\hbar\omega_1}{2} - \lambda\right)\left(\hbar\omega_2\left(n+\frac{1}{2}\right) + \frac{\hbar\omega_1}{2} - \lambda\right) - \hbar^2G^2(n+1) = 0$$

$$\hbar^2\omega_2^2\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right) + \frac{\hbar^2\omega_1\omega_2}{2}\left(n+\frac{3}{2}\right) - \hbar\omega_2\left(n+\frac{3}{2}\right)\lambda$$

$$- \frac{\hbar^2\omega_1\omega_2}{2}\left(n+\frac{1}{2}\right) - \frac{\hbar^2\omega_1^2}{4} + \frac{\hbar\omega_1}{2}\lambda$$

$$- \hbar\omega_2\left(n+\frac{1}{2}\right)\lambda + \frac{\hbar\omega_1}{2}\lambda + \lambda^2 - \hbar^2G^2(n+1) = 0.$$

$a = 1$

$b = -2\hbar\omega_2(n+1)$

$$c = \hbar^2\omega_2^2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right) + \hbar^2\frac{\omega_1\omega_2}{2}\left(n-n+\frac{3-1}{2}\right) - \frac{\hbar^2\omega_1^2}{4}$$
$$-\hbar^2 G^2(n+1)$$
$$= \hbar^2\left(\omega_2^2\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right) + \frac{\omega_1\omega_2}{2} - \frac{\omega_1^2}{4} - G^2(n+1)\right)$$

· Now we apply the quadratic formula, but first let's isolate the discriminant:

$$b^2 - 4ac = 4\hbar^2\omega_2^2(n+1)^2 - 4\hbar^2\left(\omega_2\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right) + \frac{\omega_1\omega_2}{2}\right.$$
$$\left. - \frac{\omega_1^2}{4} - G^2(n+1)\right)$$

$$= 4\hbar^2\left(\omega_2^2(n^2+2n+1) - \omega_2^2\left(n^2+2n+\frac{3}{4}\right) - \frac{\omega_1\omega_2}{2} + \frac{\omega_1^2}{4}\right.$$
$$\left. + G^2(n+1)\right)$$
$$= 4\hbar^2\left(\frac{\omega_1^2+\omega_2^2}{4} - \frac{\omega_1\omega_2}{2} + G^2(n+1)\right)$$

· Thus, we have :

$$\lambda = \frac{1}{2}\cdot\left(2\hbar\omega_2(n+1) \pm \left(4\hbar^2\left(\frac{\omega_1^2+\omega_2^2}{2} - \frac{\omega_1\omega_2}{2} + G^2(n+1)\right)\right)^{\frac{1}{2}}\right)$$

$$\lambda = \hbar\left[\omega_2(n+1) \pm \sqrt{\left(\frac{\omega_1^2+\omega_2^2}{4} - \frac{\omega_1\omega_2}{2} + G^2(n+1)\right)}\right] \quad //$$

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4. Eigenstates of matrix

· Let's label $\lambda_{+,-} = \hbar\omega_2(n+1) \pm k$ where:

$$k = \hbar\sqrt{\left(\frac{\omega_1^2+\omega_2^2}{4} - \frac{\omega_1\omega_2}{2} + a^2(n+1)\right)}$$

Thus, for λ_+:

· For $(m,n)=(1,1)$ of matrix:

$$\hbar\omega_2\left(n+\frac{3}{2}\right) - \frac{\hbar\omega_1}{2} - \hbar\omega_2(n+1) - k$$

$$= \hbar\left(\frac{\omega_2-\omega_1}{2}\right) - k$$

· For $(m,n)=(2,2)$ of matrix:

$$\hbar\omega_2\left(n+\frac{1}{2}\right) + \frac{\hbar\omega_1}{2} - \hbar\omega_2(n+1) + k$$

$$= \hbar\left(\frac{\omega_1-\omega_2}{2}\right) + k$$

· Thus:

$$\begin{pmatrix} \hbar\left(\frac{\omega_2-\omega_1}{2}\right) - k & \hbar a\sqrt{n+1} \\ \hbar a\sqrt{n+1} & \hbar\left(\frac{\omega_1-\omega_2}{2}\right) - k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

· Labelling $\hbar a\sqrt{n+1} = c$, $\hbar\left(\frac{\omega_1-\omega_2}{2}\right) = a$:

$$\begin{pmatrix} -a-k & c \\ c & a-k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

· Thus, for λ_+ :

$$-(a+k)x_1 + cx_2 = 0$$

$$x_2 = \frac{a+k}{c}x_1$$

$$\underline{v}_+ = \begin{pmatrix} 1 \\ (a+k)/c \end{pmatrix}$$

· Now, we normalize v_1:

$$\hat{v}_+ = \left(1 + \left(\frac{a+k}{c}\right)^2\right)^{-\frac{1}{2}} \begin{bmatrix} 1 \\ \frac{a+k}{c} \end{bmatrix}$$

· Now, we carry out the same procedure for λ_- :

$$\begin{pmatrix} -a+k & c \\ c & a+k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(-a+k)x_1 + cx_2 = 0$$

$$x_2 = \frac{a-k}{c}x_1$$

$$\underline{v}_- = \begin{pmatrix} 1 \\ \frac{a-k}{c} \end{pmatrix}$$

$$\hat{v}_- = \left(1 + \left(\frac{a-k}{c}\right)^2\right)^{-\frac{1}{2}} \cdot \begin{bmatrix} 1 \\ \frac{a-k}{c} \end{bmatrix}$$

4.1 Eigenstate superposition

- We now have our eigenstates of our system, with each one having an associated eigenvalue.
- $\lambda_+ : \quad \hat{v}_+ = \left(1 + \left(\frac{a+k}{c}\right)^2\right)^{-\frac{1}{2}} \begin{bmatrix} 1 \\ \frac{a+k}{c} \end{bmatrix}$
- $\lambda_- : \quad \hat{v}_- = \left(1 + \left(\frac{a-k}{c}\right)^2\right)^{-\frac{1}{2}} \begin{bmatrix} 1 \\ \frac{a-k}{c} \end{bmatrix}$
- We can now write our eigenstates as a superposition of the basis states:

$$|\psi\rangle_\pm = \left(1 + \left(\frac{a \pm k}{c}\right)^2\right)^{-\frac{1}{2}} \left(|g, n+1\rangle + \frac{a \pm k}{c} |e, n\rangle\right) \quad //$$

where $a = \hbar\left(\frac{\omega_1 - \omega_2}{2}\right)$

$$k = \hbar \sqrt{\left(\frac{\omega_1^2 + \omega_2^2}{4} - \frac{\omega_1 \omega_2}{2} + G^2(n+1)\right)}$$

$$c = \hbar G \sqrt{(n+1)}$$

5. Diagonalisation

- Previously, we looked into the simplifications $n = 1$ and $\omega_1 = \omega_2 = \omega$. If we plug these in above, we get our previous solution. Furthermore, the previous solution's matrix was confirmed to be diagonalisible, and so this generalised version is as well.

$$D = \begin{pmatrix} \hbar\omega_2(n+1) + k & 0 \\ 0 & \hbar\omega_2(n+1) - k \end{pmatrix} \quad //$$

where k is defined above.