Quantum Harmonic Oscillator and 2LS Composite System: Eigenvalues and Eigenstates

- Here, we will follow the steps below to determine the eigenvalues and states of the composite system of a Quantum Harmonic Oscillator and 2 Level System (2LS).

1. Define Hamiltonians for 2LS, QHO and the coupled interaction. Combine into total Hamiltonian.
2. Combine all into a matrix using expectation values.
3. Calculate eigenvalues of matrix.
4. Calculate eigenstates.
5. Diagonalize the matrix.

1. Hamiltonian Definitions

- The total Hamiltonian will take the form:

$$\hat{H} = \hat{H}_{2LS} + \hat{H}_o + \hat{H}_{int}$$

⇒ Let's index for ease, setting 2LS → 1, O → 2.

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12}$$

- $\hat{H}_1 = \hat{H}_{2LS} = \dfrac{\hbar \omega_1}{2} \hat{\sigma}_z$

where $\hbar = \dfrac{h}{2\pi}$

ω_1 is the frequency, such that it influences energy of the 2LS.

$\hat{\sigma}_z$ is the Pauli z-matrix, $\hat{\sigma}_z = |1\rangle\langle 1| - |0\rangle\langle 0|$

• $\hat{H}_2 = \hbar\omega_2\left(a^\dagger a + \frac{1}{2}\right)$ where ω_2 is the angular frequency of the QHO and influences energy of that system.

$a^\dagger$ is the raising operator;

$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$

a is the lowering operator

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

• Note that $\hat{H}_2$ is written sometimes as $\hbar\omega_2 a^\dagger a$, and omits the $\frac{1}{2}$. This is sometimes done to simplify calculations. However, we will use the full version as above.

• $\hat{H}_3 = \hbar G\,(a\sigma_{10} + a^\dagger\sigma_{01})$ where G is the measures coupling of system.

$\sigma_{10} = |1\rangle\langle 0|$

$\sigma_{01} = |0\rangle\langle 1|$

• $\hat{H} = \frac{\hbar\omega_1}{2}\hat{\sigma}_z + \hbar\omega_2\left(a^\dagger a + \frac{1}{2}\right) + \hbar G\left(a\sigma_{10} + a^\dagger\sigma_{01}\right)$ //

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2. Matrix Representation of Hamiltonian

- The total composite system $|\psi\rangle$ will be a tensor product of the 2LS state $|\psi_1\rangle$ and the QHO state $|\psi_2\rangle$:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

- For a 2LS state, the basis states are $|g\rangle$ and $|e\rangle$ (denoting general ground and excited states respectively).
  - Let us relabel $\hat{\sigma}_z$, $\hat{\sigma}_{01}$ and $\hat{\sigma}_{10}$ to match the 2LS basis state labels:

$$\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$$
$$\hat{\sigma}_{01} = |g\rangle\langle e|$$
$$\hat{\sigma}_{10} = |e\rangle\langle g|$$

- For a QHO state, the basis states are $\{|n\rangle\}$, where $n = 0, 1, 2, \ldots$

- As seen in $\boxed{\text{Week 2}}$ calculations, the 4×4 matrix when considering $|0\rangle$ and $|1\rangle$ becomes difficult to solve analytically, and a computational approach is much easier.

However, the 4×4 was a sparse matrix, and coupling terms only appeared for the terms $\langle e,0|\hat{H}|g,1\rangle$ and $\langle g,1|\hat{H}|e,0\rangle$. Thus, we can simplify things by considering only the joint states $|g,1\rangle$ and $|e,0\rangle$.

- Furthermore, as seen in a reference [1], to simplify calculations $\omega_1 = \omega_2 = \omega$. Thus, we shall implement these simplifications.

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(1) Hagelstein, Peter L., and Ifran Chaudary. *Two-level systems coupled to oscillators.*

$$\langle g,1|\hat{H}|g,1\rangle = -\frac{\hbar\omega}{2} + \hbar\omega\left(\langle g,1|a^\dagger a|g,1\rangle + \frac{1}{2}\right)$$
$$= \frac{3\hbar\omega}{2} - \frac{\hbar\omega}{2} = \hbar\omega$$

$$\langle e,0|\hat{H}|e,0\rangle = \frac{\hbar\omega}{2} + \hbar\omega\left(\overset{=0}{\langle e,0|a^\dagger a|e,0\rangle} + \frac{1}{2}\right)$$
$$= \hbar\omega$$

$$\langle g,1|\hat{H}|e,0\rangle = 0 + 0 + \hbar G\left(\langle g,1|a|e\rangle\overset{=0}{\langle g|e,0\rangle} + \langle g,1|a^\dagger|g\rangle\langle e|e,0\rangle\right)$$
$$= \hbar G$$

$$\langle e,0|\hat{H}|g,1\rangle = 0 + 0 + \hbar G\left(\langle e,0|a|e\rangle\langle g|g,1\rangle + 0\right)$$
$$= \hbar G$$

· Our matrix built on $|g,1\rangle, |e,0\rangle$ is:

$$H = \hbar\begin{pmatrix} \omega & G \\ G & \omega \end{pmatrix} \quad //$$

3. Eigenvalues of matrix

· Solve $(H - \lambda I)\underline{x} = 0$.

$$\det(H - \lambda I) = 0$$

$$(\hbar\omega - \lambda)^2 - \hbar^2 G^2 = 0.$$

$$\hbar^2\omega^2 - 2\lambda\hbar\omega + \lambda^2 - \hbar^2 G^2 = 0$$

$$\lambda = \frac{\cancel{2}\hbar\omega \pm \sqrt{\cancel{4}\hbar^2\omega^2 - \cancel{4}(\hbar^2\omega^2 - \hbar^2 G^2)}}{\cancel{2}}$$

$$= \hbar\omega \pm \sqrt{\hbar^2 G^2} = \hbar\omega \pm \hbar G \quad //$$

4. Eigenstates of matrix

- $\lambda_1 = \hbar\omega + \hbar G$:

$$\begin{pmatrix} -\hbar G & \hbar G \\ \hbar G & -\hbar G \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ normalized } \hat{\underline{v}}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- $\lambda_2 = \hbar\omega - \hbar G$:

$$\hbar\begin{pmatrix} G & G \\ G & G \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \hat{\underline{v}}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

5. Diagonalizing the matrix

- We can confirm H is diagonalized by calculating :

$$H = PDP^{-1}$$

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$ columns of eigenvectors

$$D = \hbar\begin{pmatrix} \omega + G & 0 \\ 0 & \omega - G \end{pmatrix}$$ diagonals of eigenvalues.

$P^{-1} = P^T$ since its columns are orthonormal.

$$P^{-1} = P^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = P$$

· Starting with PD:

$$PD = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \omega + a & 0 \\ 0 & \omega - a \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} \omega + a & \omega - a \\ \omega + a & a - \omega \end{pmatrix}$$

· Now, PDP^{-1}:

$$\frac{\hbar}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} \omega + a & \omega - a \\ \omega + a & a - \omega \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} \omega + a + \omega - a & \omega + a - \omega + a \\ \omega + a + a - \omega & \omega + a - a + \omega \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 2\omega & 2a \\ 2a & 2\omega \end{pmatrix}$$

$$= \hbar \begin{pmatrix} \omega & a \\ a & \omega \end{pmatrix} //$$

· This is equal to H, confirming it is diagonalisible. The diagonal matrix is:

$$D = \hbar \begin{pmatrix} \omega + a & 0 \\ 0 & \omega - a \end{pmatrix}$$