Quantum Error Correction and Fault Tolerance

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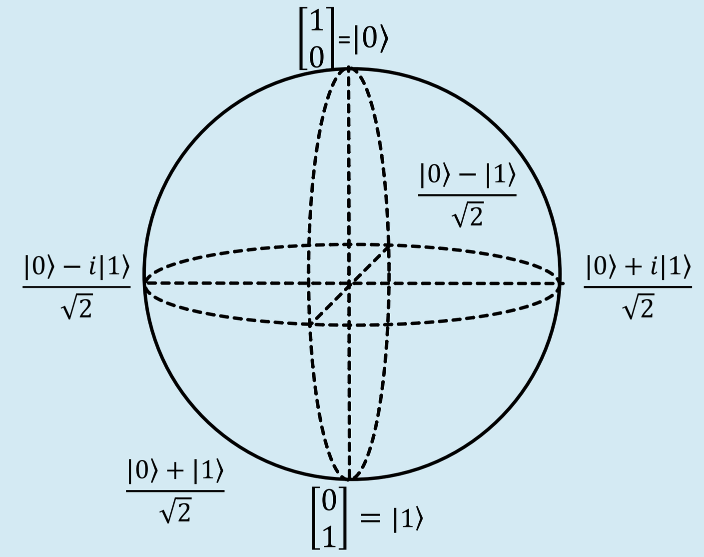
Quantum Computers

Quantum Mechanics: The state of a particle can be represented as a wavefunction, which describes the probability of finding it in a particular state. Physical qualities and measurements of the system are represented by applying operators.
The principle of quantum superposition asserts that a particle can be in a combination of many states at once – like overlapping waves on water. But when a measurement operator is applied the state collapses to a single value

$$|\varphi\rangle = a|0\rangle + b|1\rangle \to |\varphi\rangle = |0\rangle$$

Qubits: Qubits are the basic units of quantum computing: they use the principles of quantum superposition in order to represent many information states at the same time. Mathematically, they can be represented as a vector in a 2-dimensional Hilbert space. We also represent qubits as Bloch vectors, which make the spin-up and spin-down states, and the effects of various computational gates, easier to visualise – the state can be anywhere on the surface of the sphere. $|0\rangle$ and $|1\rangle$ are referred to as the computational basis



Dirac's Bra-Ket notation is a convenient way of representing the vectors in a quantum space. $|\varphi\rangle$ represents a state vector, and $\langle\omega|$ represents either a complex conjugate, or applying a linear function.

Computing Operators: Computers are operated using logic gates, like the classical case. These gates might be applied using a magnetic field, or a laser, for example. Some of these are based on the Pauli matrices, while others rotate the Bloch vector by an arbitrary angle.

Pauli X-Gate	**Pauli Y-Gate**	**Pauli Z-Gate**	**Hadamard Gate**
$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$

Surface Codes

Surface codes use an array of physical qubits to represent a logical qubit. These are arranged with alternating measurement and data qubits, so that the measurement qubits act on the data qubits adjacent to them. The measurement operators are divided between X and Z measurements to correspond to stabilisers from the Pauli group which lets the measurements be performed without collapsing the superposition

Errors are detected by looking for changes in measurements. The overall array is in a single quiescent state. Because each data qubit is measured by 2 X and 2 Z qubits, any error that affects them will anti-commute and change the result for 2 or more of the measurements.

To perform the X and Z operations on the whole logical qubit, a chain of X or Z operations are done across the array, which changes the quiescent state but not the overall measurement.

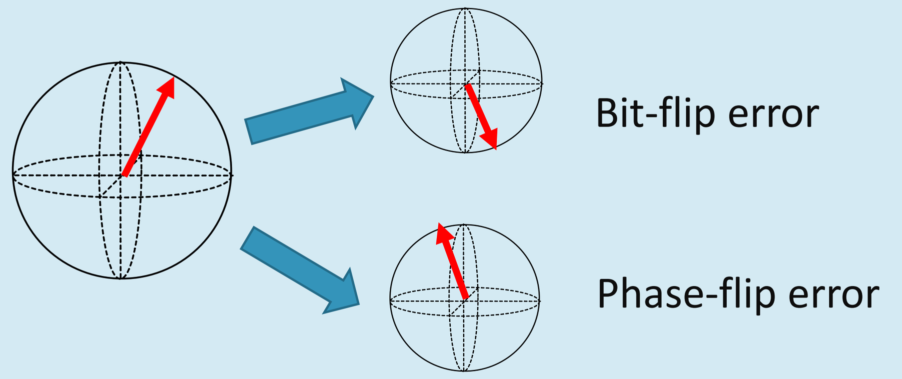
They have the advantage of having a higher error threshold than CSS codes, but also require a higher number of physical qubits to make a single logical qubit. Thousands or tens of thousands might be needed to implement a simple algorithm.

Error Models For Quantum Codes

The additional possible states for a qubit mean that there are more potential errors. As well as changing 1 to 0, the channel could also change 1 to -1, or change the amplitude of the superposition so that the chance of each state is closer to 50%.
Due to limited scope, I have been assuming that errors are a product of the Pauli matrices (X, Y, Z)and the identity, are equally likely to occur on each qubit, and the errors occur independently of each other. So for example, a bit being flipped in the third qubit would be represented

$$I \otimes I \otimes X$$

Some of the potential error causes for qubits are external heat sources that can disrupt effects like superconductivity that they rely on, and entanglement with the environment. The computer can't be completely isolated which makes an error-correcting protocol crucial.



CSS Codes

The Caldebank-Shor-Steane construction provides a way of constructing new quantum codes using classical linear codes – for example, the Steane code uses the Hamming [7,4] code and its dual code.

