Unraveling Coordination Problems*

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July 6, 2023

Abstract

This paper studies policy design in coordination problems. In a coordination game, a subsidy raises player i's incentive to play the subsidized action. This raises j's incentive to play the same action, which further incentivizes i, and so on. Building upon this "unraveling effect", we characterize the subsidies that implement a given outcome of the game as its unique equilibrium. Subsidies are (i) symmetric for identical players, (ii) globally continuous in payoff parameters, (iii) increasing in opportunity costs, (iv) decreasing in spillovers, and (v) increasing in the planner's ambition. Applications of our model include joint investment problems, participation decisions, and principal-agent contracting.

1 Introduction

In coordination problems, players face strategic uncertainty regarding the actions and beliefs of others. Strategic uncertainty forces players to second-guess the strategies of their opponents and pessimistic beliefs may become self-fulfilling and lead to coordination failure. A worthwhile project may not take off simply because investors believe others will not invest. A promising network technology may never mature only because potential adopters are pessimistic about adoption by others. An infectious disease may not get eradicated solely on the ground that governments believe other nations will not attempt to. The possibility of costly coordination failures motivates intervention. This paper studies policy design in coordination problems.

The usual rationale for policy intervention is to correct market failures introduced by externalities. All market failures are not equal, however, and it is crucial for policy design to know the type of externality an intervention targets. One kind of externality arises when there exists a gap between the private and social value of behavior. Thus, an individual household's greenhouse gas emissions may be higher than socially optimal as it ignores the effects its emissions have on others. Such externalities can be addressed though Pigouvian taxes or subsidies. Another, more complicated kind of externality arises in coordination problems where

*For valuable comments and suggestions, we would like to thank Eric van Damme, Tabaré Capitán, Reyer Gerlagh, Eirik Gaard Kristiansen, and Robert Schmidt.

individual actions are strategic complements (Bulow et al., 1985). Strategic complementarity results in multiple, Pareto-ranked equilibria and opens the door to coordination failures. Thus, renewable technologies may provide a viable replacement for fossil fuels but only if sufficient capacity is installed; there hence are multiple equilibria, and renewables might never mature despite their potential (and known) advantages. Such externalities cannot be solved through simple Pigouvian policy. The goal of this paper is to design optimal policies for coordination problems. To streamline the narrative, we focus on subsidies.

The main results in this paper characterize the subsidy scheme that induces a given outcome of a coordination game as its unique equilibrium. The policy that achieves this admits a number of intuitive properties: subsidies are (i) symmetric for identical players; (ii) globally continuous in model parameters; (iii) increasing in opportunity costs; (iv) decreasing in coordination spillovers; and (v) increasing in the planner's ambition. These properties of an optimal policy are robust to various extensions and generalizations of the main model.

One especially notable feature of an optimal subsidy scheme is that subsidies pursuant to the scheme are small: they do not make the targeted equilibrium strategy strictly dominant for any of the players. We explain this by a strategic *unravelling effect* that derives from the intricacies of policy in coordination games. A subsidy raises player i's incentive to play the subsidized action. Because players in a coordination game want to match actions, the subsidy to player i also (indirectly) increases player j's incentive to play that action. This, in turn, makes the subsidized action even more attractive for player i, and so on. If subsidies are common knowledge, one obtains an infinitely compounded feedback effect of policy that allows seemingly modest intervention to unravel coordination problems; see Figure 1 for an illustration. This finding stands in contrast to much of the literature on policy design in coordination problems, which emphasizes the need to subsidize at least a subset of player to strict dominance (*cf.* Segal, 2003; Winter, 2004; Bernstein and Winter, 2012; Sakovics and Steiner, 2012; Halac et al., 2020).

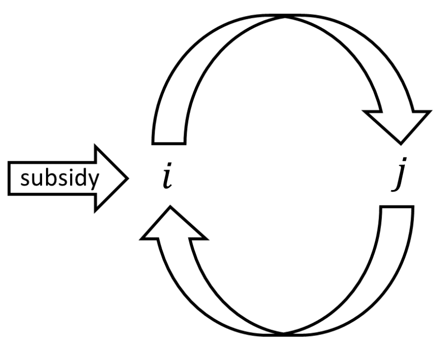

Figure 1: In coordination game, a subsidy kickstarts an infinitely compounded positive feedback loop on players' incentives to play the subsidized action: the unraveling effect.

Formally, the model in this paper consists of a planner and N (heterogeneous) players each of whom independently chooses an action from a binary set $\{0, 1\}$. If player i plays 0, his payoff is c_i. When instead player i plays 1, his payoff is the sum of two components. The first component, x, is a hidden state of Nature. The second component, $w_i : \{0, 1\}^{N-1} \to \mathbb{R}$, gives the externalities that other players' actions impose upon him. The analysis centers around coordination games, or games with strategic complementarities, in which w_i is increasing in the number of players that play 1. A planner publicly announces subsidies to players who play 1. The problem of the planner is to find the vector of subsidies $\tilde{s} = (\tilde{s}_i)$ that induces coordination on $(1, 1, ..., 1)$ for all $x > \tilde{x}$, where the critical state $\tilde{x} \in \mathbb{R}$ is chosen by the planner. The paper also explores a number of extensions and special cases of the base model, including: principal-agent models; games of regime change; asymmetric policy targets; and games with heterogeneous externalities.

Were a player informed about the actions of his opponents, his problem would be trivial. Yet players do not typically possess such information. In a coordination game with multiple Nash equilibria, the resulting strategic uncertainty forces players to second-guess the actions and beliefs of others. This complicates the planner's problem: even if a policy makes coordination on $(1, 1, ..., 1)$ *an* equilibrium of the games for all $x > \tilde{x}$, there may yet be others. Unless the planner can coordinate play on her most-preferred equilibrium – a power economists have been reluctant to grant (*cf.* Segal, 1999, 2003; Winter, 2004; Sakovics and Steiner, 2012; Bernstein and Winter, 2012; Halac et al., 2020, 2021) – the purpose of her policy is not simply to make the targeted outcome an equilibrium. Instead, she seeks to attract coordination on one, rather than another, equilibrium. She therefore cannot separate the issue of policy design from that of equilibrium selection.

