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last modified: 03/31/2014

Problem Category

- Statistical Pattern Recognition
- Supervised Learning
- Parametric Learning
- Bayes Decision Theory
- Multivariate data (2-dimensional)
- 2-class problem
- different variances
- equal prior probabilities
- Gaussian model (2 parameters)
- with conditional Risk (1-0 loss functions)

Sections

Given information:

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model: continuous univariate normal (Gaussian) model for the class-conditional densities

$p(\vec{x}|\omega_j) \sim N(\vec{\mu}|\Sigma)$

$p(\vec{x}|\omega_j) \sim \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\bigg[-\frac{1}{2}(\vec{x}-\vec{\mu})^t\Sigma^{-1}(\vec{x}-\vec{\mu})\bigg]$

Prior probabilities:

$P(\omega_1) = P(\omega_2) = 0.5$

Loss functions:

where

$\lambda(\alpha_i|\omega_j) = \lambda_{ij}$,

the loss occured if $action_i$ is taken if the actual true class is ω_j (assuming that $action_i$ classifies sample as ω_i)

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The samples are of 2-dimensional feature vectors:

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Means of the sample distributions for 2-dimensional features:

$$\vec{\mu}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \vec{\mu}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Covariance matrices for the statistically independend and identically distributed ('i.i.d') features:

$$\Sigma_i = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 \end{bmatrix}, \ \Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \Sigma_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Deriving the decision boundary

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Bayes' Rule:

$$P(\omega_j|x) = \frac{p(x|\omega_j) * P(\omega_j)}{p(x)}$$

Risk Functions:

$$R(\alpha_1|\,\vec{x}) = \lambda_{11} P(\omega_1|\,\vec{x}) + \lambda_{12} P(\omega_2|\,\vec{x})$$

$$R(\alpha_2|\,\vec{x}) = \lambda_{21} P(\omega_1|\,\vec{x}) + \lambda_{22} P(\omega_2|\,\vec{x})$$

with 1-0 loss function:

$$R(\alpha_1|\,\vec{x}) = P(\omega_2|\,\vec{x}) = 1 - P(\omega_1|\,\vec{x})$$

$$R(\alpha_2|\,\vec{x}) = P(\omega_1|\,\vec{x}) = 1 - P(\omega_2|\,\vec{x})$$

Discriminant Functions:

The goal is to maximize the discriminant function, which we define as the posterior probability here to perform a **minimum-error classification** (Bayes classifier).

$$g_1(\vec{x}) = P(\omega_1|\,\vec{x}), \quad g_2(\vec{x}) = P(\omega_2|\,\vec{x})$$

$$\Rightarrow g_1(\vec{x}) = P(\vec{x}|\,\omega_1) \ \cdot \ P(\omega_1) \quad |\, ln$$
$$g_2(\vec{x}) = P(\vec{x}|\,\omega_2) \ \cdot \ P(\omega_2) \quad |\, ln$$

We can drop the prior probabilities (since we have eual priors in this case):

$$\Rightarrow g_1(\vec{x}) = ln(P(\vec{x}|\,\omega_1))$$
$$g_2(\vec{x}) = ln(P(\vec{x}|\,\omega_2))$$

$$\Rightarrow g_1(\vec{x}) = \vec{x}^{\,t} - \frac{1}{2}\,\Sigma_1^{-1}\vec{x} + \left(\Sigma_1^{-1}\vec{\mu}_1\right)^t + \left(-\frac{1}{2}\,\vec{\mu}_1^{\,t}\Sigma_1^{-1}\vec{\mu}_1 - \frac{1}{2}\,ln(|\Sigma_1|)\right)$$

$$g_2(\vec{x}) = \vec{x}^{\,t} - \frac{1}{2}\,\Sigma_2^{-1}\vec{x} + \left(\Sigma_2^{-1}\vec{\mu}_2\right)^t + \left(-\frac{1}{2}\,\vec{\mu}_2^{\,t}\Sigma_2^{-1}\vec{\mu}_2 - \frac{1}{2}\,ln(|\Sigma_2|)\right)$$

