Unraveling Coordination Problems*

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Preliminary draft. Please do not circulate.

Abstract

We study policy design in coordination problems. We identify an unraveling effect of policies not described before. A subsidy raises player i's incentive to play the subsidized action. In coordination games, this raises j's incentive to play the same action, which further incentivizes i, and so on. Building upon this logic, we characterize the subsidies that uniquely implement the efficient Nash equilibrium of the game. Optimal subsidies are increasing in the strength of players' coordination incentives. Excessive subsidization causes inefficiency. In asymmetric games, some subsides may be negative.

1 Introduction

Coordination problems arise throughout the economy and the inability of markets to solve these problems efficiently is a fundamental motivation for policy intervention. It is not obvious, however, what such interventions ideally looks like. In this paper, we study the optimal design of policies in coordination games. Our results can be applied to a wide range of problems including network good adoption (Jensen, 2007; Björkegren, 2019), disease eradication (Barrett, 2003), standard setting (Farrell and Saloner, 1985), discrete public good provision (Palfrey and Rosenthal, 1984).

The usual rationale for policy intervention is to correct market failures introduced by externalities. All market failures are not equal, however, and it is crucial for policy design to know the type of externality an intervention should deal with. One kind of externality arises when there exists a gap between the private and social value of behavior. Thus, an individual household's greenhouse gas emissions may be higher than socially optimal as it ignores the effects its emissions have on others. Such externalities can be addressed though Pigouvian taxes or subsidies. Another, more complicated kind of externality arises in coordination problems where individual actions are strategic complements (Bulow et al., 1985). Strategic complementarity results in multiple, Pareto-ranked equilibria and opens the door to coordination failures. Thus, hydrogen technologies may provide a viable replacement for

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fossil fuels but only if sufficient capacity is installed; there hence are multiple equilibria, and hydrogen technologies might never mature despite their potential (and known) advantages. Such externalities cannot be solved through Pigouvian policy. The goal of this paper is to design optimal policies in coordination problems. To streamline the narrative, we focus on subsidies; by symmetry, the analysis also applies to tax design.

Our main result characterizes the optimal subsidy scheme in a broad class of coordination problems. Optimal subsidies induce the efficient outcome of a coordination game as its unique equilibrium. Our characterization shows that fairly modest subsidies can solve coordination problems; in the canonical example of a joint investment problem, optimal subsidies are only a fraction of full investment costs. The driving force behind this result is an *unraveling effect* that derives from the strategic effects of policies in coordination problems. We also characterize the optimal subsidies for a planner who wishes to reach a given ex ante probability, less than one, that players coordinate efficiently. We show that subsidization in excess of our optimal subsidies is itself a source of equilibrium inefficiency. In asymmetric games, symmetric players are subsidized symmetrically and optimal subsidies are continuous in the parameters of players' payoff functions.

We derive our results in the context of a global game with strategic complementarities. Pioneered by Carlsson and Van Damme (1993), global games are incomplete information games in which players do not observe the true game they play but only a private and noisy signal of it. In our model, uncertainty about payoff functions leads to prior uncertainty about the posterior efficient outcome. We consider regulatory environments in which the planner must commit to a policy before the uncertainty is resolved, opening up the realistic possibility that subsidies stimulate an ex post inefficient action. The fact that policy targets an outcome of uncertain efficiency has a moderating effect on optimal subsidies: high, invasive subsidies are too heavy handed and risks attracting coordination on an inferior outcome. Under prior uncertainty, the planner should avoid picking winners.

To make things concrete, consider a very simple example. Two players simultaneously decide to invest, or not, in a project. If both players invest, the project succeeds and player i receives a payoff $b_i - c_i > 0$. If only one player invests, the project fails and the investing player gets $-c_i < 0$. The payoff to not investing is 0. A planner publicly offers subsidies s_i to investing players in hopes of making mutual investment the unique equilibrium of the game. It is easy to see that $s_i = c_i$, making investment dominant for both players, will yield the desired outcome. Yet the cost of such an invasive policy ramps up quickly as the number of players increases.

Fortunately, our results suggest that the subsidies can be substantially less than full investment costs. In this example, our characterization of the optimal subsidies simplifies to $s_i^* = c_i - b_i/2 < c_i/2$. We observe that optimal subsidies are less than half players' investment costs: the policy does not make investment dominant for either player. We explain the sufficiency of such partial subsidization by a strategic *unraveling effect* of policies in coordination problems. When player i receives an investment subsidy, his incentive to invest increases. His opponent, player j, understands this and consequently attaches a higher probability to the event that i invests; her own investment incentive, it follows, also increases. A rational player i is aware of the increased incentive of j, which makes investment even more attractive to him. And so on. There are, in some sense, increasing returns to policy in coordination games. Given that subsidies are common knowledge, the result is an infinitely compounded feedback effect of policy that allows seemingly modest intervention to unravel coordination problems. Provided that mutual investment is achieved, the total cost our optimal subsidies is $c_1 + c_2 - (b_1 + b_2)/2$.

It is instructive to compare our subsidies with the so-called *divide and conquer* mechanism, proposed by Segal (2003). First, the planner ranks all players; given the ranking, each player is offered a subsidy that incentivizes him to play the subsidized action assuming all players who precede him in the ranking also play this action while those after him do not. Bernstein and Winter (2012) derive the optimal ranking of players in such a policy. In our joint investment problem, divide and conquer is simple: the first-ranked player (i) is offered her full investment cost c_i; the other player (j) receives $c_j - b_j < 0$ (i.e. the second-ranked player is taxed). The cost of such a policy is $c_1 + c_2 - b_j$; the optimal ranking is that which ranks second the player who gains most from project success. When players have the same project benefit, so $b_1 = b_2 = b$, the total cost of divide and conquer is $c_1 + c_2 - b$. In this case, our optimal subsidies cost the same as divide and conquer even though both players receive a subsidy (and no one is taxed). We also note that, in contrast to a divide and conquer mechanism, our optimal policy treats symmetric agents symmetrically.

In the contracting literature, the paper closest to our is Sakovics and Steiner (2012). They study the joint investment problem in a global games context. In their model, and in contrast to ours, there is no prior uncertainty about efficiency; coordinated investment is always the efficient equilibrium of the game. Instead, uncertainty pertains to the critical threshold of investments required to achieve project success. Both in our game and the problem studied by Sákovics and Steiner, externalities depend only upon the aggregate action. However, the externality structure in Sákovics and Steiner is richer as they allow asymmetries in individual players' impact on the aggregate action; we assume that players' actions affect the aggregate symmetrically. Our problem, in contrast, allows for asymmetry in players' best-responses to a given aggregate action, a possibility that Sákovics and Steiner rule out. These differences, albeit fairly subtle, lead to vastly different policy implications. In the problem of Sakovics and Steiner (2012), the planner should target only a subset of players who (in their baseline specification) are subsidized fully; in our problem, all players are targeted and always receive a partial subsidy. Subsidies are not globally continuous in a players' payoff parameters in the model of Sákovics and Steiner, as changes in parameters can affect whether or not the player gets targeted leading to a discrete jump in policy treatment; our optimal subsidies instead are globally continuous in the model's parameters. In our view, the results in this paper complement the important contribution of Sakovics and Steiner (2012).[1]

The paper proceeds as follows. Section 2 introduces the model. Section 3 presents the main analysis. In it, we study explicitly how subsidies affect incentives in a coordination game. We prove equilibrium uniqueness and characterize the unique equilibrium. We use

[1]Given the differences described, it is not especially insightful to compare our results with those due to Sakovics and Steiner (2012). One might nevertheless be interested in such a comparison. Imagine again the joint investment problem describe above and assume $b_1 < b_2$. In the Sákovics and Steiner world, an optimal subsidy exclusively targets player 1 who is offered a (partial) subsidy $c_1 + c_2 b_1/b_2 - b_1$. Assuming the policy is successful and both players invest, the cost of this policy is simply $c_1 + c_2 b_1/b_2 - b_1$. In our model, the planner offers each player a subsidy $c_i - b_i/2$; if the project succeeds, the total cost of this policy is $c_1 + c_2 - (b_1 + b_2)/2$. It easy to verify that for particular parameter values, our policy is cheaper (for example, take $c_1 = c_2 = 1/2$, $b_1 = 2/3$, and $b_2 = 3/4$).

this characterization to derive the optimal subsidy, which we show falls short of the subsidies necessary to achieve strict dominance. We also discuss the generalization to heterogeneous agents. Section 4 concludes.