This paper deals with equilibrium selection using a global games approach. Pioneered by Carlsson and Van Damme (1993), global games are incomplete information games in which players do not observe the true game they play but only a private and noisy signal of it. In the present paper, players do not know the hidden state x; rather, each player i observes a private noisy signal x_i^ε of x. We focus on regulatory environments in which the planner does not know x either or else, if she does, must commit to her policy before Nature draws x; hence, the planner's choice of policy cannot signal any private knowledge she might possess (Angeletos et al., 2006).[1] Given this information structure, it is impossible to tackle the planner's problem directly. Instead, the analysis first solves a slightly modified version of her problem: find that subsidy scheme $\tilde{s}$ subject to which the unique equilibrium strategy of each player i is to choose 1 whenever his signal x_i^ε exceeds $\tilde{x}$. In the limit as signals become arbitrarily precise, this implies coordination on $(1, 1, ..., 1)$ for all $x > \tilde{x}$ with probability 1 and thus solves the planner's original problem as well. The main result of this paper shows that the subsidy scheme $\tilde{s}$ exists, that it is unique, and provides a characterization.

In the global game studied here, uncertainty about the state x implies prior uncertainty about the efficient outcome of the game. Because the planner must commit to her policy before the true state is observed, uncertainty about x calls for policy restraint: high subsidies are too heavy handed and risks inducing coordination on an ex post inferior outcome. The possibility that the planner might end up subsidizing an inefficient outcome sets this paper apart from a number of related analyses on policy design in global coordination games (Angeletos et al., 2006; Sakovics and Steiner, 2012; Edmond, 2013). The difference between the results presented here and those due to Sakovics and Steiner (2012) in particular demonstrates that prior uncertainty about ex post efficiency matters a great deal for policy design.

Related literature.—A closely related paper is Sakovics and Steiner (2012), who study policy design in a global game of regime change. Games of regime change are coordination

[1] More specifically, the problem of the planner is not one of Bayesian persuasion or information design (cf. Kamenica and Gentzkow, 2011; Bergemann and Morris, 2016; Ely, 2017; Mathevet et al., 2020).

games in which a status quo is abandoned, causing a discrete change in payoffs, once a sufficiently large number of agents take an action against it. Sakovics and Steiner (2012) find that (in the limit as signals become arbitrarily precise) an optimal policy fully subsidizes a subset of players, targeting those who matter most for regime change and/or have least incentive to take an action against the regime. These results provide a stark counterpoint to the findings in this paper, which say that an optimal policy subsidize *all* players partially. The difference is a consequence of the distinct information structures considered. In Sakovics and Steiner (2012), uncertainty pertains to the mass of players necessary to invoke a change of regime; conditional on the realized regime, payoffs are common knowledge. In contrast, the regime change version of the present model assumes that the critical mass needed to change regime is common knowledge; however, conditional on the regime in place there is uncertainty about payoffs. Relatedly, in Sakovics and Steiner (2012) there is no uncertainty about the ex post efficient outcome of the game; in the present paper, there is.[2]

Another related paper is Halac et al. (2020), who study the problem of a firm that seeks to raise capital from multiple investors to fund a project. The project succeeds only if the caital raised exceeds a stochastic threshold; the firm offers payments contingent on project success. Halac et al. (2020) identify conditions under which larger investors receive higher per-dollar returns on investment in an optimal policy, thus perpetuating inequalities. The focus on contingent per-dollar returns in Halac et al. (2020) is different from the approach in this paper, in which actions are binary and subsidies are paid regardless of eventual outcomes.

This paper is also related to the literature on principal-agent contracting, see Winter (2004) and Halac et al. (2021) in particular. Contrasting sharply with the findings presented here, the seminal result in Winter (2004) is that optimal mechanisms are inherently discriminatory under complete information – no two agents are rewarded equally even when agents are symmetric. Halac et al. (2020) extend the model in Winter (2004) to allow for asymmetries among the agents and private contract offers; they find that symmetric agents are offered identical rewards in a optimal contract when players' ranking in the reward scheme is uncertain.

Another literature to which this paper connects is that on contracting with externalities (e.g., Segal, 1999, 2003; Segal and Whinston, 2000; Bernstein and Winter, 2012). Segal (2003) and Bernstein and Winter (2012) consider complete information contracting problems that, save for the informational environment, are essentially equivalent to the game studied in this paper. They establish optimality of the *divide and conquer* mechanism in which the planner first ranks all players; given the ranking, each player is offered a subsidy that incentivizes him to play the subsidized action assuming all players who precede him in the ranking also play this action while those after him do not. Bernstein and Winter (2012) derive the optimal ranking of players in such a policy. Like the mechanism derived in Winter (2004), an (optimal) divide and conquer scheme is inherently discriminatory and treats symmetric agents asymmetrically.

The remainder of the paper is organized as follows. Section 2 introduces the model and the concepts needed for the analysis. Section 3 introduces the planner's problem and states out

[2] At least since Morris and Shin (1998), the information structure in Sakovics and Steiner (2012) has been standard in the applied literature on global games; for other examples of regime change games with similar information structures, see Angeletos et al. (2006, 2007), Goldstein and Pauzner (2005), Basak and Zhou (2020), and Edmond (2013).

main result. Section 4 presents the core of the analysis. Various special cases and extensions of our model are discussed on Section 5. Section 6 discusses and concludes. All proofs are in the Appendix.

2 The Game

Consider a normal form game played by players in a set $\mathscr{N} = \{1, 2, ..., N\}$, indexed i, who simultaneously choose binary actions $a_i \in \{0, 1\}$. Given that a vector of actions $a \in \{0, 1\}^N$ played, let $A := \sum_{i \in \mathscr{N}} a_i$ denote the *aggregate action*. Define $a_{-i} := a \setminus \{a_i\}$, $A_{-i} := \sum_{j \neq i} a_j$, $\overline{a} := (1, 1, ..., 1)$, $\underline{a} := (0, 0, ..., 0)$, $\overline{a}_{-i} := \overline{a} \setminus \{a_i\}$, and $\underline{a}_{-i} := \underline{a} \setminus \{a_i\}$. When a is played, player i who chooses 1 in a gets payoff $x + w_i(a_{-i})$; when instead player i chooses 0 in a, his payoff is c_i. Here, $w_i(a_{-i})$ describes the externalities on player i deriving from other players' actions. To simplify the exposition, the main analysis assumes that w_i depends upon a_{-i} only through the aggregate A_{-i} which permits one to write $w_i(A_{-i})$; Section 5.3 explores generalizations of the game in which externalities depend upon the exact subset of players who play 1. The variable x is a hidden state of Nature that affects players' payoffs. Lastly, c_i is player i's payoff to playing 0, which in some interpretations of the model is best thought of as the cost of playing 1. Combining these elements, the payoff to player i is given by

$$\pi_i(a \mid x) = \begin{cases} x + w_i(a_{-i}) & \text{if} \quad a_i = 1 \text{ in } a, \\ c_i & \text{if} \quad a_i = 0 \text{ in } a. \end{cases} \tag{1}$$

We restrict attention to games with strategic complementarities meaning that $w_i(a_{-i})$ is increasing in a_{-i}, i.e. $w_i(n+1) \geq w_i(n)$ for all $n = 0, ..., N-2$. In the canonical example of a joint investment problem, the action $a_i = 1$ is interpreted as investment and c_i as the cost of investing (Sakovics and Steiner, 2012). Alternatively, actions might represent the choice to use of a particular kind of network technology and c_i is the cost differential between technologies (Björkegren, 2019). Or actions could describe the decisions to work or shirk by agents working on a common project such that c_i is agent i's cost of effort and w_i his (discrete) benefit from project success, see Winter (2004) and Halac et al. (2021).

