QEA3 Project 1: Melting Mozzarella Sticks

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Executive Summary

For this project we created a model of the temperature of a mozzarella stick being cooked in a deep fryer using first order ODEs solved using MATLAB. We were trying to answer the question of how long one should cook the stick in order to reach an ideal temperature on the inside. The model takes the second order PDE heat equation and descritizes it into a mesh which can then be solved as a coupled system. We used ODE45 in MATLAB to calculate the temperature over time of each part of the mesh. Then we visualized the temperature across the mozzarella stick on a contour plot as well as the temperature at specific "probe" points. In the end, we found that it took 9.4 minutes of deep frying for the internal temperature of the mozzarella stick to reach an ideal melting temperature.

Background

When we first started this project we were looking to model the velocity of water as it moves across a pipe. To do this we looked into the Navier-Stokes second order PDE. We looked into how to discritises this equation into a mesh of ODEs. Significant progress in the discretization was made but as the equation is nonlinear and has two variables that are differentiated with respect to multiple spatial directions, it made it complicated to turn into a system of first order ODE. We decided to pivot to analyzing the 2D flow of heat through an object as it still drew on the work we had done with the discretization of Navier-Stokes but would be easier to create an ODE from, since the second order PDE is in terms of only one unknown function: the temperature field.

Modeling

Governing Equation

Our model is centered around the 2D heat equation:

$$\frac{dT}{dt} = \alpha(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2})$$

Equation 1: The 2D heat equation [ref. 6]

This equation shows that the rate of temperature change in a temperature field is proportional to the Laplacian (sum of the second order spatial partial derivatives) of that temperature field. This is an appropriate model for heat flow through an object in two dimensions. Therefore, this equation is neither an ODE nor first order, instead, it is a second order partial differential equation. To solve this equation exactly would be outside the scope of the project as we have not learned how to solve PDEs. However, by discretizing the problem, we can decompose the equation into a large system of coupled first order linear differential equations.

Mesh

In order to discretize the problem, we began by defining a mesh of elements that make up the domain over which we will approximate the solution to the heat equations. This domain represents a rectangular cross section of the mozzarella stick when cut in half. The mesh is made up of 12,000 elements representing an area of 100mm by 30mm. Each element is 0.5mm in length and width.

Each element in the mesh has two properties: a temperature (T) and a diffusivity constant (α). The diffusivity constant allows us to define the material composition of mozzarella sticks such as having a layer of breading around the outside and mozzarella on the inside. We set the top and bottom 3mm of the mesh with a diffusivity equivalent to that of bread. To make our model more interesting, we chose to define a thicker breading (6mm) on the left and right edges of the rectangular domain. The remaining inner elements have a diffusivity equivalent to mozzarella.

The diffusivity is defined:

$$\alpha = \frac{K}{\rho C_v}$$

Equation 2: Definition of diffusivity in a discrete context, adapted from [ref. 7]

Where K is the thermal conductivity ($\frac{W}{m \cdot K}$), ρ is the density of the material ($\frac{kg}{m^3}$), and C_v is the specific heat at constant volume ($\frac{J}{kg \cdot K}$). Thus, α has units of $\frac{m^2}{s}$.

To calculate the diffusivity of the bread and mozzarella cheese, we found values for the thermal conductivity, density, and specific heat for each material. The volume of the mesh cells changes with the mesh density, so it is expressed in terms of the characteristic cell dimension, ℓ (m) and the thickness of the mozzarella stick, d (m). For our model, we used a constant mesh density, and we assumed that the mozzarella stick could be approximated as a rectangular prism, so L and d are constants across the mesh.

Material	$K\ (\frac{W}{m \cdot K})$	$\rho\ (\frac{kg}{m^3})$	$C_v (\frac{J}{kg \cdot K})$	$\frac{m^2}{s}$
Bread	0.072 [ref. 1]	590 [ref. 3]	3150 [ref. 1]	3.874×10^{-8}
Mozzarella Cheese	0.384 [ref. 4]	1100 [ref. 8]	2489 [ref. 4]	1.403×10^{-7}

Table 1: Material properties and diffusivities of bread and mozzarella cheese in SI units with ℓ and d both in meters.