Let:

$$\vec{W}_i = -\frac{1}{2}\,\Sigma_i^{-1}$$

$$\vec{w}\,i = \left(\Sigma_i^{-1}\vec{\mu}_i\right)^t$$

$$\omega_{i0} = \left(-\frac{1}{2}\,\vec{\mu}_i^{\,t}\Sigma_i^{-1}\vec{\mu}_i - \frac{1}{2}\,ln(|\Sigma_i|)\right)$$

$$\vec{W}_1 = \begin{bmatrix} (1/4) & 0 \\ 0 & (1/4) \end{bmatrix}$$

$$\vec{w}_1 = \begin{bmatrix} (1/4) & 0 \\ 0 & (1/4) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\omega_{10} = -\frac{1}{2}\begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}(1/4) & 0 \\ 0 & (1/4)\end{bmatrix} \cdot \begin{bmatrix}0 \\ 0\end{bmatrix} - ln(2) = -ln(2)$$

$$\vec{W}_2 = \begin{bmatrix}(-1/2) & 0 \\ 0 & (-1/2)\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}1 \\ 2\end{bmatrix}$$

$$\omega_{20} = -\frac{1}{2}\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}(1/4) & 0 \\ 0 & (1/4)\end{bmatrix} \cdot \begin{bmatrix}1 \\ 2\end{bmatrix} - \frac{1}{2} ln(1) = -2.5$$

$$\Rightarrow g_1(\vec{x}) = \vec{x}^{\,t}\begin{bmatrix}(1/4) & 0 \\ 0 & (1/4)\end{bmatrix}\vec{x} + \begin{bmatrix}0 \\ 0\end{bmatrix}^t - ln(2) = \vec{x}^{\,t} - \frac{1}{4}\vec{x} - ln(2)$$

$$\Rightarrow g_2(\vec{x}) = \vec{x}^{\,t}\begin{bmatrix}(-1/2) & 0 \\ 0 & (-1/2)\end{bmatrix}\vec{x} + \begin{bmatrix}1 \\ 2\end{bmatrix}^t \vec{x} - 2.5 = \vec{x}^{\,t} - \frac{1}{2}\vec{x} + \begin{bmatrix}1 & 2\end{bmatrix}\vec{x} - 2.5$$

Decision Boundary

$g_1(\vec{x}) = g_2(\vec{x})$

$$\Rightarrow \vec{x}^{\,t} - \frac{1}{4}\vec{x} - ln(2) = \vec{x}^{\,t} - \frac{1}{2}\vec{x} + \begin{bmatrix}1 & 2\end{bmatrix}\vec{x} - 2.5 \quad \Big| \cdot 4$$

$$\Rightarrow \vec{x}^{\,t} - \vec{x} - 4ln(2) = \vec{x}^{\,t} - 2\vec{x} + 4\left(\begin{bmatrix}1 & 2\end{bmatrix}\vec{x}\right) - 10$$

$$\Rightarrow \begin{bmatrix}x_1 & x_2\end{bmatrix} \cdot \left(-\begin{bmatrix}x_1 \\ x_2\end{bmatrix}\right) - 4ln(2) = \begin{bmatrix}x_1 & x_2\end{bmatrix} \cdot \left(-\begin{bmatrix}2x_1 \\ 2x_2\end{bmatrix}\right) + \begin{bmatrix}4 & 8\end{bmatrix} \cdot \begin{bmatrix}x_1 \\ x_2\end{bmatrix} - 10$$