The above elements combined describe a game of complete information $\Gamma(x)$. In $\Gamma(x)$, define a player's *incentive* to choose 1 as the gain from playing 1, rather than 0, or

$$u_i(a_{-i} \mid x) = \pi_i(1, a_{-i} \mid x) - \pi_i(0, a_{-i} \mid x) = x + w_i(a_{-i}) - c_i. \tag{2}$$

Observe that, given a_{-i}, the incentive u_i to play 1 is strictly increasing in x. Denote $x_i^0 := c_i - w_i(0)$ and $x_i^N := c_i - w_i(N-1)$. One has $u_i(\overline{a}_{-i} \mid x_i^0) = u_i(\underline{a}_{-i} \mid x_i^N) = 0$. In other words, to each player i playing 1 is strictly dominant for all $x > \overline{x}_i^0$; playing 0 is strictly dominant for $x < \underline{x}_i^N$. Define $x^N := \sup\{x_i^N \mid i \in \mathscr{N}\}$, $x^0 := \inf\{x_i^0 \mid i \in \mathscr{N}\}$, $\underline{x} = \inf\{x_i^0 \mid i \in \mathscr{N}\}$, and $\overline{x} = \sup\{x_i^N \mid i \in \mathscr{N}\}$. Let $[\underline{x}, \overline{x}]$ be nonempty so that, for all x in $[\underline{x}, \overline{x}]$, $\Gamma(x)$ is a true coordination game with multiple strict Nash equilibria.

To reflect the many uncertainties that decision makers in the real world face, we assume that the state of nature x is hidden. Instead, it is common knowledge that x is drawn from the uniform distribution on $[\underline{X}, \overline{X}]$ where $\underline{X} < \underline{x}$ and $\overline{X} > \overline{x}$ and that each player i receives a private noisy signal x_i^ε of x, given by:

$$x_i^\varepsilon = x + \varepsilon_i.$$

One can think of x_i^ε as the player's *type*. Note that, conditional on x, types are drawn independently. Let Γ^ε denote the game of incomplete information about x.

In (2), the term ε_i captures the noise in i's private signal. It is common knowledge that ε_i is an i.i.d. draw from the uniform distribution on $[-\varepsilon, \varepsilon]$. If $\varepsilon \to 0$, players' signals become arbitrarily precise. Let $x^\varepsilon = (x_i^\varepsilon)$ denote the vector of signals received by all players, and let x_{-i}^ε denote the vector of signals received by all players but i, i.e. $x_{-i}^\varepsilon = (x_j^\varepsilon)_{j \neq i}$. Note that player i observes x_i^ε but neither x nor x_{-i}^ε. Let $\Phi^\varepsilon(\cdot \mid x_i^\varepsilon)$ denote the joint probability function of $(x, x_j^\varepsilon)_{j \neq i}$ conditional on x_i^ε.

The timing of Γ^ε is as follow. First, Nature draws a true x. Second, each player i receives its private signal x_i^ε of x. Third, all players simultaneously choose their actions. Lastly, payoffs are realized according to the true x and the actions chosen by all players.

2.1 Concepts and notation

Strategies. A strategy p_i for player i in Γ^ε is a function that assigns to any $x_i^\varepsilon \in [\underline{X} - \varepsilon, \overline{X} + \varepsilon]$ a probability $p_i(x_i^\varepsilon) \geq 0$ with which the player chooses action $a_i = 1$ when they observe x_i^ε. Write $p = (p_1, p_2, ..., p_N)$ for a strategy vector for all player, and $p_{-i} = (p_j)_{j \neq i}$ for the vector of strategies for all players but i. A strategy vector p is *symmetric* is for every $i, j \in \mathscr{N}$ and every signal x^ε one has $p_i(x^\varepsilon) = p_j(x^\varepsilon)$. Conditional on the strategy vector p_{-i} and a private signal x_i^ε, the expected incentive to play 1 for player i is given by:

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) := \int u_i(p_{-i}(x_{-i}^\varepsilon) \mid x) \, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon).$$

When no confusion can arise, we refer to the expected incentive $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon)$ simply as a player's incentive.

Strict dominance. The action $a_i = 1$ is strictly dominant at x_i^ε if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) > 0$ for all p_{-i}. Similarly, the action $a_i = 0$ is strictly dominant (in the global game G^ε) at x_i^ε if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) < 0$ for all p_{-i}. When $a_i = \alpha$ is strictly dominant, the action $a_i = 1 - \alpha$ is said to be strictly dominated.

Conditional dominance. Let L and R be real numbers. The action $a_i = 1$ is said to be dominant at x_i^ε conditional on R if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) > 0$ for all p_{-i} with $p_j(x_j^\varepsilon) = 1$ for all $x_j^\varepsilon > R$, all $j \neq i$. Similarly, the action $a_i = 0$ is dominant at x_i^ε conditional on L if $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) < 0$ for all p_{-i} with $p_j(x_j^\varepsilon) = 1$ for all $x_j^\varepsilon > L$, all $j \neq i$.

Increasing strategies. For $X \in \mathbb{R}$, let p_i^X denote the particular strategy such that $p_i^X(x_i^\varepsilon) = 0$ for all $x_i^\varepsilon < X$ and $p_i^X(x_i^\varepsilon) = 1$ for all $x_i^\varepsilon \geq X$. The strategy p_i^X is called an *increasing strategy with switching point* X. Let $p^X = (p_1^X, p_2^X, ..., p_N^X)$ denote the strategy vector of increasing strategies with switching point X, and $p_{-i}^X = (p_j^X)_{j \neq i}$. Note that $a_i = 1$ is strictly dominant at x_i^ε conditional on R if and only if $u_i^\varepsilon(p_{-i}^R \mid x_i^\varepsilon) > 0$. Similarly, if $a_i = 0$ is strictly dominant at x_i^ε conditional on L then it must hold that $u_i^\varepsilon(p_{-i}^L \mid x_i^\varepsilon) < 0$. Generally, for a vector of real numbers (y_i) let $p^y = (p_i^{y_i})$ be a (possibly asymmetric) increasing strategy vector, and $p_{-i}^y = (p_j^{y_j})_{j \neq i}$.

