Correcting the Reduced Density Matrix Calculation.

- Previously I had written the reduced density matrices by tracing out over the respective eigenstates of the system. This, however, has no physical meaning: I should instead trace out over the subsystems (TLS, HO) of the composite system itself.

Rewriting the density matrix

- Previously, I had the following solutions to the time-evolved state and the eigenstates.

$$|\psi(t)\rangle = e^{-i\left(\omega_2 + \frac{k'}{\hbar}\right)t}\left(1 + \left(\frac{a+k'}{c'}\right)^2\right)^{-\frac{1}{2}}\left(\frac{a+k'}{c'}\right)|V_+\rangle$$
$$+ e^{-i\left(\omega_2 - \frac{k'}{\hbar}\right)t}\left(1 + \left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}}\cdot\left(\frac{a-k'}{c'}\right)|V_-\rangle$$

$$|V_+\rangle = \left(1 + \left(\frac{a+k'}{c'}\right)^2\right)^{-\frac{1}{2}}\cdot\left(|g,n+1\rangle + \frac{a+k'}{c'}|e,n\rangle\right)$$

$$|V_-\rangle = \left(1 + \left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}}\cdot\left(|g,n+1\rangle + \frac{a-k'}{c'}|e,n\rangle\right)$$

where $a = \hbar\left(\frac{\omega_1 - \omega_2}{2}\right)$, ~~dh=~~

$$k' = \hbar\sqrt{\left(\frac{\omega_1^2+\omega_2^2}{2} - \frac{\omega_1\omega_2}{2} + G^2\right)}$$

~~$k' = k$ where $n = 0$~~

$$c' = \hbar G$$

- In order to transform $|\psi(t)\rangle$ back into the basis original of joint state; we insert the identity operator:

$$\mathbb{1} = \sum_i |\phi_i\rangle\langle\phi_i| = |g,n+1\rangle\langle g,n+1| + |e,n\rangle\langle e,n|$$

which 'selects' those bases.

$$|\psi(t)\rangle = \mathbb{1}\cdot\sum_i c_i |v_i\rangle = \left(|g,n+1\rangle\langle g,n+1| + |e,n\rangle\langle e,n|\right)$$
$$\times\left(c_+|v_+\rangle + c_-|v_-\rangle\right)$$

$$|\psi(t)\rangle = \alpha|g,n+1\rangle + \beta|e,n\rangle$$

- Recalling that our time evolved state has $n=0$, let's find the ~~coeff~~ factors α and β.

$$\alpha(t) = e^{-i(\omega_2+\frac{k'}{\hbar})t}\left(1+\left(\frac{k'+a}{c'}\right)^2\right)^{-\frac{1}{2}}\cdot\left(\frac{a+k'}{c'}\right)\cdot\left(1+\left(\frac{a+k'}{c'}\right)^2\right)^{-\frac{1}{2}}$$
$$+e^{-i(\omega_2-\frac{k'}{\hbar})t}\left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}}\cdot\left(\frac{a-k'}{c'}\right)\cdot\left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}}$$

$$= e^{-i(\omega_2+\frac{k'}{\hbar})t}\left(1+\left(\frac{a+k'}{c'}\right)^2\right)^{-1}\cdot\left(\frac{a+k'}{c'}\right)$$
$$+ e^{-i(\omega_2-\frac{k'}{\hbar})t}\left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-1}\cdot\left(\frac{a-k'}{c'}\right).$$

$$\beta(t) = e^{-i(\omega_2+\frac{k'}{\hbar})t}\left(1+\left(\frac{a+k'}{c'}\right)^2\right)^{-1}\cdot\left(\frac{a+k'}{c'}\right)^2$$
$$+e^{-i(\omega_2+\frac{k'}{\hbar})t}\left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-1}\left(\frac{a-k'}{c'}\right)^2$$

- We previously derived that for $|\psi\rangle = \alpha|g,n+1\rangle + \beta|e,n\rangle$:

$$\rho_{\text{HO}} = \alpha\alpha^*|n+1\rangle\langle n+1| + \beta\beta^*|n\rangle\langle n|$$

$$\rho_{\text{TLS}} = \alpha\alpha^*|g\rangle\langle g| + \beta\beta^*|e\rangle\langle e|$$

