Unraveling Coordination Problems*

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Preliminary draft. Please do not circulate.

Abstract

We study policy design in coordination problems. Our results reveal an unraveling effect of policies not documented before. A subsidy raises player i's incentive to choose the subsidized action. In coordination games, this raises j's incentive to play the same action, which further incentivizes i, and so on. Building on this logic, we characterize the optimal subsidy scheme that implement the efficient Nash equilibrium of the game as its unique equilibrium. Optimal subsidies are unique and increasing in the strength of players' coordination incentives. In asymmetric games, some subsides may be negative. Excessive intervention is a source of equilibrium inefficiency.

1 Introduction

Coordination problems arise throughout the economy and the inability of markets to solve these problems efficiently is a fundamental motivation for policy intervention. It is not obvious, however, what such interventions ideally looks like. In this paper, we study the optimal design of policies in coordination games. Our results can be applied to a wide range of problems including economic development (Jensen, 2007; Björkegren, 2019), disease eradication (Barrett, 2003), standard setting (Farrell and Saloner, 1985), discrete public good provision (Palfrey and Rosenthal, 1984).

Policy design in coordination games is difficult due to the strategic considerations that do not arise in other types of games. A policy must create incentives for players to pursue the desired strategy; however, in a coordination game with multiple Nash equilibria the incentive to play any given strategy depends upon the strategies played by all players. What obtains is a situation in which each player must second-guess other players' strategies to decide upon their own. Because players are largely free in their choice of strategic beliefs, this reasoning suggests that policy intervention should be extreme: it must make the desired course of action strictly dominant, that is, attractive under even the most pessimistic strategic expectations. Though intuitively appealing, we argue that such arguments are wrong.

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This paper characterizes the optimal subsidies that sustain coordination on the efficient Nash equilibrium of a coordination game as its unique equilibrium. Our results illustrate that the strategic nature of coordination problems does *not* motivate extreme intervention; rather, we show that coordination incentives can be put to good use in the design of policies. Optimal subsidies are unique and strictly less that those needed to achieve dominance – the difference is increasing in the strength of players' coordination incentives. In symmetric games, subsidies do not depend on opportunity costs (*c.f.* Sakovics and Steiner, 2012); in asymmetric games, only the difference in opportunity costs matters. Symmetric players are subsidized symmetrically (*c.f.* Bernstein and Winter, 2012). Finally, a subsidy in excess of our optimal subsidy is inefficient because it incentivizes players to coordinate on a Pareto dominated equilibrium.

Why can relatively modest subsidies solve the coordination problem? The essence of our argument is simple. Consider a set of players who must choose to participate, or not, in some joint activity. The individual incentives to participate are increasing in overall participation such that both full participation and full non-participation are equilibria of the game. A planner considers offering subsidies to stimulate participation. How do these subsidies affect incentives? To begin with, the subsidy raises each player i's incentive to participate; this is the direct effect. Due to the direct effect, a subsidy paid to player i also increase the incentive to participate of every player $j \neq i$ because of the increasing returns to participation; this is the indirect effect. The increase in j's incentive further raises the incentive of player i, and so on. This positive feedback loop makes participation increasingly attractive to all players – much more so than the direct effect alone would suggest. Even fairly minor subsidies can therefore be sufficient to unravel a coordination problem and sustain coordination on its efficient equilibrium. Figure 1 gives a graphical illustration of the *unraveling effect*. To our knowledge, this paper is the first to describe this effect.

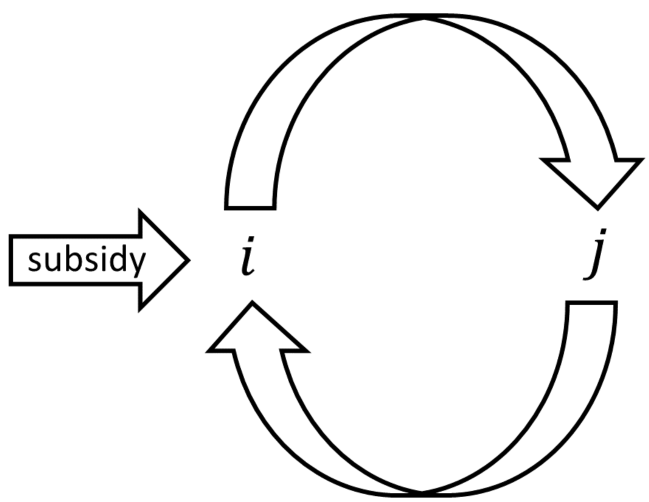

Figure 1: In coordination problems, subsidies kickstart a positive feedback loop that drives the unraveling effect.

The formal argument is a bit more complicated and calls for additional technicalities. Most importantly, the "direct effect" of subsidies on incentives requires qualification. In a game with multiple Nash equilibria, a small subsidy does not necessarily raise players' participation incentive; indeed, upon being offered subsidies incentives may even go down. The reason is that players are free to adjust their beliefs about opponents' strategies and, in particular, how these respond to the subsidy. It is therefore possible that players expect their opponents to participate without subsidies but not to participate with subsidies (that is, modest subsidies may not be enough to prevent both outcomes from being a Nash equilibrium of the game). The only way to work around this problem is to offer subsidies that make participation incentives independent of players' strategic beliefs; that is, subsidies that make participation strictly dominant.

Of course, it is not intuitively plausible that players would choose to participate without subsidies yet choose not to with subsidies. That this possibility arises is an artifact of the equilibrium multiplicity of the game. To rule such counterintuitive strategies out, we must address equilibrium selection. We do so using the methodology of global games (Carlsson and Van Damme, 1993; Frankel et al., 2003). In a global game, players do not observe their true payoff functions; rather, each player receives a private noisy signal of it. Under some minor technical assumption, a process of iteratively eliminating strictly dominated strategies then leads to selection of a unique Bayesian Nash equilibrium. Equilibrium uniqueness severely restricts players' strategic beliefs and, as we establish, prevents the kind of counterintuive expectations about other players' responses to a policy. Specifically, we can show that subsidies unambiguously raise each player's incentive to play the subsidized action in a global game (and the more so the higher the subsidy). This means the direct effect of subsidies is at play in our global game. The *unraveling effect* can then be trusted to solve the coordination problem with fairly modest intervention.

Where the unraveling effect of policy allows subsidies to be modest without materially affecting incentives, another force necessitates moderating policy. Intuitively, prior uncertainty about payoff functions warrants conservative policy intervention to avoid picking possibly inefficient winners. Consider again the participation problem discussed above. In our game, there is an (unconditional) benefit parameter x that determines the strength of players' incentives to participate. It follows that the value of x also determines whether full participation is the efficient outcome of the game, or not. Generally, there will be some value $\underline{x}$ such that for $x < \underline{x}$, full participation is Pareto inferior to full non-participation. When offering too high a subsidy, the planner stimulates participation even when $x < \underline{x}$, leading to policy-induces inefficiency. Under prior uncertainty about x, the planner should shy away from picking winners and offer modest subsidies to avoid sustaining coordination on an inefficient equilibrium. This type of moderating effect is not present in the participation problems studied by Sakovics and Steiner (2012) and Bernstein and Winter (2012), where full participation is by assumption the efficient outcome of the game (or, at least, assumed to be the planner's desired outcome.)

Interestingly, optimal subsidies depend critically on the strength of players' coordination incentives. Opportunity costs have only a minor (or, in symmetric games, none at all) effect on optimal subsidies. This finding differs from the policies proposed by Sakovics and Steiner (2012) and Bernstein and Winter (2012), where contracts target opportunity costs.