2 Model

Consider a normal form game played by players in a set $\mathscr{N} = \{1, 2, ..., N\}$, indexed i, who simultaneously choose binary actions $a_i \in \{0, 1\}$. We assume for now that players are symmetric to develop the intuition in the simplest possible environment; Section 3.2 discusses the case of asymmetric players.

Let $a \in \{0, 1\}^N$ denote a vector of actions by all players, and define $a_{-i} := a \setminus \{a_i\}$. We define $\overline{x} = (1, 1, ..., 1)$ and $\underline{x} = (0, 0, ..., 0)$; $\overline{x}_{-i}$ and $\underline{x}_{-i}$ are defined in similar ways. Let $n(a_{-i}) = \sum_{j \neq i} a_j$ denote the number of ones in a_{-i}. When a is played, player i who chooses 1 in a gets payoff $w(n(a_{-i})) + x$; when instead player i chooses 0 in a, their payoff is c_i. Here, $w(n(a_{-i}))$ describes the externalities on player i deriving from other players' actions. At times we may write $w(a_{-i})$, where it is understood that w depends upon a_{-i} only through $n(a_i)$. The variable x is a benefit parameter in players' payoff functions, and c is the opportunity cost of playing 1. Combining these elements, a player's payoff is given by

$$\pi_i(a \mid x) = \begin{cases} w(a_{-i}) + x & \text{if} \quad a_i = 1, \\ c & \text{if} \quad a_i = 0. \end{cases} \tag{1}$$

We study coordination problems in which the externality $w(a_{-i})$ is increasing in a_{-i}, i.e. $w(n + 1) \geq w(n)$ for all $n = 0, ..., N - 2$.[2] In the canonical example of a joint investment problem, the action $a_i = 1$ is interpreted as investment and c as the cost of investing (Sakovics and Steiner, 2012). Alternatively, actions might represent the choice to use of a particular kind of network technology and c is the cost differential between technologies (Björkegren, 2019). Or actions could denote participation in discrete public good provision, in which case c denotes the cost of provision (Palfrey and Rosenthal, 1984). Throughout this paper, to reflect the generality of our logic we shall mostly shy away from concrete interpretations and talk in terms of generic actions instead.

The above elements combined describe a game of complete information $G(x)$. In $G(x)$, we define a player's incentive $u_i(a_{-i})$ to choose 1 as the gain from playing 1, rather than 0, or

$$u_i(a_{-i} \mid x) = \pi_i(1, a_{-i} \mid x) - \pi_i(0, a_{-i} \mid x) = w(a_{-i}) + x - c. \tag{2}$$

Observe that, given a_{-i}, the incentive u_i to play 1 is strictly increasing in x. Denote $\underline{x} := c - w(N - 1)$ and $\overline{x} := c - w(0)$. One has $u_i(\overline{x}_{-i} \mid \underline{x}) = u_i(\underline{x}_{-i} \mid \overline{x}) = 0$. In other words, playing 1 is strictly dominant for all $x > \overline{x}$; playing 0 is strictly dominant for $x < \underline{x}$. The following is now immediate.

Lemma 1. *In $G(x)$,*

[2] A special case of our model are so-called threshold or regime change games, in which there exists $n^* \in \{1, 2, ..., N\}$ such that $w(n) = \underline{w}$ for all $n < n^*$ and $w(n) = \overline{w}$ otherwise, where $\underline{w} < \overline{w}$. For examples, see Sakovics and Steiner (2012) and Halac et al. (2020).

(i) $\overline{a}$ is a Nash equilibrium for all $x \geq \underline{x}$. It is the unique Nash equilibrium for all $x > \overline{x}$.

(ii) $\underline{a}$ is a Nash equilibrium for all $x \leq \overline{x}$. It is the unique Nash equilibrium for all $x < \underline{x}$.

(iii) $\overline{a}$ is the strictly welfare-maximizing outcome for all $x > \underline{x}$.

Rather than assume common knowledge of x, we model the problem as a global game G^ε in which players do not observe x. Instead, it is assumed that x is drawn from the uniform distribution on $[\underline{X}, \overline{X}]$ where $\underline{X} < \underline{x}$ and $\overline{X} > \overline{x}$ and that each player i receives a private noisy signal x_i^ε of x, given by:[3]

$$x_i^\varepsilon = x + \varepsilon_i. \tag{3}$$

We occasionally refer to x_i^ε as the player's type. Note that, conditional on the true game played, types are drawn independently.

In (3), the term ε_i captures idiosyncratic noise in i's private signal. It is common knowledge that ε_i is an i.i.d. draw from the uniform distribution on $[-\varepsilon, \varepsilon]$. We assume that ε is sufficiently small, in a way made precise in Section 3.1. Let $x^\varepsilon = (x_i^\varepsilon)$ denote the vector of signals received by all players, and let x_{-i}^ε denote the vector of signals received by all players but i, i.e. $x_{-i}^\varepsilon = (x_j^\varepsilon)_{j \neq i}$. Note that player i observes x_i^ε but neither x nor x_{-i}^ε. We write $\Phi^\varepsilon(\cdot \mid x_i^\varepsilon)$ for the joint probability function of $(x, x_j^\varepsilon)_{j \neq i}$ conditional on x_i^ε.

The timing of G^ε is as follow. First, Nature draws a true x. Second, each player i receives its private signal x_i^ε of x. Third, all players simultaneously choose their actions. Lastly, payoffs are realized according to the true x and the actions chosen by all players.

Strategies and strict dominance. A strategy p_i for player i in G^ε is a function that assigns to any $x_i^\varepsilon \in [\underline{X} - \varepsilon, \overline{X} + \varepsilon]$ a probability $p_i(x_i^\varepsilon) \geq 0$ with which the player chooses action $a_i = 1$ when they observe x_i^ε. We write $p = (p_1, p_2, ..., p_N)$ for a strategy vector. Similarly, we write $p_{-i} = (p_j)_{j \neq i}$ for the vector of strategies for all players but i. Conditional on the strategy vector p_{-i} and a private signal x_i^ε, the expected incentive to play 1 for player i is given by:

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) := \int u_i(p_{-i}(x_{-i}^\varepsilon) \mid x) \, d\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon). \tag{4}$$

We say that the action $a_i = 1$ is strictly dominant at x_i^ε if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) > 0$ for all p_{-i}. Similarly, the action $a_i = 0$ is strictly dominant (in the global game G^ε) at x_i^ε if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) < 0$ for all p_{-i}. When $a_i = a$ is strictly dominant, we say that $a_i = 1 - a$ is strictly dominated.

Conditional dominance. Let L and R be real numbers. The action $a_i = 1$ is said to be dominant at x_i^ε conditional on R if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) > 0$ for all p_{-i} with $p_j(x_j^\varepsilon) = 1$ for all $x_j^\varepsilon > R$, all $j \neq i$. Similarly, the action $a_i = 0$ is dominant at x_i^ε conditional on L if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) < 0$ for all p_{-i} with $p_j(x_j^\varepsilon) = 1$ for all $x_j^\varepsilon > L$, all $j \neq i$.

Iterated elimination of strictly dominated strategies. Eliminate all pure strategies that are strictly dominated, as rational players may be assumed never to pursue such strategies. Next, eliminate a player's pure strategies that are strictly dominated if all other players are known to play only strategies that survived the prior round of elimination; and so on. The set of

[3] In game theory, it is assumed that the game (in this case G^ε) is common knowledge; hence, the structure of the uncertainty (the joint distribution of x and all the signals x_j^ε), the possible actions and all the payoff functions are commonly known. For a formal treatment of common knowledge, see Aumann (1976).

strategies that survive infinite rounds of elimination are said to survive iterated elimination of strictly dominated strategies, which is the solution concept we use in this paper.

