Von Neumann Entropy and the Relative Entropy of Entanglement Revised:

As given by reference [1] (see footnote), for a system in a pure state, the Von Neumann Entropy of <u>either</u> of the two subsystems is:

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

$$S = -\mathrm{Tr}\left(\rho_{TLS}\ln(\rho_{TLS})\right) = -\mathrm{Tr}\left(\rho_{HO}\ln(\rho_{HO})\right)$$

Furthermore, if the density matrix being considered is diagonalised then the Von Neumann Entropy can be written as:

$$S = \sum_i -\lambda_i \ln(\lambda_i)$$ where λ_i are the eigenvalues of the diagonalized $\rho_A(t)$.

· In our case, both ρ_{TLS}, ρ_{HO} are diagonalized and so we can proceed with the above; using $\rho_{HO}(t)$.

$$\rho_{HO}(t) = \alpha(t)\alpha^*(t)\,|1\rangle\langle 1| + \beta(t)\beta^*(t)\,|0\rangle\langle 0|$$

$$S(t) = -|\alpha(t)|^2\ln(|\alpha(t)|^2) - |\beta(t)|^2\ln(|\beta(t)|^2)$$

where $\alpha(t)\alpha^*(t) = \eta_+^2 + \eta_-^2 + 2\eta_+\eta_-\cos\left(\frac{K'}{\hbar}t\right)$

$$\beta(t)\beta^*(t) = \eta_+^2\left(\frac{a+k'}{c'}\right)^2 + \eta_-\left(\frac{a-k'}{c'}\right)^2 *$$
$$+ 2\eta_+\eta_-\left(\frac{a+k'}{c'}\right)\left(\frac{a-k'}{c'}\right)\cos\left(\frac{k'}{\hbar}t\right)$$

Resonance Condition ~~PTO~~

· In ref. [1], $\omega_1 = \omega_2$ resonance condition was looked at, and it was found that $S(t)$ oscillates between 0 and $\ln(2)$.

PTO

[1] E. Hernández-Concepción, D. Alonso and S. Brouard. 'Entanglement in a continously measured TLS-HO." Physical Review Journals 2009.

- In our case, the Von Neumann Entropy is dependent on time, and has a maximum and minimum that is dependent on $\cos(Gt)$.

- Let's plug in $\omega_1 = \omega_2$ and simplify $|\alpha(t)|^2$ and $|\beta(t)|^2$.

$$\eta_\pm = \left(1 + \left(\frac{a \pm k'}{c'}\right)^2\right)^{-1} \left(\frac{a \pm k'}{c'}\right)$$

At resonance, $a = 0$ where G is the interaction coupling constant.

$$k' = \hbar G$$
$$c' = \hbar G$$

$$\eta_\pm = \pm \frac{1}{2}, \quad \cos\left(\frac{k'}{\hbar}t\right) = \cos(Gt)$$

At $\cos(Gt) = 1$:

$$|\alpha(t)|^2 = \frac{1}{4} + \frac{1}{4} + 2\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = 0$$

$$|\beta(t)|^2 = \frac{1}{4} + \frac{1}{4} + 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1.$$

$$S(t) = -0\ln(0) - 1\ln(1) = 0. //$$

This aligns well with our initial state $|\psi(0)\rangle = |e,0\rangle$ because at $t=0$ $\cos(Gt) = 1$ and there is no entanglement. $|\psi(0)\rangle$ can be written as a product state:

$$|\psi(0)\rangle = |e\rangle \otimes |0\rangle$$

which should have <u>zero</u> entanglement.

- Similarly, at $\cos(Gt) = -1$, $|\alpha(t)|^2 = 1$, $|\beta(t)|^2 = 0$.

$$S(t) = 0. //$$

Finally, at $\cos(\alpha t) = 0$:

$$|\alpha(t)|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$|\beta(t)|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$S(t) = -\frac{1}{2}\ln\left(\frac{1}{2}\right) - \frac{1}{2}\ln\left(\frac{1}{2}\right) = \ln(2) \; //$$

- Thus, as expected, we can see that the maximum entanglement is $\ln(2)$ at $\cos(\alpha t) = 0$.

- The entanglement value oscillates between 0, $\ln(2)$ and is dependent only on time. //

Relative Entropy of Entanglement

- The relative entropy of entanglement (REE) is defined [1][2] as:

$$E(\rho) := \min_{\sigma \in \mathcal{D}} D(\rho \| \sigma)$$

$$= \mathrm{Tr}(\rho(\ln\rho - \ln\sigma))$$

where $\mathcal{D}$ is the set of all the disentangled states
ρ is our density matrix
σ is a separable (product) state that minimises the relative entropy between ρ and σ.

- In words, it is the 'minimum of a distance' between the density matrix of the system ρ and the set of all disentangled density matrices; σ represents the <u>chosen</u> minimum density matrix.

[2] V. Vedral and M. B. Plenio. "Entanglement measures and purification procedures". Physical Review A vol. 57, 1998.

- In order to work out σ, it is most conveniently done via numerical methods. Howere, there is a simplijication and trivial answe for our system.

- [2] states that for pure states (that is, $\rho = |\psi\rangle\langle\psi|$), the REE reduces to the entropy of entanglement (in our case, the Von Neumann entropy of entanglement) for reduced density operators.

- Thus: $$E(\rho) = \min_{\sigma \in D} D(\rho \| \sigma) = -Tr(\rho_{TLS} \ln \rho_{TLS}) = -Tr(\rho_{HO} \ln \rho_{HO})$$

- In our case:

$$\cancel{E(\rho)} = E(\rho(t)) = -|\alpha(t)|^2 \ln(|\alpha(t)|^2) - |\beta(t)|^2 \ln(|\beta(t)|^2)$$

- And so the REE, since it equals the Von Neumann entropy, oscillates between $0, \ln(2)$. //