Iterated elimination of strictly dominated strategies. The solution concept in this paper is iterated elimination of strictly dominated strategies (IESDS). Eliminate all pure strategies that are strictly dominated, as rational players may be assumed never to pursue such strategies. Next, eliminate a player's pure strategies that are strictly dominated if all other players are known to play only strategies that survived the prior round of elimination; and so on. The set of strategies that survive infinite rounds of elimination are said to survive IESDS.

3 Optimal Subsidies

3.1 The Planner's Problem

Next we introduce a social planner whose problem is to implement (in a way made precise shortly) coordination on $\overline{a}$ whenever $x > \tilde{x}$, where $\tilde{x} \in \mathbb{R}$ is the *critical state* which she – the planner – chooses. The planner faces two constraints. First, she cannot condition her policy on the realization of x or players' signals thereof. One interpretation is that the planner must commit to her policy before Nature draws a true x and cannot change her policy afterward.[3]

The second constraint upon the planner's problem has to do with the kinds of policies she can use. It is assumed that the planner cannot coordinate players on her preferred equilibrium in a multiple equilibria setting. Instead, she has to rely on simple subsidies (or taxes) to create the appropriate incentives. The focus on simple instruments also means that policies cannot condition directly upon other players' actions. These are standard assumptions in the literature on policy or contract design for coordination problems (Segal, 2003; Winter, 2004; Bernstein and Winter, 2012; Sakovics and Steiner, 2012; Halac et al., 2020).

To streamline the narrative, the anslysis henceforth focuses on subsidies as the planner's policy instrument. Let s_i denote the subsidy paid to a player i who chooses $a_i = 1$. Conditional on the subsidy s_i, player i's incentive to choose 1 becomes

$$u_i(a_{-i} \mid x, s_i) = u_i(a_{-i} \mid x) + s_i = x + w_i(a_{-i}) - c_i + s_i,$$

and the expected incentive, given the signal x_i^ε, is

$$\begin{aligned} u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s_i) &= \int u_i(p_{-i}(x_{-i}^\varepsilon) \mid x, s_i)\, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\ &= \int \big[u_i(p_{-i}(x_{-i}^\varepsilon) \mid x) + s_i\big]\, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\ &= u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) + s_i. \end{aligned} \tag{3}$$

It is clear that a tax equal to s_i on playing 0 has the same effect on incentives. Note that (3) assumes observability of a_i; this assumption is maintained throughout most of the analysis. Section 5 considers an extension of the base model to principal-agent problems in which the vector of actions a is unobserved.

[3] Laffont and Maskin (1982); Myerson and Satterthwaite (1983)

3.2 Unique Implementation

Given a vector of subsidies $s = (s_i)$, let $\Gamma^\varepsilon(s)$ denote the game Γ^ε in which the planner publicly commits to paying each player i who plays 1 a subsidy $s_i \in s$. Since the planner cannot condition her policy on x, and because players choose their actions before learning the true value of x, one can only define implementation in terms of players' signals. Henceforth, the vector of subsidies $\tilde{s}$ is said to *implement* coordination on $\overline{a}$ for all $x > \tilde{x}$ if $p^{\tilde{x}} = (p_i^{\tilde{x}})$ is the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(\tilde{s})$. The focus on unique equilibrium implementation is in keeping with the broader literature on policy design in coordination games (*cf.* Segal, 1999, 2003; Segal and Whinston, 2000; Sakovics and Steiner, 2012; Bernstein and Winter, 2012; Halac et al., 2020, 2021, 2022). Note that, in the limit as $\varepsilon \to 0$, the working definition of implementation also solves the planner's problem as originally formulated: if $\varepsilon \to 0$ then for all $x > \tilde{x}$ each player i receives a signal $x_i^\varepsilon > \tilde{x}$ which in the unique equilibrium $p^{\tilde{x}}$ implieas that players coordinate on $\overline{a}$ for all $x > \tilde{x}$.

Take some critical state $\tilde{x} \in \mathbb{R}$. Given $\tilde{x}$, let $\tilde{s} = (\tilde{s}_i)$ denote the subsidy scheme such that each $\tilde{s}_i \in \tilde{s}$ is given by

$$\tilde{s}_i = c_i - \tilde{x} - \sum_{n=0}^{N-1} \frac{w_i(n)}{N}. \tag{$*$}$$

The main result of this paper is Theorem 1, which establishes that $\tilde{s}$ is the unique subsidy scheme that solves the planner's problem.

Theorem 1. *Given is $\tilde{x} \in \mathbb{R}$.*

(i) There exists a unique subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $p^{\tilde{x}}$.

(ii) For each $i \in \mathscr{N}$, the subsidy $\tilde{s}_i$ pursuant to the scheme is given by $(*)$.

The analysis will reveal that Theorem 1 remains valid for a slightly more general definition of implementation.

Note that, as in Sandholm (2002, 2005), we do not impose that players must play an equilibrium of $\Gamma^\varepsilon(s)$. Instead, we depart from more primitive assumptions on players' strategic sophistication by requiring that none play a strategy that is iteratively dominated. We ultimately prove, in Proposition 5, that the strategy vector so obtained is also the unique equilibrium of the global game. Equilibrium play is hence obtained as a result rather than an assumption of our analysis.

Section 4 presents the building blocks of this result and develops some of the intuition. It also discusses the properties of $\tilde{s}$ and compares those to other proposals in the literature. Extensions and special cases are presented in Section 5.

4 Analysis

The plan for this section is as follows. We first show that for any vector of subsidies s there exists a unique vector of real numbers $x(s) = (x_i(s))$ such that the increasing strategy vector $p^{x(s)}$ is the unique strategy vector that survives IESDS in $\Gamma^\varepsilon(s)$. Then we demonstrate that the strategy vector $p^{x(s)}$ is also the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(s)$. We use this, and some minor technical results, to derive the unique subsidy scheme $\tilde{s}$ that implements $p^{\tilde{x}}$.

The analysis relies heavily on two monotonicity properties of players' expected incentives.

Lemma 1. *Given is a vector of real numbers $y = (y_i)$ and the associated increasing strategy vector $p^y = (p_i^{y_i})$. Then,*

(i) $u_i^\varepsilon(p_{-i}^y \mid x_i^\varepsilon)$ is monotone increasing in x_i^ε;

(ii) $u_i^\varepsilon(p_{-i}^y \mid x_i^\varepsilon)$ is monotone decreasing in y_j, all $j \in \mathscr{N} \setminus \{i\}$.