2 Model

Consider a normal form game played by players in a set $\mathscr{N} = \{1, 2, ..., N\}$, indexed i, who simultaneously choose binary actions $a_i \in \{0, 1\}$. We assume for now that players are symmetric to develop the intuition in the simplest possible environment; Section 3.3 discusses the case of asymmetric players.

Let $a \in \{0, 1\}^N$ denote a vector of actions by all players, and define $a_{-i} := a \setminus \{a_i\}$. We define $\overline{x} = (1, 1, ..., 1)$ and $\underline{x} = (0, 0, ..., 0)$; $\overline{x}_{-i}$ and $\underline{x}_{-i}$ are defined in similar ways. Let $n(a_{-i}) = \sum_{j \neq i} a_j$ denote the number of ones in a_{-i}. When a is played, player i who chooses 1 in a gets payoff $w(n(a_{-i})) + x$; when instead player i chooses 0 in a, their payoff is c_i. Here, $w(n(a_{-i}))$ describes the externalities on player i deriving from other players' actions. At times we may write $w(a_{-i})$, where it is understood that w depends upon a_{-i} only through $n(a_i)$. The variable x is a benefit parameter in players' payoff functions, and c is the opportunity cost of playing 1. Combining these elements, a player's payoff is given by

$$\pi_i(a \mid x) = \begin{cases} w(a_{-i}) + x & \text{if} \quad a_i = 1, \\ c & \text{if} \quad a_i = 0. \end{cases} \tag{1}$$

We study coordination problems in which the externality $w(a_{-i})$ is increasing in a_{-i}, i.e. $w(n + 1) \geq w(n)$ for all $n = 0, ..., N - 2$.[1] In the canonical example of a joint investment problem, the action $a_i = 1$ is interpreted as investment and c as the cost of investing (Sakovics and Steiner, 2012). Alternatively, actions might represent the choice to use of a particular kind of network technology and c is the cost differential between technologies (Björkegren, 2019). Or actions could denote participation in discrete public good provision, in which case c denotes the cost of provision (Palfrey and Rosenthal, 1984). Throughout this paper, to reflect the generality of our logic we shall mostly shy away from concrete interpretations and talk in terms of generic actions instead.

The above elements combined describe a game of complete information $G(x)$. In $G(x)$, we define a player's incentive $u_i(a_{-i})$ to choose 1 as the gain from playing 1, rather than 0, or

$$u_i(a_{-i} \mid x) = \pi_i(1, a_{-i} \mid x) - \pi_i(0, a_{-i} \mid x) = w(a_{-i}) + x - c. \tag{2}$$

Observe that, given a_{-i}, the incentive u_i to play 1 is strictly increasing in x. Denote $\underline{x} := c - w(N - 1)$ and $\overline{x} := c - w(0)$. One has $u_i(\overline{x}_{-i} \mid \underline{x}) = u_i(\underline{x}_{-i} \mid \overline{x}) = 0$. In other words, playing 1 is strictly dominant for all $x > \overline{x}$; playing 0 is strictly dominant for $x < \underline{x}$. The following is now immediate.

Lemma 1. *In $G(x)$,*

(i) $\overline{a}$ is a Nash equilibrium for all $x \geq \underline{x}$. It is the unique Nash equilibrium for all $x > \overline{x}$.

(ii) $\underline{a}$ is a Nash equilibrium for all $x \leq \overline{x}$. It is the unique Nash equilibrium for all $x < \underline{x}$.

(iii) $\overline{a}$ is the strictly welfare-maximizing outcome for all $x > \underline{x}$.

[1] Our model includes so-called threshold or regime change games, in which there exists $n^* \in \{1, 2, ..., N\}$ such that $w(n) = \underline{w}$ for all $n < n^*$ and $w(n) = \overline{w}$ otherwise, where $\underline{w} < \overline{w}$.

Proof. Follows from the preceding dominance results and direct payoff comparisons. $\square$

Rather than assume common knowledge of x, we consider a global game G^ε in which players do not observe x. Instead, it is assumed that x is drawn from the uniform distribution on $[\underline{X}, \overline{X}]$ where $\underline{X} < \underline{x}$ and $\overline{X} > \overline{x}$ and that each player i receives a private noisy signal x_i^ε of x, given by:[2]

$$x_i^\varepsilon = x + \varepsilon_i. \tag{3}$$

The term ε_i captures idiosyncratic noise in i's private signal. It is common knowledge that ε_i is an i.i.d. draw from the uniform distribution on $[-\varepsilon, \varepsilon]$. We assume that ε is sufficiently small, in a way made precise in Section 3.2. Let $x^\varepsilon = (x_i^\varepsilon)$ denote the vector of signals received by all players, and let x_{-i}^ε denote the vector of signals received by all players but i, i.e. $x_{-i}^\varepsilon = (x_j^\varepsilon)_{j\neq i}$. Note that player i observes x_i^ε but neither x nor x_{-i}^ε. We write $\Phi^\varepsilon(\cdot \mid x_i^\varepsilon)$ for the joint probability function of $(x, x_j^\varepsilon)_{j\neq i}$ conditional on x_i^ε.

The timing of G^ε is as follow. First, Nature draws a true x. Second, each player i receives its private signal x_i^ε of x. Third, all players simultaneously choose their actions. Lastly, payoffs are realized according to the true x and the actions chosen by all players.

Strategies and strict dominance. A strategy p_i for player i in G^ε is a function that assigns to any $x_i^\varepsilon \in [\underline{X} - \varepsilon, \overline{X} + \varepsilon]$ a probability $p_i(x_i^\varepsilon) \geq 0$ with which the player chooses action $a_i = 1$ when they observe x_i^ε. We write $p = (p_1, p_2, ..., p_N)$ for a strategy vector. Similarly, we write $p_{-i} = (p_j)_{j\neq i}$ for the vector of strategies for all players but i. Conditional on the strategy vector p_{-i} and a private signal x_i^ε, the expected incentive to play 1 for player i is given by:

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) := \int u_i(p_{-i}(x_{-i}^\varepsilon) \mid x)\, d\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon). \tag{4}$$

We say that the action $a_i = 1$ is strictly dominant at x_i^ε if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) > 0$ for all p_{-i}. Similarly, the action $a_i = 0$ is strictly dominant (in the global game G^ε) at x_i^ε if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) < 0$ for all p_{-i}. When $a_i = a$ is strictly dominant, we say that $a_i = 1 - a$ is strictly dominated.

Conditional dominance. Let L and R be real numbers. The action $a_i = 1$ is said to be dominant at x_i^ε conditional on R if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) > 0$ for all p_{-i} with $p_j(x_j^\varepsilon) = 1$ for all $x_j^\varepsilon > R$, all $j \neq i$. Similarly, the action $a_i = 0$ is dominant at x_i^ε conditional on L if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) < 0$ for all p_{-i} with $p_j(x_j^\varepsilon) = 1$ for all $x_j^\varepsilon > L$, all $j \neq i$.

Iterated elimination of strictly dominated strategies. Eliminate all pure strategies that are strictly dominated, as rational players may be assumed never to pursue such strategies. Next, eliminate a player's pure strategies that are strictly dominated if all other players are known to play only strategies that survived the prior round of elimination; and so on. The set of strategies that survive infinite rounds of elimination are said to survive iterated elimination of strictly dominated strategies, which is the solution concept we use in this paper.

