Unraveling Coordination Problems*

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Abstract

We study policy design in coordination problems. In a global game, a subsidy raises player i's incentive to play the subsidized action. This raises j's incentive to play the same action, which further incentivizes i, and so on. Building upon this "unraveling effect", we characterize the subsidies that implement a given outcome of the game as its unique equilibrium. Subsidies are (i) symmetric for identical players, (ii) globally continuous in payoff parameters, (iii) increasing in opportunity costs, (iv) decreasing in spillovers, and (v) increasing in the planner's ambition. Applications of our model include joint investment projects, participation decisions, and principal-agent problems.

1 Introduction

Coordination problems arise throughout the economy and the inability of markets to solve these problems efficiently is a fundamental motivation for policy intervention. It is not obvious, however, what such interventions ideally looks like. This paper studies the optimal design of policies in coordination games.

The usual rationale for policy intervention is to correct market failures introduced by externalities. All market failures are not equal, however, and it is crucial for policy design to know the type of externality an intervention targets. One kind of externality arises when there exists a gap between the private and social value of behavior. Thus, an individual household's greenhouse gas emissions may be higher than socially optimal as it ignores the effects its emissions have on others. Such externalities can be addressed though Pigouvian taxes or subsidies. Another, more complicated kind of externality arises in coordination problems where individual actions are strategic complements (Bulow et al., 1985). Strategic complementarity results in multiple, Pareto-ranked equilibria and opens the door to coordination failures. Thus, renewable technologies may provide a viable replacement for fossil fuels but only if sufficient capacity is installed; there hence are multiple equilibria, and renewables might never mature despite their potential (and known) advantages. Such externalities cannot be solved through simple Pigouvian policy. The goal of this paper is to design optimal policies for coordination problems. To streamline the narrative, we focus on subsidies.

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The main result of this paper characterizes the subsidy scheme that induces coordination on a given outcome of the game as its unique equilibrium. We focus on the problem of a planner who offers subsidies to implement a symmetric equilibrium in which all players pursue identical strategies. The policy that achieves this admits a number of intuitive properties: subsidies are (i) symmetric for identical players; (ii) globally continuous in model parameters; (iii) increasing in opportunity costs; (iv) decreasing in coordination spillovers; (v) increasing in the planner's ambition. These properties carry over to principal-agent problems in which actions are not contractable, providing a counterpoint to the well-known results in the literature on principal-agent contracting.

The problem we consider consists of a planner and N (heterogeneous) players each of whom independently chooses an action from a binary action set $\{0, 1\}$. If player i plays 0, his payoff is c_i. When instead player i plays 1, his payoff is the sum of two components. The first component, x, depends only upon the player's own action and is uncertain. The second component, $w_i : \{0, 1\}^{N-1} \to \mathbb{R}$, describes the externalities that other players impose upon player i. We focus on coordination games and assume that w_i is increasing in the total number of players that play 1. The planner publicly announces subsidies to players who play 1. The problem of the planner is to find the vector of subsidies $\tilde{s} = (\tilde{s}_i)$ that induces coordination on $(1, 1, ..., 1)$ for all $x > \tilde{x}$, where $\tilde{x} \in \mathbb{R}$ is chosen by the planner.

Were a player informed about the actions played by his opponents, his problem would be trivial. Yet players are not typically informed about this, which complicates policy design as in coordination games as the incentives a policy creates depend critically upon a player's strategic beliefs. What obtains is a game with multiple Nash equilibria in which *strategic uncertainty* makes it hard for players to determine their payoff-maximizing action. Equilibrium multiplicity and the implied strategic uncertainty complicate the problem of the planner: even if a policy s makes coordination on $(1, 1, ..., 1)$ *an* equilibrium, there may yet be others. Unless the planner can coordinate play on her most-preferred equilibrium – a power economists have been reluctant to grant (*c.f.* Segal, 1999, 2003; Winter, 2004; Sakovics and Steiner, 2012; Bernstein and Winter, 2012; Halac et al., 2020, 2021) – the purpose of her policy is not simply to make the targeted outcome an equilibrium. Instead, it is to attract coordination on one, rather than another, equilibrium. It is hence impossible to separate the issue of policy design from that of equilibrium selection.

The approach toward equilibrium selection pursued here is to model our coordination problem as a global game (Carlsson and Van Damme, 1993). A global game is an incomplete information game in which players do not observe the true game they play but only a private and noisy signal of it. In our model, players do not observe the payoff parameter x; rather, each player i receives a private noisy signal x_i^ε of x. We consider regulatory environments in which the planner does not know x either, if she does, must commit to its policy before Nature draws x; hence, the planner's choice of policy cannot signal any private information the planner might have (Angeletos et al., 2006).[1] Given any vector of subsidies s, we show that rational players are forced to coordinate on a unique Bayesian Nash equilibrium in the global game. We use this result to characterize the (also unique) vector of subsidies $\tilde{s} = (\tilde{s}_i)$ that induces each player i to play 1 whenever x_i^ε. In the limit as signals become arbitrarily

[1]More specifically, the problem of our planner is not one of Bayesian persuasion or information design (c.f. Kamenica and Gentzkow, 2011; Bergemann and Morris, 2016; Ely, 2017; Mathevet et al., 2020).

precise, this implies that $\tilde{s}$ induces players to coordinate on $(1, 1, ..., 1)$ for all $x > \tilde{x}$, fulfilling the planner's ambition.

Our analysis highlights the strategic intricacies of policy in coordination games that drive an *unraveling effect*. When player i receives an subsidy for playing 1, his incentive to play 1 increases. His opponent, player j, understands this and consequently attaches a higher probability to the event that i plays 1; her own incentive to play 1, it follows, also increases. A rational player i is aware of the increased incentive of j, which makes playing 1 even more attractive to him. And so on. There are, in some sense, increasing returns to policy in coordination games. Given that subsidies are common knowledge, this results in an infinitely compounded feedback effect of policy that allows seemingly modest intervention to unravel coordination problems.

The unraveling effect has implications for policy design. In particular, we show that the subsidies necessary to implement the planner's targeted equilibrium are (sometimes substantially) less than the subsidies that would make the associated equilibrium strategy strictly dominant for every player. This result stands in contrast to the literature on policy design in coordination problems, which frequently emphasizes the need to subsidize at least a subset of player all the way toward strict dominance (*c.f.* Segal, 2003; Winter, 2004; Bernstein and Winter, 2012; Sakovics and Steiner, 2012; Halac et al., 2020).

In our model, uncertainty about payoff functions leads to prior uncertainty about the set of Nash equilibria in the game drawn and, indeed, the posterior efficient outcome of the game. Because the planner must commit to its subsidies before the uncertainty is resolved, this calls for a degree of policy restraint: high subsidies are too heavy handed and risks inducing coordination on an ex post inferior outcome. Under prior uncertainty, the planner should avoid picking winners.

Related literature.—A closely related paper is Sakovics and Steiner (2012). These authors study the joint investment problem in a global game where agents differ in their influence over the aggregate action, their benefit from project success, and their cost of investment. They find that an optimal policy fully subsidizes a subset of players, targeting those who matter most for project success and/or have least incentive to invest absent the policy. Another related paper is Halac et al. (2020), who study the problem of a firm that offers payments contingent on project success to differently sized investors; their results identify conditions under which larger investors receive higher per-dollar returns on investment, thus perpetuating inequalities. The focus on contingent per-dollar returns in Halac et al. (2020) is different from the approaches in Sakovics and Steiner (2012) and our paper, in which investment decisions are binary and subsidies are paid regardless of project success.

The investment problems studied by Sakovics and Steiner (2012) and Halac et al. (2020) are examples of so-called *games of regime change* – for other notable examples, see Morris and Shin (1998), Angeletos et al. (2007), Goldstein and Pauzner (2005), Basak and Zhou (2020), and Edmond (2013). Games or regime change are coordination games in which a status quo is abandoned, causing a discrete change in payoffs, once a sufficiently large number of agents take an action against it. Our problem reduces to a game of regime game under some additional assumptions on the externality structure, connecting our paper to this literature. Global games of regime change typically assume that model uncertainty pertains to the number of player necessary to overthrow the regime; conditional on the realized regime, payoffs are common knowledge (*c.f.* Morris and Shin, 1998; Goldstein and Pauzner, 2005; Angeletos et al., 2007; Sakovics and Steiner, 2012; Edmond, 2013; Basak and Zhou, 2020). Our analysis complements this approach. In the regime change version of our model, the critical number of players needed to change regime is common knowledge; however, conditional on the regime in place there is uncertainty about (relative) payoffs.

Our paper is also related to the literature on principal-agent contracting, see Winter (2004) and Halac et al. (2021) in particular. In the complete information environment considered by Winter (2004), an optimal mechanism is discriminatory – no two agents are rewarded equally even when agents are symmetric. Halac et al. (2020) extend the model in Winter (2004) to allow for asymmetries among the agents and private contract offers. As in our model, Halac et al. (2020) find that symmetric agents are offered identical rewards in a optimal contract.

Another literature to which this paper is related is the literature on contracting with externalities (e.g., Segal, 1999; Segal and Whinston, 2000; Bernstein and Winter, 2012). Segal (2003) and Bernstein and Winter (2012) consider complete information contracting problems that, save for the informational environment, are essentially equivalent to the game studied here. They establish optimality of the *divide and conquer* mechanism in which the planner first ranks all players; given the ranking, each player is offered a subsidy that incentivizes him to play the subsidized action assuming all players who precede him in the ranking also play this action while those after him do not. Bernstein and Winter (2012) derive the optimal ranking of players in such a policy. Like the mechanism derived in Winter (2004), an (optimal) divide and conquer scheme treats symmetric agents asymmetrically.

The rest of the paper is organized as follows. Section 2 introduces the model and the concepts needed for the analysis. Section 3 introduces the planner's problem and states out main result. Section 4 presents the core of the analysis. Various special cases and extensions of our model are discussed on Section 5. Section 6 discusses and concludes. All proofs are in the Appendix.

