Entropy of Entanglement for pure states

The von Neumann Entropy can be written in terms of the density matrix ρ as:

$$S = -\,\mathrm{tr}(\rho \ln(\rho))$$

If the density matrix is written in the eigenbasis, then:

$$S = -\sum_j c_j \ln(c_j)$$

where c_j are the eigenvalues ~~coefficients~~ of the density matrix ~~eigenvectors~~.

· Thus, for our system where:

$$\rho(t) = \begin{pmatrix} \eta_+\eta_+^* & \eta_+\eta_-^* \\ \eta_+^*\eta_- & \eta_-\eta_-^* \end{pmatrix} = \begin{pmatrix} |\eta_+|^2 & \eta_+\eta_-^* \\ \eta_+^*\eta_- & |\eta_-|^2 \end{pmatrix}$$

where $\eta_\pm = e^{-i(\omega_2 \pm k'/\hbar)t}\left(1+\left(\frac{a\pm k'}{c'}\right)^2\right)^{-\frac{1}{2}} \cdot \left(\frac{a\pm k'}{c'}\right)$

we can calculate its eigenvalues by $\det(\rho - \lambda I) = 0$.

$$(|\eta_+|^2 - \lambda)(|\eta_-|^2 - \lambda) - |\eta_+|^2|\eta_-|^2 = 0$$

$$\lambda^2 - \lambda(|\eta_+|^2 + |\eta_-|^2) = 0.$$

$$\lambda(\lambda - (|\eta_+|^2 + |\eta_-|^2)) = 0.$$

$$\lambda_1 = 0, \quad \lambda_2 = |\eta_+|^2 + |\eta_-|^2$$

Thus: $S = -\lambda_2 \ln \lambda_2$.

PTO

· Let's first simplify $|\eta_+|^2 + |\eta_-|^2$:

$$|\eta_+|^2 = \eta_+\eta_+^* = e^{-i(\omega_2+k')t} e^{i(\omega_2+k')t}$$
$$\times \left(1 + \left(\frac{a+k'}{c'}\right)^2\right)^{-1}\left(\frac{a+k'}{c'}\right)^2$$

$$\therefore \quad |\eta_\pm|^2 = \left(1+\left(\frac{a\pm k'}{c'}\right)^2\right)^{-1}\left(\frac{a\pm k'}{c'}\right)^2$$

· Thus, we can write that the Von Neumann Entropy is:

$$S = -\lambda_2 \ln \lambda_2 = -(|\eta_+|^2+|\eta_-|^2)\ln(|\eta_+|^2+|\eta_-|^2) \; //$$

Verification

· In the reference [1], the maximally entangled state is given as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0,e\rangle + |1,g\rangle)$$

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\det(\rho - \lambda I) = 0$$

$$\left(\tfrac{1}{2}-\lambda\right)^2 - \tfrac{1}{4} = 0$$

$$\lambda_{1,2} = 0, 1$$

$$S = -\lambda_2 \ln \lambda_2 = -1 \cdot \ln 1 = 0.$$

· In ~~these~~ [1], $n=0$ and $\omega_1 = \omega_2$ i.e. resonance condition is worked at.

[1] E. Hernández-Concepción, D. Alonso, and S. Brouard. "Entanglement in a continuously measured TLS coupled to a HO." Physical Review Journals 2009

- It can also be noted that, since the Von Nuemann Entropy is $-\mathrm{Tr}(\rho \ln \rho)$, the trace operation is invariant under basis change.

- Thus, if we have the same conditions, $n=0$ and resonance, then the general S we defined should equal that of the maximally entangled state. ~~Both~~

- Furthermore, since the density matrix is a pure state $\rho = |\psi\rangle\langle\psi|$, we have an eigenvalue $=1$. Thus, for pure states:

$$S = 0. \quad \text{for pure states.}$$

which further motivates our solution.

- Let's now verify this for our general S.

$$S = (|\eta_+|^2 + |\eta_-|^2)\ln(|\eta_+|^2 + |\eta_-|^2)$$

- ~~In~~ In our case, $n=0$ already. Thus, set $\omega_1 = \omega_2$. We have the following definitions.

$$a = \hbar\left(\frac{\omega_1 - \omega_2}{2}\right) = 0.$$

$$c' = \hbar G$$

$$k' = \hbar\sqrt{\left(\frac{\omega_1^2 + \omega_2^2}{4} - \frac{\omega_1\omega_2}{2} + G^2\right)} = \hbar G$$

$$|\eta_\pm|^2 = \left(1 + \left(\frac{a \pm k'}{c'}\right)^2\right)^{-1}\left(\frac{a \pm k'}{c'}\right)^2$$

$$= \frac{1}{2}$$

· Finally, for our general S:

$$|\eta_+|^2 = |\eta_-|^2 = \frac{1}{2}$$

$$S = \left(\frac{1}{2}+\frac{1}{2}\right)\ln\left(\frac{1}{2}+\frac{1}{2}\right) = 0 \, . \, //$$