Increasing strategies. For some $X \in \mathbb{R}$, let p_i^X denote the particular strategy such that $p_i^X(x_i^\varepsilon) = 0$ for all $x_i^\varepsilon < X$ and $p_i^X(x_i^\varepsilon) = 1$ for all $x_i^\varepsilon \geq X$. We will call p_i^X the *increasing strategy with switching point* X. By $p^X = (p_1^X, p_2^X, ..., p_N^X)$ we denote the strategy vector of increasing strategies with switching point X, and $p_{-i}^X = (p_j^X)_{j\neq i}$. Note that $a_i = 1$ is strictly

[2] In game theory, it is assumed that the game (in this case G^ε) is common knowledge; hence, the structure of the uncertainty (the joint distribution of x and all the signals x_j^ε), the possible actions and all the payoff functions are commonly known. For a formal treatment of common knowledge, see Aumann (1976).

dominant at x_i^ε conditional on R if and only if $u_i^\varepsilon(p_{-i}^R \mid x_i^\varepsilon) > 0$. Similarly, if $a_i = 0$ is strictly dominant at x_i^ε conditional on L then it must hold that $u_i^\varepsilon(p_{-i}^L \mid x_i^\varepsilon) < 0$.

In general, for a vector of real numbers (y_i) let $p^y = (p_i^{y_i})$ be a (possibly asymmetric) increasing strategy vector, and $p_{-i}^y = (p_j^{y_j})_{j \neq i}$. The following Lemma will prove highly useful throughout the analysis.

Lemma 2. *Given is a vector of real numbers (y_i) and the associated strategy vector $p^y = (p_i^{y_i})$. Then,*

(i) $u_i^\varepsilon(p_{-i}^y \mid x_i^\varepsilon)$ is monotone increasing in x_i^ε;

(ii) $u_i^\varepsilon(p_{-i}^y \mid x_i^\varepsilon)$ is monotone decreasing in y_j, all $j \neq i$.

Proof. First, $u_i^\varepsilon(n \mid x_i^\varepsilon)$ is increasing in both n and x_i^ε; and second, $\int n(p_{-i}^y(x_{-i}^\varepsilon))d\Phi^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon)$ is increasing in x_i^ε and decreasing in y_j. The result follows. □

It is important to note that monotonicity of $u_i^\varepsilon(p_{-i}^y \mid x_i^\varepsilon)$ in x_i^ε depends upon p_{-i}^y being increasing. For generic p_{-i}, $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon)$ can be locally decreasing in x_i^ε.

Subsidies. Let the planner commit to paying subsidy s on playing 1 to player i. Conditional on the subsidy s, player i's incentive to choose 1 becomes

$$u_i(a_{-i} \mid x, s) = u_i(a_{-i} \mid x) + s = w(a_{-i}) + x - c + s, \tag{5}$$

and the expected incentive (given the signal x_i^ε) is

$$\begin{aligned} u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s) &= \int u_i(p_{-i}(x_{-i}^\varepsilon) \mid x, s) d\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\ &= \int \left[u_i(p_{-i}(x_{-i}^\varepsilon) \mid x) + s\right] d\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\ &= u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) + s. \end{aligned} \tag{6}$$

A tax equal to s on playing 0 has the same effect on incentives; for consistency, we shall narrate the analysis in terms of subsidies.

All concepts and notation are now in place to proceed with the analysis.

3 The Unraveling Effect

3.1 Increasing Strategy Equilibrium

Suppose that, for some real number X, there exists an increasing strategy vector $p^X = (p_i^X)$ that is a Bayesian Nash equilibrium of G^ε. This assumption raises two questions: is there indeed an equilibrium increasing strategy equilibrium and, if so, what is X? Answers to both of these questions are provided in Section 3.2. For now, we simply maintain this assumption and use it to identify the unraveling effect of policy as simply as possible.

Since p^X is an equilibrium by hypothesis, we observe that for any player i it must hold that

$$u_i^\varepsilon(p_{-i}^X \mid X) = 0.$$

When (only) player i is offered a subsidy $s > 0$ on playing 1, their incentive to play 1 (in the assumed equilibrium and conditional on observing signal $x_i^\varepsilon = X$) becomes:

$$u_i^\varepsilon(p_{-i}^X \mid X, s) = u_i^\varepsilon(p_{-i}^X \mid X) + s = s > 0.$$

Thus, upon being offered this subsidy, p_i^X no longer is a best-response to p_{-i}^X. Instead, there is a new point X_i^1 that solves

$$u_i^\varepsilon(p_{-i}^X \mid X_i^1, s) = u_i^\varepsilon(p_{-i}^X \mid X_i^1) + s = 0, \tag{7}$$

such that $p_i^{X_i^1}$ is the best response of player i against p_{-i}^X given the subsidy s.[3] Because $u_i^\varepsilon(p_{-i}^X \mid x_i^\varepsilon)$ is increasing in x_i^ε, it follows that $X_i^1 < X$. Conditional on the subsidy s offered to player i, a candidate Bayesian Nash equilibrium is now the strategy vector $(p_i^{X_i^1}, p_{-i}^X)$.

Even though only player i is offered the subsidy s, $(p_i^{X_i^1}, p_{-i}^X)$ is not a Bayesian Nash equilibrium of the game thus obtained. The reason is that for each player j, p_j^X is a best response to p_{-j}^X; it is not a best response to $(p_i^{X_i^1}, p_{-ij}^X)$, where $p_{-ij}^X := (p_l^X)_{l\neq i,j}$. Define X_j^1 as the unique solution to

$$u_j^\varepsilon\left(p_i^{X_i^1}, p_{-ij}^X \mid X_j^1\right). \tag{8}$$

The increasing strategy $p_j^{X_j^1}$ is player j's best response to the strategy vector $(p_i^{X_i^1}, p_{-ij}^X)$ for every player $j \neq i$. From $X_i^1 < X$ follows that $X_j^1 < X$. Define $X_{-i}^1 := (X_j^1)_{j\neq i}$. Thus, a second candidate equilibrium would be $(p_i^{X_i^1}, p_{-i}^{X_{-i}^1})$.

We are still not done. If player i plays $p_i^{X^1}$ then all other players j will respond with $p_j^{X_j^1}$, and $p_i^{X^1}$ is not a best response to this. Define $X_i^0 = X_j^0 = X$. For all natural numbers $k \geq 0$, recursively define

$$u_i^\varepsilon(p_{-i}^{X_{-i}^k} \mid X_i^{k+1}, s) = 0 \tag{9}$$

and

$$u_j^\varepsilon(p_i^{X_i^k}, p_{-ij}^{X_{-i}^k} \mid X_j^{k+1}) = 0, \tag{10}$$

for all $j \neq i$. We observe that $X_i^{k+1} < X_i^k$ for all k and all $i = 1, 2, ..., N$ so that (X_i^k) and (X_j^k) are monotone decreasing sequences. Let X_j^s and X_j^s, all $j \neq i$, denote the limits of these sequences. By construction, we have:

$$u_i^\varepsilon(p_{-i}^{X_{-i}^s} \mid X_i^s, s) = u_j^\varepsilon(p_i^{X_i^s}, p_{-ij}^{X_{-i}^s} \mid X_j^s) = 0.$$

Hence, the increasing strategy vector $(p_i^{X_i^s}, p_{-i}^{X_{-i}^s})$ is a Bayesian Nash equilibrium of the game when only player i is offered the subsidy s on playing 1.

Even though only player i was subsidized, this subsidy increases other players' incentives to play 1 due to their desire to coordinate. The increased incentives of the other players in turn raises player i's incentive, and so on. In coordination problems, a policy thus kickstarts

We discussed the case in which only a single player is offered the subsidy to make explicit that a policy targeting one player also, and similarly, affects all other players in a coordination

[3] Clearly X_i^1 depends upon s; we do not write out this dependence explicitly to avoid notational clutter.

game. The more natural policy would have each player offered the subsidy s. We analyze this scenario in the next section, where we also establish that the game with subsidies has only one Bayesian Nash equilibrium, which is in increasing strategies. Finally, we characterize the equilibrium and solve for the optimal subsidy s^*.