Part (i) of Lemma 1 says that a player's incentive to play 1 is increasing in his type x_i^ε when his opponents play increasing strategies. There are two sides to this. First, taking as given the vector of actions a_{-i}, a player's expected payoff to playing 1 is linearly increasing in x_i^ε; hence, his expected incentive is increasing in his signal x_i^ε. Second, as x_i^ε increases player i's posterior distribution on the hidden state x and, therefore, the signals of his opponents shifts to the right. If his opponents play increasing strategies, this also shifts his distribution of the aggregate action to the right which, because externalities are increasing in the aggregate action, further raises his incentive to play 1. Note that monotonicity of $u_i^\varepsilon(p_{-i}^y \mid x_i^\varepsilon)$ in x_i^ε depends upon p_{-i}^y being increasing; for generic p_{-i}, $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon)$ can be locally decreasing in x_i^ε.

Part (ii) of Lemma 1 says that the incentive to play 1 of a player i whose opponents play increasing strategies is decreasing in the switching point of each of these increasing strategies. For given signal x_i^ε, the probability player i attaches to the event that his opponent j receives a signal $x_j^\varepsilon > y_j$ and thus, in $p_j^{y_j}$, plays 1 is decreasing in y_j. Therefore player i's incentive to play 1 is decreasing in the switching y_j.

4.1 Subsidies, Strategies, Selection

Recall that x_i^N and x_i^0 demarcate strict dominance regions for player i: when $x > x_i^N$ [$x < x_i^0$], playing 1 [playing 0] is strictly dominant for player i in $\Gamma(x)$. A subsidy s_i to player i shifts these boundaries to $x_i^N - s_i$ and $x_i^0 - s_i$, respectively. In the game of incomplete information Γ^ε, the boundaries for strict dominance in terms of a player's signals instead are $x_i^N - s_i + \varepsilon$ and $x_i^0 - s_i - \varepsilon$, respectively.

Lemma 2. *Given a subsidy policy $s = (s_i)$, consider the game $\Gamma^\varepsilon(s)$. (i) For each player i, the action $a_i = 1$ is strictly dominant at all $x_i^\varepsilon > \overline{x} - s_i + \varepsilon$. (ii) For each player i, the action $a_i = 0$ is strictly dominant at $x_i^\varepsilon < \underline{x} - s_i - \varepsilon$.*

Per Lemma 2, $u_i^\varepsilon(p_{-i} \mid \overline{X}, s_i) > 0$ for all p_{-i}. In particular, therefore, one has

$$u_i^\varepsilon(p_{-i}^{\overline{X}} \mid \overline{X}, s_i) > 0.$$

Let r_i^1 be the solution to

$$u_i^\varepsilon(p_{-i}^{\overline{X}} \mid r_i^1, s_i) = 0.$$

To any player i, the action $a_i = 1$ is strictly dominant at all $x_i^\varepsilon > r_i^1$ conditional on $\overline{X}$; denote $r^1 := (r_i^1)$. It is clear that r_i^1 depends upon the subsidy s_i, but for brevity we leave this dependence out of the notation for now. From Lemma 1 follows that $r_i^1 < \overline{X}$ for all i.

Player i knows that no player j will pursue a strategy $p_j < p_j^{r_j^1}$ since such a strategy is iteratively strictly dominated. Now define

$$u_i^\varepsilon(p_{-i}^{r^1} \mid r_i^2, s_i) = 0,$$

for all i. Because $p_i^{\overline{X}}$ is strictly dominated for every i, the any strategy $p_i < p_i^{r_i^1}$ is iteratively strictly dominated for all i, which in turn implies that any $p_i < p_i^{r_i^2}$ is iteratively dominated. This argument can – and should – be repeated indefinitely. We obtain a sequence $\overline{X} = r_i^0, r_i^1, ...,$ all i. For any k and r_i^k such that $u_i^\varepsilon(p^{r^k_{-i}} \mid r_i^k, s_i) > 0$, there exists r_i^{k+1} that solves $u_i^\varepsilon(p_{-i}^{r^k} \mid r_i^{k+1}, s_i) = 0$. Induction on k, using Lemma 1, reveals that $r_i^{k+1} < r_i^k$ for all $k \geq 0$. Per Lemma 2, we know that $r_i^k \geq \underline{X}$ for all k. It follows that the sequence (r_i^k) is monotone and bounded. Such a sequence must converge; let r_i denote its limit and define $r := (r_i)$. By construction, R solves

$$u_i^\varepsilon\left(p_{-i}^r \mid r_i, s_i\right) = 0.$$

A perfectly symmetric procedure should be carried out starting from low signals, eliminating ranges of x_i^ε for which playing 1 is strictly (iteratively) dominated. For every player i this yields an increasing and bounded sequence (l_i^k) whose limit is l_i, and $l := (l_i)$. The limit L solves

$$u_i^\varepsilon\left(p_{-i}^l \mid l_i, s_i\right) = 0.$$

From the construction of $l = (l_i)$ and $r = (r_i)$, the following is true by definition.

Lemma 3. *To each player i, a strategy p_i survives iterated elimination of strictly dominated strategies if and only if $p_i^{r_i}(x_i^\varepsilon) \leq p_i(x_i^\varepsilon) \leq p_i^{l_i}(x_i^\varepsilon)$ for all x_i^ε.*

The set of signals for which player i does not have a unique strategy that survives IESDS is $[l_i, r_i]$. We next show that this set has measure zero.

Lemma 4. *For each $i \in \mathscr{N}$, $l_i = r_i$.*

So far, the subsidies $s = (s_i)$ were taken as given and therefore left out of the notation. To make the dependence of strategies on subsidies more explicit we henceforth write $x_i(s) = l_i = r_i$ for the unique switching point (given s), and $x(s) = (x_i(s))$, i.e. $(x_i(s))$ solves

$$u_i^\varepsilon\left(p_{-i}^{x(s)} \mid x_i(s), s_i\right) = 0$$

for all $i \in \mathscr{N}$.

4.2 Implementation and Characterization

Recall that a strategy vector $p = (p_1, p_2, ..., p_N)$ is a Bayesian Nash Equilibrium (BNE) of $\Gamma^\varepsilon(s)$ if for any p_i and x_i^ε it holds that:

$$p_i(x_i^\varepsilon) \in \underset{a_i \in \{0,1\}}{\arg\max}\, \pi_i^\varepsilon(a_i, p_{-i} \mid x_i^\varepsilon, s_i), \tag{4}$$

where $\pi_i^\varepsilon(a_i, p_{-i} \mid x_i^\varepsilon) := \int \pi_i(a_i, p_{-i}(x_{-i}^\varepsilon) \mid x)\, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon)$. It follows immediately that $p^{x(s)}$ is a BNE of $\Gamma^\varepsilon(s)$. Lemma 5 strengthens this result and establishes that $p^{x(s)}$ is the *only* BNE of $\Gamma^\varepsilon(s)$.

