14.19 Cheatsheet

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Welfare Theorem

First Welfare Theorem: Competitive equilibrium is Pareto optimal. Assumptions: 1. No externalities 2. Perfect competition 3. Perfect Information 4. Rational Agents **Second Welfare Theorem**: Any Pareto optimal allocation can be achieved by a competitive equilibrium with lump-sum transfers.

Coase and Pareto

A **competitive equilibrium** is a price and allocation pair (p,x) such that individuals maximize their utility given prices and prices clear markets (demand = supply) An allocation x is **Pareto optimal** if there is no other allocation x' such that one of the consumers is made strictly better off without hurting the other consumer. **Market failure** is When the competitive provision is not Pareto optimal.

Coase: if property rights are clearly specified and there are no transaction costs, bargaining will lead to an efficient outcome no matter how the rights are allocated

Arrow's Theorem

Universal Domain: Admits all of Π^n. **Pareto Efficiency:** $\forall a \neq b$ if all $a \succ_i b$ then $a \succ b$. **Independence of Irrelevant Alternatives:** $a \succ b$ is a function of $(a \succ_i b)_i$ only.

Arrow's Theorem: If $n \geq 3$, UD, PE, and IIA, then the only social welfare function is a dictatorship.

Social Choice Functions and Mechanisms

We now introduce a new formalism. Central planner would like to choose an alternative in A. Each agent i has type $t_i \in T_i$. The type of an agent is private information. Each agent i gets utility $u_i(t_i,A)$ if alternative A is chosen. Refer to $t=(t_i)_i \in \times_i T_i \equiv T$ as the **state of the world**, unknown by the central planner. A **social choice function** (SCF) is $f:T \to A$, which the planner would like to implement.

A **mechanism** is a pair (M,ϕ) where $M=(M_i)_i$ is a message space and $\phi: \times_i M_i \to A$ is a function. Each agent i submits a message $m_i \in M_i$. The outcome of the mechanism is $\phi(m)$. A mechanism **implements** f if for all t, there exists a **NASH EQUILIBRIUM** m such that $\phi(m)=f(t)$; one can also say "implements in dominant strategies," which is strictly stronger.

If each message space satisfies $M_i=T_i$, then we refer to ϕ as a **direct mechanism**; the set of messages a player can send is simply their set of possible types. A direct mechanism ϕ is **incentive compatible** if for all $t \in T$, t is a **NASH EQUILIBRIUM** of the mechanism game.

Weakly-dominant strategy: weakly dominant independent of deviations by other players; strictly stronger than Nash equilibrium.

Strategy Proof: if it is weakly-dominant-strategy incentive-compatible; revealing your type is a weakly dominant equilibrium (robust against deviations by yourself AND OTHER AGENTS). Typical examples of SP mechanisms are:

- a majority vote between two alternatives;
- a second-price auction when participants have quasilinear utility;
- a VCG mechanism when participants have quasilinear utility.

Typical examples of mechanisms that are not SP are:

- any deterministic non-dictatorial election between three or more alternatives;
- a first-price auction.

In particular, not that SP is not the same as BNIC (Bayesian Nash Incentive Compatible).

Revelation Principle: Any SCF which can be implemented by a Bayesian-Nash equilibrium can also be implemented by a BNIC mechanism. Any SCF which can be implemented in dominant-strategies can also be implemented by a SP mechanism.

Pareto Efficient if for any state of the world, it chooses a social outcome such that there exists no other social outcome that would make everybody weakly better off and at least one agent strictly better off. **Dictatorial** if there exists an agent i such that for all states $t \in T$, $\phi(t)$ is the top social alternative of i when his type is t_i.

Gibbard-Satterthwaite Theorem: If $|A| \geq 3$ is finite and types of agents are such that the utility functions represent *all strict preference rankings* on A, then any Pareto-efficient and strategy-proof mechanism is necessarily dictatorial.

Notably, utilities which allow for indifference between alternatives are not covered by this theorem.

