Time Evolution of a TLS - HO composite system

- I will be finding the state at any t $|\psi(t)\rangle$ for an initial state $|\psi(0)\rangle = |e,0\rangle$. For this state, the ~~are~~ following steps will be followed :

  1. In the initial state, the number of phonons is 1 (1 - phonon manifold). Thus, I will set $n=0$ in the diagonalised Hamiltonian.

  2. Write the initial state in terms of the ~~bas~~ eigenstate basis.

  3. Calculate the evolution operator.

  4. Find $|\psi(t)\rangle$ using $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$

1. Setting up the diagonalised Hamiltonian

- Since our initial state is in the 1-phonon manifold, $n=0$. Our diagonalised Hamiltonian is as follows:

$$D = \begin{bmatrix} \hbar\omega_2(n+1) + k & 0 \\ 0 & \hbar\omega_2(n+1) - k \end{bmatrix}$$

where $k = \hbar\sqrt{\left(\frac{\omega_1^2+\omega_2^2}{4} - \frac{\omega_1\omega_2}{2} + a^2(n+1)\right)}$

- Setting $n=0$, we get :

$$D = \begin{bmatrix} \hbar\omega_2 + k' & 0 \\ 0 & \hbar\omega_2 - k' \end{bmatrix} \quad //$$

where $k' = \hbar\sqrt{\left(\frac{\omega_1^2+\omega_2^2}{4} - \frac{\omega_1\omega_2}{2} + a^2\right)}$.

2. Transforming basis of initial state to eigenbasis

- Knowing that a general state $|\psi\rangle$ is a linear combination of $|e,n\rangle$ and $|g,n+1\rangle$:

$$|\psi\rangle = \alpha|g,n+1\rangle + \beta|e,n\rangle$$

- We can write our initial state as:

$$|\psi(0)\rangle = |e,0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{where } \alpha = 0,\ \beta = 1.$$

- In order to transform this state's basis into our eigenbasis of diag H we do:

$$P^{\dagger}|\psi(0)\rangle = |\psi'(0)\rangle$$

where $P^{\dagger}$ is the conjugate transpose of the matrix formed of eigenstates

$|\psi'(0)\rangle$ is the state in the eigenbasis

- Noting that $P^{\dagger} = P^{T}$ since P elements are real:

$$P^{T} = \begin{bmatrix} \left(1+\left(\frac{a+k'}{c'}\right)^2\right)^{-\frac{1}{2}} & \left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}} \\ \left(1+\left(\frac{a+k'}{c'}\right)^2\right)^{-\frac{1}{2}}\left(\frac{a+k'}{c'}\right) & \left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}}\left(\frac{a-k'}{c'}\right) \end{bmatrix}^{T}$$

$$= \begin{bmatrix} \left(1+\left(\frac{a+k'}{c'}\right)^2\right)^{-\frac{1}{2}} & \left(1+\left(\frac{a+k'}{c'}\right)^2\right)^{-\frac{1}{2}}\left(\frac{a+k'}{c'}\right) \\ \left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}} & \left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}}\left(\frac{a-k'}{c'}\right) \end{bmatrix}$$

where $c' = \hbar G$ and $a = \hbar\left(\frac{\omega_1 - \omega_2}{2}\right)$.

· Doing $P^{\dagger}|\psi(0)\rangle$ we get:

$$|\psi'(0)\rangle = \begin{bmatrix} \left(1+\left(\frac{a+k'}{c'}\right)^2\right)^{-\frac{1}{2}} \cdot \left(\frac{a+k'}{c'}\right) \\ \left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}} \cdot \left(\frac{a-k'}{c'}\right) \end{bmatrix}$$

3. Evolution Operator

· The evolution operator is defined as $\hat{U}(t) = e^{\frac{-iHt}{\hbar}}$. We have our diagonalised Hamiltonian, which makes our calculation of $\hat{U}$ easy.

$$\hat{U}(t) = \begin{bmatrix} e^{-i(\omega_2+k')t} & 0 \\ 0 & e^{-i(\omega_2-k')t} \end{bmatrix}$$

4. Calculating $|\psi(t)\rangle$

· We know $|\psi(t)\rangle = \hat{U}(t)|\psi'(0)\rangle$, so:

$$|\psi(t)\rangle = \begin{bmatrix} e^{-i(\omega_2+k')t} & 0 \\ 0 & e^{-i(\omega_2-k')t} \end{bmatrix} \begin{bmatrix} \left(1+\left(\frac{a+k'}{c'}\right)^2\right)^{-\frac{1}{2}} \cdot \left(\frac{a+k'}{c'}\right) \\ \left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}} \cdot \left(\frac{a-k'}{c'}\right) \end{bmatrix}$$

$$|\psi(t)\rangle = \begin{bmatrix} e^{-i(\omega_2+k')t}\left(1+\left(\frac{a+k'}{c'}\right)^2\right)^{-\frac{1}{2}} \cdot \left(\frac{a+k'}{c'}\right) \\ e^{-i(\omega_2-k')t}\left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}} \cdot \left(\frac{a-k'}{c'}\right) \end{bmatrix} //$$