Lemma 5. *Given is s. The essentially unique Bayesian Nash equilibrium of $\Gamma^{\varepsilon}(s)$ is $p^{x(s)}$. In particular, if p a BNE of $\Gamma^{\varepsilon}(s)$ then any $p_i \in p$ satisfies $p_i(x_i^{\varepsilon}) = p_i^{x_i(s)}(x_i^{\varepsilon})$ for all $x_i^{\varepsilon} \neq x_i(s)$ and all i.*

If the strategy vector $p = (p_i)$ is a BNE of $\Gamma^{\varepsilon}(s)$, then for each p_i it must hold that $p_i(x_i^{\varepsilon}) = p_i^{x_i(s)}(x_i^{\varepsilon})$ for all $x_i^{\varepsilon} \neq x_i(s)$. Hence, the vector of subsidies s implements the increasing strategy vector $p^{x(s)}$. We next establish that the reverse is also true: given an increasing strategy vector $p^{\hat{x}}$, $\hat{x} = (\hat{x}_i)$, there is a unique vector of subsidies $\hat{s} = (\hat{s}_i)$ that implements $p^{\hat{x}}$.

Lemma 6. *Given is vector of real numbers $\hat{x} = (\hat{x}_i)$. There is a unique vector of subsidies $\hat{s} = (\hat{s}_i)$ such that $x(\hat{s}) = \hat{x}$.*

Lemma 6 implpies that there exists a unique subsidy scheme $\tilde{s}$ that implements $p^{\tilde{x}}$. This proves part (i) of Theorem 1. In the final step of the analysis, we characterize $\tilde{s}$.

Lemma 7. *For all X such that $\underline{X} + \varepsilon \leq X \leq \overline{X} - \varepsilon$, one has*

$$u_i^{\varepsilon}\left(p_{-i}^{X} \mid X, s_i\right) = X + \sum_{n=0}^{N-1} \frac{w_i(n)}{N} - c_i + s_i. \tag{5}$$

If his opponents all play the same increasing strategy p_j^X, then upon observing the threshold signal $x_i^{\varepsilon} = X$ player i's belief over the aggregate action A_{-i} is uniform. Convergence to uniform strategic beliefs is a common property in global games; see Lemma 1 in Sakovics and Steiner (2012) for a particularly relevant reference.

Recall that, if $x(s)$ is the vector of switching points such that $p^{x(s)}$ is the unique BNE of $\Gamma^{\varepsilon}(s)$, then $x_i(s)$ solves (4.1) for all i. Imposing now that $\tilde{s}$ be such that $x_i(\tilde{s}) = \tilde{x}$ for all $i \in \mathscr{N}$, one obtains

$$u_i^{\varepsilon}\left(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}_i\right) = 0 \tag{6}$$

as the N identifying conditions for the subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $p^{\tilde{x}}$. By Lemma 7, (6) gives

$$u_i^{\varepsilon}\left(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}_i\right) = \tilde{x} + \sum_{n=0}^{N-1} \frac{w_i(n)}{N} - c_i + \tilde{s}_i = 0,$$

or, solving for $\tilde{s}_i$,

$$\tilde{s}_i = c_i - \tilde{x} - \sum_{n=0}^{N-1} \frac{w_i(n)}{N},$$

which is $(*)$. This proves part (ii) of Theorem 1 and concludes the main analysis.

4.3 Discussion

Our results characterize the subsidy scheme $\tilde{s}$ a planner must commit to when seeking to implement $p^{\tilde{x}}$ among rational players. Let us discuss several properties of this policy.

First, optimal subsidies are modest relative to the planner's goal: $\tilde{s}_i$ does not make $p_i^{\tilde{x}}$ strictly dominant for any player i. The sufficiency of modest subsidies is the consequence of a strategic *unraveling effect* of subsidies in coordination games. A subsidy to player i raises his incentive to play 1. In a coordination game, the increased incentive of player i raises the incentive of player j to play 1. The increase in j's incentive in turn makes playing 1 even more attractive to player i, and so on. Under common knowledge of the subsidy, what obtains is a indefinitely compounded positive feedback look, the unraveling effect; see Figure 1 in the Introduction. Because of the unraveling effect, even seemingly minor subsides can go a long way toward solving the planner's problem. This feature of $\tilde{s}$ is a key counterpoint to several well-known results in the literature on policy design in coordination problems that stress optimality of subsidizing at least some players to strict dominance (Segal, 2003; Winter, 2004; Bernstein and Winter, 2012; Sakovics and Steiner, 2012).[4]

Second, symmetric players are offered identical subsidies. The scheme $\tilde{s}$ shares this property with the policies derived in Sakovics and Steiner (2012) and Halac et al. (2021). The symmetric treatment of identical players deviates from a number of other notable proposals including a divide-and-conquer policy (*cf.* Segal, 2003; Bernstein and Winter, 2012) and the incentive schemes studied in Winter (2004) and Halac et al. (2020).[5]

Third, subsidies target all players and are globally continuous in model parameters. The characterization in ($*$) establishes global continuity of $\tilde{s}_i$ in all the parameters upon which it depends $(c_i, w_i(n), \tilde{x})$. While conditional on policy treatment the optimal subsidies in Sakovics and Steiner (2012) are continuous in the relevant model parameters as well, changes in one player's parameters could affect whether or not said player is targeted, causing a discrete jump in subsidies received. Similarly, subsidies are continuous conditional on a player's position in the policy ranking in a divide and conquer mechanism (Segal, 2003; Bernstein and Winter, 2012); however, a player's position in the optimal ranking is affected by a change in its parameters, which can lead to discrete jumps in subsidy entitlement.

Fourth, subsidies are increasing in c_i, the opportunity cost of playing 1. Given x, the cost of playing 1 is inreasing in c_i; hence, to induce coordination on 1 subsidies should increase as the cost c_i rises. This property is intuitive and shared (conditional on policy treatment and/or ranking) by many recent contributions on policy design in coordination problems (Winter, 2004; Sakovics and Steiner, 2012; Bernstein and Winter, 2012; Halac et al., 2020, 2021).

Fifth, subsidies are decreasing in $\tilde{x}$, the threshold for coordination on 1 targeted by the planner. All else equal, a player's (expected) incentive to play 1 is increasing in his signal x_i^{ε}. Hence, for higher signals a player needs less subsidy to induce him to play 1. One can interpret $\tilde{x}$ as an inverse measure of the planner's ambition: the higher is $\tilde{x}$, the lower is the prior probability that coordination on 1 will be achieved. In this interpretation, being ambitious is costly: assuming coordination on 1 is indeed achieved, total spending on subsidies

[4]Observe that the unraveling effect does not rely on strategic complementarities being direct. Under "indirect cyclical" strategic complementarities (e.g. player i generates a (positive) externality on player j, who generates an externality on k, who in turn generates an externality on i), subsidies clearly have similar strategic effects.