Vickrey-Clarke-Groves Mechanism

The Vickrey-Clarke-Groves (VCG) mechanism is a method for achieving efficient outcomes in social choice problems where participants have private information about their preferences. The mechanism ensures that individuals truthfully report their preferences, leading to a socially efficient outcome. Here's how the VCG mechanism works, with key equations:

1. **Social Choice Setting** - Let there be a set of agents $N=\{1,2,...,n\}$. - Let D represent the set of possible social outcomes (or decisions). - Each agent $i \in N$ has a private type t_i, which determines their utility function $v_i(t_i,d)$, where $d \in D$ is the social outcome.
2. **Efficient Outcome****** The goal of the mechanism is to choose a social outcome $d \in D$ that maximizes the total utility (social welfare) of all agents. The efficient decision $d^*(t)$ is the one that maximizes the sum of the utilities of all agents:
$$d^*(t) = \operatorname*{argmax}_{d \in D} \sum_{i=1}^{n} v_i(t_i,d)$$
where $t=(t_1,t_2,...,t_n)$ is the vector of agents' types.
3. **Transfers (Payments)** Each agent receives a monetary transfer designed to incentivize truthful reporting of their preferences. The transfer for agent i is determined as follows:
$$y_i(t) = \sum_{j \neq i} v_j(t_j,d^*(t)) - \sum_{j \neq i} v_j(t_j,d^*(t_{-i}))$$
where: - $d^*(t)$ is the efficient outcome when all agents' types are reported. - $d^*(t_{-i})$ is the efficient outcome when agent i is excluded from the decision-making process (i.e., $t_{-i}=(t_1,t_2,...,t_{i-1},t_{i+1},...,t_n)$). - The transfer $y_i(t)$ represents the difference in the welfare of the other agents when agent i is included versus when agent i is excluded. This is called the **marginal contribution** of agent i to society.
4. **Individual Utility** The utility that agent i derives from participating in the mechanism is:
$$u_i(t_i,t) = v_i(t_i,d^*(t)) + y_i(t)$$
This utility consists of two parts: the direct utility from the chosen outcome $d^*(t)$, and the transfer $y_i(t)$.
5. **Truthful Reporting as a Dominant Strategy** In the VCG mechanism, truthful reporting of an agent's type t_i maximizes their utility, as their payment y_i is designed to

account for their contribution to the overall welfare. Thus, truth-telling is a **dominant strategy** for each agent.

The mechanism is **efficient** because it is strategy-proof (it is not GTO to deviate from telling the truth if nobody else deviates) and conditioned on truth-telling it selects the utilitarian alternative (trivially).

Groves: Adding some $x_i(t_{-i})$ to the transfer rule y_i does not affect the efficiency of VCG; hence all of these are strategy-proof. **Holmstrom-Green-Laffont**: If $d: T \to A$ is efficient, (d,y) is strategy-proof, and all type spaces are complete, then the transfer rule must be VCG (plus constants).

Housing Market

1. **Housing Market** (Shapley - Scarf, 1974):
$$\langle I, H, \succ, \mu \rangle$$
where: • I: set of agents. • H: set of houses with $|H| = |I|$. • $\succ$: list of strict preferences over houses. • μ: initial endowment matching. • **Pareto Efficiency**: Matching μ is Pareto efficient if there is no matching ν such that:
$$\mu(i) \succeq_i \nu(i) \quad \forall i \in I, \quad \text{and} \quad \mu(i) \succ_i \nu(i) \quad \text{for some } i \in I$$
2. **Efficiency Equilibrium**: A matching μ is efficient if there exists a price vectors π such that $\forall i,j$ $\mu(j) \prec_i \mu(i)$ implies $p(\mu(j)) > p(\mu(i))$ i.e. nobody can afford a better house.
A **competitive equilibrium** is just an efficient matching ν which respects the initial endowment; $p(\nu(i)) = p(\mu(i))$ for all i.
3. **Core of the Housing Market**: A matching η is in the core if there is no coalition $T \subseteq I$ and matching $\nu \neq \mu$ such that:
$$\nu(i) \in \{h_i\}_{i \in T}, \quad \nu(i) \succeq_i \eta(i) \quad \forall i \in T$$
i.e. a subset which can weakly improve through internal collusion. (If we only consider subsets of size 1, then this is called **individually rational**).
4. **Gale's Top Trading Cycles (TTC) Algorithm**: • Agents point to the owner of their favorite house. • A cycle is formed, and each agent in the cycle is assigned their most preferred house. • This process is repeated until all agents are assigned a house.
5. **Core Properties**: • The outcome of Gale's TTC algorithm is the unique matching in the core. • This matching is the unique competitive allocation; there exists a price vector π that supports this outcome and respetcs the initial endowment. • A direct mechanism is **strategy-proof** if truth-telling is a dominant strategy. Core is SP.
6. **Ma 1994**: Core is the only mechanism which is PE, IR, and SP.
7. **Serial Dictatorship**: A serial dictatorship mechanism ϕ_f assigns houses based on a priority ordering f. The agent with highest priority gets their top choice, followed by others in sequence:
$$\phi_f[\succ] = \mu.$$
SD is obviously PE and SP.