Increasing strategies. For $X \in \mathbb{R}$, let p_i^X denote the particular strategy such that $p_i^X(x_i^\varepsilon) = 0$ for all $x_i^\varepsilon < X$ and $p_i^X(x_i^\varepsilon) = 1$ for all $x_i^\varepsilon \geq X$. We will call p_i^X the *increasing strategy with switching point* X. By $p^X = (p_1^X, p_2^X, ..., p_N^X)$ we denote the strategy vector of increasing strategies with switching point X, and $p_{-i}^X = (p_j^X)_{j \neq i}$. Note that $a_i = 1$ is strictly dominant at x_i^ε conditional on R if and only if $u_i^\varepsilon(p_{-i}^R \mid x_i^\varepsilon) > 0$. Similarly, if $a_i = 0$ is strictly dominant at x_i^ε conditional on L then it must hold that $u_i^\varepsilon(p_{-i}^L \mid x_i^\varepsilon) < 0$.

In general, for a vector of real numbers (y_i) let $p^y = (p_i^{y_i})$ be a (possibly asymmetric) increasing strategy vector, and $p_{-i}^y = (p_j^{y_j})_{j \neq i}$. The following Lemma will prove highly useful throughout the analysis.

Lemma 2. *Given is a vector of real numbers (y_i) and the associated strategy vector $p^y = (p_i^{y_i})$. Then,*

(i) $u_i^\varepsilon(p_{-i}^y \mid x_i^\varepsilon)$ is monotone increasing in x_i^ε;

(ii) $u_i^\varepsilon(p_{-i}^y \mid x_i^\varepsilon)$ is monotone decreasing in y_j, all $j \neq i$.

It is important to note that monotonicity of $u_i^\varepsilon(p_{-i}^y \mid x_i^\varepsilon)$ in x_i^ε depends upon p_{-i}^y being increasing. For generic p_{-i}, $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon)$ can be locally decreasing in x_i^ε.

Subsidies. Let the planner commit to paying subsidy s on playing 1 to player i. Conditional on the subsidy s, player i's incentive to choose 1 becomes

$$u_i(a_{-i} \mid x, s) = u_i(a_{-i} \mid x) + s = w(a_{-i}) + x - c + s, \tag{5}$$

and the expected incentive (given the signal x_i^ε) is

$$\begin{aligned} u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s) &= \int u_i(p_{-i}(x_{-i}^\varepsilon) \mid x, s) d\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\ &= \int \left[u_i(p_{-i}(x_{-i}^\varepsilon) \mid x) + s\right] d\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\ &= u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) + s. \end{aligned} \tag{6}$$

A tax equal to s on playing 0 has the same effect on incentives; for consistency, we shall narrate the analysis in terms of subsidies.

All concepts and notation are now in place to proceed with the formal analysis.

3 Formal Analysis

3.1 Equilibrium Policy Propagation

Let $G^\varepsilon(s)$ denote the game G^ε in which the planner commits to paying each player who plays 1 a subsidy equal to $s > 0$. We assume throughout that $2\varepsilon < \min\{\underline{x} - s - \underline{X}, \overline{X} - \overline{x} + s\}$.

The goal of this section is as follows. First, we show that $G^\varepsilon(s)$ has a unique Bayesian Nash equilibrium. We characterize the unique equilibrium, showing that it is in increasing strategies and depends upon s. We use our characterization to solve for the optimal subsidy

s^*; here, optimality is defined as the subsidy that makes $p^{\underline{x}}$ the unique equilibrium of the game, which as $\varepsilon \to 0$ implies coordination on the efficient outcome of the game almost surely.

The following lemma is a useful starting point for our analysis.

Lemma 3. *Given a subsidy $s > 0$, consider the game $G^\varepsilon(s)$. (i) For each player i, the action $a_i = 1$ is strictly dominant at all $x_i^\varepsilon > \overline{x} - s + \varepsilon$. (ii) For each player i, the action $a_i = 0$ is strictly dominant at $x_i^\varepsilon < \underline{x} - s - \varepsilon$.*

Per Lemma 3, $u_i^\varepsilon(p_{-i} \mid \overline{X}, s) > 0$ for all p_{-i}. In particular, therefore, we have

$$u_i^\varepsilon(p_{-i}^{\overline{X}} \mid \overline{X}, s) > 0. \tag{7}$$

Let R^1 be the solution to

$$u_i^\varepsilon(p_{-i}^{\overline{X}} \mid R^1, s) = 0. \tag{8}$$

To any player i, the action $a_i = 1$ is strictly dominant at all $x_i^\varepsilon > R^1$ conditional on $\overline{X}$.

This argument can be repeated and we obtain a sequence $\overline{X} = R^0, R^1, \ldots$. For any k and R^k such that $u_i^\varepsilon(p_{-i}^{R^k} \mid R^k, s) > 0$, there exists R^{k+1} that solves $u_i^\varepsilon(p_{-i}^{R^k} \mid R^{k+1}, s) = 0$. Induction on k reveals that $R^{k+1} < R^k$ for all $k \geq 0$. Per Lemma 3, we know that $R^k \geq \underline{X}$ for all k. It follows that the sequence (R^k) is monotone and bounded. Such a sequence must converge; let R^* denote its limit. By construction, R^* solves

$$u_i^\varepsilon(p_{-i}^{R^*} \mid R^*, s) = 0. \tag{9}$$

A similar procedure should be carried out starting from low signals, eliminating ranges of x_i^ε for which playing 1 is strictly (iteratively) dominated. This yields an increasing and bounded sequence (L^k) whose limit is L^*. The limit L^* solves

$$u_i^\varepsilon(p_{-i}^{L^*} \mid L^*, s) = 0. \tag{10}$$

Lemma 4. *To each player i, a strategy p_i survives iterated elimination of strictly dominated strategies if and only if $p_i^{R^*}(x_i^\varepsilon) \leq p_i(x_i^\varepsilon) \leq p_i^{L^*}(x_i^\varepsilon)$ for all x_i^ε.*

It should be clear at this point that here, too, the unraveling effect of policy is at play. Lest any doubts remain, consider an alternative subsidy $s' > s$. By (8), we have $u_i^\varepsilon(p_{-i}^{\overline{X}} \mid R^1, s') > 0$. There hence exists $R^{1'} < R^1$ such that $u_i^\varepsilon(p_{-i}^{\overline{X}} \mid R^{1'}, s') = 0$, making $p_i^{R^{1'}}$ the best response to $p_{-i}^{\overline{X}}$ given the subsidy s'. Since $p_i^{R^{1'}}(x_i^\varepsilon) \geq p_i^{R^1}(x_i^\varepsilon)$, with a strict inequality for all $x_i^\varepsilon \in (R^{1'}, R^1)$, the higher subsidy s' affects the best-response strategies of all other players $j \neq i$ even if offered the original subsidy s; ceteris paribus, it raises their incentive to play 1. The increased incentives for all $j \neq i$ in turn raise i's incentive to play 1, and so on.

Lemma 4 restricts the set of strategies that survive iterated dominance, which can differ in the action they prescribe only for signals $x_i^\varepsilon \in (L^*, R^*)$. Lemma 5 will be used to established that this interval has measure zero.

Lemma 5. *For all X such that $\underline{X} + \varepsilon \leq X \leq \overline{X} - \varepsilon$, one has the following:*

$$u_i^\varepsilon(p_{-i}^X \mid X, s) = X + \sum_{n=0}^{N-1} \frac{w(n)}{N} - c + s. \tag{11}$$

From Lemma 5 follows that $u_i^\varepsilon(p_{-i}^X \mid X, s)$ is strictly increasing in X. Moreover, by definition the limits L^* and R^* solve $u_i^\varepsilon(p_{-i}^{L^*} \mid L^*, s) = u_i^\varepsilon(p_{-i}^{R^*} \mid R^*, s) = 0$. It follows that $L^* = R^*$. We henceforth write $x^* = L^* = R^*$ (for given s).

