Density Matrix and Reduced Density Matrix of a TLS-HO Composite System

- Previously, we calculated the evolved state written in the eigenbasis.

- Now, we will calculate the density and reduced density matrices for the unevolved state and evolved state.

1. Density Matrix for basis states

- Recall $|\psi\rangle = \alpha|g,n+1\rangle + \beta|e,n\rangle$.

- The density matrix is the sum of the states written as projectors. Since we have a pure state (only one state associated with system):

$$\rho = |\psi\rangle\langle\psi|$$

$$= \alpha\alpha^*|g,n+1\rangle\langle g,n+1| + \beta\beta^*|e,n\rangle\langle e,n|$$

$$+ \alpha\beta^*|g,n+1\rangle\langle e,n| + \alpha^*\beta|e,n\rangle\langle g,n+1|$$

- Written in matrix form:

$$\rho = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \alpha^*\beta & \beta\beta^* \end{pmatrix} //$$

- Now, for the reduced density matrices, we 'trace' out a subsystem to get a density matrix in terms of the other.

$$\rho_{HO} = Tr_{TLS}(\rho) = \langle g|\rho|g\rangle + \langle e|\rho|e\rangle$$

$$= \alpha\alpha^*\langle g|g,n+1\rangle\langle g,n+1|g\rangle$$

$$+ \beta\beta^*\langle e|e,n\rangle\langle e,n|e\rangle = \alpha\alpha^*|n+1\rangle\langle n+1| + \beta\beta^*|n\rangle\langle n|$$

· And for the subsystem for TLS:

$$\rho_{TLS} = Tr_{HO}(\rho) = \langle n|\rho|n\rangle + \langle n+1|\rho|n+1\rangle$$
$$= \alpha\alpha^* |g\rangle\langle g| + \beta\beta^* |e\rangle\langle e|$$

· So, in summary:

$$\rho = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \alpha^*\beta & \beta\beta^* \end{pmatrix}$$

$$\rho_{HO} = \alpha\alpha^* |n+1\rangle\langle n+1| + \beta\beta^* |n\rangle\langle n|$$

$$\rho_{TLS} = \alpha\alpha^* |g\rangle\langle g| + \beta\beta^* |e\rangle\langle e|$$

2. Density Matrix for evolved state

· We derived $|\psi(t)\rangle$ previously to be:

$$|\psi(t)\rangle = \begin{bmatrix} e^{-i(\omega_2+k')t}\left(1+\left(\frac{a+k'}{c'}\right)^2\right)^{-\frac{1}{2}}\cdot\left(\frac{a+k'}{c'}\right) \\ e^{-i(\omega_2-k')t}\left(1+\left(\frac{a-k'}{c'}\right)^2\right)^{-\frac{1}{2}}\cdot\left(\frac{a-k'}{c'}\right) \end{bmatrix}$$

where $k' = \hbar\sqrt{\left(\frac{\omega_1^2+\omega_2^2}{4} - \frac{\omega_1\omega_2}{2} + G^2\right)}$

$$a = \hbar\left(\frac{\omega_1-\omega_2}{2}\right)$$

$$c' = \hbar G$$

ω_1 is the frequency associated to the TLS
ω_2 is the frequency associated to the HO.

· The evolved state is written in terms of the eigenbasis $\{|V_+\rangle, |V_-\rangle\}$. Furthermore, it is associated with an initial state of $|\psi(0)\rangle = |e,0\rangle$.

· We can thus write $|\psi(t)\rangle$ as a superposition in the eigenbasis.

$$|\psi(t)\rangle = \eta_+ |v_+\rangle + \eta_- |v_-\rangle$$

where $\eta_\pm = e^{-i(\omega_2 \pm k')t} \left(1 + \left(\frac{a \pm k'}{c'}\right)^2\right)^{-\frac{1}{2}} \cdot \left(\frac{a \pm k'}{c'}\right)$

· We now carry out the same steps as for the basis states,

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

$$= \eta_+\eta_+^* |v_+\rangle\langle v_+| + \eta_-\eta_-^* |v_-\rangle\langle v_-|$$

$$+ \eta_+\eta_-^* |v_+\rangle\langle v_-| + \eta_+^*\eta_- |v_-\rangle\langle v_+|$$

where $\eta_\pm^* = e^{i(\omega_2 \pm k')t} \left(1 + \left(\frac{a \pm k'}{c'}\right)\right)^{-\frac{1}{2}} \cdot \left(\frac{a \pm k'}{c'}\right)$

· In matrix form:

$$\rho(t) = \begin{pmatrix} \eta_+\eta_+^* & \eta_+\eta_-^* \\ \eta_+^*\eta_- & \eta_-\eta_-^* \end{pmatrix} //$$

· And the subsystems are:

$$\rho_+(t) = \eta_+\eta_+^* |v_+\rangle\langle v_+|$$

$$\rho_-(t) = \eta_-\eta_-^* |v_-\rangle\langle v_-| \quad //$$