Quantum Harmonic Oscillator and 2LS Composite System: Eigenvalues and Eigenstates

- Here, we will follow the steps below to determine the eigenvalues and states of the composite system of a Quantum Harmonic Oscillator and 2 Level System (2LS).

    1. Define Hamiltonians for 2LS, QHO and the coupled interaction. Combine into total Hamiltonian.

    2. Combine all into a matrix using expectation values.

    3. Calculate eigenvalues of matrix.

    4. Calculate eigenstates.

    5. Diagonalize the matrix.

1. Hamiltonian Definitions

- The total Hamiltonian will take the form:

$$\hat{H} = \hat{H}_{2LS} + \hat{H}_0 + \hat{H}_{int}$$

⇒ Let's index for ease, setting $2LS \to 1$, $0 \to 2$.

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12}$$

- $$\hat{H}_1 = \hat{H}_{2LS} = \frac{\hbar \omega_1}{2} \hat{\sigma}_z$$

where $\hbar = \dfrac{h}{2\pi}$

ω_1 is the frequency, such that it influences energy of the 2LS.

$\hat{\sigma}_z$ is the Pauli z-matrix, $\hat{\sigma}_z = |1\rangle\langle 1| - |0\rangle\langle 0|$

- $\hat{H}_2 = \hbar\omega_2 \left(a^\dagger a + \frac{1}{2}\right)$ where ω_2 is the angular frequency of the QHO and influences energy of that system.

  $a^\dagger$ is the raising operator;

  $$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

  a is the lowering operator

  $$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

- Note that $\hat{H}_2$ is written sometimes as $\hbar\omega_2 a^\dagger a$, and omits the $1/2$. This is sometimes done to simplify calculations. However, we will use the full version as above.

- $\hat{H}_3 = \hbar G (a\sigma_{10} + a^\dagger \sigma_{01})$ where G ~~is~~ the measures coupling of system.

  $\sigma_{10} = |1\rangle\langle 0|$

  $\sigma_{01} = |0\rangle\langle 1|$

- $$\hat{H} = \frac{\hbar\omega_1}{2}\hat{\sigma}_z + \hbar\omega_2\left(a^\dagger a + \frac{1}{2}\right) + \hbar G\left(a\sigma_{10} + a^\dagger\sigma_{01}\right) \quad //$$

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$n + \frac{1}{2}$ $\langle n | n | n-1 \rangle = \sqrt{n-1+1}\,|n\rangle$

$\langle 1 | = \sqrt{n}\,|n\rangle$

2. Matrix Representation of Hamiltonian

- The total composite system $|\psi\rangle$ will be a tensor product of the 2LS state ($|\psi_1\rangle$) and the QHO state ($|\psi_2\rangle$):

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

- For a 2LS state, the basis states are $|0\rangle$ and $|1\rangle$.

- For a QHO state, the basis states are $\{|n\rangle\}$, where $n = 0, 1, 2, \ldots$ For the sake of an easier analytical calculation, we shall look at the $|0\rangle$ and $|1\rangle$ basis states.

- Generally, we can write the composite system as:

entangled state

$$|\psi\rangle = c_1|0,0\rangle + c_2|1,0\rangle + c_3|0,1\rangle + c_4|1,1\rangle$$

where c_n are complex coefficients (probability amplitudes)

- We can write our Hamiltonian as a matrix by using the expectation value, $\langle\psi|\hat{H}|\psi\rangle$. We write the rows and columns as the basis state tensor products seen above.

- To begin, we see that for our Hamiltonian, the interaction part disappears for diagonal elements ~~since~~ due to orthonormality.

- Diagonals:

$$\langle 0,0|\hat{H}|0,0\rangle = -\frac{\hbar\omega_1}{2}\langle 0|0\rangle\langle 0|0\rangle + \hbar\omega_2\left(\overset{\langle 0,0|}{a^\dagger}\sqrt{0}\,|0-1\rangle + \frac{1}{2}\right)$$

$$= \cancel{\frac{\hbar\omega_2}{2}}\ \frac{\hbar}{2}(\omega_2 - \omega_1)$$

$$\langle 1,0|\hat{H}|1,0\rangle = \frac{\hbar\omega_1}{2} + \hbar\omega_2\left(\langle 1,0|a^\dagger\sqrt{0}\,|0-1\rangle + \frac{1}{2}\right)$$

$$= \frac{\hbar}{2}(\omega_1 + \omega_2)$$

$$\langle 0,1|\hat{H}|0,1\rangle = -\frac{\hbar\omega_1}{2} + \hbar\omega_2\left(\langle 1|a^\dagger|0\rangle + \frac{1}{2}\right)$$

$$= \frac{\hbar}{2}(3\omega_2 - \omega_1)$$

$$\langle 1,1|\hat{H}|1,1\rangle = \frac{\hbar\omega_1}{2} + \hbar\omega\left(\frac{3}{2}\right) = \frac{\hbar}{2}(3\omega_2 + \omega_1)$$

- For the off-diagonals, only ~~4~~ 2 elements are non-zero, and due to orthonormality only interaction terms survive.

- Specifically, the QHO basis states need to be opposite to one another. We get:

$$\langle 1,0|\hat{H}|0,1\rangle$$
$$\langle 0,1|\hat{H}|1,0\rangle$$
$$\cancel{\langle 0,0|\hat{H}|1,1\rangle}$$
$$\cancel{\langle 1,1|\hat{H}|0,0\rangle}$$

- $\langle 1,0|\hat{H}|0,1\rangle = \hbar G\left(\langle 1,0|a|1\rangle\langle 0|0,1\rangle + \overset{=0}{\langle 1,0|a^\dagger|0\rangle}\langle 1|0,1\rangle\right)$

$$= \hbar G$$

- In fact, all these off-diagonal terms are equal to $\hbar G$.

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· Our matrix is :

$$\begin{pmatrix} \frac{\hbar}{2}(\omega_2-\omega_1) & 0 & 0 & \cancel{\hbar G}\ 0 \\ 0 & \frac{\hbar}{2}(\omega_1+\omega_2) & \hbar G & 0 \\ 0 & \hbar G & \frac{\hbar}{2}(3\omega_2-\omega_1) & 0 \\ \cancel{\hbar G}\ 0 & 0 & 0 & \frac{\hbar}{2}(3\omega_2+\omega_1) \end{pmatrix}$$

B. Eigenvalues

· After some analysis, we get the determinant:

$$\det(H) = \frac{\hbar}{2}\Big[\Big(9\omega_2^4+\omega_1^4-10\omega_1^2\omega_2^2+8\omega_1^2\omega_2-\frac{G^2}{4}(3\omega_2+\omega_1)\Big) + \lambda^4 + \cancel{12\lambda\lambda} - 8\lambda^3\omega_2 + \lambda^2(21\omega_2^2-2\omega_1^2) + \lambda\Big(\frac{G^2}{4}-2\omega_2^3\Big)\Big] = 0$$