Lemma 6. *The strategy vector p^{x^*} is the essentially unique strategy vector surviving iterated elimination of strictly dominated strategies in $G^\varepsilon(s)$. In particular, if, for any player i, the strategy p_i survives iterated elimination of strictly dominated strategies, then p_i must satisfy $p_i(x_i^\varepsilon) = p_i^{x^*}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x^*$.*

The planner commits to paying subsidies s hoping to affect the equilibrium of the game. Given the assumed information structure, Bayesian Nash equilibrium (BNE) is the relevant equilibrium concept. We recall that a strategy vector $p = (p_1, p_2, ..., p_N)$ is a BNE of $G^\varepsilon(s)$ if for any p_i and any x_i^ε it holds that:

$$p_i(x_i^\varepsilon) \in \underset{a_i \in \{0,1\}}{\arg\max}\, \pi_i^\varepsilon((p_{-i}) \mid x_i^\varepsilon), \tag{12}$$

where $\pi_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid x_i^\varepsilon) := \int \pi_i(p_{-i}(x_{-i}^\varepsilon) \mid x)\, d\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon)$. It is therefore immediate that p^{x^*} is a BNE of $G^\varepsilon(s)$. Theorem 1 substantially strengthens this result: if the strategy vector $p = (p_i)$ is a BNE of $G^\varepsilon(s)$, then for each p_i it must hold that $p_i(x_i^\varepsilon) = p^{x^*}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x^*$. We say that $G^\varepsilon(s)$ has an essentially unique BNE.

Theorem 1. *The essentially unique Bayesian Nash equilibrium of $G^\varepsilon(s)$ is p^{x^*}. In particular, if p an equilibrium of $G^\varepsilon(s)$ then any $p_i \in p$ satisfies $p_i(x_i^\varepsilon) = p_i^{x^*}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x^*$ and all i.*

We note that Theorem 1 implies full equilibrium coordination in *strategies*. A strategy is not an action. In particular, though players coordinate on $p^{\underline{x}}$ in equilibrium, when $x_i^\varepsilon > \underline{x}$ for some players i while $x_j^\varepsilon < \underline{x}$ for some other players j, coordination on $\overline{a}$ is not achieved in equilibrium. When $\varepsilon \to 0$, such signal distributions are possible only if $x = \underline{x}$, which has prior probability zero.

Corollary 1.

(i) For all $x > a^* + \varepsilon$ it holds that $\Pr\left[p^{x^*}(x^\varepsilon) = \mathbf{1} \mid x\right] = 1$.

(ii) For all $x < x^* - \varepsilon$ it holds that $\Pr\left[p^{x^*}(x^\varepsilon) = \mathbf{0} \mid x\right] = 1$.

(iii) For $x \in [x^* - \varepsilon, x^* + \varepsilon]$, $\Pr\left[p^{x^*}(x^\varepsilon) = \mathbf{1} \mid x\right]$ *is strictly increasing in x.*

Theorem 1, as well as Corollary 1, are graphically illustrated in Figure 1. Except in a vanishing neighborhood around x^*, coordination of actions is achieved.

Thus far, the analysis took $s \geq 0$ as given, which allowed us to leave it out of the notation when discussing the limits L^* and R^*. It is clear, however, that these limits depends upon s. Moreover, the subsidy is set by the planner and so is a variable we can tinker with. To make this more explicit, we henceforth write $x^*(s)$ as the common limit, i.e. $x^*(s)$ is the solution to

$$u_i^\varepsilon(p_{-i}^{x^*(s)} \mid x^*(s), s) = 0$$

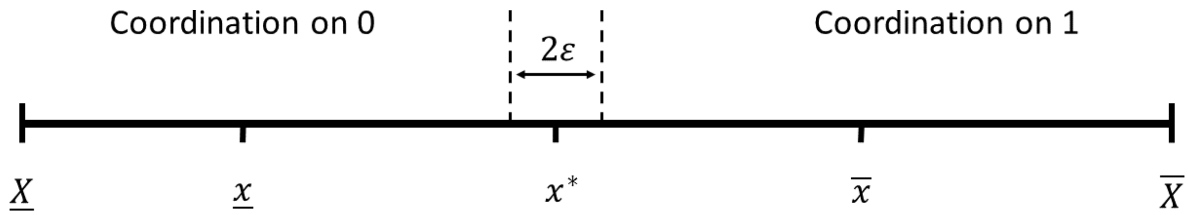

Figure 1: Illustration of the unique Bayesian Nash equilibrium of $G^\varepsilon(s)$, given s.

which, using (11), gives

$$x^*(s) = c - s - \sum_{n=0}^{N-1} \frac{w(n)}{N}. \tag{13}$$

We want to find s^* that allows the planner to implement $x^*(s^*) = \underline{x}$, i.e. to coordinate players on playing 1 whenever their signal tells them coordination on 1 is the efficient outcome of the game. Solving (13) for $x^*(s^*) = \underline{x}$ characterizes the optimal subsidy s^*.

Theorem 2. *The optimal subsidy s^* is given by*

$$s^* = w(N-1) - \sum_{n=0}^{N-1} \frac{w(n)}{N}. \tag{14}$$

In the limit as $\varepsilon \to 0$, s^* sustains coordination on the efficient equilibrium of $G(x)$ as the essentially unique Bayesian Nash equilibrium of $G^\varepsilon(s^*)$ almost surely.

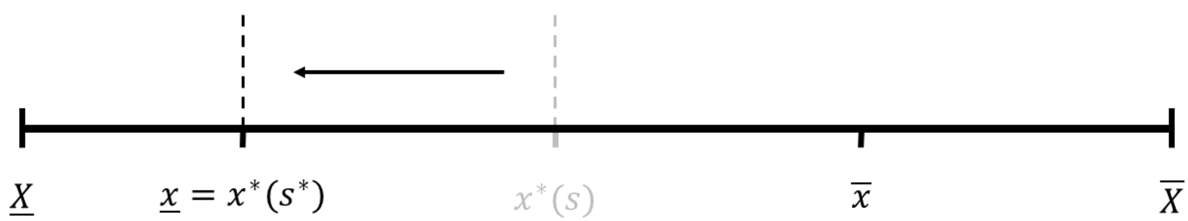

Figure 2: The optimal subsidy s^* makes coordination on $p^{\underline{x}}$ is the unique Bayesian Nash equilibrium of $G^\varepsilon(s^*)$.

Some properties of s^* are noteworthy.

First, optimal subsidies do not depend upon c, the outside option or cost of playing 1. In the example of a joint investment subsidy, subsidies typically target investment costs to make investment a safe option for some or all players. In our framework, subsidies instead target the spillovers from coordination. The basic intuition is similar: in a coordination problem, playing 1 (investing) is risky because (i) investment is costly and (ii) pays off only if sufficiently many other players do the same. A subsidy that covers opportunity costs takes care of (i) and makes players effectively indifferent to the payoff uncertainty that derives from (ii).[4] An optimal subsidy equal to s^* instead targets the payoff uncertainty that derives from the (ii), removing just enough of said uncertainty to incentive players to play 1. When opportunity costs vary across players, they do affect optimal subsidies, though even then it is only the *difference* in costs that matters; see Section 3.2.

Second, and related, optimal subsidies are increasing in $w(N-1) - w(n)$, all $n < N-1$. Intuitively, $w(N-1) - w(n)$ is a measure for the strength of players' coordination incentives. This is most easily seen by considering the special case of a regime change game (see footnote 2). In such games, a player who chooses to play 1 gains a one-off amount $\overline{w} - \underline{w} > 0$ when the critical number of player who choose 1 is reached. The higher is $\overline{w} - \underline{w}$, the greater the player's potential benefit from coordination, that is, the stronger their coordination incentives.

Third, subsidies in excess of s^* are inefficient. This inefficiency is twofold. First, the planner is spending more money than strictly necessary, money that – outside of our simple model – could be put to other uses. More interestingly, however, is that subsidies in excess of s^* make equilibrium play inefficient. If $\tilde{s} > s^*$, then using (13) to solve for $\tilde{x} = x^*(\tilde{s})$ gives $x^*(\tilde{s}) < \underline{x}$. By Theorem 1, the unique equilibrium of $G^\varepsilon(\tilde{s})$ is $p^{\tilde{x}}$. Letting $\varepsilon \to 0$, this makes $\overline{a}$ the unique equilibrium outcome of $G^\varepsilon(\tilde{x})$ for all $x \in [\tilde{x}, \underline{x})$ even though $\underline{a}$ would be the efficient outcome of $G(x)$ for those x. This second type of inefficiency does not arise in the participation problems studied by Sakovics and Steiner (2012) or Bernstein and Winter (2012), where full participation is by assumption the efficient outcome of the game. When there is prior uncertainty about the efficient course of action, the planner should subsidize conservatively to avoid picking inefficient winners.