3.2 Formal Analysis

Let $G^\varepsilon(s)$ denote the game G^ε in which the planner commits to paying each player who plays 1 a subsidy equal to $s > 0$. We assume throughout that $2\varepsilon < \min\{\underline{x} - s - \underline{X}, \overline{X} - \overline{x} + s\}$.

The goal of this section is as follows. First, we show that $G^\varepsilon(s)$ has a unique Bayesian Nash equilibrium. We characterize the unique equilibrium, showing that it is in increasing strategies and depends upon s. We use our characterization to solve for the optimal subsidy s^*; here, optimality is defined as the subsidy that makes $p^{\underline{x}}$ the unique equilibrium of the game, which as $\varepsilon \to 0$ implies coordination on the efficient outcome of the game almost surely.

The following lemma is a useful starting point for our analysis.

Lemma 3. *Given a subsidy $s > 0$, consider the game $G^\varepsilon(s)$. (i) For each player i, the action $a_i = 1$ is strictly dominant at all $x_i^\varepsilon > \overline{x} - s + \varepsilon$. (ii) For each player i, the action $a_i = 0$ is strictly dominant at $x_i^\varepsilon < \underline{x} - s - \varepsilon$.*

Proof. When $x_i^\varepsilon > \overline{x} - s + \varepsilon$, the entire support of player i's conditional distribution on x lies in a region where $x > \overline{x}$. For any such x, playing 1 is strictly dominant. (i) follows. Part (ii) is proven in a similar way. $\square$

Per Lemma 3, $u_i^\varepsilon(p_{-i} \mid \overline{X}, s) > 0$ for all p_{-i}. In particular, therefore, we have

$$u_i^\varepsilon(p_{-i}^{\overline{X}} \mid \overline{X}, s) > 0. \tag{11}$$

Let R^1 be the solution to

$$u_i^\varepsilon(p_{-i}^{\overline{X}} \mid R^1, s) = 0. \tag{12}$$

To any player i, the action $a_i = 1$ is strictly dominant at all $x_i^\varepsilon > R^1$ conditional on $\overline{X}$.

This argument can be repeated and we obtain a sequence $\overline{X} = R^0, R^1, ...$ For any k and R^k such that $u_i^\varepsilon(p_{-i}^{R^k} \mid R^k, s) > 0$, there exists R^{k+1} that solves $u_i^\varepsilon(p_{-i}^{R^k} \mid R^{k+1}, s) = 0$. Induction on k reveals that $R^{k+1} < R^k$ for all $k \geq 0$. Per Lemma 3, we know that $R^k \geq \underline{X}$ for all k. It follows that the sequence (R^k) is monotone and bounded. Such a sequence must converge; let R^* denote its limit. By construction, R^* solves

$$u_i^\varepsilon(p_{-i}^{R^*} \mid R^*, s) = 0. \tag{13}$$

A similar procedure should be carried out starting from low signals, eliminating ranges of x_i^ε for which playing 1 is strictly (iteratively) dominated. This yields an increasing and bounded sequence (L^k) whose limit is L^*. The limit L^* solves

$$u_i^\varepsilon(p_{-i}^{L^*} \mid L^*, s) = 0. \tag{14}$$

Lemma 4. *To each player i, a strategy p_i survives iterated elimination of strictly dominated strategies if and only if $p_i^{R^*}(x_i^\varepsilon) \leq p_i(x_i^\varepsilon) \leq p_i^{L^*}(x_i^\varepsilon)$ for all x_i^ε.*

It should be clear at this point that here, too, the unraveling effect of policy is at play. Lest any doubts remain, consider an alternative subsidy $s' > s$. By (12), we have $u_i^\varepsilon(p_{-i}^{\overline{X}} \mid R^1, s') > 0$. There hence exists $R^{1'} < R^1$ such that $u_i^\varepsilon(p_{-i}^{\overline{X}} \mid R^{1'}, s') = 0$, making $p_i^{R^{1'}}$ the best response to $p_{-i}^{\overline{X}}$ given the subsidy s'. Since $p_i^{R^{1'}}(x_i^\varepsilon) \geq p_i^{R^1}(x_i^\varepsilon)$, with a strict inequality for all $x_i^\varepsilon \in (R^{1'}, R^1)$, the higher subsidy s' affects the best-response strategies of all other players $j \neq i$ even if offered the original subsidy s; ceteris paribus, it raises their incentive to play 1. The increased incentives for all $j \neq i$ in turn raise i's incentive to play 1, and so on.

Lemma 4 restricts the set of strategies that survive iterated dominance, which can differ in the action they prescribe only for signals $x_i^\varepsilon \in (L^*, R^*)$. Lemma 5 will be used to established that this interval has measure zero.

Lemma 5. *For all X such that $\underline{X} + \varepsilon \leq X \leq \overline{X} - \varepsilon$, one has the following:*

$$u_i^\varepsilon(p_{-i}^X \mid X, s) = X + \sum_{n=0}^{N-1} \frac{w(n)}{N} - c + s. \tag{15}$$

Proof. First fix $x \in [\underline{X}+\varepsilon, \overline{X}-\varepsilon]$. Each player $j \neq i$ is assumed to play p_j^X, so the probability that $x_j = 1$ is given by

$$\Pr[x_j^\varepsilon > X \mid x] = \Pr[\varepsilon_j > X - x] = \frac{x + \varepsilon - X}{2\varepsilon}, \tag{16}$$

for all $X \in [x - \varepsilon, x + \varepsilon]$ while $\Pr[x_j^\varepsilon > X \mid x]$ is either 0 or 1 otherwise. Clearly, $a_j = 0$ is played with the complementary probability (given x and X). Since each ε_j is drawn independently from the same distribution, the probability that m given players $j \neq i$ play $a_j = 1$ while the remaining $N - m - 1$ players play $a_j = 0$ (given p_{-i}^X and x) is:

$$\left[\frac{x + \varepsilon - X}{2\varepsilon}\right]^m \left[\frac{X + \varepsilon - x}{2\varepsilon}\right]^{N-m-1}. \tag{17}$$

As there are $\binom{N-1}{m}$ unique ways in which m out of $N - 1$ players j can choose $a_j = 1$, the total probability of this happening, as a function of a, is:

$$\binom{N-1}{m} \left[\frac{x + \varepsilon - X}{2\varepsilon}\right]^m \left[\frac{X + \varepsilon - x}{2\varepsilon}\right]^{N-m-1}. \tag{18}$$