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· Thus, our time-evolved state $|\psi(t)\rangle = \alpha(t)|g, n+1\rangle + \beta(t)|e,n\rangle$ has subspaces:

$$\rho_{HO}(t) = \alpha(t)\alpha^*(t)|1\rangle\langle 1| + \beta(t)\beta^*(t)|0\rangle\langle 0|$$

$$\rho_{TLS}(t) = \alpha(t)\alpha^*(t)|g\rangle\langle g| + \beta(t)\beta^*(t)|e\rangle\langle e|$$

· Let's calculate $\alpha(t)\alpha^*(t)$ and $\beta(t)\beta^*(t)$:

$$\alpha(t)\alpha^*(t) = \left(e^{-i(\omega_2 + \frac{k'}{\hbar})t} \cdot \eta_+ + e^{-i(\omega_2 - \frac{k'}{\hbar})t} \cdot \eta_-\right)$$
$$\times \left(e^{i(\omega_2 + k'/\hbar)t} \cdot \eta_+ + e^{i(\omega_2 - k'/\hbar)t} \cdot \eta_-\right)$$

where $\eta_\pm = \left(1 + \left(\frac{a \pm k'}{c'}\right)^2\right)^{-1} \cdot \left(\frac{a \pm k'}{c'}\right)$

$$\alpha(t)\alpha^*(t) = \eta_+^2 + \eta_-^2 + e^{-ik'/\hbar\, t}\eta_+\eta_- + e^{i\frac{k'}{\hbar}t}\eta_+\eta_-$$
$$= \eta_+^2 + \eta_-^2 + 2\eta_+\eta_- \cos\left(\frac{k't}{\hbar}\right)$$

$$\beta(t)\beta^*(t) = \left(e^{-i(\omega_2 + \frac{k'}{\hbar})t}\eta_+\left(\frac{a+k'}{c'}\right) + e^{-i(\omega_2 - \frac{k'}{\hbar})t}\eta_-\left(\frac{a-k'}{c'}\right)\right)$$
$$\times \left(e^{i(\omega_2 + \frac{k'}{\hbar})t}\eta_+\left(\frac{a+k'}{c'}\right) + e^{i(\omega_2 - \frac{k'}{\hbar})t}\eta_-\left(\frac{a-k'}{c'}\right)\right)$$

$$= \eta_+^2\left(\frac{a+k'}{c'}\right)^2 + \eta_-^2\left(\frac{a-k'}{c'}\right)^2 + e^{-i\frac{k'}{\hbar}t}\eta_+\eta_-\left(\frac{a+k'}{c'}\right)\left(\frac{a-k'}{c'}\right)$$
$$+ e^{i\frac{k'}{\hbar}t}\eta_+\eta_-\left(\frac{a-k'}{c'}\right)\left(\frac{a+k'}{c'}\right)$$

$$= \eta_+^2\left(\frac{a+k'}{c'}\right)^2 + \eta_-^2\left(\frac{a-k'}{c'}\right)^2 + 2\cos\left(\frac{k'}{\hbar}t\right)\eta_+\eta_-\left(\frac{a+k'}{c'}\right)\left(\frac{a-k'}{c'}\right)$$

In summary:

$$\alpha(t)\alpha^*(t) = \eta_+^2 + \eta_-^2 + 2\eta_+\eta_- \cos\left(\frac{k'}{\hbar}t\right)$$

$$\beta(t)\beta^*(t) = \eta_+^2\left(\frac{a+k'}{c'}\right)^2 + \eta_-^2\left(\frac{a-k'}{c'}\right)^2 + 2\eta_+\eta_-\left(\frac{a+k'}{c'}\right)\left(\frac{a-k'}{c'}\right)\cos\left(\frac{k'}{\hbar}t\right)$$

For the reduced states:

$$\rho_{HO}(t) = \alpha(t)\alpha^*(t)\,|1\rangle\langle 1| + \beta(t)\beta^*(t)\,|0\rangle\langle 0|$$

$$\rho_{TLS}(t) = \alpha(t)\alpha^*(t)\,|g\rangle\langle g| + \beta(t)\beta^*(t)\,|e\rangle\langle e|$$

which were derived from an initial state $|\psi(0)\rangle = |e,0\rangle$ where $n=0$.

where $$\eta_\pm = \left(1 + \left(\frac{a \pm k'}{c'}\right)^2\right)^{-1}\left(\frac{a \pm k'}{c'}\right)$$

$$a = \hbar\left(\frac{\omega_1 - \omega_2}{2}\right)$$

$$k' = \hbar\sqrt{\left(\frac{\omega_1^2 + \omega_2^2}{4} - \frac{\omega_1\omega_2}{2} + G^2\right)}$$

$$c' = \hbar G$$