Finally, we emphasize that a modest s^* solves the coordination problem due to the *strategic* repercussions of subsidies in coordination problems, that is, the unraveling effect. To see this, consider a global game in which players have only first-order knowledge of the subsidy; that is, each player i knows they will receive a subsidy equal to s upon playing 1, but this is all they know.[5] We assume that $x^*(0) - \underline{x} > 2\varepsilon$. Let $\tilde{G}^\varepsilon(s)$ denote the global game in which players have only first-order knowledge of the subsidy s, and let $\tilde{u}_i^\varepsilon$ denote the (conditional) expected incentive to play 1 for player i in $\tilde{G}^\varepsilon(s)$. Since no player i is assumed to be aware of the subsidies paid to other players, i's belief about the equilibrium strategies of its opponents is uniquely given by $p_{-i} = p_{-i}^{x^*(0)}$. Based on this belief, is playing $p_i^{\underline{x}}$ the equilibrium strategy of player i? No. Indeed, one has

$$\tilde{u}_i^\varepsilon(p_{-i}^{x^*(0)} \mid \underline{x}, s^*) = \underline{x} + s^* + w(0) - c = w(0) - \sum_{n=0}^{N-1} \frac{w(n)}{N} < 0.$$

Absent common knowledge of the policy, a modest subsidy like s^* is insufficient to solve the coordination problem. Indeed, with only first-order knowledge of the policy a planner would need to offer subsidies equal to $\tilde{s} = w(N-1) - w(0) > s^*$ to make $p^{\underline{x}}$ the unique BNE of $\tilde{G}^\varepsilon(\tilde{s})$, which is the subsidy that makes playing 1 strictly dominant for all $x > \underline{x}$. The reason, as this example was meant to illustrate, is strategic: assuming that other players do not know

[4] We say effectively indifferent because players would still prefer full coordination; however, whether or not coordination is achieved does not affect the sign of their incentive to play 1.

[5] This contrasts strongly with the game $G^\varepsilon(s)$ in which s is common knowledge. Any game in which there is at most n-th order knowledge about s, for finite n, would illustrate our point; the case $n = 1$ is taken for simplicity.

about the subsidy offered to player i, he by assumption believes that the associated increase in his incentive to play 1 is not known to his opponents. Hence, i cannot believe his increased incentive also raises his opponents' incentives to play 1, and the subsidy affects i's incentive only directly. Absent the indirect, strategic effect of subsidies in player i's incentives, he must be subsidized up to strict dominance to have an incentive to play $p_i^{\underline{x}}$. The modest subsidy s^* works only when players are able to figure out how others respond to subsidies and the feedback effects these responses trigger.

So far, we assumed that the planner offers subsidies to implement $p^{\underline{x}}$ as the unique BNE of the game. While our results indicate that the subsidies required to achieve this can be relatively small, the planner might still lack the resources to offer the policy. This possibility motivates the analysis in Sakovics and Steiner (2012), who consider the problem of a planner seeking to achieve a given *ex ante* probability of successful efficient coordination; a probability less than 1 allows for lower subsidies, accommodating a weak planner's limited budget. We briefly discuss this care here.

Let $\tilde{x} \in [\underline{x}, x^*(0)]$ denote the chosen value of x such that the planner seeks to implement $p^{\tilde{x}}$ as the unique BNE of the game. The prior distribution of x directly implies a probability that $x \in [\underline{x}, \tilde{x}]$ is drawn and thus gives the ex ante probability of inefficient coordination. What subsidy $\tilde{s}$ is necessary and sufficient to implement $p^{\tilde{x}}$?

Corollary 2. *For $\tilde{x} \in [\underline{x}, x^*(0)]$, let $\tilde{s}$ denote the subsidy such that $p^{\tilde{x}}$ is the unique Bayesian Nash equilibrium of $G^{\varepsilon}(\tilde{s})$. Then $\tilde{s}$ is given by*

$$\tilde{s} = c - \sum_{n=0}^{N-1} \frac{w(n)}{N} - \tilde{x} \leq s^*. \tag{15}$$

The inequality is strict for all $\tilde{x} > \underline{x}$. The prior (before x is drawn) that players coordinate on the efficient equilibrium of $G(x)$ is

$$1 - \frac{\tilde{x} - \underline{x}}{\overline{X} - \underline{X}}, \tag{16}$$

which is less than 1.

One may use Corollary 2 to investigate optimal subsidies in joint investment problems, also called games of regime change. In those games, a player's payoff to playing 1 increases by some discrete amount once a critical degree of coordination 1 is reached but otherwise remains constant. Formally, let there be some I such that $0 < I < N$. Let $w(a_{-i}) = 0$ for all a_{-i} such that $n(a_{-i}) < I$ and $w(a_{-i}) = b > c$ otherwise. Let n^* denote the largest integer smaller than I. Joint investment problems implicitly assume $x = 0$ (if $x > 0$, it can be subsumed by b anyway) and focus on the problem of making coordination on $\overline{a}$ the unique equilibrium of the game. Imposing therefore that $\tilde{x} = 0$, we seek the subsidy $\tilde{s}$ that implements p^0 in strictly dominant strategies. By Corollary 2 the optimal subsidy $\tilde{s}$ for this game is given by

$$\tilde{s} = c - \frac{n^*}{N} \cdot b. \tag{17}$$

We will compare three possibilities toward subsidization the planner might pursue in such a game. To keep things simple, we focus on two-player games. In a two-player game, there is a coordination problem only if $n^* = 1$, which we will assume.

Option 1: subsidize according to the optimal subsidies derived in this paper. From (17), $\tilde{s} = c - b/2$. The total cost of this policy, for the planner in equilibrium, is $2c - b$.

Option 2: fully subsidize only one player to make investment dominant; then, taking as given investment by this player, his opponent will invest even without being offered a subsidy. This option is based on the example discussed in Sakovics and Steiner (2012). The total cost of this approach would be c. Noting that $b > c$, it follows that $2c - b < c$.

Option 3: fully subsidize one player, and tax the other playing up to indifference. This is a so-called *divide and conquer strategy*, discussed at length in Segal (2003) and Bernstein and Winter (2012). To make investment dominant for one player, the necessary subsidy is again c. Taking as given investment by the subsidized player, the other can be taxed by $t < b - c$ without affecting the equilibrium of the game. For some $\delta > 0$, let the chosen tax be given by $\tilde{t} = b - c - \delta$.[6] The total cost of this policy would be $\tilde{s} - \tilde{t} = 2c + \delta - b \searrow 2c - b$ as $\delta \to 0$.

Across the three policy strategies compared, the cheapest option is to rely on the optimal subsidies we develop in this paper.

The foregoing analysis maintained the strong assumption of symmetric players. In the next section, we discuss asymmetric games. We will show that the key insight developed so far – that policies have compounded effects in coordination games – generalize completely to asymmetric problems.

3.2 Generalization

In this section, we briefly discuss the extension to coordination games played among asymmetric players. Let there again be N players, indexed i, who must choose binary actions $a_i \in \{0, 1\}$. The payoff to player i is given by:

$$\pi_i(a \mid x) = \begin{cases} w_i(n(a_{-i})) + x & \text{if} \quad a_i = 1, \\ c_i & \text{if} \quad a_i = 0. \end{cases} \tag{18}$$

In (18), we allow c_i and $w_i(a_{-i})$ to vary across players; that is, players can have different opportunity costs to playing 1. They may also differ in terms of their coordination incentive, that is, their sensitivity to externalities (through w_i). We maintain the assumption that actions are strategic complements, so $w_i(n + 1) \geq w_i(n)$ for all n and $i \in \mathscr{N}$.[7]

The conditional incentive to play 1 is

$$u_i(a_{-i} \mid x) = \pi_i(1, a_{-i} \mid x) - \pi_i(0, a_{-i} \mid x) = w_i(n(a_{-i})) + x - c_i. \tag{19}$$

Define $\underline{x}_i := c_i - w_i(N - 1)$ and $\overline{x}_i := c_i - w_i(0)$. Whenever $x \in [\underline{x}_i, \overline{x}_i]$ player i's payoff-maximizing action may be either 0 or 1, depending on the a_{-i} played. Without loss of

[6] We need $\delta > 0$ to implement coordination on investment in *strictly* dominant strategies; if $\delta = 0$, investment is only weakly dominant for the taxed player.