The derivation so far took x as known and given. We next take account of the fact that player i does not observe x directly but only the noisy signal x_i^ε. Given $p_{-i} = p_{-i}^X$ and $x_i^\varepsilon = X$, the expected incentive for player i to play $a_i = 1$ becomes:

$$\begin{aligned} u_i^\varepsilon(p_{-i}^X \mid X) = &\frac{1}{2\varepsilon} \int_{X-\varepsilon}^{X+\varepsilon} x \, dx - c \\ &+ \sum_{m=0}^{N-1} w(m+1) \binom{N-1}{m} \frac{1}{2\varepsilon} \int_{X-\varepsilon}^{X+\varepsilon} \left[\frac{x + \varepsilon - X}{2\varepsilon}\right]^m \left[\frac{X + \varepsilon - x}{2\varepsilon}\right]^{N-m-1} dx \end{aligned} \tag{19}$$

$$=X - c + \sum_{m=0}^{N-1} w(m+1) \binom{N-1}{m} \int_0^1 q^m (1-q)^{N-m-1} dq \tag{20}$$

$$=X - c + \sum_{m=0}^{N-1} w(m+1) \frac{(N-1)!}{m!\,(N-m-1)!} \frac{m!\,(N-m-1)!}{N!} \tag{21}$$

$$=X - c + \sum_{m=0}^{N-1} \frac{w(m+1)}{N}. \tag{22}$$

Equation (19) takes the expression for $u_i(m \mid x)$ given in (2) and integrates out x and m, given $x_i^\varepsilon = X$ and $p_{-i} = p_{-i}^X$. Equation (20) uses integration by substitution (using $q = 1/2 - (X-x)/2\varepsilon$) to rewrite the second integral in (19). Equation (21) rewrites both the integral in (20) and the binomial coefficient $\binom{N-1}{m}$ in terms of factorials. Equation (22) simplifies. Finally, we know that $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s) = u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) + s$. $\square$

From Lemma 5 follows that $u_i^\varepsilon(p_{-i}^X \mid X, s)$ is strictly increasing in X. Moreover, by definition the limits L^* and R^* solve $u_i^\varepsilon(p_{-i}^{L^*} \mid L^*, s) = u_i^\varepsilon(p_{-i}^{R^*} \mid R^*, s) = 0$. It follows that $L^* = R^*$. We henceforth write $x^* = L^* = R^*$ (for given s).

Lemma 6. *The strategy vector p^{x^*} is the essentially unique strategy vector surviving iterated elimination of strictly dominated strategies in $G^\varepsilon(s)$. In particular, if, for any player i, the strategy p_i survives iterated elimination of strictly dominated strategies, then p_i must satisfy $p_i(x_i^\varepsilon) = p_i^{x^*}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x^*$.*

Proof. Combine the definitions of R^* and L^* in (13) and (13), respectively, with Lemmas 4 and 5. $\square$

The planner commits to paying subsidies s hoping to affect the equilibrium of the game. Given the assumed information structure, Bayesian Nash equilibrium (BNE) is the relevant equilibrium concept. We recall that a strategy vector $p = (p_1, p_2, ..., p_N)$ is a BNE of $G^\varepsilon(s)$ if for any p_i and any x_i^ε it holds that:

$$p_i(x_i^\varepsilon) \in \arg\max_{a_i \in \{0,1\}} \pi_i^\varepsilon((p_{-i}) \mid x_i^\varepsilon), \tag{23}$$

where $\pi_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid x_i^\varepsilon) := \int \pi_i(p_{-i}(x_{-i}^\varepsilon) \mid x)\, d\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon)$. It is therefore immediate that p^{x^*} is a BNE of $G^\varepsilon(s)$. Theorem 1 substantially strengthens this result: if the strategy vector $p = (p_i)$ is a BNE of $G^\varepsilon(s)$, then for each p_i it must hold that $p_i(x_i^\varepsilon) = p^{x^*}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x^*$. We say that $G^\varepsilon(s)$ has an essentially unique BNE.

Theorem 1. *The essentially unique Bayesian Nash equilibrium of $G^\varepsilon(s)$ is p^{x^*}. In particular, if p an equilibrium of $G^\varepsilon(s)$ then any $p_i \in p$ satisfies $p_i(x_i^\varepsilon) = p_i^{x^*}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x^*$ and all i.*

Proof. Let $p = (p_i)$ be a BNE of G^ε. For any player i, define

$$\underline{\underline{x}}_i = \inf\{x_i^\varepsilon \mid p_i(x_i^\varepsilon) > 0\}, \tag{24}$$

and

$$\overline{\overline{x}}_i = \sup\{x_i^\varepsilon \mid p_i(x_i^\varepsilon) < 1\}. \tag{25}$$

Observe that $\underline{\underline{x}}_i \leq \overline{\overline{x}}_i$. Now define

$$\underline{\underline{x}} = \min\{\underline{\underline{x}}_i\}, \tag{26}$$

and

$$\overline{\overline{x}} = \max\{\overline{\overline{x}}_i\}. \tag{27}$$

By construction, $\overline{\overline{x}} \geq \overline{\overline{x}}_i \geq \underline{\underline{x}}_i \geq \underline{\underline{x}}$. Observe that p is a BNE of $G^\varepsilon(s)$ only if, for each i, it holds that $u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i) \geq 0$. Consider then the expected incentive $u_i^\varepsilon(p_{-i}^{\underline{\underline{x}}}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i)$. It follows from the definition of $\underline{\underline{x}}$ that $p^{\underline{\underline{x}}}(x^\varepsilon) \geq p(x^\varepsilon)$ for all x^ε. The implication is that, for each i, $u_i^\varepsilon(p_{-i}^{\underline{\underline{x}}}(x_i - i^\varepsilon) \mid \underline{\underline{x}}_i) \geq u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i) \geq 0$. From Proposition 5 then follows that $\underline{\underline{x}} \geq x^*$.

Similarly, if p is a BNE of $G^\varepsilon(s)$ then, for each i, it must hold that $u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \overline{x}_i) \leq 0$. Consider the expected incentive $u_i^\varepsilon(p_{-i}^{\overline{x}}(x_{-i}^\varepsilon) \mid \overline{\overline{x}}_i)$. It follows from the definition of $\overline{\overline{x}}$ that $p^{\overline{\overline{x}}}(x^\varepsilon) \leq p(x^\varepsilon)$ for all x^ε. For each i it therefore holds that $u_i^\varepsilon(p_{-i}^{\overline{\overline{x}}}(x_{-i}^\varepsilon) \mid \overline{\overline{x}}_i) \leq u_i^\varepsilon(p_{-i}(x_i - i^\varepsilon) \mid \overline{\overline{x}}_i) \leq 0$. Hence $\overline{\overline{x}} \leq x^*$.

Since $\underline{\underline{x}} \leq \overline{\overline{x}}$ while also $\underline{\underline{x}} \geq x^*$ and $\overline{\overline{x}} \leq x^*$ it must hold that $\underline{\underline{x}} = \overline{\overline{x}} = x^*$. Moreover, since $p^{\underline{\underline{x}}} \geq p$ while also $p^{\overline{\overline{x}}} \leq p$, given $\underline{\underline{x}} = \overline{\overline{x}} = x^*$, it follows that $p_i(s_i^\varepsilon) = p_i^{x^*}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x^*$ and all i (recall that for each player i one has $u_i^\varepsilon(p_{-i}^{x^*} \mid x^*) = 0$, explaining the singleton exeption at $x_i^\varepsilon = x^*$). Thus, if $p = (p_i)$ is a BNE of $G^\varepsilon(s)$ then it must hold that $p_i(x_i^\varepsilon) = p_i^{x^*}(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x^*$ and all i, as we needed to prove. □

We note that Theorem 1 implies full equilibrium coordination in *strategies*. A strategy is not an action. In particular, though players coordinate on $p^{\underline{x}}$ in equilibrium, when $x_i^\varepsilon > \underline{x}$ for some players i while $x_j^\varepsilon < \underline{x}$ for some other players j, coordination on $\overline{a}$ is not achieved in equilibrium. When $\varepsilon \to 0$, such signal distributions are possible only if $x = \underline{x}$, which has prior probability zero.

Corollary 1.

(i) *For all* $x > a^* + \varepsilon$ *it holds that* $\Pr\left[p^{x^*}(x^\varepsilon) = \mathbf{1} \mid x\right] = 1$.