[5]Onuchic and Ray (2023) also show that "identical agents" may be compensated asymmetrically in equilibrium; however, though identical in the payoff-relevant sense their players may still vary in payoff-irrelevant "identifies". Asymmetries in policy treatment derive from asymmetric identities.

is increasing in the planner's ambition (decreasing in $\tilde{x}$).

Sixth, subsidies are decreasing in spillovers, i.e. $\partial \tilde{s}_i / \partial w_i(n) < 0$. When observing the threshold signal $\tilde{x}$, a player i's belief over the aggregate action A_{-i} is uniform; in particular, therefore, he assigns strictly positive probability to the event that $A_{-i} = n$ for all $n = 0, 1, ..., N-1$. If $w_i(n)$ increases, the *expected* spillover a player expects to enjoy upon playing 1 is hence greater. This raises his incenive to play 1 and, for given $\tilde{x}$, the subsidy required to make him willing to do so is smaller. Given a ranking of players, subsides for each player (except the first-ranked) are also decreasing in spillovers in a divide-and-conquer policy (Segal, 2003; Bernstein and Winter, 2012). The optimal subsidies in Sakovics and Steiner (2012) are not generally decreasing in spillovers, except insofar as players who benefit less from project success are more likely to be targeted.

Seventh, though players coordinate on a symmetric *equilibrium*, they do not (necessarily) have symmetric *payoffs* in equilibrium.

5 Special Cases and Extensions

Generic properties of $\tilde{s}$ were discussed above. Several additional features become apparant upon considering special cases of the model.

5.1 Games of Regime Change

There is a project in which N investors can invest. The cost of investment to investor i is $c_i > 0$. If the project succeeds, an investing investor i realizes benefit $b_i > c_i$. The project is successful if and only if a critical mass of investors invests; specifically, there exists $I \in (0, N)$ such that the project succeeds if $A \geq I$ and fails otherwise.[6] The payoff to not investing, the outside option, is given by $-x$. The uncertainty about x, or more generally about $x - c_i$, can be thought of as any kind of (fundamental) uncertainty that pertains to the cost or benefit of investment; see Abel (1983) or Pindyck (1993) for possible interpretations.

Hoping to attract investment, a planner offers each investor i an investment subsidy s_i. For comparability with the literature, we are particularly interested in the subsidy scheme $s^0 = (s_i^0)$ that implements p^0 (as regime change games usually normalize the payoff to the outside option to 0, see Sakovics and Steiner (2012) or Halac et al. (2020)).

Proposition 1. *Consider a joint investment problem in which $I \in (0, N)$ is the critical threshold for project success. Let $N - n^*$ be the smallest integer greater than I. The subsidy scheme $s^0 = (s_i^0)$ that makes coordination on p^0 the unique Bayesian Nash equilibrium of this game is given by*

$$s_i^0 = c_i - \frac{n^*}{N} \cdot b_i \tag{7}$$

for every $i \in \mathcal{N}$.

In the subsidy scheme s^0, all investors are subsidized; subsidies are a fraction of their investment costs. The latter is explained through the unraveling effect of policies: if investor

[6] In the notation of (1), we thus have $w_i(n) = \underline{b}_i$ for all $n < I$ and $w_i(n) = \overline{b}_i$ for all $n \geq I$, where $\overline{b}_i > \underline{b}_i$ and $b_i := \overline{b}_i - \underline{b}_i$. Observe that, for $\underline{b}_i > 0$, the model allows for free-riding.

i receives an investment subsidy, he is more likely to invest. Anticipating the increased likelihood that i invests, project success becomes more likely and this attracts investment by investor j. The greater likelihood that j invests in turn makes investment even more interesting for i, and so on. This feedback effect is strong: in (non-trivial) two-player joint investment problems, subsidies are less than half players' investment costs.

While the problem here bears close resemblance to the global game in Sakovics and Steiner (2012), the models differ in fundamental ways that make a direct comparison complicated. Most notably, the game in Sakovics and Steiner (2012) does not have no prior uncertainty about the efficient outcome of the game; coordinated investment is always the efficient equilibrium of the game. Instead, uncertainty pertains to the critical threshold of investments required to achieve project success. Similarly, and related, conditional on the regime in place, there is no uncertainty about payoffs in Sakovics and Steiner (2012); we instead assume uncertainty about payoffs even conditional on the regime.[7] It is interesting that these differences, albeit fairly subtle, lead to vastly different policy implications.

An important and, in our view, realistic possibility in the investment problem studied here is that joint investment need *not* be ex post efficient: if x is very low, it can be efficient for all players to not invest and take the outside option. We return to this issue in Section 5.5.

5.2 Principal-Agent Problems

The main analysis assumes that actions are contractible. Here we discuss the implications of our results for principal-agent problems in which subsidies cannot condition on individual actions but only on aggregate outcomes, see for example Winter (2004) and Halac et al. (2021).

There is an organizational project that involves N tasks each performed by one agent $i \in \mathcal{N}$. Each agent i decides whether to work ($a_i = 1$) towards completing his task or shirk ($a_i = 0$). The cost of working to agent i is given by $c_i > 0$. Success of the project depends upon the decisions of all agents through a production technology $q : \{0, 1, ..., N\} \to [0, 1]$, where $q(n)$ is the probability of success given that n agents work. As in Winter (2004) and Halac et al. (2021), we assume that $q(n+1) > q(n)$ for all $n \leq N-1$.

A principal offers contracts that specify rewards $v = (v_i)$ to agents contingent on project success; if the project fails, all agents receive zero. We assume that agents' work effort is their private knowledge – any rewards the principal offers can condition only upon project success. An agent who shirks gets payoff $-x$. We interpret x generally as an uncertain fundamental that determines agents' payoffs, see also Halac et al. (2022) for a model of contracting under fundamental uncertainty.

Proposition 2. *Consider a principal-agent problem in which the principal offers rewards $(\tilde{v}_i)$ to implement $\mathrm{p}^{\tilde{x}}$ as the unique equilibrium of the game. For each $i \in \mathcal{N}$, the reward $\tilde{v}_i$ is given by*

$$\bar{q} \cdot \tilde{v}_i = c_i - \tilde{x}, \tag{8}$$

where $\bar{q} := \sum_{n=0}^{N-1} \frac{q(n+1)-q(n)}{N}$.