2 The Game

We consider a normal form game played by players in a set $\mathscr{N} = \{1, 2, ..., N\}$, indexed i, who simultaneously choose binary actions $a_i \in \{0, 1\}$. Given that a vector of actions $a \in \{0, 1\}^N$ played, let $A := \sum_{i \in \mathscr{N}} a_i$ denote the *aggregate action*. We similarly define $a_{-i} := a \setminus \{a_i\}$ and $A_{-i} = \sum_{j \neq i} a_j$. We define $\overline{a} = (1, 1, ..., 1)$ and $\underline{a} = (0, 0, ..., 0)$; $\overline{a}_{-i}$ and $\underline{a}_{-i}$ are defined in similar ways. When a is played, player i who chooses 1 in a gets payoff $w_i(a_{-i}) + x$; when instead player i chooses 0 in a, his payoff is c_i. Here, $w_i(a_{-i})$ describes the externalities on player i deriving from other players' actions. In the main analysis, we assume that w_i depends upon a_{-i} only through the aggregate A_{-i} and may occasionally write $w_i(A_{-i})$; Section 5.3 explores generalizations of our model to games in which externalities depend upon the exact subset of players who play 1. The variable x is a benefit parameter in players' payoff functions, and c_i is player i's the opportunity cost of playing 1. Combining these elements, the payoff to player i is given by

$$\pi_i(a \mid x) = \begin{cases} w_i(a_{-i}) + x & \text{if} \quad a_i = 1 \text{ in } a, \\ c_i & \text{if} \quad a_i = 0 \text{ in } a. \end{cases} \tag{1}$$

We study games with strategic complementarities meaning that $w_i(a_{-i})$ is increasing in a_{-i}, i.e. $w_i(n+1) \geq w_i(n)$ for all $n = 0, ..., N-2$. In the canonical example of a joint investment problem, the action $a_i = 1$ is interpreted as investment and c_i as the cost of investing (Sakovics and Steiner, 2012). Alternatively, actions might represent the choice to use of a particular kind of network technology and c_i is the cost differential between technologies (Björkegren, 2019). Or actions could describe the decisions to work or shirk by agents working on a common project such that c_i is agent i's cost of effort and w_i his (discrete) benefit from project success, see Winter (2004) and Halac et al. (2021). To reflect the generality of our approach, we shall mostly shy away from concrete interpretations and talk in terms of generic actions instead.

The above elements combined describe a game of complete information $\Gamma(x)$. In $\Gamma(x)$, we define a player's incentive $u_i(a_{-i})$ to choose 1 as the gain from playing 1, rather than 0, or

$$u_i(a_{-i} \mid x) = \pi_i(1, a_{-i} \mid x) - \pi_i(0, a_{-i} \mid x) = w_i(a_{-i}) + x - c_i. \tag{2}$$

Observe that, given a_{-i}, the incentive u_i to play 1 is strictly increasing in x. Denote $x_i^0 := c_i - w_i(0)$ and $x_i^N := c_i - w_i(N-1)$. One has $u_i(\overline{a}_{-i} \mid x_i^0) = u_i(\underline{a}_{-i} \mid x_i^N) = 0$. In other words, to each player i playing 1 is strictly dominant for all $x > \overline{x}_i^0$; playing 0 is strictly dominant for $x < \underline{x}_i^N$. We define $x^N := \sup\{x_i^N \mid i \in \mathscr{N}\}$ and $x^0 := \inf\{x_i^0 \mid i \in \mathscr{N}\}$. We also define $\underline{x} = \inf\{x_i^0 \mid i \in \mathscr{N}\}$ and $\overline{x} = \sup\{x_i^N \mid i \in \mathscr{N}\}$. We assume that $[\underline{x}, \overline{x}]$ is nonempty. For all x in $[\underline{x}, \overline{x}]$, $\Gamma(x)$ is a coordination game with multiple strict Nash equilibria.

Rather than assume common knowledge of x, we model the problem as a global game Γ^ε in which players do not observe x.[2] Instead, it is assumed that x is drawn from the uniform distribution on $[\underline{X}, \overline{X}]$ where $\underline{X} < \underline{x}$ and $\overline{X} > \overline{x}$ and that each player i receives a private noisy signal x_i^ε of x, given by:[3]

$$x_i^\varepsilon = x + \varepsilon_i. \tag{3}$$

We occasionally refer to x_i^ε as the player's type. Note that, conditional on the true game played, types are drawn independently.

In (3), the term ε_i captures the noise in i's private signal. It is common knowledge that ε_i is an i.i.d. draw from the uniform distribution on $[-\varepsilon, \varepsilon]$. We assume that ε is sufficiently small, in a way made precise in Section 4.1. Let $x^\varepsilon = (x_i^\varepsilon)$ denote the vector of signals received by all players, and let x_{-i}^ε denote the vector of signals received by all players but i, i.e. $x_{-i}^\varepsilon = (x_j^\varepsilon)_{j \neq i}$. Note that player i observes x_i^ε but neither x nor x_{-i}^ε. We write $\Phi^\varepsilon(\cdot \mid x_i^\varepsilon)$ for the joint probability function of $(x, x_j^\varepsilon)_{j \neq i}$ conditional on x_i^ε.

The timing of Γ^ε is as follow. First, Nature draws a true x. Second, each player i receives its private signal x_i^ε of x. Third, all players simultaneously choose their actions. Lastly, payoffs are realized according to the true x and the actions chosen by all players.

[2]Note that one could just as well make c_i uncertain, or $c_i - x$. The choice of x as unobserved variable is essentially random.

[3]In game theory, it is assumed that the game (in this case G^ε) is common knowledge; hence, the structure of the uncertainty (the joint distribution of x and all the signals x_j^ε), the possible actions and all the payoff functions are commonly known. For a formal treatment of common knowledge, see Aumann (1976).

2.1 Concepts and notation

Strategies and strict dominance. A strategy p_i for player i in Γ^ε is a function that assigns to any $x_i^\varepsilon \in [\underline{X} - \varepsilon, \overline{X} + \varepsilon]$ a probability $p_i(x_i^\varepsilon) \geq 0$ with which the player chooses action $a_i = 1$ when they observe x_i^ε. We write $p = (p_1, p_2, ..., p_N)$ for a strategy vector. Similarly, we write $p_{-i} = (p_j)_{j \neq i}$ for the vector of strategies for all players but i. A strategy vector p is *symmetric* is for every $i, j \in \mathscr{N}$ and every signal x^ε one has $p_i(x^\varepsilon) = p_j(x^\varepsilon)$. Conditional on the strategy vector p_{-i} and a private signal x_i^ε, the expected incentive to play 1 for player i is given by:

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) := \int u_i(p_{-i}(x_{-i}^\varepsilon) \mid x) \, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon). \tag{4}$$

We say that the action $a_i = 1$ is strictly dominant at x_i^ε if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) > 0$ for all p_{-i}. Similarly, the action $a_i = 0$ is strictly dominant (in the global game G^ε) at x_i^ε if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) < 0$ for all p_{-i}. When $a_i = a$ is strictly dominant, we say that $a_i = 1 - a$ is strictly dominated.

Conditional dominance. Let L and R be real numbers. The action $a_i = 1$ is said to be dominant at x_i^ε conditional on R if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) > 0$ for all p_{-i} with $p_j(x_j^\varepsilon) = 1$ for all $x_j^\varepsilon > R$, all $j \neq i$. Similarly, the action $a_i = 0$ is dominant at x_i^ε conditional on L if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) < 0$ for all p_{-i} with $p_j(x_j^\varepsilon) = 1$ for all $x_j^\varepsilon > L$, all $j \neq i$.

Increasing strategies. For $X \in \mathbb{R}$, let p_i^X denote the particular strategy such that $p_i^X(x_i^\varepsilon) = 0$ for all $x_i^\varepsilon < X$ and $p_i^X(x_i^\varepsilon) = 1$ for all $x_i^\varepsilon \geq X$. We will call p_i^X the *increasing strategy with switching point* X. By $p^X = (p_1^X, p_2^X, ..., p_N^X)$ we denote the strategy vector of increasing strategies with switching point X, and $p_{-i}^X = (p_j^X)_{j \neq i}$. Note that $a_i = 1$ is strictly dominant at x_i^ε conditional on R if and only if $u_i^\varepsilon(p_{-i}^R \mid x_i^\varepsilon) > 0$. Similarly, if $a_i = 0$ is strictly dominant at x_i^ε conditional on L then it must hold that $u_i^\varepsilon(p_{-i}^L \mid x_i^\varepsilon) < 0$. Generally, for a vector of real numbers (y_i) let $p^y = (p_i^{y_i})$ be a (possibly asymmetric) increasing strategy vector, and $p_{-i}^y = (p_j^{y_j})_{j \neq i}$.

Iterated elimination of strictly dominated strategies. Our solution concept is iterated elimination of strictly dominated strategies (IESDS). Eliminate all pure strategies that are strictly dominated, as rational players may be assumed never to pursue such strategies. Next, eliminate a player's pure strategies that are strictly dominated if all other players are known to play only strategies that survived the prior round of elimination; and so on. The set of strategies that survive infinite rounds of elimination are said to survive IESDS.

3 Optimal Subsidies

3.1 The Planner's Problem

We now introduce a social planner who wishes to implement coordination on $(1, 1, ..., 1)$ whenever $x > \tilde{x}$ for some real number $\tilde{x}$ which she – the planner – chooses. We call $\tilde{x}$ the *critical state.* The planner faces two constraints. First, she cannot condition her policy on the realization of x or players' signals thereof. One interpretation is that the planner must commit to her policy before Nature draws a true x and cannot change her policy afterward.[4]

[4] Laffont and Maskin (1982); Myerson and Satterthwaite (1983)

The second constraint upon the planner's problem has to do with the kinds of policies she can use. We consider situations in which the planner is unable to coordinate agents on her preferred equilibrium in a multiple equilibria setting. Instead, she has to rely on simple subsidies (or taxes) to create the appropriate incentives. The focus on simple instruments also means that policies cannot condition directly upon other players' actions. These are standard assumptions in the literature on policy or contract design for coordination problems (Segal, 2003; Winter, 2004; Bernstein and Winter, 2012; Sakovics and Steiner, 2012; Halac et al., 2020).

To streamline the narrative, we henceforth focus of subsidies as the planner's policy instrument. Let s_i denote the subsidy paid to a player i who chooses $a_i = 1$. Conditional on the subsidy s_i, player i's incentive to choose 1 becomes

$$u_i(a_{-i} \mid x, s_i) = u_i(a_{-i} \mid x) + s_i = w_i(a_{-i}) + x - c + s_i,$$

and the expected incentive, given the signal x_i^ε, is

$$\begin{aligned} u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s_i) &= \int u_i(p_{-i}(x_{-i}^\varepsilon) \mid x, s_i)\, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\ &= \int \left[u_i(p_{-i}(x_{-i}^\varepsilon) \mid x) + s_i\right] \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\ &= u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) + s_i. \end{aligned} \tag{5}$$

It is clear that a tax equal to s_i on playing 0 has the same effect on incentives. We note that (5) assumes observability of a_i; we maintain this assumption throughout most of the analysis. We consider the case of a principal-agent problem in which the vector of actions a is unobserved in Section 5.

