Quantum Harmonic Oscillator and 2LS Composite System: Eigenvalues and Eigenstates

- Here, we will follow the steps below to determine the eigenvalues and states of the composite system of a Quantum Harmonic Oscillator and 2 Level System (2LS).

    1. Define Hamiltonians for 2LS, QHO and the coupled interaction. Combine into total Hamiltonian.

    2. Combine all into a matrix using expectation values.

    3. Calculate eigenvalues of matrix.

    4. Calculate eigenstates.

    5. Diagonalize the matrix.

1. Hamiltonian Definitions

- The total Hamiltonian will take the form:

$$\hat{H} = \hat{H}_{2LS} + \hat{H}_0 + \hat{H}_{int}$$

⇒ Let's index for ease, setting $2LS \rightarrow 1$, $0 \rightarrow 2$.

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12}$$

- $\hat{H}_1 = \hat{H}_{2LS} = \frac{\hbar\omega_1}{2}\hat{\sigma}_z$

where $\hbar = \frac{h}{2\pi}$

ω_1 is the frequency, such that it influences energy of the 2LS.

$\hat{\sigma}_z$ is the Pauli z-matrix, $\hat{\sigma}_z = |1\rangle\langle 1| - |0\rangle\langle 0|$

- $\hat{H}_2 = \hbar\omega_2 \left(a^\dagger a + \frac{1}{2}\right)$ where ω_2 is the angular frequency of the QHO and influences energy of that system.

  $a^\dagger$ is the raising operator;

  $$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

  a is the lowering operator

  $$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

- Note that $\hat{H}_2$ is written sometimes as $\hbar\omega_2 a^\dagger a$, and omits the $\frac{1}{2}$. This is sometimes done to simplify calculations. However, we will use the full version as above.

- $\hat{H}_3 = \hbar G\left(a\sigma_{10} + a^\dagger \sigma_{01}\right)$ where G measures coupling of system.

  $\sigma_{10} = |1\rangle\langle 0|$

  $\sigma_{01} = |0\rangle\langle 1|$

- $$\hat{H} = \frac{\hbar\omega_1}{2}\hat{\sigma}_z + \hbar\omega_2\left(a^\dagger a + \frac{1}{2}\right) + \hbar G\left(a\sigma_{10} + a^\dagger\sigma_{01}\right) \quad //$$

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2. Matrix Representation of Hamiltonian

- The total composite system $|\psi\rangle$ will be a tensor product of the 2LS state $|\psi_1\rangle$ and the QHO state $|\psi_2\rangle$:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

- For a 2LS state, the basis states are $|0\rangle$ and $|1\rangle$.

- For a QHO state, the basis states are $\{|n\rangle\}$, where $n = 0, 1, 2 \ldots$ For the sake of simplicity, we shall only consider one state, $|n\rangle$.

- We can write the composite system as:

product state

$$|\psi\rangle = c_1|0,n\rangle + c_2|1,n\rangle$$
$$= (c_1|0\rangle + c_2|1\rangle)|n\rangle$$

where c_n are complex coefficients (probability amplitude).

- We can write our Hamiltonian matrix by using the expectation value, such that the rows and columns are basis states.

- We can see that the diagonal terms have no contribution from the interaction part due to the orthonormality condition of the inner product.

- Similarly, off diagonals are 0 due to orthonormality.

$$\langle 0,n|\hat{H}|0,n\rangle = \hbar\omega_2\left(n + \frac{1}{2}\right) - \frac{\hbar\omega_1}{2}$$

$$\langle 1,n|\hat{H}|1,n\rangle = \hbar\omega_2\left(n + \frac{1}{2}\right) + \frac{\hbar\omega_1}{2}$$

· Thus, our matrix is $\hat{H} = \hbar \begin{pmatrix} \omega_2(n+\frac{1}{2}) - \frac{\omega_1}{2} & 0 \\ 0 & \omega_2(n+\frac{1}{2}) + \frac{\omega_1}{2} \end{pmatrix}$

3, 4. Eigenvalues, Eigenstates

· Our eigenvalues are $\lambda_1 = \hbar\omega_2(n+\frac{1}{2}) - \frac{\hbar\omega_1}{2}$, $\lambda_2 = \hbar\omega_2(n+\frac{1}{2}) + \frac{\hbar\omega_1}{2}$

· For λ_1 : $\begin{pmatrix} 0 & 0 \\ 0 & 2\hbar\omega \end{pmatrix}\begin{pmatrix} x_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \cancel{y_1}\, 0 \\ \cancel{y_2}\, 0 \end{pmatrix}$, thus our first eigenstate

$$|u_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

· For λ_2 :

$$|u_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

5. Diagonalisation

· This matrix is already diagonal.