(ii) *For all* $x < x^* - \varepsilon$ *it holds that* $\Pr\left[p^{x^*}(x^\varepsilon) = \mathbf{0} \mid x\right] = 1$.

(iii) *For* $x \in [x^* - \varepsilon, x^* + \varepsilon]$, $\Pr\left[p^{x^*}(x^\varepsilon) = \mathbf{1} \mid x\right]$ *is strictly increasing in* x.

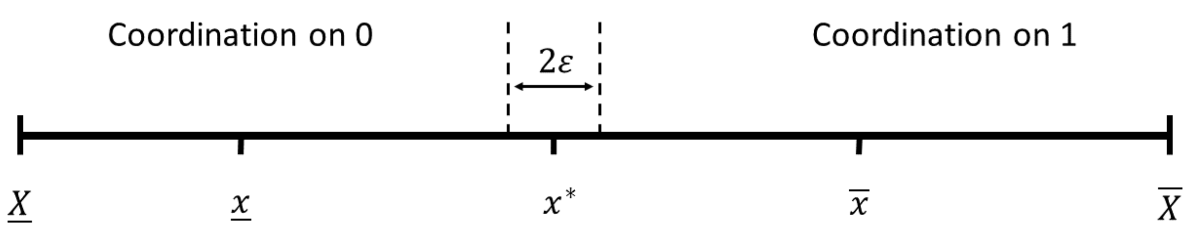

Figure 2: Illustration of the unique Bayesian Nash equilibrium of $G^\varepsilon(s)$, given s.

Theorem 1, as well as Corollary 1, are graphically illustrated in Figure 2. Except in a vanishing neighborhood around x^*, coordination of actions is achieved.

Thus far, the analysis took $s \geq 0$ as given, which allowed us to leave it out of the notation when discussing the limits L^* and R^*. It is clear, however, that these limits depends upon s. Moreover, the subsidy is set by the planner and so is a variable we can tinker with. To make this more explicit, we henceforth write $x^*(s)$ as the common limit, i.e. $x^*(s)$ is the solution to

$$u_i^\varepsilon(p_{-i}^{x^*(s)} \mid x^*(s), s) = 0$$

which, using (15), gives

$$x^*(s) = c - s - \sum_{n=0}^{N-1} \frac{w(n)}{N}. \tag{28}$$

We want to find s^* that allows the planner to implement $x^*(s^*) = \underline{x}$, i.e. to coordinate players on playing 1 whenever their signal tells them coordination on 1 is the efficient outcome of the game. Solving (28) for $x^*(s^*) = \underline{x}$ characterizes the optimal subsidy s^*.

Theorem 2. *The optimal subsidy s^* is given by*

$$s^* = w(N-1) - \sum_{n=0}^{N-1} \frac{w(n)}{N}. \tag{29}$$

In the limit as $\varepsilon \to 0$, s^* sustains coordination on the efficient equilibrium of $G(x)$ as the essentially unique Bayesian Nash equilibrium of $G^\varepsilon(s^*)$ almost surely.

Proof. Solving (28) for $x^*(s) = \underline{x} = c - w(N-1)$ gives

$$x^*(s^*) = c - s^* - \sum_{n=0}^{N-1} \frac{w(n)}{N} = c - w(N-1) \implies s^* = w(N-1) - \sum_{n=0}^{N-1} \frac{w(n)}{N}.$$

By Corollary 1, coordination on the efficient outcome of $G(x)$ thus happens with probability 1 for all $x \notin [\underline{x} - \varepsilon, \underline{x} + \varepsilon]$ in $G^\varepsilon(s^*)$. As $\varepsilon \to 0$, the probability that $x \notin [\underline{x} - \varepsilon, \underline{x} + \varepsilon]$ goes to zero and coordination on the efficient outcome of $G(x)$, see Lemma 1, is sustained almost surely. $\square$

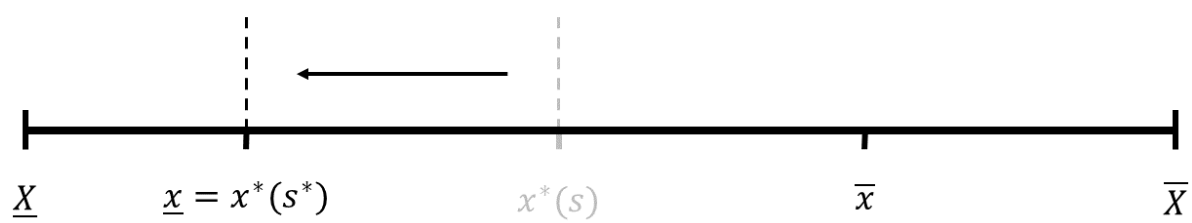

Figure 3: The optimal subsidy s^* makes coordination on $p^{\underline{x}}$ is the unique Bayesian Nash equilibrium of $G^\varepsilon(s^*)$.

Some properties of s^* are noteworthy.

First, optimal subsidies do not depend upon c, the outside option or cost of playing 1. In the canonical example of a joint investment subsidy (*c.f.* Sakovics and Steiner, 2012; Bernstein and Winter, 2012), subsidies typically target investment costs to make investment a safe option for some or all players. In our framework, subsidies instead target the spillovers from coordination. The basic intuition is similar: in a coordination problem, playing 1 (investing) is risky because (i) investment is costly and (ii) pays off only if sufficiently many other players do the same. A subsidy that covers opportunity costs takes care of (i) and makes players effectively indifferent to the payoff uncertainty that derives from (ii).[4] An optimal subsidy equal to s^* instead targets the payoff uncertainty that derives from the (ii), removing just enough of said uncertainty to incentive players to play 1. When opportunity costs vary across players, they do affect optimal subsidies, though even then it is only the *difference* in costs that matters; see Section 3.3.

Second, and related, optimal subsidies are increasing in $w(N-1) - w(n)$, all $n < N-1$. Intuitively, $w(N-1) - w(n)$ is a measure for the strength of players' coordination incentives. This is most easily seen by considering the special case of a regime change game (see footnote 1). In such games, a player who chooses to play 1 gains a one-off amount $\overline{w} - \underline{w} > 0$ when the critical number of player who choose 1 is reached. The higher is $\overline{w} - \underline{w}$, the greater the player's potential benefit from coordination, that is, the stronger their coordination incentives.

Third, like the subsidization problem studied in Sakovics and Steiner (2012) and unlike a divide-and-conquer strategy (*c.f.* Segal, 2003; Bernstein and Winter, 2012), our optimal policy subsidizes symmetric players symmetrically. We think of such equal treatment under the policy as both intuitively appealing and empirically plausible.

Fourth, subsidies in excess of s^* are inefficient. This inefficiency is twofold. First, the planner is spending more money than strictly necessary, money that – outside of our simple model – could be put to other uses. More interestingly, however, is that subsidies in excess of s^* make equilibrium play inefficient. If $\tilde{s} > s^*$, then using (28) to solve for $\tilde{x} = x^*(\tilde{s})$ gives $x^*(\tilde{s}) < \underline{x}$. By Theorem 1, the unique equilibrium of $G^\varepsilon(\tilde{s})$ is $p^{\tilde{x}}$. Letting $\varepsilon \to 0$, this makes $\overline{a}$ the unique equilibrium outcome of $G^\varepsilon(\tilde{x})$ for all $x \in [\tilde{x}, \underline{x})$ even though $\underline{a}$ would be the efficient outcome of $G(x)$ for those x. This second type of inefficiency does not arise in the participation problems studied by Sakovics and Steiner (2012) or Bernstein and Winter (2012), where full participation is by assumption the efficient outcome of the game. When there is prior uncertainty about the efficient course of action, the planner should subsidize conservatively to avoid picking inefficient winners.