3.2 Unique Implementation

We next define what is meant by implementing coordination on $(1, 1., , , 1)$ for all $x > \tilde{x}$. Given a vector of subsidies $s = (s_i)$, let $\Gamma^\varepsilon(s)$ denote the game Γ^ε in which the planner publicly commits to paying each player i who plays 1 a subsidy $s_i \in s$. Since the planner cannot condition her policy on x, and because players choose their actions before learning the true value of x, we can only define implementation in terms of players' signals. We will show that, for any s, there is a unique vector of real numbers $x(s) = (x_i(s))$ such that the vector of increasing strategies $p^{x(s)}$ is the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(s)$. If $\tilde{s}$ is the policy such that $x_i(\tilde{s}) = \tilde{x}$ for all $i \in \mathscr{N}$, we say that $\tilde{s}$ implements $(1, 1., , , 1)$ for all $x > \tilde{x}$. We call $\tilde{s}$ the optimal subsidies. The focus on unique equilibrium implementation is standard in the literature (*c.f.* Segal, 2003; Winter, 2004; Bernstein and Winter, 2012; Sakovics and Steiner, 2012; Halac et al., 2020, 2021).

Note that, as in Sandholm (2002, 2005), we do not impose that players must play an equilibrium of $\Gamma^\varepsilon(s)$. Instead, we depart from more primitive assumptions on players' strategic sophistication by requiring that none play a strategy that is iteratively dominated. We ultimately prove, in Proposition 5, that the strategy vector so obtained is also the unique equilibrium of the global game. Equilibrium play is hence obtained as a result rather than an assumption of our analysis.

Take some real number $\tilde{x}$. Given $\tilde{x}$, let $\tilde{s} = (\tilde{s}_i)$ denote the subsidy scheme such that each $\tilde{s}_i \in \tilde{s}$ is given by

$$\tilde{s}_i = c_i - \sum_{n=0}^{N-1} \frac{w_i(n)}{N} - \tilde{x}. \tag{$*$}$$

Under this scheme, the subsidy player i receives is equal to her opportunity cost of playing 1 in the critical state $(c_i - \tilde{x})$ minus the expected externality other players impose upon her assuming the aggregate action is uniformly distributed.

Theorem 1. *Given is $\tilde{x} \in \mathbb{R}$.*

(i) There exists a unique subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $p^{\tilde{x}}$.

(ii) For each $i \in \mathscr{N}$, the subsidy $\tilde{s}_i$ pursuant to the scheme is given by $(*)$.

Section 4 presents the building blocks of this result and develops some of the intuition. It also discusses the properties of $\tilde{s}$ and compares those to other proposals in the literature. Extensions and special cases are presented in Section 5.

4 Analysis

The goal of this section is as follows. First, we show that there is a unique increasing strategy vector that survives IESDS in $\Gamma^{\varepsilon}(s)$ for any vector of subsidies s. We then establish that this strategy vector is also the unique Bayesian Nash equilibrium of the game. We use these results to characterize the unique subsidy scheme $\tilde{s}$ that implements $p^{\tilde{x}}$.

4.1 Subsidies, Strategies, Selection

The following Lemma will prove highly useful throughout the analysis.

Lemma 1. *Given is a vector of real numbers (y_i) and the associated strategy vector $p^y = (p_i^{y_i})$. Then,*

(i) $u_i^{\varepsilon}(p_{-i}^y \mid x_i^{\varepsilon})$ is monotone increasing in x_i^{ε};

(ii) $u_i^{\varepsilon}(p_{-i}^y \mid x_i^{\varepsilon})$ is monotone decreasing in y_j, all $j \neq i$.

It is important to note that monotonicity of $u_i^{\varepsilon}(p_{-i}^y \mid x_i^{\varepsilon})$ in x_i^{ε} depends upon p_{-i}^y being increasing. For generic p_{-i}, $u_i^{\varepsilon}(p_{-i} \mid x_i^{\varepsilon})$ can be locally decreasing in x_i^{ε}.

The following lemma is a useful starting point for our analysis.

Lemma 2. *Given a subsidy policy $s = (s_i)$, consider the game $\Gamma^{\varepsilon}(s)$. (i) For each player i, the action $a_i = 1$ is strictly dominant at all $x_i^{\varepsilon} > \overline{x} - s_i + \varepsilon$. (ii) For each player i, the action $a_i = 0$ is strictly dominant at $x_i^{\varepsilon} < \underline{x} - s_i - \varepsilon$.*

Per Lemma 2, $u_i^\varepsilon(p_{-i} \mid \overline{X}, s) > 0$ for all p_{-i}. In particular, therefore, we have

$$u_i^\varepsilon(p_{-i}^{\overline{X}} \mid \overline{X}, s_i) > 0. \tag{6}$$

Let r_i^1 be the solution to

$$u_i^\varepsilon(p_{-i}^{\overline{X}} \mid r_i^1, s_i) = 0. \tag{7}$$

To any player i, the action $a_i = 1$ is strictly dominant at all $x_i^\varepsilon > r_i^1$ conditional on $\overline{X}$; denote $r^1 := (r_i^1)$. It is clear that r_i^1 depends upon the subsidy s_i, but for brevity we leave this dependence out of the notation for now. From Lemma 1 follows that $r_i^1 < \overline{X}$ for all i.

Player i knows that no player j will pursue a strategy $p_j < p_j^{r_j^1}$ since such a strategy is iteratively strictly dominated. Now define

$$u_i^\varepsilon(p_{-i}^{r^1} \mid r_i^2, s_i) = 0, \tag{8}$$

for all i. Because $p_i^{\overline{X}}$ is strictly dominated for every i, the any strategy $p_i < p_i^{r_i^1}$ is iteratively strictly dominated for all i, which in turn implies that any $p_i < p_i^{r_i^2}$ is iteratively dominated. This argument can – and should – be repeated indefinitely. We obtain a sequence $\overline{X} = r_i^0, r_i^1, ...,$ all i. For any k and r_i^k such that $u_i^\varepsilon(p^{r^k_{-i}} \mid r_i^k, s_i) > 0$, there exists r_i^{k+1} that solves $u_i^\varepsilon(p_{-i}^{r^k} \mid r_i^{k+1}, s_i) = 0$. Induction on k, using Lemma 1, reveals that $r_i^{k+1} < r_i^k$ for all $k \geq 0$. Per Lemma 2, we know that $r_i^k \geq \underline{X}$ for all k. It follows that the sequence (r_i^k) is monotone and bounded. Such a sequence must converge; let r_i denote its limit and define $r := (r_i)$. By construction, R solves

$$u_i^\varepsilon\left(p_{-i}^r \mid r_i, s_i\right) = 0. \tag{9}$$

A perfectly symmetric procedure should be carried out starting from low signals, eliminating ranges of x_i^ε for which playing 1 is strictly (iteratively) dominated. For every player i this yields an increasing and bounded sequence (l_i^k) whose limit is l_i, and $l := (l_i)$. The limit L solves

$$u_i^\varepsilon\left(p_{-i}^l \mid l_i, s_i\right) = 0. \tag{10}$$

From the construction of $l = (l_i)$ and $r = (r_i)$, the following is true by definition.

Lemma 3. *To each player i, a strategy p_i survives iterated elimination of strictly dominated strategies if and only if $p_i^{r_i}(x_i^\varepsilon) \leq p_i(x_i^\varepsilon) \leq p_i^{l_i}(x_i^\varepsilon)$ for all x_i^ε.*

The set of signals for which player i does not have a unique strategy that survives IESDS is $[l_i, r_i]$. We next show that this set has measure zero.

Lemma 4. *For each $i \in \mathscr{N}$, $l_i = r_i$.*

The next section uses Lemma 4 to characterize the subsidy scheme that, for given $\tilde{x} \in \mathbb{R}$, implements $p^{\tilde{x}}$. First, however, it is instructive to discuss the strategic effects of subsidies a bit more deeply.

So far, the subsidies $s = (s_i)$ were taken as given and therefore left out of the notation. To make the dependence of strategies on subsidies more explicit we henceforth write $x_i(s) = l_i = r_i$ for the unique switching point (given s), and $x(s) = (x_i(s))$, i.e. $(x_i(s))$ solves

$$u_i^\varepsilon\left(p_{-i}^{x(s)} \mid x_i(s), s_i\right) = 0 \tag{11}$$

for all $i \in \mathscr{N}$.

Consider the effect of s_i on $x_i(s)$. Given a player's signal x_i^ε and his opponents' strategies $p_{-i}^{x(s)}$, an increase in s_i raises player i's incentive to play 1; see (5). Because $x_i(s)$ must, by definition, solve (11), an increase in s_i causes $x_i(s)$ to decrease. Intuitively, a higher subsidy makes it more likely that player i will play 1. This is the immediate, or direct, effect of a subsidy.

The higher likelihood that i plays 1 has repercussion on other players' strategies; each player $j \neq i$ understands the effect of s_i on $x_i(s)$. Moreover, as the only strategy a rational player i can play is $p_i^{x(s)}$, the increased incentive of player i to play 1 also raises any player j' incentive to play 1; see Lemma 1. As $x_j(s)$ must, by definition, solve (11), an increase in s_i indirectly – through the associated reduction in $x_i(s)$ – leads to a decrease in $x_j(s)$. This is the indirect, or strategic, effect of a subsidy.

Just as a decrease in $x_i(s)$ leads to a decrease in $x_j(s)$, so – by the same logic – a decrease in $x_j(s)$ triggers a fall in $x_i(s)$. Hence, the increased incentive of player j to play 1 further raises i's incentive to play 1. What obtains is a positive feedback loop of indirect effects that, in a coordination game, keeps on compounding. Because of the compounded strategic effect of subsidies, even apparently modest subsidies can go a long way in unraveling coordination problems. See also Figure 1.