In order to minimize errors due to Matlab's handling of very small numbers, we converted the diffusivity into units of $\frac{mm^2}{s}$ with ℓ and d now being in millimeters.

Material	$\alpha\ \frac{mm^2}{s}$
Bread	3.874×10^{-2}
Mozzarella Cheese	1.403×10^{-1}

Table 2: Diffusivities of bread and mozzarella cheese in $\frac{mm^2}{s}$ with ℓ and d both in millimeters.

Discretization

In order to take advantage of the discrete mesh, we can break down the second order partial derivatives from heat equation into first order equations using a clever approximation.

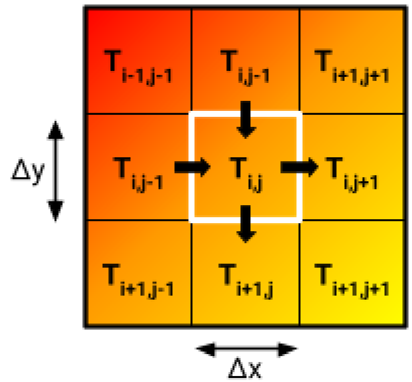

Figure 1: Mesh Discretization Diagram

$$\frac{\partial^2 T_{i,j}}{\partial x^2} \approx \frac{(T_{i,j+1} - T_{i,j}) - (T_{i,j} - T_{i,j-1})}{\Delta x^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta x^2}$$

$$\frac{\partial^2 T_{i,j}}{\partial y^2} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta y^2}$$

Equation 3: Discrete Derivative Approximation

By choosing a small enough mesh size (Δx and Δy), we can approximate the true solution to a high degree. The benefit of this simplification is that it allows us to express the rate of change of temperature of an element in terms of the current temperatures of the 4 neighboring elements. This effectively breaks down the problem into a scaled up version of the "thermal house" model where we use coupled ODES to model the temperature flow between two "elements" (floor and wall). The only difference is now our system has twelve thousand elements and therefore twelve thousand coupled equations to solve simultaneously rather than just two.

$$\frac{dT_{i,j}}{dt} = \alpha(\frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta x^2} + \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta y^2})$$

Equation 4: ODE describing the Temperature evolution over time of a typical element. This is a linear first order ODE.

MATLAB Implementation

We chose to use MATLAB's ODE45 to solve our coupled system. This was achieved by expressing the system as a series of matrices. The temperature matrix, for example, stores the temperature for each element while the K-matrix stores the thermal diffusivity for each element. Inside ODE45, we use these two matrices to evaluate a new matrix of derivatives for each element. ODE45 then uses these derivatives to update the temperature matrix as time increases.

In our model, we initialized the temperature of the inner elements to 273 K while the outer layer of elements are set to 450 K. This emulates the temperature of the mozzarella stick coming out of the freezer and being dropped in the deep fryer. Inside ODE45, we apply a "zero-flux" boundary condition to the outer layer of elements. This decision is embedded in the assumption that at the surface boundary of the mozzarella stick, the temperature will be equal to the temperature of the deep frying oil it is in contact with. In our discrete approximation of the mozzarella stick, these zero derivatives coupled with the initial temperatures means that the outer elements are held constant at 450K.

We also used an event function to halt the integration process once the temperature of the center element reached the “ideal” melting temperature of 325K.

The output of ODE45 is a tensor of different “temperature frames” over time. We can visualize the complete 2D temperature evolution by plotting/animating subsequent frames of data using the function `contorf`. We also chose to plot the temperature over time at specific probe points which helps to better understand how the mozzarella stick heats up.