[7] We note that the payoff function (18) is similar to that studied by Sakovics and Steiner (2012); however, the games considered differ in important ways. While Sakovics and Steiner (2012) allow players to differ in their private benefits and costs associated with playing 1, the critical mass of players needed to make a player willing to play 1 is the same for all. In contrast, our game allows each player i to have its own critical threshold n_i^*. Relatedly, it is, for some values of x, possible that some or all players prefer not to coordinate on playing 1, or that coordination on 1 is the inefficient outcome of the game, neither of which can occur in the game studied by Sakovics and Steiner (2012).

generality, relabel the players in decreasing order of $\underline{x}_i$, i.e. so that $\underline{x}_1 \geq \underline{x}_2 \geq \ldots \geq \underline{x}_N$. Define $\underline{x} := \sup\{\underline{x}_i \mid i \in \mathscr{N}\} = c_1 - w_1(N-1)$ and $\overline{x} = \inf\{\overline{x}_i \mid i \in \mathscr{N}\}$. We assume that $[\underline{x}, \overline{x}]$ is nonempty to ensure that, for some values of x at least, the players face a genuine coordination problem. That is, given common knowledge of x both $\overline{a}$ and $\underline{a}$ are a strict Nash equilibrium for all $x \in (\underline{x}, \overline{x})$.

We do not assume common knowledge of x, however. As before, we consider a global game Γ^ε in which the parameter x is not observed. The information structure is the same as in G^ε. Let $\Gamma^\varepsilon(s)$, $s = (s_i)$, denote the global game Γ^ε in which player i is offered a subsidy equal to $s_i \geq 0$ on playing 1.

In $\Gamma^\varepsilon(s)$, the expected conditional incentive to play 1 for player i is

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) = \int u_i(p_{-i}(x_{-i}^\varepsilon) \mid x_i^\varepsilon)\, d\Phi(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \tag{20}$$

and

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s_i) = u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) + s_i, \tag{21}$$

per the same argument as (6).

The goal of the planner is to find the optimal subsidy scheme $s^* = (s_i^*)$ such that coordination on $p^{\underline{x}}$ is the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(s^*)$. In the limit as $\varepsilon \to 0$, the unique equilibrium outcome of $\Gamma^\varepsilon(s^*)$ will be coordination on $\overline{a}$ whenever $x > \underline{x}$. That is, coordination on 1 will be the unique equilibrium of the global game whenever it is also a Nash equilibrium of the true game $\Gamma(x)$.[8] Since the argument is very similar to that for the case of symmetric players, we relegate it to the appendix.

Theorem 3. *In $\Gamma^\varepsilon(s)$, the optimal subsidy s_i^* is given by*

$$s_i^* = c_i - c_1 + w_1(N-1) - \sum_{n=0}^{N-1} \frac{w_i(n)}{N}, \tag{22}$$

for all i. In the limit as $\varepsilon \to 0$, the optimal subsidy scheme $s^* = (s_i^*)$ implements coordination on $\overline{a}$ whenever that is the efficient Nash equilibrium of $\Gamma(x)$ almost surely.

In the special case of symmetric players, (22) simplifies to (14), as expected. We observe that s_i^* admits the same qualitative properties discussed in Section 3.1.

It is worth pointing out that in the asymmetric game, the optimal subsidy for player i is increasing in i's opportunity cost c_i. An interesting insight emerges here by re-introducing some symmetry. Suppose that $w_i(n) = w_j(n)$ for all n and all $i, j \in \mathscr{N}$, so any asymmetries derive exclusively from differences in players' opportunity costs. In fact, from the way we relabeled players we have $c_1 \geq c_2 \geq \ldots \geq c_N$. The highest subsidy goes to 1, and subsidies subsequently decrease as players' opportunity costs decrease. This is intuitive: the costlier playing 1 is, the higher should a subsidy be to incentive a player to do so. Moreover, except for player 1 and other players for whom $c_i = c_1$ optimal subsidies are lower than in the symmetric game. This happens because those players are not subsidized all the way toward their individual $\underline{x}_j$, but only to the higher $\underline{x}_1$. We think of the case in which a planner offers

[8] In which case it is also the efficient outcome.

subsidies to make $p_i^{\underline{x}_i}$ the unique Bayesian Nash equilibrium as less interesting since it leads away from the *coordination* problem, which is the topic of this paper. For completeness, we describe the optimal subsidies for this policy objective in the Appendix.

In asymmetric games, an optimal subsidies policy admits two additional features that are both intuitively plausible and practically desirable.

First, symmetric players receive the same subsidy. Our subsidies shares this property with the policy in Sakovics and Steiner (2012) but not with the divide and conquer mechanism, in which symmetric players will be ranked differently resulting in asymmetric policy treatment (*c.f.* Segal, 2003; Bernstein and Winter, 2012).

Second, optimal subsidies are globally continuous in the model parameters. To see what we mean by this, observe that our characterization in (22) establishes global continuity of s_i^* in all the parameters upon which it depends $(c_i, c_1, w_1(N-1), w_i(n))$. Moreover, and importantly, each player $i \in \mathscr{N}$ is offered a subsidy. While conditional on policy treatment the optimal subsidies in Sakovics and Steiner (2012) are continuous in the relevant model parameters as well, changes in one player's parameters could affect whether or not said player is targeted, causing a discrete jump in subsidies received. Similarly, subsidies are continuous conditional on a player's position in the policy ranking in a divide and conquer mechanism (Segal, 2003; Bernstein and Winter, 2012); however, a player's position in the optimal ranking is affected by a change in its parameters, which cna lead to discrete jumps in subsidy entitlement.

In the spirit of our discussion leading up to Corollary 2, consider a less ambitious planner who wants to implement coordination on $p^{\tilde{x}}$, where $\tilde{x} > \underline{x}$. The subsidy $\tilde{s}_i$ that implements this equilibrium should solve

$$u_i^\varepsilon(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}_i) = \tilde{x} + \sum_{n=0}^{N-1} \frac{w_i(n)}{N} - c_i + \tilde{s}_i = 0, \tag{23}$$

for all i. The characterization of $\tilde{s}$ follows immediately

Corollary 3. *For $\tilde{x} \geq \underline{x}$, let $\tilde{s} = (\tilde{s}_i)$ denote the subsidy such that $p^{\tilde{x}}$ is the unique Bayesian Nash equilibrium of $G^\varepsilon(\tilde{s})$. Then $\tilde{s}_i$ is given by*

$$\tilde{s} = c_i - \sum_{n=0}^{N-1} \frac{w_i(n)}{N} - \tilde{x}, \tag{24}$$

for all i. The inequality is strict if $\tilde{x} > \underline{x}$.

In the special case of an asymmetric two-player joint investment problem, let the cost of investment to player i be c_i while his potential benefit is $b_i > c_i$. To each player i, the benefit from investment b_i is realized iff $a_i = a_j = 1$. We normalize the payoff from playing $a_i = 0$ to 0. A planner seeks to implement p^0. In this environment, the total equilibrium cost to a planner who offers our optimal subsidies is $\tilde{s}_1 + \tilde{s}_2 = c_1 + c_2 - (b_1 + b_2)/2$. Total equilibrium costs of fully subsidizing only a single player are $\min\{c_1, c_2\}$. If $2b_i > c_1 + c_2$ for $i = 1, 2$, partially subsidizing *all* investing players through our optimal subsidies is cheaper, in equilibrium, than fully subsidizing only a single player.

To conclude this discussion on optimal subsidies in asymmetric games with, let us relax the requirement that $\underline{x}$ must be strictly less than $\overline{x}$. That is, while $\underline{x}_i < \overline{x}_i$ for every i, it is possible that $\overline{x}_j < \underline{x}_i$ for some pairs of players $i, j \in \mathscr{N}$. Assume that there is at least one player i^* for whom this is true, i.e. $\overline{x}_{i^*} < \underline{x}$. The optimal subsidy to this player is negative: $s^*_{i*} < 0$.