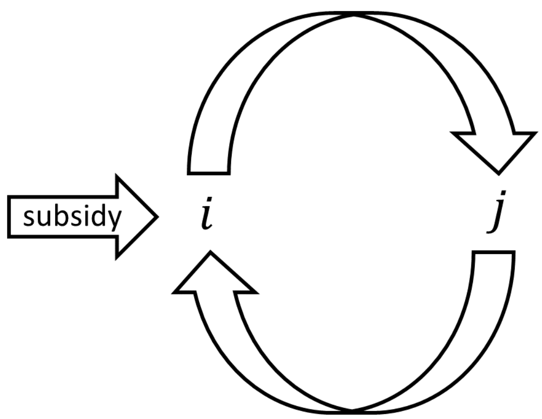

Figure 1: In coordination game, a subsidy kickstarts an infinitely compounded positive feedback loop on players' incentives to play the subsidized action: the unraveling effect.

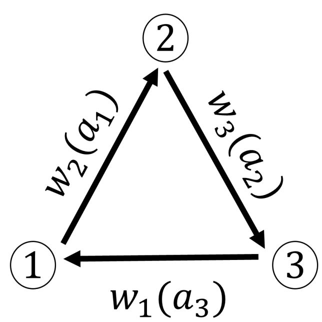

Figure 2:

In the next section, it is demonstrated that the unraveling effect of subsidies also describes the unique *equilibrium effect* of policy in a coordination game. We build upon this result to solve the planner's implementation problem.

4.2 Implementation and Characterization

Recall that a strategy vector $p = (p_1, p_2, ..., p_N)$ is a Bayesian Nash Equilibrium (BNE) of $\Gamma^\varepsilon(s)$ if for any p_i and any x_i^ε it holds that:

$$p_i(x_i^\varepsilon) \in \underset{a_i \in \{0,1\}}{\arg\max}\, \pi_i^\varepsilon(a_i, p_{-i} \mid x_i^\varepsilon, s_i), \tag{12}$$

where $\pi_i^\varepsilon(a_i, p_{-i} \mid x_i^\varepsilon) := \int \pi_i(a_i, p_{-i}(x_{-i}^\varepsilon) \mid x)\, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon)$. It follows immediately that $p^{x(s)}$ is a BNE of $\Gamma^\varepsilon(s)$. Lemma 5 strengthens this result and establishes that $p^{x(s)}$ is the *only* BNE of $\Gamma^\varepsilon(s)$.

Lemma 5. *Given is s. The essentially unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(s)$ is $p^{x(s)}$. In particular, if p a BNE of $\Gamma^\varepsilon(s)$ then any $p_i \in p$ satisfies $p_i(x_i^\varepsilon) = p_i^{x_i(s)}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x_i(s)$ and all i.*

If the strategy vector $p = (p_i)$ is a BNE of $\Gamma^\varepsilon(s)$, then for each p_i it must hold that $p_i(x_i^\varepsilon) = p_i^{x_i(s)}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x_i(s)$. Consequently, iterated dominance implementation implies implementation as a unique Bayesian Nash equilibrium. While we do not depart from the assumption that subsidies must make the desired outcome an equilibrium, equilibrium play is a result of our analysis (Sandholm, 2002, 2005).

One may use Lemma 5 to backward engineer the optimal subsidy scheme $\tilde{s}$ that implements $p^{\tilde{x}}$, for any $\tilde{x} \in \mathbb{R}$. The following Lemma is key.

Lemma 6. *For all X such that $\underline{X} + \varepsilon \leq X \leq \overline{X} - \varepsilon$, one has*

$$u_i^\varepsilon\left(p_{-i}^X \mid X, s_i\right) = X + \sum_{n=0}^{N-1} \frac{w_i(n)}{N} - c_i + s_i. \tag{13}$$

If his opponents all play the same increasing strategy p_j^X, then upon observing the threshold signal $x_i^\varepsilon = X$ player i's belief over the aggregate action A_{-i} is uniform. Convergence to uniform strategic beliefs is a common property in global games; see Lemma 1 in Sakovics and Steiner (2012) for a particularly relevant reference.

We recall that, if $x(s)$ is the vector of switching points such that $p^{x(s)}$ is the unique BNE of $\Gamma^\varepsilon(s)$, then $x_i(s)$ solves (11) for all i. Imposing now that $\tilde{s}$ be such that $x_i(\tilde{s}) = \tilde{x}$ for all $i \in \mathscr{N}$, one obtains

$$u_i^\varepsilon\left(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}_i\right) = 0 \tag{14}$$

as the N identifying conditions for the subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $p^{\tilde{x}}$. By Lemma 6, (14) gives

$$u_i^\varepsilon\left(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}_i\right) = \tilde{x} + \sum_{n=0}^{N-1} \frac{w_i(n)}{N} - c_i + \tilde{s}_i = 0,$$

or, solving for $\tilde{s}_i$,

$$\tilde{s}_i = c_i - \sum_{n=0}^{N-1} \frac{w_i(n)}{N} - \tilde{x},$$

which is $(*)$. This proves Theorem 1 and concludes the main analysis.

4.3 Discussion

Our results characterize the subsidy scheme $\tilde{s}$ a planner must commit to when seeking to implement $p^{\tilde{x}}$ among rational players. Let us discuss several properties of this policy.

First, subsidies are increasing in c_i, the opportunity cost of playing 1. Given x, the cost of playing 1 is inreasing in c_i; hence, to induce coordination on 1 subsidies should increase as the cost c_i rises. This property is intuitive and shared (conditional on policy treatment and/or ranking) by many recent contributions on policy design in coordination problems (Winter, 2004; Sakovics and Steiner, 2012; Bernstein and Winter, 2012; Halac et al., 2020, 2021).

Second, symmetric players are offered identical subsidies. The scheme $\tilde{s}$ shares this property with the policies derived in Sakovics and Steiner (2012) and Halac et al. (2021). The symmetric treatment of identical players deviates from a number of other notable proposals including a divide-and-conquer policy (*c.f.* Segal, 2003; Bernstein and Winter, 2012) and the incentive schemes studied in Winter (2004) and Halac et al. (2020).[5]

Third, optimal subsidies are relatively small in the sense that $\tilde{s}_i$ does not make $p_i^{\tilde{x}}$ strictly dominant.[6] The sufficiency of modest subsidies can be explained by a strategic *unraveling effect* of subsidies in coordination games. A subsidy to player i raises his incentive to play 1. In a coordination game, the increased incentive of player i raises the incentive of player j to play 1. The increase in j's incentive in turn makes playing 1 even more attractive to player i, and so on. Under common knowledge of the subsidy, what obtains is a indefinitely compounded positive feedback look, the unraveling effect. Because of the compounded effect subsidies have in a coordination game, even seemingly minor subsides can go a long way toward solving the planner's problem. This feature of $\tilde{s}$ is a key counterpoint to several well-known results on the optimality of subsidizing some subset of players to strict dominance (Segal, 2003; Winter, 2004; Bernstein and Winter, 2012; Sakovics and Steiner, 2012).

Fourth, subsidies are decreasing in $\tilde{x}$, the threshold for coordination on 1 targeted by the planner. All else equal, a player's (expected) incentive to play 1 is increasing in his signal x_i^{ε}. Hence, for higher signals a player needs less subsidy to induce him to play 1. One can interpret $\tilde{x}$ as an inverse measure of the planner's ambition: the higher is $\tilde{x}$, the lower is the prior probability that coordination on 1 will be achieved. In this interpretation, being ambitious is costly: assuming coordination on 1 is indeed achieved, total spending on subsidies is increasing in the planner's ambition (decreasing in $\tilde{x}$).

[5] Onuchic and Ray (2023) also show that "identical agents" may be compensated asymmetrically in equilibrium; however, though identical in the payoff-relevant sense their players may still vary in payoff-irrelevant "identifies". Asymmetries in policy treatment derive from asymmetric identities.

[6] Indeed, the subsidy s_i^d that makes $p_i^{\tilde{x}}$ strictly dominant for player i is given by $c_i - w_i(0) - \tilde{x}$, which is strictly greater than $\tilde{s}_i$ as $w_i(n)$ is increasing in n.

Fifth, subsidies target all players and are globally continuous in model parameters. The characterization in $(*)$ establishes global continuity of $\tilde{s}_i$ in all the parameters upon which it depends $(c_i, w_i(n), \tilde{x})$. Our policy shares this property with the bonus scheme proposed by Halac et al. (2021). While conditional on policy treatment the optimal subsidies in Sakovics and Steiner (2012) are continuous in the relevant model parameters as well, changes in one player's parameters could affect whether or not said player is targeted, causing a discrete jump in subsidies received. Similarly, subsidies are continuous conditional on a player's position in the policy ranking in a divide and conquer mechanism (Segal, 2003; Bernstein and Winter, 2012); however, a player's position in the optimal ranking is affected by a change in its parameters, which can lead to discrete jumps in subsidy entitlement.

Sixth, subsidies are decreasing in spillovers, i.e. $\partial\tilde{s}_i/\partial w_i(n) < 0$. When observing the threshold signal $\tilde{x}$, a player i's belief over the aggregate action A_{-i} is uniform; in particular, therefore, he assigns strictly positive probability to the event that $A_{-i} = n$ for all $n = 0, 1, ..., N-1$. If $w_i(n)$ increases, the *expected* spillover a player expects to enjoy upon playing 1 is hence greater. This raises his incenive to play 1 and, for given $\tilde{x}$, the subsidy required to make him willing to do so is smaller. Given a ranking of players, subsides for each player (except the first-ranked) are also decreasing in spillovers in a divide-and-conquer policy (Segal, 2003; Bernstein and Winter, 2012). The optimal subsidies in Sakovics and Steiner (2012) are not generally decreasing in spillovers, except insofar as players who benefit less from project success are more likely to be targeted.

Seventh, though players coordinate on a symmetric *equilibrium*, they do not (necessarily) have symmetric *payoffs* in equilibrium. Focusing on the case $\varepsilon \to 0$ for simplicity, if $x < \tilde{x}$ then each player i has payoff c_i. If $x > \tilde{x}$, then each player i has payoff $x - \tilde{x} + w_i(N-1) - \sum_{n=0}^{N-1} w_i(n)/N$ inclusive of subsidies. Neither payoff is necessarily symmetric across players, except for identical ones.

5 Special Cases and Extensions

Generic properties of $\tilde{s}$ were discussed above. Several additional features become apparant upon considering special cases of the model.