Finally, we emphasize that a modest s^* solves the coordination problem due to the *strategic* repercussions of subsidies in coordination problems, that is, the unraveling effect. To see this, consider a global game in which players have only first-order knowledge of the subsidy; that is, each player i knows they will receive a subsidy equal to s upon playing 1, but this is all they know.[5] We assume that $x^*(0) - \underline{x} > 2\varepsilon$. Let $\tilde{G}^\varepsilon(s)$ denote the global game in which players have only first-order knowledge of the subsidy s, and let $\tilde{u}_i^\varepsilon$ denote the (conditional) expected incentive to play 1 for player i in $\tilde{G}^\varepsilon(s)$. Since no player i is assumed to be aware of

[4]We say effectively indifferent because players would still prefer full coordination; however, whether or not coordination is achieved does not affect the sign of their incentive to play 1.

[5]This contrasts strongly with the game $G^\varepsilon(s)$ in which s is common knowledge. Any game in which there is at most n-th order knowledge about s, for finite n, would illustrate our point; the case $n = 1$ is taken for simplicity.

the subsidies paid to other players, i's belief about the equilibrium strategies of its opponents is uniquely given by $p_{-i} = p_{-i}^{x^*(0)}$. Based on this belief, is playing $p_i^{\underline{x}}$ the equilibrium strategy of player i? No. Indeed, one has

$$\tilde{u}_i^{\varepsilon}(p_{-i}^{x^*(0)} \mid \underline{x}, s^*) = \underline{x} + s^* + w(0) - c = w(0) - \sum_{n=0}^{N-1} \frac{w(n)}{N} < 0.$$

Absent common knowledge of the policy, a modest subsidy like s^* is insufficient to solve the coordination problem. Indeed, with only first-order knowledge of the policy a planner would need to offer subsidies equal to $\tilde{s} = w(N-1) - w(0) > s^*$ to make $p^{\underline{x}}$ the unique BNE of $\tilde{G}^{\varepsilon}(\tilde{s})$, which is the subsidy that makes playing 1 strictly dominant for all $x > \underline{x}$. The reason, as this example was meant to illustrate, is strategic: assuming that other players do not know about the subsidy offered to player i, he by assumption believes that the associated increase in his incentive to play 1 is not known to his opponents. Hence, i cannot believe his increased incentive also raises his opponents' incentives to play 1, and the subsidy affects i's incentive only directly. Absent the indirect, strategic effect of subsidies in player i's incentives, he must be subsidized up to strict dominance to have an incentive to play $p_i^{\underline{x}}$. The modest subsidy s^* works only when players are able to figure out how others respond to subsidies and the feedback effects these responses trigger, see Figure 1.

3.3 Generalization

In this section, we briefly discuss the extension to coordination games played among asymmetric players. Let there again be N players, indexed i, who must choose binary actions $a_i \in \{0, 1\}$. The payoff to player i is given by:

$$\pi_i(a \mid x) = \begin{cases} w_i(n(a_{-i})) + x & \text{if} \quad a_i = 1, \\ c_i & \text{if} \quad a_i = 0. \end{cases} \tag{30}$$

In (30), we allow c_i and $w_i(a_{-i})$ to vary across players; that is, players can have different opportunity costs to playing 1. They may also differ in terms of their coordination incentive, that is, their sensitivity to externalities (through w_i). We maintain the assumption that actions are strategic complements, so $w_i(n+1) \geq w_i(n)$ for all n and $i \in \mathscr{N}$.[6]

The conditional incentive to play 1 is

$$u_i(a_{-i} \mid x) = \pi_i(1, a_{-i} \mid x) - \pi_i(0, a_{-i} \mid x) = w_i(n(a_{-i})) + x - c_i. \tag{31}$$

Define $\underline{x}_i := c_i - w_i(N-1)$ and $\overline{x}_i := c_i - w_i(0)$. Whenever $x \in [\underline{x}_i, \overline{x}_i]$ player i's payoff-maximizing action may be either 0 or 1, depending on the a_{-i} played. Without loss of

[6] We note that the payoff function (30) is similar to that studied by Sakovics and Steiner (2012); however, the games considered differ in important ways. While Sakovics and Steiner (2012) allow players to differ in their private benefits and costs associated with playing 1, the critical mass of players needed to make a player willing to play 1 is the same for all. In contrast, our game allows each player i to have its own critical threshold n_i^*. Relatedly, it is, for some values of x, possible that some or all players prefer not to coordinate on playing 1, or that coordination on 1 is the inefficient outcome of the game, neither of which can occur in the game studied by Sakovics and Steiner (2012).

generality, relabel the players in decreasing order of $\underline{x}_i$, i.e. so that $\underline{x}_1 \geq \underline{x}_2 \geq \ldots \geq \underline{x}_N$. Define $\underline{x} := \sup\{\underline{x}_i \mid i \in \mathscr{N}\} = c_1 - w_1(N-1)$ and $\overline{x} = \inf\{\overline{x}_i \mid i \in \mathscr{N}\}$. We assume that $[\underline{x}, \overline{x}]$ is nonempty to ensure that, for some values of x at least, the players face a genuine coordination problem. That is, given common knowledge of x both $\overline{a}$ and $\underline{a}$ are a strict Nash equilibrium for all $x \in (\underline{x}, \overline{x})$.

We do not assume common knowledge of x, however. As before, we consider a global game Γ^ε in which the parameter x is not observed. The information structure is the same as in G^ε. Let $\Gamma^\varepsilon(s)$, $s = (s_i)$, denote the global game Γ^ε in which player i is offered a subsidy equal to $s_i \geq 0$ on playing 1.

In $\Gamma^\varepsilon(s)$, the expected conditional incentive to play 1 for player i is

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) = \int u_i(p_{-i}(x_{-i}^\varepsilon) \mid x_i^\varepsilon)\, d\Phi(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \tag{32}$$

and

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s_i) = u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) + s_i, \tag{33}$$

per the same argument as (6).

The goal of the planner is to find the optimal subsidy scheme $s^* = (s_i^*)$ such that coordination on $p^{\underline{x}}$ is the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(s^*)$. In the limit as $\varepsilon \to 0$, the unique equilibrium outcome of $\Gamma^\varepsilon(s^*)$ will be coordination on $\overline{a}$ whenever $x > \underline{x}$. That is, coordination on 1 will be the unique equilibrium of the global game whenever it is also a Nash equilibrium of the true game $\Gamma(x)$.[7] Since the argument is very similar to that for the case of symmetric players, we relegate it to the appendix.

Theorem 3. *In $\Gamma^\varepsilon(s)$, the optimal subsidy s_i^* is given by*

$$s_i^* = c_i - c_1 + w_1(N-1) - \sum_{n=0}^{N-1} \frac{w_i(n)}{N}, \tag{34}$$

for all i. In the limit as $\varepsilon \to 0$, the optimal subsidy scheme $s^* = (s_i^*)$ implements coordination on $\overline{a}$ whenever that is the efficient Nash equilibrium of $\Gamma(x)$ almost surely.

In the special case of symmetric players, (34) simplifies to (29), as expected. We observe that s_i^* admits the same qualitative properties discussed in Section 3.2.