Results

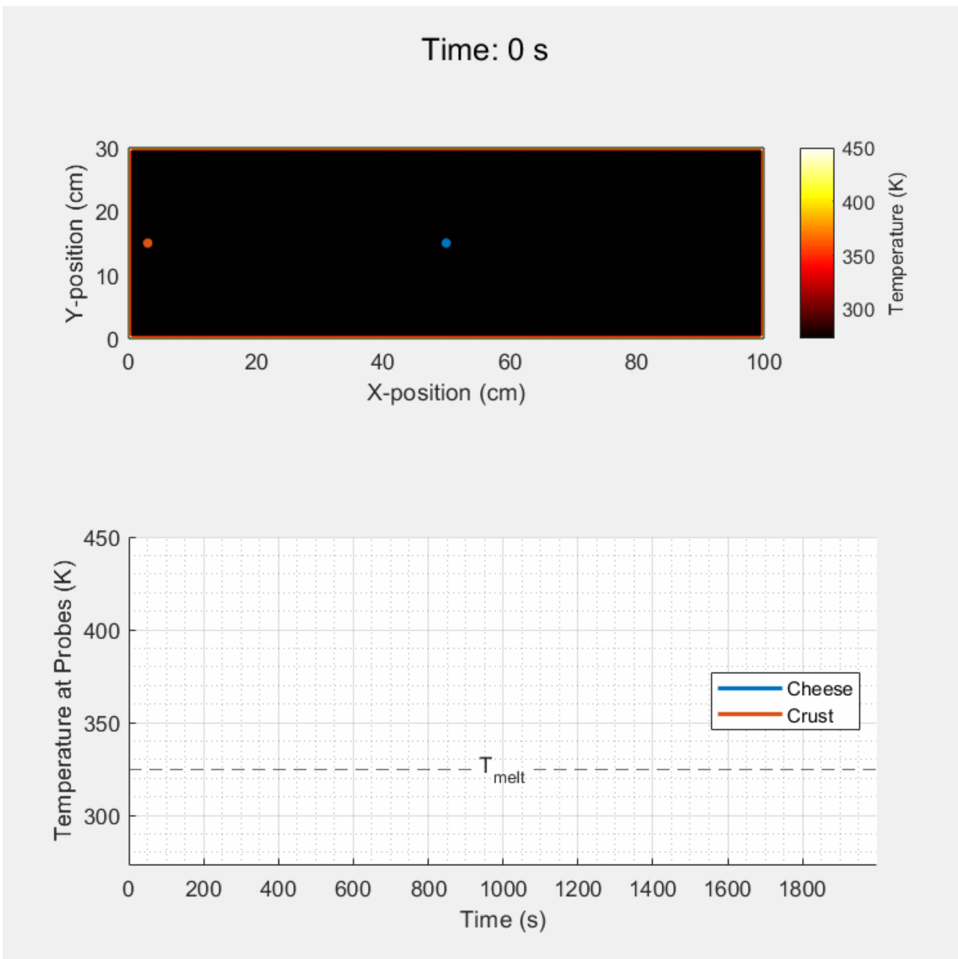

Figure 2: [Animation](#) of the contour and temperature plots of the mozzarella cheese and crust over time. The innermost mozzarella reached the target temperature after 564 seconds (9.4 minutes)

Mesh Refinement

One notable source of error in our modeling technique arises from the nature of using a discrete mesh to approximate the domain of the mozzarella stick. One way that we can ensure that this error is small is by performing a mesh refinement study.