5.1 Games of Regime Change

There is a project in which N can invest. The cost of investment to investor i is $c_i > 0$. If the project succeeds, an investing investor i realizes benefit $b_i > c_i$. The project is successful if and only if a critical mass of investors invests; specifically, there exists $I \in (0, N)$ such that the project succeeds if $A \geq I$ and fails otherwise.[7] The payoff to not investing, the outside option, is given by $-x$. The uncertainty about x, or more generally about $x - c_i$, can be thought of as any kind of (fundamental) uncertainty that pertains to the cost or benefit of investment; see Abel (1983) or Pindyck (1993) for possible interpretations.

Hoping to attract investment, a planner offers each investor i an investment subsidy s_i. For comparability with the literature, we are particularly interested in the subsidy scheme

[7] In the notation of (1), we thus have $w_i(n) = \underline{b}_i$ for all $n < I$ and $w_i(n) = \overline{b}_i$ for all $n \geq I$, where $\overline{b}_i > \underline{b}_i$ and $b_i := \overline{b}_i - \underline{b}_i$. Observe that, for $\underline{b}_i > 0$, the model allows for free-riding.

$s^0 = (s_i^0)$ that implements p^0 (as regime change games usually normalize the payoff to the outside option to 0, see Sakovics and Steiner (2012); Halac et al. (2020)).

Proposition 1. *Consider a joint investment problem in which $I \in (0, N)$ is the critical threshold for project success. Let $N - n^*$ be the smallest integer greater than I. The subsidy scheme $s^0 = (s_i^0)$ that makes coordination on p^0 the unique Bayesian Nash equilibrium of this game is given by*

$$s_i^0 = c_i - \frac{n^*}{N} \cdot b_i \tag{15}$$

for every $i \in \mathscr{N}$.

In the subsidy scheme s^0, all investors are subsidized; subsidies are a fraction of their investment costs. The latter is explained through the unraveling effect of policies: if investor i receives an investment subsidy, he is more likely to invest. Anticipating the increased likelihood that i invests, project success becomes more likely and this attracts investment by investor j. The greater likelihood that j invests in turn makes investment even more interesting for i, and so on. This feedback effect is strong: in two-player joint investment problems, subsidies are less than half players' investment costs.

While the problem here bears close resemblance to the global game in Sakovics and Steiner (2012), the models differ in fundamental ways making a direct comparison complicated. Most notably, Sakovics and Steiner (2012) do not have no prior uncertainty about the efficient outcome of the game; coordinated investment is always the efficient equilibrium of the game. Instead, uncertainty pertains to the critical threshold of investments required to achieve project success. Similarly, and related, conditional on the regime in place, there is no uncertainty about payoffs in Sakovics and Steiner (2012); the present paper instead assumes uncertainty about payoffs even conditional on the regime.[8] It is interesting that these differences, albeit fairly subtle, lead to vastly different policy implications. In this sense, Proposition 1 complements the important contribution of Sakovics and Steiner (2012).

An important and, in our view, realistic possibility in the investment problem studied here is that joint investment need *not* be ex post efficient: if x is very low, it can be efficient for all players to not invest and take the outside option. We return to this issue in Section 5.6.

5.2 Principal-Agent Problems

The main analysis assumes that actions are contractible. Here we discuss the implications of our results for principal-agent problems in which subsidies cannot condition on individual actions but only on aggregate outcomes, see for example Winter (2004) and Halac et al. (2021).

There is an organizational project that involves N tasks each performed by one agent $i \in \mathscr{N}$. Each agent i decides whether to work ($a_i = 1$) towards completing his task or shirk ($a_i = 0$). The cost of working to agent i is given by $c_i > 0$. Success of the project depends upon the decisions of all agents through a production technology $q : \{0, 1\}^N \to [0, 1]$, where

[8]This distinction applies more generally to the literature on global games of regime change, see Morris and Shin (1998), Angeletos et al. (2007), Goldstein and Pauzner (2005), Basak and Zhou (2020), and Edmond (2013).

$q(n)$ is the probability of success given that n agents work.[9] As in Winter (2004) and Halac et al. (2021), we assume that $q(n+1) > q(n)$ for all $n \leq N-1$.

A principal offers contracts that specify rewards $v = (v_i)$ to agents contingent on project success; if the project fails, all agents receive zero. We assume that agents' work effort is not observable. Hence, any rewards the principal offers can condition only upon project success. An agent who shirks gets payoff $-x$. We interpret x generally as an uncertain fundamental that determines agents' payoffs, see also Halac et al. (2022) for a model of contracting under fundamental uncertainty.

Proposition 2. *Consider a principal-agent problem in which the principal offers rewards $(\tilde{v}_i)$ to implement $\mathrm{p}^{\tilde{x}}$ as the unique equilibrium of the game. For each $i \in \mathscr{N}$, the reward $\tilde{v}_i$ is given by*

$$\bar{q} \cdot \tilde{v}_i = c_i - \tilde{x}, \tag{16}$$

where $\bar{q} := \sum_{n=0}^{N-1} \frac{q(n+1)-q(n)}{N}$.

The characterization in (16) has an intuitive interpretation. Consider the threshold type agent i who observes signal $x_i^\varepsilon = \tilde{x}$. On the one hand, her (expected) *net cost* of working is $c_i - \tilde{x}$; it is the sum of the cost of work c_i and her expected benefit of having free time $-\tilde{x}$. On the other hand, assuming that n fellow agents work a threshold agent i who shirks expects to receive the reward $\tilde{v}_i$ with probability $q(n)$. If instead agent i works, she increases this probability to $q(n+1)$. The threshold type agent i who observes $x_i^\varepsilon = \tilde{x}$ therefore believes that working, rather than shirking, increases the probability of receiving her reward by $q(n+1) - q(n)$. Moreover, since the threshold type agent has the uniform belief over the aggregate action (see Lemma 6), her *expected* marginal contribution to project success is $\sum_{n=0}^{N-1} \frac{q(n+1)-q(n)}{N}$, or $\bar{q}$. Hence, agent i's expected benefit of working is simply $\bar{q} \cdot \tilde{v}_i$. Given that the principal targets $p^{\tilde{x}}$, such an agent should be just indifferent between working and shirking – for her the expected benefit of working should be equal to her net cost of working. This gives (16).

Note that for $x = 0$, the payoffs in this model exactly replicate those in the canonical principal-agent problem (Winter, 2004). Interestingly, the analysis under uncertainty fails to yield Winter's prescription that optimal contracts are inherently discriminatory and should reward identical agents asymmetrically. In this paper, symmetric agents receive identical rewards. It is similarly noteworthy that, in contrast to Halac et al. (2021), this symmetry does not rely on contracts being private – indeed, Corollary 2 critically relies upon the scheme $\tilde{v}$ being common knowledge.

5.3 Heterogeneous Externalities

The main analysis assumes that only the aggregate action A_{-i} matters for the externality other players impose upon player i. We relax this unrealistic assumption here. In particular, we allow that the externality $w_i(a_{-i})$ depends upon the specific vector a_{-i} played. We maintain a focus on games with strategic complementarities and assume that if $a''_{-i} \geq a_{-i}$,

[9] Here, we proceed under the simplifying assumption that only the total number of agents who work matters for project success. One could well imagine cases in which also the identity of the working agents matters. See Section 5.3 for such an extension.

then $w_i(a''_{-i}) \geq w_i(a_{-i})$. Observe that this externality structure encompasses the games in Bernstein and Winter (2012) and Halac et al. (2021), where externalities are allowed to depend upon the subset $M \subseteq \mathscr{N}$ of players who play 1. It also nests the approach in Sakovics and Steiner (2012) where externalities depend upon the aggregate action, but some players have a stronger impact on the aggregate action than others. Finally, this representation can describe games among players in a network in which the strenght of externalities depend on players' "social connectedness" and location in the network, see Leister et al. (2022).

Let a^n_{-i} denote an action vector a_{-i} in which exactly n players play 1 (and the remaining $N-n-1$ players play 0). We write A^n_{-i} for the set of all (unique) action vectors a^n_{-i}. Note that there are exactly $\binom{N-1}{n}$ vectors a^n_{-i} in A^n_{-i}. For all i, define

$$w_i^n := \frac{\sum_{a^N_{-i} \in A^N_{-i}} w_i\left(a^n_{-i}\right)}{\binom{N-1}{n}}.$$

In words, w_i^n is the expected externality imposed upon player i who expects that n opponents play 1 and believes that every player $j \neq i$ is equally likely to one of those n.

Proposition 3. *Consider a game Γ^ε with heterogeneous externalities. Given is $\tilde{x} \in \mathbb{R}$. There exists a unique subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $p^{\tilde{x}}$. For each $i \in \mathscr{N}$, the subsidy $\tilde{s}_i$ pursuant to the scheme is given by*

$$\tilde{s}_i = c_i - \tilde{x} - \sum_{n=0}^{N-1} \frac{w_i^n}{N}. \tag{17}$$

In the game with heterogeneous externalities, too, symmetric players receive identical subsidies. This conclusion remains valid if one considers the "limit of uncertainty" as $\varepsilon \to 0$. Thus, even a little bit of uncertainty can upset the canonical results by Segal (2003), Winter (2004), and Bernstein and Winter (2012) that optimal contracts are fundamentally discriminatory in coordination games.

5.4 Spillovers on Directed Graphs

5.5 Asymmetric Targets

It was so far maintained that the planner seeks to implement a symmetric equilibrium. In many practical situations policies may instead target asymmetric outcomes. We consider such instances here.

Given are two real numbers $\tilde{x}_1$ and $\tilde{x}_2$. Without loss, let $\tilde{x}_1 < \tilde{x}_2$. The planner partitions the player set $\mathscr{N}$ into two subsets $\mathscr{N}_1$ and $\mathscr{N}_2$ such that $\mathscr{N}_1 \cup \mathscr{N}_2 = \mathscr{N}$. There are N_1 players in $\mathscr{N}_1$ and $N_2 = N - N_1$ player in $\mathscr{N}_2$.

Suppose the planner seeks to implement the asymmetric equilibrium $(p_1^{\tilde{x}_1}, p_2^{\tilde{x}_2})$ according to which (with a slight abuse of notation) each player $i \in \mathscr{N}_1$ plays the increasing strategy $p_1^{\tilde{x}_1}$ while each $j \in \mathscr{N}_2$ plays $p_1^{\tilde{x}_1}$. We are agnostic as to the motivations behind such an asymmetric policy goal.