It is constructed using two linear codes, where C_2 is a subset of C_1. The codewords are cosets in C_1:

$$|c_i\rangle = \frac{1}{\sqrt{2^{k'}}}\sum_{x\in C_2}|x + c_i\rangle$$

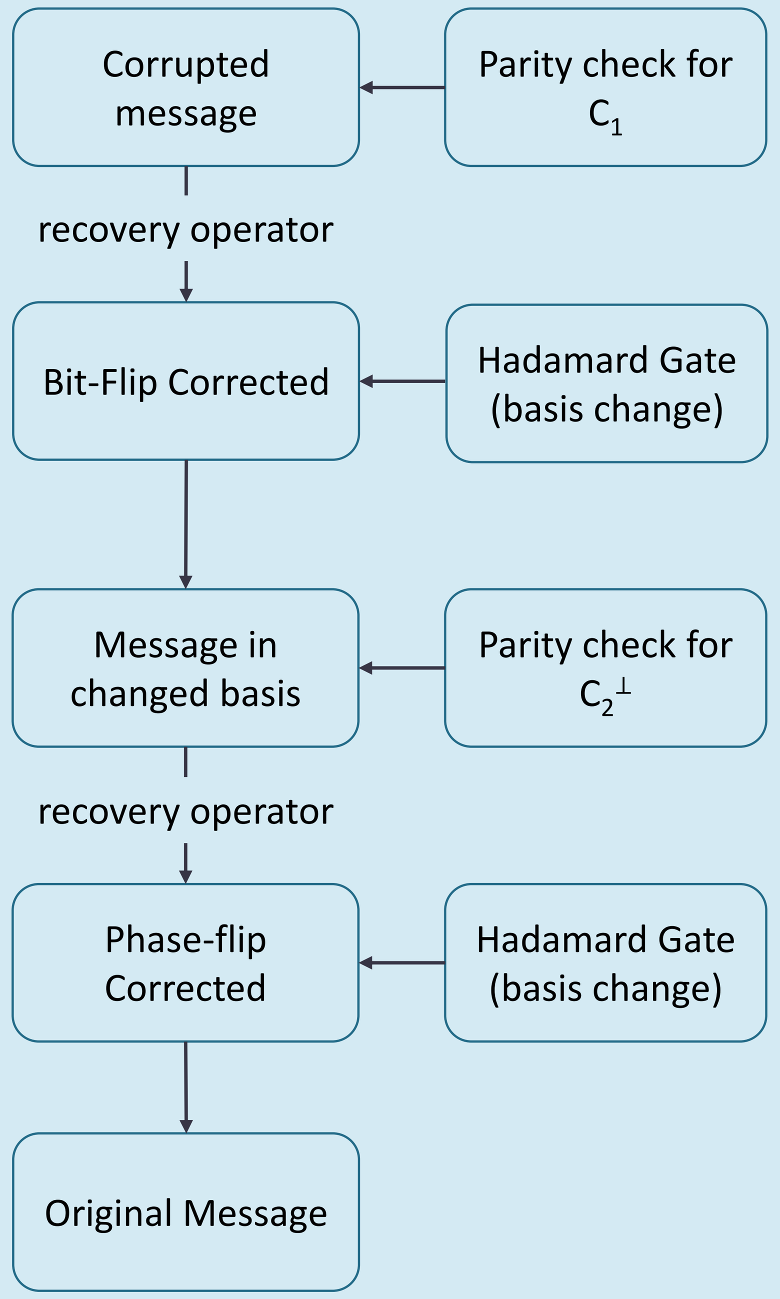
This allows both bit- and phase-flip errors to be corrected. The final state is

$$\frac{1}{\sqrt{2^{k'}}}\sum_{x\in C_2}(-1)^{(c+x)e_p}|c + x + e_b\rangle$$

Then the parity check matrices for C_1 and the dual code of C_2 are used to find and correct the errors as shown in the diagram.

The role of the Hadamard gate in this process if to switch the x and z axes in the Bloch vector, allowing the effect of the phase-flip error to be treated in the same way as the bit-flip error, rather than as an exponential.

These aren't the only possible ECCs for quantum computers. They also do not account for more complicated types of error.



References
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Webpage
rowanmrsm.github.io/index.html

(not coded responsively so I wouldn't recommend mobile viewing)

Fault Tolerance

Fault tolerance in a quantum computer is the property that, if a single error occurs due to a component failure, at most one error occurs in each qubit block. For example, if an error occurs during storage of a state, then applying a gate to the error state won't proliferate errors.
Designing fault tolerant circuits using error-correcting procedures allows quantum computing to be more robust when noise is encountered, and is therefore an important application of the development of quantum error correcting codes.
Logic gates are implemented fault-tolerantly by encoding them to match the encoded qubits. For the Steane [7,1] code, we would encode the universal logic gates as shown in the table, apart from the $\pi/8$ which has a more complex encoding

X	XXXXXXX
Z	ZZZZZZZ
CNOT	CNOT applied to each qubit in turn
Hadamard (H)	HHHHHHH
Phase (P)	(ZP)(ZP)(ZP)(ZP) (ZP)(ZP)(ZP)

For computing to be fault tolerant, we also need to be able to measure outcomes without propagating errors. To do this, extra qubits are used. Then a measurement operator is applied depending on the state of the extra qubit. This procedure is repeated and the majority result is accepted.

Accuracy Threshold Theorem

Suppose Q is the circuit being used to compute a function. Then for any ε>0, it is possible to find another circuit Q' which computes a function ε-close to C provided its error rate is below a certain threshold