Relative Entropy of Coherence Calculation

- Recall that for a composite system:

$$|\psi(t)\rangle = \alpha(t)|g,1\rangle + \beta(t)|e,0\rangle$$

we obtained a density matrix:

$$\rho(t) = \alpha(t)\alpha^*(t)|g,1\rangle\langle g,1| + \alpha(t)\beta^*(t)|g,1\rangle\langle e,0|$$

$$+ \alpha^*(t)\beta(t)|e,0\rangle\langle g,1| + \beta^*(t)\beta|e,0\rangle\langle e,0|$$

$$= \begin{bmatrix} \alpha(t)\alpha^*(t) & \alpha(t)\beta^*(t) \\ \alpha^*(t)\beta(t) & \beta(t)\beta^*(t) \end{bmatrix}$$

- The relative entropy of coherence is:

$$C_r = S(\Delta(\rho)) - S(\rho)$$

where $\Delta(\rho) = \rho_{diag}$, which removes all non-diagonal entries of the matrix ρ.

- The Von Neumann entropy of a matrix $\in$ $S(\rho)$ is:

$$S(\rho) = -Tr(\rho \ln(\rho)).$$

If ρ is diagonalisable, then:

$$e^{\ln \rho} = \rho$$

$$\Rightarrow S(\rho) = -\sum_i \lambda_i \ln \lambda_i$$

Let us calculate $C_r(t)$ for both subsystems, and for the system as a whole.

PTO

1. Rel. Entropy of Coherence for the subsystems

· Previously, we found:

$$\rho_{HO}(t) = \alpha(t)\alpha^*(t)|1\rangle\langle 1| + \beta(t)\beta^*(t)|0\rangle\langle 0|$$

$$\rho_{TLS}(t) = \alpha(t)\alpha^*(t)|g\rangle\langle g| + \beta(t)\beta^*(t)|e\rangle\langle e|$$

· ~~Since either sub~~ Focussing on $\rho_{HO}(t)$ for now, we can see that it is a diagonal matrix. Hence:

$$\rho_{HO}(t) = \rho_{HO,diag}(t) = \Delta(\rho_{HO}(t))$$

· Thus;

$$C_r(t) = S(\rho_{HO}(t)) - S(\rho_{HO}(t)) = 0. //$$

· This is also applicable to the $\rho_{TLS}(t)$.

2. Rel. Entropy of Coherence for the total system

· It is important to note that the Von Neumann entropy is basis independent.

· However, since $\Delta(\rho)$ is always diagonal, and its values on the diagonal are basis dependent; $S(\Delta(\rho))$ is in effect dependent on basis choice.

- Let us start by diagonalising the matrix $\rho(t)$:

$$\det(\rho(t) - \lambda I) = 0$$

$$(\alpha\alpha^* - \lambda)(\beta\beta^* - \lambda) - \alpha\alpha^*\beta\beta^* = 0$$

$$\lambda_1 = 0$$
$$\lambda_2 = \alpha\alpha^* + \beta\beta^* = 1$$

- Note two things: we have labelled $\alpha(t), \beta(t) \rightarrow \alpha, \beta$ for convenience.
$\alpha\alpha^* + \beta\beta^* = 1$ since all probabilites must sum to 1 i.e. $Tr(\rho) = 1$ for a pure density matrix.

- Thus:

$$C_r(t) = S(\Delta(\rho(t))) - S(\rho(t))$$

$$S(\rho(t)) = -0 \ln 0 - 1 \ln 1 = 0.$$

- So:

$$C_r(t) = S(\Delta(\rho(t)))$$

$$= S\left(\begin{pmatrix} \alpha(t)\alpha^*(t) & 0 \\ 0 & \beta(t)\beta^*(t) \end{pmatrix}\right)$$

$$= -|\alpha(t)|^2 \ln(|\alpha(t)|^2) - |\beta(t)|^2 \ln(|\beta(t)|^2)$$

- When studying entanglement, we found that:

$$|\alpha(t)|^2 = \eta_+^2 + \eta_-^2 + 2\eta_+\eta_- \cos(at)$$

$$|\beta(t)|^2 = \eta_+^2 \left(\frac{a+k'}{c'}\right)^2 + \eta_- \left(\frac{a-k'}{c'}\right)^2 + 2\eta_+\eta_- \left(\frac{a+k'}{c'}\right)\left(\frac{a-k'}{c'}\right)\cos(at)$$

- For $\cos(at) = 1$; $|\alpha|^2 = 0$, $|\beta|^2 = 1$.

$$C_r(t) = 0 - 1 \cdot \ln(1) = 0 \ //$$

- For $\cos(at) = 0$, $|\alpha|^2 = 1/2$, $|\beta|^2 = 1/2$.

$$C_r(t) = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \ln(2) \ //$$

- As we can see, our $C_r(t)$ oscillates between 0 and $\ln(2)$.