As before, $\tilde{s} = (\tilde{s}_i)$ denotes the vector of subsidies that implements $(p_1^{\tilde{x}_1}, p_2^{\tilde{x}_2})$, i.e. $(p_1^{\tilde{x}_1}, p_2^{\tilde{x}_2})$ is the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(\tilde{s})$. Proposition 4 establishes that $\tilde{s}$ is unique and given by

$$\begin{aligned} \tilde{s}_i &= c_i - \tilde{x}_1 - \sum_{n=0}^{N_1-1} \frac{w_i(n)}{N_1} \quad \text{for all } i \in \mathscr{N}_1, \\ \tilde{s}_j &= c_j - \tilde{x}_2 - \sum_{n=N_1}^{N-1} \frac{w_j(n)}{N_2} \quad \text{for all } j \in \mathscr{N}_2. \end{aligned} \tag{18}$$

We focus on the case $2\varepsilon < \tilde{x}_2 - \tilde{x}_1$.

Proposition 4. *Given are $\tilde{x}_1$ and $\tilde{x}_2$.*

(i) There exists a unique subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $\left(p_1^{\tilde{x}_1}, p_2^{\tilde{x}_2}\right)$.

(ii) For each $i \in \mathscr{N}$, the subsidy $\tilde{s}_i$ pursuant to the scheme is given by (18)*.*

The characterization of optimal subsidies for asymmetric policy targets in (18) underlines the importance of strategic uncertainty for policy design. Compared to the symmetric equilibrium case in which all players coordinate on the same strategy vector, under an asymmetric equilibrium target players in $\mathscr{N}_1$ receive higher subsidies while players in $\mathscr{N}_2$ receive lower subsidies. The reason is the variation in players' strategic equilibrium beliefs between subsets. In equilibrium, a player i in $\mathscr{N}_1$ who observes signal $x_i^\varepsilon = \tilde{x}_1$ knows that no other player will have received a signal greater than $\tilde{x}_2$. Hence, player i also knows that all players who play strategy $p^{\tilde{x}_2}$ will play 0. Upon observing the signal $\tilde{x}_1$, there consequently is no uncertainty about the actions played by all players in $\mathscr{N}_2$; the only remaining strategic uncertainty pertains to i's opponents in $\mathscr{N}_1$. This is reflected in $\tilde{s}_i$ in (18). Similarly, a player $j \in \mathscr{N}_2$ who observes signal $x_j^\varepsilon = \tilde{x}_2$ knows that no player has received a signal below $\tilde{x}_1$. Hence, in equilibrium player j takes as given that all players $i \in \mathscr{N}_1$ play 1; the only remaining strategic uncertainty pertains to the actions of his opponents in $\mathscr{N}_2$. This is reflected in $\tilde{s}_j$ in (18).

5.6 Induced Coordination Failure

Because the planner must commit to its policy before Nature draws the parameter x, subsidies can end up stimulating an action that is ex post inefficient. This possibility warrants policy moderation, as is most simply illustrated in a symmetric game.

Consider a coordination problem among symmetric players such that $c_i = c$ and $w_i(n) = w(n)$ for all $i \in \mathscr{N}$. In symmetric games, $\overline{a}$ is the efficient Nash equilibrium of the complete information game $\Gamma(x)$ for all $x > \underline{x}$ where $\underline{x} = c - w(N-1)$. Let s^* denote the *optimal subsidy* in a symmetric game in the sense that s^* induces coordination on $p^{\underline{x}}$ as the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(s^*)$. Per the characterization in Theorem 1, we have

$$s^* = w(N-1) - \sum_{n=0}^{N-1} \frac{w(n)}{N}. \tag{19}$$

The following corollary says that s^* is not only sufficient to induce coordination on an efficient equilibrium; subsidization in excess of s^* causes equilibrium inefficiency.

Corollary 1. *In symmetric games, subsidies $\hat{s}$ such that $\hat{s} > s^*$ are inefficient. Specifically, the subsidy $\hat{s}$ induces coordination on $p^{\hat{x}}$ where $\hat{x} < \underline{x}$. In the limit as $\varepsilon \to 0$, players thus coordinate on an inefficient outcome of $\Gamma(x)$ for all $x \in (\hat{x}, \underline{x})$ with probability 1.*

The possibility of policy-induced coordination failure due to excessive subsidization is illustrated in Figure 3. Though Corollary 1 follows trivially from Theorem 1, we single

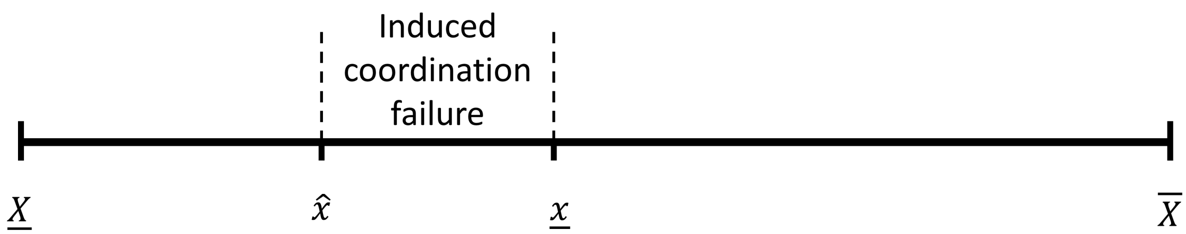

Figure 3: If $\hat{s} > s^*$, the unique equilibrium of $\Gamma^{\varepsilon}(\hat{s})$ has players coordinate on $p^{\hat{x}}$ with $\hat{x} < \underline{x}$. For $\varepsilon \to 0$, the policy $\hat{s}$ thus induces coordination on an inefficient outcome of the game – a policy-induced coordination failure – for all $x \in (\hat{x}, \underline{x})$.

it out to emphasize an important economic implication of our results. Prior uncertainty about x implies prior uncertainty about the efficient outcome of $\Gamma(x)$. If the planner must commit to its policy s before the random variable x is drawn, such uncertainty warrants policy moderation as high subsidies risk stimulating coordination on $\overline{a}$ even when $\underline{a}$ turns out to be the efficient outcome. Intuitively, under prior uncertainty about payoff functions the planner should subsidize conservatively to avoid picking inefficient winners; the planner should not be wedded to the idea that coordination on $\overline{a}$ must always be achieved.

In our view, the notion that policy intervention can itself be a source of inefficiency is realistic and consistent with historical evidence (*c.f.* Cowan, 1990). The literature on coordination games has not always emphasized this possibility. In the coordination problems studied by Sakovics and Steiner (2012), Bernstein and Winter (2012), and Halac et al. (2020, 2021) for example, coordination on 1 is always the strictly Pareto-dominant outcome of the game. The inefficiency associated with excessive subsidization in those models is that the planner ends up spending more on subsidies than would be strictly necessary. Our analyis complements those results by making explicit another, non-budgetary source of policy inefficiency: the possibility that excessive intervention induces coordination on an inefficient outcome.

6 Concluding Remarks

The results in this paper suggest that a planner needs only modest policy intervention to solve coordination problems. The core mechanism driving this result is an *unraveling* effect of policies in coordination games. A subsidy raises a player's incentive to play the subsidized action. Her opponents know this and, given their desire to match actions, experience an increase in their own incentives, which in turn raises the player's incentive to pursue the subsidized action even further, and so on. Under common knowledge of the policy, this positive feedback loop compounds indefinitely. Compounding means that seemingly modest policy intervention can provide enough stimulus to steer rational agents toward efficiency. This unraveling effect is illustrated in Figure 1.

We derive our results in a global coordination game. We use a global games approach to address equilibrium selection explicitly. Coordination games tend to have multiple strict Nash equilibria. Policies need to work conditional on players' beliefs about their opponents' equilibrium strategies, which depend upon the equilibrium they believe others will be playing. With multiple equilibria, strategic beliefs therefore are not uniquely defined and this complicates policy design. It also motivates invasive policymaking, as policies must work even against the most pessimistic (strategic) beliefs. However, certain expectations about equilibrium play would seem highly counterintuitive; for example, one would not expect a subsidy to decrease incentives to play the subsidizes action. That such outcomes are consistent with equilibrium play is an artefact of the equilibrium multiplicity in complete information coordination games. We therefore address equilibrium selection explicitly, and we do so in a global game. We provide conditions under which rational agents will be able to select a unique equilibrium of the coordination game they are playing. We characterize the unique equilibrium as a function of the subsidy in place and use this characterization to derive the optimal subsidy, that is, the subsidy that makes the Pareto efficient strategy the unique Bayesian Nash equilibrium of the global game.

A useful feature of global games is that these games are solved by iterated elimination of strictly dominated strategies. Iterated dominance depends upon players' ability to form higher-order beliefs about the strategies pursued by their opponents and, consequently, their strategic beliefs. This is useful because our analysis of subsidy design and the unraveling effect presupposes the same ability with respect to the effect of subsidies on strategies. For this reason, and though other theories of equilibrium selection exist, we think of global games as the best suited framework to study our problem.

The unraveling effect makes it *possible* to solve coordination problems through modest policy intervention. Prior uncertainty about the efficient outcome of the game makes modest intervention necessary. Subsidies that are too high risk stimulating players to pursue a particular action even if coordination on that action is an inefficient, or Pareto-dominated, outcome of the game. This warrants a conservative approach toward policy as the planner should avoid picking possibly Pareto-inferior winners. This possibility naturally complicates policy design compared to cases in which the efficient outcome is assumed to be known a priori (*c.f.* Bernstein and Winter, 2012; Sakovics and Steiner, 2012; Halac et al., 2020) and favors an noninvasive planner who subsidizes modestly.

A Proofs

PROOF OF LEMMA 1

Proof. First, observe that

$$\begin{aligned}
u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) &= \int u_i\left(p_{-i}\left(x_{-i}^\varepsilon\right) \mid x\right)\, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\
&= \int w_i\left(p_{-i}\left(x_{-i}^\varepsilon\right)\right) + x\, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) - c_i \\
&= \int w_i\left(p_{-i}\left(x_{-i}^\varepsilon\right)\right)\, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) + x_i^\varepsilon - c_i,
\end{aligned}$$

for any strategy vector p_{-i}.