Proposition 1. *Consider the game $\Gamma^\varepsilon(s)$ and all players i^* for whom $\overline{x}_{i^*} < \underline{x}$. If $s^* = (s^*_i)$ is the subsidy scheme that implements $p^{\underline{x}}$ as the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(s^*)$, then player i^* is taxed for playing $a_{i^*} = 1$: $s_{i^*} < 0$.*

If we allow players to be sufficiently asymmetric, then the optimal subsidy scheme s^* shares common features with a divide-and-conquer policy (*c.f.* Segal, 2003; Bernstein and Winter, 2012): some players are subsidized to play 1 while for others the same action is taxed. The economic motivation is different from that in Segal (2003) and Bernstein and Winter (2012), however. In our problem, players with low opportunity costs are "too eager" to play 1. Trying to achieve complete coordination (of strategies), the planner must temper these players' incentives to play 1, which is done through a tax. Thus, while a divide-and-conquer policy may be optimal in our problem, it only is when the planner has a somewhat unreasonable goal: achieve complete coordination at a all costs, including inefficiency.

3.3 Do We Really Need Equilibrium Selection?

Do we really need equilibrium selection to draw our above conclusions? Or is our global games approach rather a needless technical complication? In this section, we present an example to answer the first question affirmatively. The same example also illustrates why the answer to question number two is "no".

Consider again the game of complete information $G(x)$ described in Section 2. By Lemma 1, for any $\hat{x} \in [\underline{x}, \overline{x}]$ the strategy vector $a(x) = (a_i(x))$ in which

$$a_i(x) = \begin{cases} 1 & \text{if} \quad x < \hat{x} \\ 0 & \text{if} \quad x \geq \hat{x} \end{cases} \tag{25}$$

for every $i \in \mathscr{N}$ is a Nash equilibrium of the game. Similarly, in the game of complete information with subsidies to playing 1 equal to s, denoted $G(x, s)$, any strategy vector $a(x, s) = (a_i(x, s))$ in which

$$a_i(x, s) = \begin{cases} 1 & \text{if} \quad x + s < \hat{x} \\ 0 & \text{if} \quad x + s \geq \hat{x} \end{cases} \tag{26}$$

for every $i \in \mathscr{N}$ is a Nash equilibrium. Assume that every player $i \neq j$ is known to play according to (25) in $G(x)$ and according to (26) in $G(x, s)$. This arguably would not be intuitive but nothing prevents such beliefs from being held in equilibrium, so it cannot be ruled out that player j truly thinks this way.[9] Then, however, player j's incentive to play 1

[9]To be clear, it is not needed that players $i \neq j$ actually pursue such a weird strategy. All we need is that player j *believes* they do.

goes *down* upon being offered the subsidy s for all $x \in (\hat{x} - s, \hat{x}]$. In other words, the direct effect of subsidies is not guaranteed in a game of complete information and multiple Nash equilibria. Since the unraveling effect builds upon the direct effect of subsidies, it breaks down. This example illustrates equilibrium selection is necessary for our analysis.

4 Discussion and conclusions

The results in this paper suggest that a planner needs only modest policy intervention to solve coordination problems. The core mechanism driving this result is an *unraveling* effect of policies in coordination games. A subsidy raises a player's incentive to play the subsidized action. Her opponents know this and, given their desire to match actions, experience an increase in their own incentives, which in turn raises the player's incentive to pursue the subsidized action even further, and so on. Under common knowledge of the policy, this positive feedback loop compounds indefinitely. Compounding means that seemingly modest policy intervention can provide enough stimulus to steer rational agents toward efficiency. This unraveling effect is illustrated in Figure **??**.

We derive our results in a global coordination game. We use a global games approach to address equilibrium selection explicitly. Coordination games tend to have multiple strict Nash equilibria. Policies need to work conditional on players' beliefs about their opponents' equilibrium strategies, which depend upon the equilibrium they believe others will be playing. With multiple equilibria, strategic beliefs therefore are not uniquely defined and this complicates policy design. It also motivates invasive policymaking, as policies must work even against the most pessimistic (strategic) beliefs. However, certain expectations about equilibrium play would seem highly counterintuitive; for example, one would not expect a subsidy to decrease incentives to play the subsidizes action. That such outcomes are consistent with equilibrium play is an artefact of the equilibrium multiplicity in complete information coordination games (as we shown in Section 3.3). We therefore address equilibrium selection explicitly, and we do so in a global game. We provide conditions under which rational agents will be able to select a unique equilibrium of the coordination game they are playing. We characterize the unique equilibrium as a function of the subsidy in place and use this characterization to derive the optimal subsidy, that is, the subsidy that makes the Pareto efficient strategy the unique Bayesian Nash equilibrium of the global game.

A useful feature of global games is that these games are solved by iterated elimination of strictly dominated strategies. Iterated dominance depends upon players' ability to form higher-order beliefs about the strategies pursued by their opponents and, consequently, their strategic beliefs. This is useful because our analysis of subsidy design and the unraveling effect presupposes the same ability with respect to the effect of subsidies on strategies. For this reason, and though other theories of equilibrium selection exist, we think of global games as the best suited framework to study our problem.

The unraveling effect makes it *possible* to solve coordination problems through modest policy intervention. Prior uncertainty about the efficient outcome of the game makes modest intervention necessary. Subsidies that are too high risk stimulating players to pursue a particular action even if coordination on that action is an inefficient, or Pareto-dominated, outcome of the game. This warrants a conservative approach toward policy as the planner should avoid picking possibly Pareto-inferior winners. This possibility naturally complicates policy design compared to cases in which the efficient outcome is assumed to be known a priori (*c.f.* Bernstein and Winter, 2012; Sakovics and Steiner, 2012; Halac et al., 2020) and favors an noninvasive planner who subsidizes modestly.

Our analysis leaves unanswered a number of interesting questions. How are optimal subsidies affected when players can choose between more than two actions? How do heterogeneous spillovers (*c.f.* Bernstein and Winter, 2012; Sakovics and Steiner, 2012) affect policy design? And, perhaps most interestingly, how do our results hold up when used to solve coordination problems among real people? Given the degree of strategic sophistication that underlies our analysis, experimental testing of our results would seem to be an important next step. We hope to take this up in future research.

A Proofs

PROOF OF LEMMA 1

Proof. Follows from the preceding dominance results and direct payoff comparisons. □

PROOF OF LEMMA 2

Proof. First, $u_i^\varepsilon(n \mid x_i^\varepsilon)$ is increasing in both n and x_i^ε; and second, $\int n(p_{-i}^y(x_{-i}^\varepsilon))d\Phi^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon)$ is increasing in x_i^ε and decreasing in y_j. The result follows. □

PROOF OF LEMMA 3

Proof. When $x_i^\varepsilon > \overline{x} - s + \varepsilon$, the entire support of player i's conditional distribution on x lies in a region where $x > \overline{x}$. For any such x, playing 1 is strictly dominant. (i) follows. Part (ii) is proven in a similar way. □

PROOF OF LEMMA 5

Proof. First fix $x \in [\underline{X}+\varepsilon, \overline{X}-\varepsilon]$. Each player $j \neq i$ is assumed to play p_j^X, so the probability that $x_j = 1$ is given by

$$\Pr[x_j^\varepsilon > X \mid x] = \Pr[\varepsilon_j > X - x] = \frac{x + \varepsilon - X}{2\varepsilon}, \tag{27}$$

for all $X \in [x - \varepsilon, x + \varepsilon]$ while $\Pr[x_j^\varepsilon > X \mid x]$ is either 0 or 1 otherwise. Clearly, $a_j = 0$ is played with the complementary probability (given x and X). Since each ε_j is drawn independently from the same distribution, the probability that m given players $j \neq i$ play $a_j = 1$ while the remaining $N - m - 1$ players play $a_j = 0$ (given p_{-i}^X and x) is:

$$\left[\frac{x + \varepsilon - X}{2\varepsilon}\right]^m \left[\frac{X + \varepsilon - x}{2\varepsilon}\right]^{N-m-1}. \tag{28}$$