It is worth pointing out that in the asymmetric game, the optimal subsidy for player i is increasing in i's opportunity cost c_i. An interesting insight emerges here by re-introducing some symmetry. Suppose that $w_i(n) = w_j(n)$ for all n and all $i, j \in \mathscr{N}$, so any asymmetries derive exclusively from differences in players' opportunity costs. In fact, from the way we relabeled players we have $c_1 \geq c_2 \geq \ldots \geq c_N$. The highest subsidy goes to 1, and subsidies subsequently decrease as players' opportunity costs decrease. This is intuitive: the costlier playing 1 is, the higher should a subsidy be to incentive a player to do so. Moreover, except for player 1 and other players for whom $c_i = c_1$ optimal subsidies are lower than in the symmetric game. This happens because those players are not subsidized all the way toward their individual $\underline{x}_j$, but only to the higher $\underline{x}_1$. We think of the case in which a planner offers

[7] In which case it is also the efficient outcome.

subsidies to make $p_i^{\overline{x}_i}$ the unique Bayesian Nash equilibrium as less interesting since it leads away from the *coordination* problem, which is the topic of this paper. For completeness, we describe the optimal subsidies for this policy objective in the Appendix.

To conclude this discussion on optimal subsidies in asymmetric games with, let us relax the requirement that $\underline{x}$ must be strictly less than $\overline{x}$. That is, while $\underline{x}_i < \overline{x}_i$ for every i, it is possible that $\overline{x}_j < \underline{x}_i$ for some pairs of players $i, j \in \mathscr{N}$. Assume that there is at least one player i^* for whom this is true, i.e. $\overline{x}_{i^*} < \underline{x}$. The optimal subsidy to this player is negative: $s^*_{i*} < 0$.

Proposition 1. *Consider the game $\Gamma^\varepsilon(s)$ and all players i^* for whom $\overline{x}_{i^*} < \underline{x}$. If $s^* = (s^*_i)$ is the subsidy scheme that implements $p^{\underline{x}}$ as the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(s^*)$, then player i^* is taxed for playing $a_{i^*} = 1$: $s_{i^*} < 0$.*

Proof. We are given that $p^{\underline{x}}$ is the unique BNE of $\Gamma^\varepsilon(s^*)$, so for player i^* we have

$$u^\varepsilon_{i^*}(p^{\underline{x}}_{-i} \mid \underline{x}, s^*_{i^*}) = u^\varepsilon_{i^*}(p^{\underline{x}}_{-i} \mid \underline{x}) + s^*_{i^*} = \underline{x} + s^*_{i^*} + \sum_{n=0}^{N-1} \frac{w_{i^*}(n)}{N} = 0,$$

where the second equality follows from Lemma 2. Per (34), the optimal subsidy $s^*_{i^*}$ to player i^* is equal to

$$s^*_{i^*} = c_{i^*} - c_1 + w_1(N-1) - \sum_{n=0}^{N-1} \frac{w_{i^*}(n)}{N}.$$

By hypothesis, $c_{i^*} \leq \underline{x}$ (i.e. because $\overline{x}_{i^*} > \underline{x}$). Combining these observations, and using that by definition $\underline{x} = c_1 - w_1(N-1)$,

$$s^*_{i^*} = \underbrace{c_{i^*} - \underline{x}}_{<0} - \underbrace{\sum_{n=0}^{N-1} \frac{w_{i^*}(n)}{N}}_{>0} < 0,$$

as claimed. $\square$

If we allow players to be sufficiently asymmetric, then the optimal subsidy scheme s^* resembles a divide-and-conquer policy (*c.f.* Segal, 2003; Bernstein and Winter, 2012): some players are subsidized to play 1 while for others the same action is taxed. The economic motivation is different from that in Segal (2003) and Bernstein and Winter (2012), however. In our problem, players with low opportunity costs are "too eager" to play 1. Trying to achieve complete coordination (of strategies), the planner must temper these players' incentives to play 1, which is done through a tax. Thus, while a divide-and-conquer policy may be optimal for our problem, it only is when the planner has a somewhat unreasonable goal: achieve complete coordination at a all costs, including inefficiency.

3.4 Do We Really Need Equilibrium Selection?

Do we really need equilibrium selection to draw our above conclusions? Or is our global games approach rather a needless technical complication? In this section, we present an example to answer the first question affirmatively. The same example also illustrates why the answer to question number two is “no”.

Consider again the game of complete information $G(x)$ described in Section 2. By Lemma 1, for any $\hat{x} \in [\underline{x}, \overline{x}]$ the strategy vector $a(x) = (a_i(x))$ in which

$$a_i(x) = \begin{cases} 1 & \text{if} \quad x < \hat{x} \\ 0 & \text{if} \quad x \geq \hat{x} \end{cases} \tag{35}$$

for every $i \in \mathscr{N}$ is a Nash equilibrium of the game. Similarly, in the game of complete information with subsidies to playing 1 equal to s, denoted $G(x, s)$, any strategy vector $a(x, s) = (a_i(x, s))$ in which

$$a_i(x, s) = \begin{cases} 1 & \text{if} \quad x + s < \hat{x} \\ 0 & \text{if} \quad x + s \geq \hat{x} \end{cases} \tag{36}$$

for every $i \in \mathscr{N}$ is a Nash equilibrium. Assume that every player $i \neq j$ is known to play according to (35) in $G(x)$ and according to (36) in $G(x, s)$. This would not be intuitive but nothing prevents such beliefs from being consistent with equilibrium play, so it cannot be ruled out that player j truly thinks this way.[8] Then, however, player j's incentive to play 1 goes *down* upon being offered the subsidy s for all $x \in (\hat{x} - s, \hat{x}]$. In other words, the direct effect of subsidies is not guaranteed in a game of complete information and multiple Nash equilibria. Since the unraveling effect builds upon the direct effect of subsidies, it breaks down. This example illustrates equilibrium selection is necessary for our analysis.

4 Discussion and conclusions

The results in this paper suggest that a planner can, and should, use only modest subsidies to solve coordination problems. We demonstrate

In coordination problems, policies have higher order effects that all point in the same direction.

References

Aumann, R. J. (1976). Agreeing to disagree. *The annals of statistics*, pages 1236–1239.

Barrett, S. (2003). Global disease eradication. *Journal of the European Economic Association*, 1(2-3):591–600.

Bernstein, S. and Winter, E. (2012). Contracting with heterogeneous externalities. *American Economic Journal: Microeconomics*, 4(2):50–76.

Björkegren, D. (2019). The adoption of network goods: Evidence from the spread of mobile phones in rwanda. *The Review of Economic Studies*, 86(3):1033–1060.

[8]To be clear, it is not needed that players $i \neq j$ actually pursue such a weird strategy. All we need is that player j *believes* they do.

Carlsson, H. and Van Damme, E. (1993). Global games and equilibrium selection. *Econometrica*, pages 989–1018.

Farrell, J. and Saloner, G. (1985). Standardization, compatibility, and innovation. *the RAND Journal of Economics*, pages 70–83.

Frankel, D. M., Morris, S., and Pauzner, A. (2003). Equilibrium selection in global games with strategic complementarities. *Journal of Economic Theory*, 108(1):1–44.

Jensen, R. (2007). The digital provide: Information (technology), market performance, and welfare in the south indian fisheries sector. *The quarterly journal of economics*, 122(3):879–924.

Palfrey, T. R. and Rosenthal, H. (1984). Participation and the provision of discrete public goods: a strategic analysis. *Journal of public Economics*, 24(2):171–193.

Sakovics, J. and Steiner, J. (2012). Who matters in coordination problems? *American Economic Review*, 102(7):3439–61.

Segal, I. (2003). Coordination and discrimination in contracting with externalities: Divide and conquer? *Journal of Economic Theory*, 113(2):147–181.