For part (i), given is $y = (y_j)_{j\neq i}$. Given also are two real numbers X_1 and X_2 such that $X_2 > X_1$. Obviously, one has $X_2 - y_j > X_1 - y_j$ for all y_j and all j, which implies that

First, $u_i^\varepsilon(n \mid x_i^\varepsilon)$ is increasing in both n and x_i^ε; and second, $\int n(p_{-i}^y(x_{-i}^\varepsilon))\mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon)$ is increasing in x_i^ε and decreasing in y_j. The result follows. □

PROOF OF LEMMA 2

Proof. When $x_i^\varepsilon > \overline{x} - s + \varepsilon$, the entire support of player i's conditional distribution on x lies in a region where $x > \overline{x}$. For any such x, playing 1 is strictly dominant. (i) follows. Part (ii) is proven in a similar way. □

PROOF OF LEMMA 4

Proof. By construction, $l_i^* \leq r_i^*$. Define $\epsilon_i := r_i^* - l_i^*$, so $\epsilon_i \geq 0$.

We first state, and prove, a useful claim. Using this claim, we can then prove the Lemma by establishing a contradiction if $\epsilon_i \neq 0$ for all $i \in \mathcal{N}$.

Claim 1. *If $\epsilon_i = \epsilon$ for all $i \in \mathcal{N}$, then $\epsilon = 0$.*

Proof of the claim. With a slight abuse of notation, we write $z_{-i} + \epsilon = (z_j + \epsilon)_{j\neq i}$. Observe that $\Phi(z_{-i} \mid x_i^\varepsilon) = \Phi(z_{-i} + \epsilon \mid x_i^\varepsilon + \epsilon)$ for all z_{-i} and x_i^ε. If $\epsilon_i = \epsilon$ for all $i \in \mathcal{N}$, this implies

$$\int w_i(p_{-i}^{l_{-i}^*})\, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid l_i^*) = \int w_i(p_{-i}^{l_{-i}^*+\epsilon})\, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid l_i^* + \epsilon) = \int w_i(p_{-i}^{r_{-i}^*})\, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid r_i^*).$$

Hence,

$$\begin{aligned}
u_i^\varepsilon(p_{-i}^{r_{-i}^*} \mid r_i^*, s_i) &= r_i^* + s_i - c_i + \int w_i(p_{-i}^{r_{-i}^*})\, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid r_i^*) \\
&= r_i^* + s_i - c_i + \int w_i(p_{-i}^{l_{-i}^*})\, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid l_i^*) \\
&= l_i^* + \epsilon + s_i - c_i \int w_i(p_{-i}^{l_{-i}^*})\, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid l_i^*) \\
&= u_i^\varepsilon(p_{-i}^{l_{-i}^*} \mid l_i^*, s_i) + \epsilon.
\end{aligned}$$

By definition, $u_i^\varepsilon(p_{-i}^{r_{-i}^*} \mid r_i^*, s_i) = u_i^\varepsilon(p_{-i}^{l_{-i}^*} \mid l_i^*, s_i)$, and it follows that $\epsilon = 0$. □

Now suppose that $\epsilon_i \neq \epsilon_j$ for at least one pair of players $i, j \in \mathscr{N}$. Let $\epsilon_i = \sup\{\epsilon_j \mid j \in \mathscr{N}\}$. Because $\epsilon_i \geq \epsilon_j$ for all $j \neq i$, we have

$$\int_{l_i^*-2\varepsilon}^{z} x_j^\varepsilon \, \mathrm{d}\Phi^\varepsilon(x_j^\varepsilon \mid l_i^*) \leq \int_{r_i^*-2\varepsilon}^{z+\epsilon_i} x_j^\varepsilon \, \mathrm{d}\Phi^\varepsilon(x_j^\varepsilon \mid r_i^*)$$

for all $z \in [l_i^* - 2\varepsilon, l_i^* + 2\varepsilon]$. The inequality is strict if $\epsilon_j < \epsilon_i$ (and by assumption there is at least one such j). The immediate implication is that

$$\int p_{-i}^{l_{-i}^*}(x_{-i}^\varepsilon) \, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid l_i^*) < \int p_{-i}^{r_{-i}^*}(x_{-i}^\varepsilon) \, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid r_i^*)$$

and, because $w_i(n)$ is increasing in n, we have

$$\int w_i\left(p_{-i}^{l_{-i}^*}(x_{-i}^\varepsilon)\right) \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid l_i^*) < \int w_i\left(p_{-i}^{r_{-i}^*}(x_{-i}^\varepsilon)\right) \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid r_i^*). \tag{I}$$

By definition, l^* satisfies $u_i^\varepsilon(p_{-i}^{l_{-i}^*} \mid l_i^*, s_i) = u_i^\varepsilon(p_{-i}^{r_{-i}^*} \mid r_i^*) = 0$ and

$$\begin{aligned}
u_i^\varepsilon(p_{-i}^{l_{-i}^*} \mid l_i^*) &= l_i^* + s_i - c_i + \int w_i\left(p_{-i}^{l_{-i}^*}(x_{-i}^\varepsilon)\right) \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid l_i^*) \\
&< r_i^* + s_i - c_i + \int w_i\left(p_{-i}^{l_{-i}^*}(x_{-i}^\varepsilon)\right) \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid l_i^*) \\
&< r_i^* + s_i - c_i + \int w_i\left(p_{-i}^{r_{-i}^*}(x_{-i}^\varepsilon)\right) \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid r_i^*) \\
&= u_i^\varepsilon(p_{-i}^{r_{-i}^*} \mid r_i^*).
\end{aligned}$$

The first inequality follows from the assumption that $r_i^* > l_i^*$, the second from (I). We have now derived that $u_i^\varepsilon(p_{-i}^{l_{-i}^*} \mid l_i^*, s_i) \neq u_i^\varepsilon(p_{-i}^{r_{-i}^*} \mid r_i^*)$, contradicting at least one of the definitions of r^* and l^* in (9) and (10), respectively. It follows there can be no player i such that $\epsilon_i \geq \epsilon_j$ for all $j \neq i$ and with a strict inequality for at least one j. By implication, $\epsilon_i = \epsilon$ for all $i \in \mathscr{N}$. The claim at the beginning of the proof established that in this case $\epsilon = 0$. We therefore have $l_i^* = r_i^*$ for all $i \in \mathscr{N}$. $\square$

PROOF OF PROPOSITION 5

Proof. Let $p = (p_i)$ be a BNE of $\Gamma^\varepsilon(s)$. For any player i, define

$$\underline{\underline{x}}_i = \inf\{x_i^\varepsilon \mid p_i(x_i^\varepsilon) > 0\}, \tag{20}$$

and

$$\overline{\overline{x}}_i = \sup\{x_i^\varepsilon \mid p_i(x_i^\varepsilon) < 1\}. \tag{21}$$

Observe that $\underline{\underline{x}}_i \leq \overline{\overline{x}}_i$. Now define

$$\underline{\underline{x}} = \min\{\underline{\underline{x}}_i\}, \tag{22}$$

and

$$\overline{\overline{x}} = \max\{\overline{\overline{x}}_i\}. \tag{23}$$

By construction, $\overline{\overline{x}} \geq \overline{\overline{x}}_i \geq \underline{\underline{x}}_i \geq \underline{\underline{x}}$. Observe that p is a BNE of $\Gamma^\varepsilon(s)$ only if, for each i, it holds that $u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i) \geq 0$. Consider then the expected incentive $u_i^\varepsilon(p_{-i}^{\underline{\underline{x}}}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i)$. It follows from the definition of $\underline{\underline{x}}$ that $p^{\underline{\underline{x}}}(x^\varepsilon) \geq p(x^\varepsilon)$ for all x^ε. The implication is that, for each i, $u_i^\varepsilon(p_{-i}^{\underline{\underline{x}}}(x_i - i^\varepsilon) \mid \underline{\underline{x}}_i) \geq u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i) \geq 0$. From Proposition 6 then follows that $\underline{\underline{x}} \geq x$.

Similarly, if p is a BNE of $\Gamma^\varepsilon(s)$ then, for each i, it must hold that $u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \overline{x}_i) \leq 0$. Consider the expected incentive $u_i^\varepsilon(p_{-i}^{\overline{\overline{x}}}(x_{-i}^\varepsilon) \mid \overline{\overline{x}}_i)$. It follows from the definition of $\overline{\overline{x}}$ that $p^{\overline{\overline{x}}}(x^\varepsilon) \leq p(x^\varepsilon)$ for all x^ε. For each i it therefore holds that $u_i^\varepsilon(p_{-i}^{\overline{\overline{x}}}(x_{-i}^\varepsilon) \mid \overline{\overline{x}}_i) \leq u_i^\varepsilon(p_{-i}(x_i - i^\varepsilon) \mid \overline{\overline{x}}_i) \leq 0$. Hence $\overline{\overline{x}} \leq x$.

Since $\underline{\underline{x}} \leq \overline{\overline{x}}$ while also $\underline{\underline{x}} \geq x$ and $\overline{\overline{x}} \leq x$ it must hold that $\underline{\underline{x}} = \overline{\overline{x}} = x$. Moreover, since $p^{\underline{\underline{x}}} \geq p$ while also $p^{\overline{\overline{x}}} \leq p$, given $\underline{\underline{x}} = \overline{\overline{x}} = x$, it follows that $p_i(s_i^\varepsilon) = p_i^x(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x$ and all i (recall that for each player i one has $u_i^\varepsilon(p_{-i}^x \mid x) = 0$, explaining the singleton exeption at $x_i^\varepsilon = x$). Thus, if $p = (p_i)$ is a BNE of $\Gamma^\varepsilon(s)$ then it must hold that $p_i(x_i^\varepsilon) = p_i^x(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x$ and all i, as we needed to prove. $\square$

PROOF OF LEMMA 6

Proof. First fix $x \in [\underline{X} + \varepsilon, \overline{X} - \varepsilon]$. Each player $j \neq i$ is assumed to play p_j^X, so the probability that $x_j = 1$ is given by

$$\Pr[x_j^\varepsilon > X \mid x] = \Pr[\varepsilon_j > X - x] = \frac{x + \varepsilon - X}{2\varepsilon}, \tag{24}$$

for all $X \in [x - \varepsilon, x + \varepsilon]$ while $\Pr[x_j^\varepsilon > X \mid x]$ is either 0 or 1 otherwise. Clearly, $a_j = 0$ is played with the complementary probability (given x and X). Since each ε_j is drawn independently from the same distribution, the probability that m given players $j \neq i$ play $a_j = 1$ while the remaining $N - m - 1$ players play $a_j = 0$ (given p_{-i}^X and x) is:

$$\left[\frac{x + \varepsilon - X}{2\varepsilon}\right]^m \left[\frac{X + \varepsilon - x}{2\varepsilon}\right]^{N-m-1}. \tag{25}$$