As there are $\binom{N-1}{m}$ unique ways in which m out of $N - 1$ players j can choose $a_j = 1$, the total probability of this happening, as a function of a, is:

$$\binom{N-1}{m} \left[\frac{x + \varepsilon - X}{2\varepsilon}\right]^m \left[\frac{X + \varepsilon - x}{2\varepsilon}\right]^{N-m-1}. \tag{29}$$

The derivation so far took x as known and given. We next take account of the fact that player i does not observe x directly but only the noisy signal x_i^ε. Given $p_{-i} = p_{-i}^X$ and $x_i^\varepsilon = X$, the expected incentive for player i to play $a_i = 1$ becomes:

$$u_i^\varepsilon(p_{-i}^X \mid X) = \frac{1}{2\varepsilon}\int\limits_{X-\varepsilon}^{X+\varepsilon} x\,dx - c$$

$$+ \sum_{m=0}^{N-1} w(m+1)\binom{N-1}{m}\frac{1}{2\varepsilon}\int\limits_{X-\varepsilon}^{X+\varepsilon}\left[\frac{x+\varepsilon-X}{2\varepsilon}\right]^m\left[\frac{X+\varepsilon-x}{2\varepsilon}\right]^{N-m-1} dx \quad (30)$$

$$= X - c + \sum_{m=0}^{N-1} w(m+1)\binom{N-1}{m}\int\limits_0^1 q^m(1-q)^{N-m-1}dq \quad (31)$$

$$= X - c + \sum_{m=0}^{N-1} w(m+1)\frac{(N-1)!}{m!\,(N-m-1)!}\frac{m!\,(N-m-1)!}{N!} \quad (32)$$

$$= X - c + \sum_{m=0}^{N-1}\frac{w(m+1)}{N}. \quad (33)$$

Equation (30) takes the expression for $u_i(m \mid x)$ given in (2) and integrates out x and m, given $x_i^\varepsilon = X$ and $p_{-i} = p_{-i}^X$. Equation (31) uses integration by substitution (using $q = 1/2 - (X-x)/2\varepsilon$) to rewrite the second integral in (30). Equation (32) rewrites both the integral in (31) and the binomial coefficient $\binom{N-1}{m}$ in terms of factorials. Equation (33) simplifies. Finally, we know that $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s) = u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) + s$. □

PROOF OF LEMMA 6

Proof. Combine the definitions of R^* and L^* in (9) and (10), respectively, with Lemmas 4 and 5. □

PROOF OF THEOREM 1

Proof. Let $p = (p_i)$ be a BNE of G^ε. For any player i, define

$$\underline{\underline{x}}_i = \inf\{x_i^\varepsilon \mid p_i(x_i^\varepsilon) > 0\}, \quad (34)$$

and

$$\overline{\overline{x}}_i = \sup\{x_i^\varepsilon \mid p_i(x_i^\varepsilon) < 1\}. \quad (35)$$

Observe that $\underline{\underline{x}}_i \leq \overline{\overline{x}}_i$. Now define

$$\underline{\underline{x}} = \min\{\underline{\underline{x}}_i\}, \quad (36)$$

and

$$\overline{\overline{x}} = \max\{\overline{\overline{x}}_i\}. \quad (37)$$

By construction, $\overline{\overline{x}} \geq \overline{\overline{x}}_i \geq \underline{\underline{x}}_i \geq \underline{\underline{x}}$. Observe that p is a BNE of $G^\varepsilon(s)$ only if, for each i, it holds that $u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i) \geq 0$. Consider then the expected incentive $u_i^\varepsilon(p_{-i}^{\underline{\underline{x}}}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i)$. It follows from the definition of $\underline{\underline{x}}$ that $p^{\underline{\underline{x}}}(x^\varepsilon) \geq p(x^\varepsilon)$ for all x^ε. The implication is that, for each i, $u_i^\varepsilon(p_{-i}^{\underline{\underline{x}}}(x_i - i^\varepsilon) \mid \underline{\underline{x}}_i) \geq u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i) \geq 0$. From Proposition 5 then follows that $\underline{\underline{x}} \geq x^*$.

Similarly, if p is a BNE of $G^\varepsilon(s)$ then, for each i, it must hold that $u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \overline{x}_i) \leq 0$. Consider the expected incentive $u_i^\varepsilon(p_{-i}^{\overline{x}}(x_{-i}^\varepsilon) \mid \overline{\overline{x}}_i)$. It follows from the definition of $\overline{\overline{x}}$ that $p^{\overline{\overline{x}}}(x^\varepsilon) \leq p(x^\varepsilon)$ for all x^ε. For each i it therefore holds that $u_i^\varepsilon(p_{-i}^{\overline{\overline{x}}}(x_{-i}^\varepsilon) \mid \overline{\overline{x}}_i) \leq u_i^\varepsilon(p_{-i}(x_i - i^\varepsilon) \mid \overline{\overline{x}}_i) \leq 0$. Hence $\overline{\overline{x}} \leq x^*$.

Since $\underline{\underline{x}} \leq \overline{\overline{x}}$ while also $\underline{\underline{x}} \geq x^*$, see Lemma 6, and $\overline{\overline{x}} \leq x^*$ it must hold that $\underline{\underline{x}} = \overline{\overline{x}} = x^*$. Moreover, since $p^{\underline{\underline{x}}} \geq p$ while also $p^{\overline{\overline{x}}} \leq p$, given $\underline{\underline{x}} = \overline{\overline{x}} = x^*$, it follows that $p_i(s_i^\varepsilon) = p_i^{x^*}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x^*$ and all i (recall that for each player i one has $u_i^\varepsilon(p_{-i}^{x^*} \mid x^*) = 0$, explaining the singleton exeption at $x_i^\varepsilon = x^*$). Thus, if $p = (p_i)$ is a BNE of $G^\varepsilon(s)$ then it must hold that $p_i(x_i^\varepsilon) = p_i^{x^*}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x^*$ and all i, as we needed to prove. □

PROOF OF THEOREM 2

Proof. Solving (13) for $x^*(s) = \underline{x} = c - w(N-1)$ gives

$$x^*(s^*) = c - s^* - \sum_{n=0}^{N-1} \frac{w(n)}{N} = c - w(N-1) \implies s^* = w(N-1) - \sum_{n=0}^{N-1} \frac{w(n)}{N}.$$

By Corollary 1, coordination on the efficient outcome of $G(x)$ thus happens with probability 1 for all $x \notin [\underline{x} - \varepsilon, \underline{x} + \varepsilon]$ in $G^\varepsilon(s^*)$. As $\varepsilon \to 0$, the probability that $x \notin [\underline{x} - \varepsilon, \underline{x} + \varepsilon]$ goes to zero and coordination on the efficient outcome of $G(x)$, see Lemma 1, is sustained almost surely. □

PROOF OF PROPOSITION 1

Proof. We are given that $p^{\underline{x}}$ is the unique BNE of $\Gamma^\varepsilon(s^*)$, so for player i^* we have

$$u_{i^*}^\varepsilon(p_{-i}^{\underline{x}} \mid \underline{x}, s_{i^*}^*) = u_{i^*}^\varepsilon(p_{-i}^{\underline{x}} \mid \underline{x}) + s_{i^*}^* = \underline{x} + s_{i^*}^* + \sum_{n=0}^{N-1} \frac{w_{i^*}(n)}{N} = 0,$$

where the second equality follows from Lemma 2. Per (22), the optimal subsidy $s_{i^*}^*$ to player i^* is equal to

$$s_{i^*}^* = c_{i^*} - c_1 + w_1(N-1) - \sum_{n=0}^{N-1} \frac{w_{i^*}(n)}{N}.$$

By hypothesis, $c_{i^*} \leq \underline{x}$ (i.e. because $\overline{x}_{i^*} > \underline{x}$). Combining these observations, and using that by definition $\underline{x} = c_1 - w_1(N-1)$,

$$s_{i^*}^* = \underbrace{c_{i^*} - \underline{x}}_{<0} - \underbrace{\sum_{n=0}^{N-1} \frac{w_{i^*}(n)}{N}}_{>0} < 0,$$

as claimed. □

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