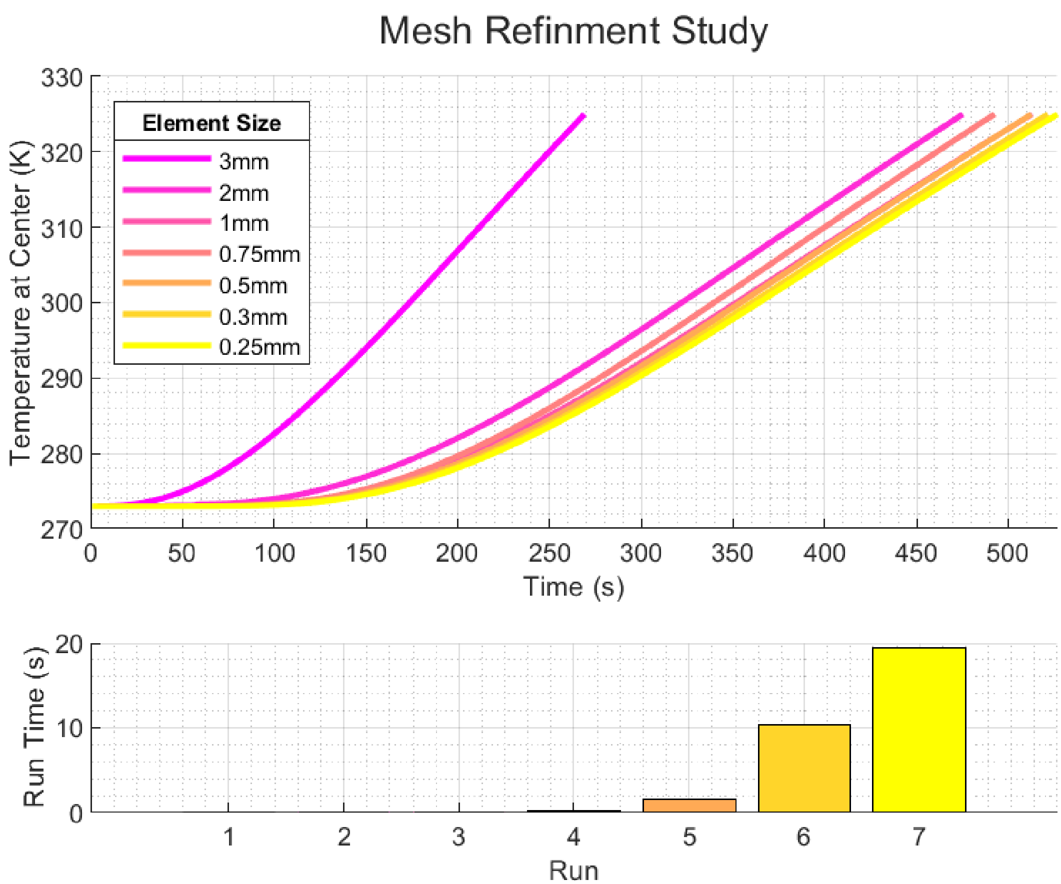

Figure 3: Mesh Refinement Study Results.

The results of the study show that a mesh size of 0.5mm is a good choice, producing results that are within a few seconds of a superfine mesh while being a fraction of the computational cost.

Discussion

The result we obtained from our model was that it took 9.4 minutes of deep frying the mozzarella stick for the inside to reach the melting temperature of mozzarella. This value is within the right order of magnitude of the expected solution (real recipes recommend about 2 minutes). There are several sources of error that could cause this discrepancy. The first source of error could be the diffusivity-value for the different ingredients of the mozzarella stick. It is more difficult to quantify the thermal properties of irregular materials like mozzarella and bread.

Another source of uncertainty comes from the physical size of the mozzarella stick. A larger stick will clearly take longer to heat up than a small one. Defining a more accurately sized domain could therefore help to improve the results. The discrete mesh contributes to some uncertainty but our mesh-refinement study indicates that this error is very small compared to the discrepancy we are seeing in cooking time.

It is also noteworthy that the crust gets hotter faster compared to the cheese. This is in agreement with the observation that the bread on a mozzarella stick is more cooked than the cheese inside.

Overall, this model was a great success as we had reasonable results and were able to refine our accuracy. We accomplished our goal of taking a second-order PDE and discretizing it into a mesh of first-order ODEs. More importantly, we gained a greater understanding of ODEs through the unique process of using them as a tool to understand a more complicated system.

Code

GitHub: https://github.com/rowan-jansens/QEA3_project1

References

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