As there are $\binom{N-1}{m}$ unique ways in which m out of $N - 1$ players j can choose $a_j = 1$, the total probability of this happening, as a function of x, is:

$$\binom{N-1}{m} \left[\frac{x + \varepsilon - X}{2\varepsilon}\right]^m \left[\frac{X + \varepsilon - x}{2\varepsilon}\right]^{N-m-1}. \tag{26}$$

The derivation so far took x as known and given. We next take account of the fact that player i does not observe x directly but only the noisy signal x_i^ε. Given $p_{-i} = p_{-i}^X$ and $x_i^\varepsilon = X$, the expected incentive for player i to play $a_i = 1$ becomes:

$$\begin{aligned} u_i^\varepsilon(p_{-i}^X \mid X) =& \frac{1}{2\varepsilon} \int_{X-\varepsilon}^{X+\varepsilon} x \, \mathrm{d}x - c_i \\ &+ \sum_{m=0}^{N-1} w(m+1) \binom{N-1}{m} \frac{1}{2\varepsilon} \int_{X-\varepsilon}^{X+\varepsilon} \left[\frac{x + \varepsilon - X}{2\varepsilon}\right]^m \left[\frac{X + \varepsilon - x}{2\varepsilon}\right]^{N-m-1} \mathrm{d}x \end{aligned} \tag{27}$$

$$=X - c_i + \sum_{m=0}^{N-1} w(m+1)\binom{N-1}{m}\int_0^1 q^m(1-q)^{N-m-1}\mathrm{d}q \tag{28}$$

$$=X - c_i + \sum_{m=0}^{N-1} w(m+1)\frac{(N-1)!}{m!\,(N-m-1)!}\frac{m!\,(N-m-1)!}{N!} \tag{29}$$

$$=X - c_i + \sum_{m=0}^{N-1} \frac{w(m+1)}{N}. \tag{30}$$

Equation (27) takes the expression for $u_i(m \mid x)$ given in (2) and integrates out x and m, given $x_i^\varepsilon = X$ and $p_{-i} = p_{-i}^X$. Equation (28) uses integration by substitution (using $q = 1/2 - (X-x)/2\varepsilon$) to rewrite the second integral in (27). Equation (29) rewrites both the integral in (28) and the binomial coefficient $\binom{N-1}{m}$ in terms of factorials. Equation (30) simplifies. Finally, we know that $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s) = u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) + s$. □

PROOF OF PROPOSITION 1

Proof. Given a vector of investment subsidies s, the payoff to investor i is

$$\pi_i(a \mid x) = \begin{cases} b_i + s_i - c_i & \text{if} \quad a_i = 1 \text{ and } A \geq I \\ s_i - c_i & \text{if} \quad a_i = 1 \text{ and } A < I \\ -x & \text{if} \quad a_i = 0, \end{cases}$$

and an investor's incentive to invest is given by

$$u_i(a_{-i} \mid x) = x + b_i \cdot \mathbb{1}_I(A) + s_i - c_i,$$

where $\mathbb{1}_I$ is the indicator function such that $\mathbb{1}_I(A_{-i}) = 1$ if $A \geq I$ and $\mathbb{1}_I(A) = 0$ otherwise. Because investors do not observe x but only a signal x_i^ε, the expected incentive to investment (given the subsidies s) is

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s_i) = \int b_i \cdot \mathbb{1}_I(\| \, p_{-i}(x_{-i}^\varepsilon) \, \|) + s_i - c_i + x \,\mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon),$$

where $\| \cdot \|$ denotes the norm of a vector.

If $\tilde{s}$ implements $p^{\tilde{x}}$, then by construction $\tilde{s}_i$ must solve

$$u_i^\varepsilon(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}_i) = b_i \cdot \int \mathbb{1}_I(\| \, p_{-i}(x_{-i}^\varepsilon) \, \|) \,\mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid \tilde{x}) + \tilde{s}_i - c_i + \tilde{x} = 0.$$

By Lemma 6, we know that

$$\int \mathbb{1}_I(\| \, p_{-i}(x_{-i}^\varepsilon) \, \|) \,\mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid \tilde{x}) = \sum_{n=1}^{N} \frac{\mathbb{1}_I(n)}{N} = \frac{n^*}{N}$$

since, by definition, $N - n^*$ is the smallest integer greater than I. Plugging $\int \mathbb{1}_I(\| \, p_{-i}(x_{-i}^\varepsilon) \, \|) \,\mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid \tilde{x}) = \frac{n^*}{N}$ back into the condition that $u_i^\varepsilon(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}_i) = 0$ we obtain

$$\frac{n^*}{N} \cdot b_i + \tilde{s}_i + \tilde{x} - c_i = 0,$$

giving

$$\tilde{s}_i = c_i - \tilde{x} - \frac{n^*}{N} \cdot b_i.$$

Setting $\tilde{x} = 0$ gives the Proposition. □

PROOF OF PROPOSITION 2

Proof. Given a_{-i} and the reward scheme v, the payoff to agent i is given by:

$$\pi_i(a_i, a_{-i} \mid x) = \begin{cases} v_i - c_i & \text{if the project succeeds and } a_i = 1 \\ v_i - x & \text{if the project succeeds and } a_i = 0 \\ -c_i & \text{if the project does not succeed and } a_i = 1 \\ -x & \text{if the project does not succeed and } a_i = 0 \end{cases} \tag{31}$$

Since project success is stochastic and agents do not observe x, their (conditional) expected payoff is:

$$\pi_i^\varepsilon(a_i, a_{-i} \mid x_i^\varepsilon) = \begin{cases} q(A_{-i} + 1) \cdot v_i - c_i & \text{if } a_i = 1 \\ q(A_{-i}) \cdot v_i - x_i^\varepsilon & \text{if } a_i = 0, \end{cases} \tag{32}$$

yielding her expected incentive to work:

$$u_i^\varepsilon(a_{-i} \mid x_i^\varepsilon) = (q(A_{-i} + 1) - q(A_{-i})) \cdot v_i - c_i + x_i^\varepsilon. \tag{33}$$

The planner seeks to implement $p^{\tilde{x}}$. We know that $\tilde{v}$ implements $p^{\tilde{x}}$ iff

$$u_i^\varepsilon(p_{-i}^{\tilde{x}} \mid \tilde{x}) = \tilde{v}_i \int \left(q\left(p_{-i}^{\tilde{x}}(x_{-i}^\varepsilon) + 1\right) - q\left(p_{-i}^{\tilde{x}}(x_{-i}^\varepsilon)\right)\right) \,\mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid \tilde{x}) - c_i + \tilde{x} = 0, \tag{34}$$

for all $i \in \mathscr{N}$. Invoking Lemma 6, we know that

$$\int \left(q\left(p_{-i}^{\tilde{x}}(x_{-i}^\varepsilon) + 1\right) - q\left(p_{-i}^{\tilde{x}}(x_{-i}^\varepsilon)\right)\right) \,\mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid \tilde{x}) = \sum_{n=0}^{N-1} \frac{q(n+1) - q(n)}{N} = \overline{q}.$$

Therefore, $\tilde{v}_i$ solves

$$\overline{q} \cdot \tilde{v}_i - c_i + \tilde{x} = 0 \implies \overline{q} \cdot \tilde{v}_i = c_i - \tilde{x},$$

as given. □

PROOF OF PROPOSITION 3

Proof. Proof here. □

PROOF OF PROPOSITION 4

Proof. Consider player $i \in \mathscr{N}_1$ who receives a signal $x_i^\varepsilon = \tilde{x}_1$. It is known from Lemma 5 that $\tilde{s}$ implements $p^* = (p_1^{\tilde{x}_1}, p_2^{\tilde{x}_2})$ iff $\tilde{s}_i$ solves

$$u_i^\varepsilon(p_{-i}^* \mid \tilde{x}_1, \tilde{s}_i) = u_i^\varepsilon(p_{-i}^* \mid \tilde{x}_1) + \tilde{s}_i = 0.$$

Because, by assumption, $\tilde{x}_2 > \tilde{x}_1 + 2\varepsilon$, we also know that $x_j^\varepsilon < \tilde{x}_2$ for all $j \in \mathscr{N}$ and, in particular, for all $j \in \mathscr{N}_2$, the probability that $n > N_1 - 1$ players $j \neq i$ play 1 is 0 – i.e. there is no strategic uncertainty about the actions of players in $\mathscr{N}_2$.

Given $x \in [\tilde{x}_1 - \varepsilon, \tilde{x}_1 + \varepsilon]$, the probability that each player $j \in \mathscr{N}_1 \setminus i$ (who plays strategy $p_j = p^{\tilde{x}_1}$) plays $a_j = 1$ is given by

$$\Pr[x_j^\varepsilon > \tilde{x}_1 \mid x] = \frac{x + \varepsilon - \tilde{x}_1}{2\varepsilon},$$

and the total probability that exactly $n \in \{0, 1, ..., N_1 - 1\}$ players play 1 is:

$$\binom{N_1 - 1}{n} \left[\frac{x + \varepsilon - \tilde{x}_1}{2\varepsilon}\right]^n \left[\frac{\tilde{x}_1 + \varepsilon - x}{2\varepsilon}\right]^{N_1 - n - 1}.$$

Hence, following the exact steps as in the proof of Lemma 6 but replacing N for N_1, one has

$$u_i^\varepsilon(p_{-i}^* \mid \tilde{x}_1, \tilde{s}_i) = \tilde{x}_1 + \sum_{n=0}^{N_1 - 1} \frac{w_i(n)}{N_1} - c_i + \tilde{s}_i = 0.$$

Solving for $\tilde{s}_i$ yields the result for all $i \in \mathscr{N}_1$. A similar procedure can be carried our for players $i \in \mathscr{N}_2$. In their case, we note that upon observing signal $x_i^\varepsilon = \tilde{x}_2$ the probability that $x_j^\varepsilon \leq \tilde{x}_1$ for any $j \neq i$ is 0. In particular, therefore, $x_j^\varepsilon > \tilde{x}_1$ for all $j \in \mathscr{N}_1$ and, as all those j play $p_j = p^{\tilde{x}_1}$, it can be taken as given that they all play 1; the only remaining strategic uncertainty pertains to i's opponents in $\mathscr{N}_2$. □

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