$$\Rightarrow -x_1^2 - x_2^2 - 4ln(2) = -2x_1^2 - 2x_2^2 + 4x_1 + 8x_2 - 10$$

$$\Rightarrow x_1^2 + x_2^2 - 4x_1 - 8x_2 - 4ln(2) + 10 = 0$$

Classifying some random example data

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In [1]:

```
%pylab inline

import numpy as np
from matplotlib import pyplot as plt

def decision_boundary(x_1):
    """ Calculates the x_2 value for plotting the decision boundary."""
    return 4 - np.sqrt(-x_1**2 + 4*x_1 + 6 + np.log(16))

def decision_rule(x_vec)
x_1^2 + x_2^2 - -4x_1 - 8x_2 - 4ln(2) + 10 = 0


# Generate 100 random patterns for class1
mu_vec1 = np.array([0,0])
cov_mat1 = np.array([[2,0],[0,2]])
x1_samples = np.random.multivariate_normal(mu_vec1, cov_mat1, 100)
mu_vec1 = mu_vec1.reshape(1,2).T # to 1-col vector

# Generate 100 random patterns for class2
mu_vec2 = np.array([1,2])
cov_mat2 = np.array([[1,0],[0,1]])
x2_samples = np.random.multivariate_normal(mu_vec2, cov_mat2, 100)
mu_vec2 = mu_vec2.reshape(1,2).T # to 1-col vector

# Scatter plot
f, ax = plt.subplots(figsize=(7, 7))
ax.scatter(x1_samples[:,0], x1_samples[:,1], marker='o', color='green', s=40, alpha=0.5)
ax.scatter(x2_samples[:,0], x2_samples[:,1], marker='^', color='blue', s=40, alpha=0.5)
plt.legend(['Class1 (w1)', 'Class2 (w2)'], loc='upper right')
plt.title('Densities of 2 classes with 100 bivariate random patterns each')
plt.ylabel('x2')
plt.xlabel('x1')
ftext = 'p(x|w1) ~ N(mu1=(0,0)^t, cov1=2*I)\np(x|w2) ~ N(mu2=(1,2)^t, cov2=I)'
plt.figtext(.15,.8, ftext, fontsize=11, ha='left')
```

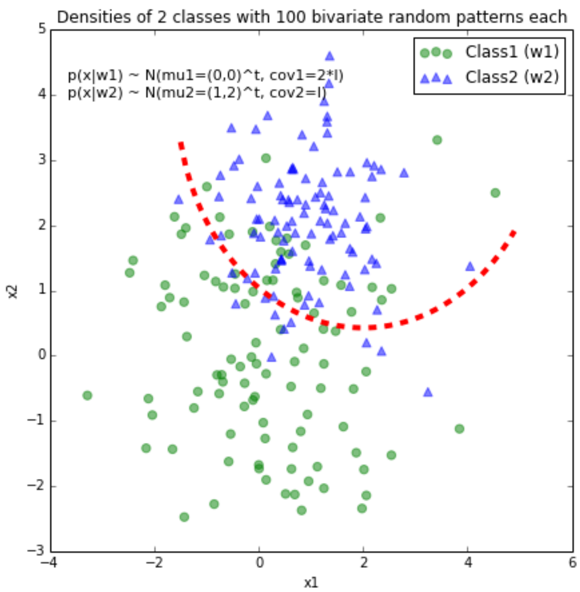
```
# Plot decision boundary
x_1 = np.arange(-5, 5, 0.1)
bound = decision_boundary(x_1)
plt.plot(x_1, bound, 'r--', lw=4)

x_vec = np.linspace(*ax.get_xlim())
x_1 = np.arange(0, 100, 0.05)

plt.show()
```

```
Populating the interactive namespace from numpy and matplotlib

-c:8: RuntimeWarning: invalid value encountered in sqrt
```



Calculating the empirical error rate

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In [17]:
```
def decision_rule(x_vec):
    """ Returns value for the decision rule of 2-d row vectors """
    x_1 = x_vec[0]
    x_2 = x_vec[1]
    return x_1**2 + x_2**2 -4*x_1 - 8*x_2 - 4*np.log(2) + 10

w1_as_w2, w2_as_w1 = 0, 0

for x in x1_samples:
    if decision_rule(x) < 0:
        w1_as_w2 += 1
for x in x2_samples:
    if decision_rule(x) > 0:
        w2_as_w1 += 1

emp_err = (w1_as_w2 + w2_as_w1) / float(len(x1_samples) + len(x2_samples))

print('Empirical Error: {}%'.format(emp_err * 100))
```

```
Empirical Error: 19.5%
```