[7]This distinction applies more generally to the literature on global games of regime change, see Morris and Shin (1998), Angeletos et al. (2007), Goldstein and Pauzner (2005), Basak and Zhou (2020), and Edmond (2013).

The characterization in (8) has an intuitive interpretation. Consider the threshold type agent i who observes signal $x_i^\varepsilon = \tilde{x}$. On the one hand, her (expected) *net cost* of working is $c_i - \tilde{x}$; it is the sum of the cost of work c_i and her expected benefit of having free time $-\tilde{x}$. On the other hand, assuming that n fellow agents work a threshold agent i who shirks expects to receive the reward $\tilde{v}_i$ with probability $q(n)$. If instead agent i works, she increases this probability to $q(n+1)$. The threshold type agent i who observes $x_i^\varepsilon = \tilde{x}$ therefore believes that working, rather than shirking, increases the probability of receiving her reward by $q(n+1) - q(n)$. Moreover, since the threshold type agent has the uniform belief over the aggregate action (see Lemma 7), her *expected* marginal contribution to project success is $\sum_{n=0}^{N-1} \frac{q(n+1)-q(n)}{N}$, or $\bar{q}$. Hence, agent i's expected benefit of working is simply $\bar{q} \cdot \tilde{v}_i$. Given that the principal targets $p^{\tilde{x}}$, such an agent should be just indifferent between working and shirking – for her the expected benefit of working should be equal to her net cost of working. This gives (8).

Note that for $x = 0$, the payoffs in this model exactly replicate those in the canonical principal-agent problem (Winter, 2004). Interestingly, the analysis under uncertainty fails to yield Winter's prescription that optimal contracts are inherently discriminatory and should reward identical agents asymmetrically. In this paper, symmetric agents receive identical rewards. It is similarly noteworthy that, in contrast to Halac et al. (2021), this symmetry does not rely on contracts being private – indeed, Corollary 2 critically relies upon the scheme $\tilde{v}$ being common knowledge.

5.3 Heterogeneous Externalities

The main analysis assumes that only the aggregate action A_{-i} matters for the externality other players impose upon player i. We relax this unrealistic assumption here. In particular, we allow that the externality $w_i(a_{-i})$ depends upon the specific vector a_{-i} played. We maintain a focus on games with strategic complementarities and assume that if $a''_{-i} \geq a_{-i}$, then $w_i(a''_{-i}) \geq w_i(a_{-i})$. Observe that this externality structure encompasses the games in Bernstein and Winter (2012) and Halac et al. (2021), where externalities are allowed to depend upon the subset $M \subseteq \mathscr{N}$ of players who play 1. It also nests the approach in Sakovics and Steiner (2012) where externalities depend upon the aggregate action, but some players have a stronger impact on the aggregate action than others. Finally, heterogeneous externalities may arise in coordination games on (directed) graphs (Leister et al., 2022).

Let a_{-i}^n denote an action vector a_{-i} in which exactly n players play 1 (and the remaining $N - n - 1$ players play 0). We write A_{-i}^n for the set of all (unique) action vectors a_{-i}^n. Note that there are exactly $\binom{N-1}{n}$ vectors a_{-i}^n in A_{-i}^n. For all i, define

$$w_i^n := \frac{\sum_{a_{-i}^N \in A_{-i}^N} w_i\left(a_{-i}^n\right)}{\binom{N-1}{n}}.$$

In words, w_i^n is the expected externality imposed upon player i who expects that n opponents play 1 and believes that every player $j \neq i$ is equally likely to one of those n.

Proposition 3. *Consider a game Γ^ε with heterogeneous externalities. Given is $\tilde{x} \in \mathbb{R}$. There exists a unique subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $p^{\tilde{x}}$. For each $i \in \mathscr{N}$, the subsidy $\tilde{s}_i$*

pursuant to the scheme is given by

$$\tilde{s}_i = c_i - \tilde{x} - \sum_{n=0}^{N-1} \frac{w_i^n}{N}. \tag{9}$$

In the game with heterogeneous externalities, too, symmetric players receive identical subsidies. This conclusion remains valid if one considers the "limit of uncertainty" as $\varepsilon \to 0$. Thus, even a little bit of uncertainty can upset the canonical results by Segal (2003), Winter (2004), and Bernstein and Winter (2012) that optimal contracts are fundamentally discriminatory in coordination games.

5.4 Asymmetric Targets

It was so far maintained that the planner seeks to implement a symmetric equilibrium. In many practical situations policies may instead target asymmetric outcomes. We consider such instances here.

Given are two real numbers $\tilde{x}_1$ and $\tilde{x}_2$. Without loss, let $\tilde{x}_1 < \tilde{x}_2$. The planner partitions the player set $\mathscr{N}$ into two subsets $\mathscr{N}_1$ and $\mathscr{N}_2$ such that $\mathscr{N}_1 \cup \mathscr{N}_2 = \mathscr{N}$. There are N_1 players in $\mathscr{N}_1$ and $N_2 = N - N_1$ player in $\mathscr{N}_2$.

Suppose the planner seeks to implement the asymmetric equilibrium $(p_1^{\tilde{x}_1}, p_2^{\tilde{x}_2})$ according to which (with a slight abuse of notation) each player $i \in \mathscr{N}_1$ plays the increasing strategy $p_1^{\tilde{x}_1}$ while each $j \in \mathscr{N}_2$ plays $p_1^{\tilde{x}_1}$. We are agnostic as to the motivations behind such an asymmetric policy goal. We call $\tilde{x}_1$ the low critical state and $\tilde{x}_2$ the high critical state; similarly, we label players in $\mathscr{N}_1$ as low state and players in $\mathscr{N}_2$ as high state.

As before, $\tilde{s} = (\tilde{s}_i)$ denotes the vector of subsidies that implements $(p_1^{\tilde{x}_1}, p_2^{\tilde{x}_2})$, i.e. $(p_1^{\tilde{x}_1}, p_2^{\tilde{x}_2})$ is the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(\tilde{s})$. Proposition 4 establishes that $\tilde{s}$ is unique and given by

$$\begin{aligned} \tilde{s}_i &= c_i - \tilde{x}_1 - \sum_{n=0}^{N_1-1} \frac{w_i(n)}{N_1} \quad \text{for all } i \in \mathscr{N}_1, \\ \tilde{s}_j &= c_j - \tilde{x}_2 - \sum_{n=N_1}^{N-1} \frac{w_j(n)}{N_2} \quad \text{for all } j \in \mathscr{N}_2. \end{aligned} \tag{10}$$

We focus on the case $2\varepsilon < \tilde{x}_2 - \tilde{x}_1$.

Proposition 4. *Given are $\tilde{x}_1$ and $\tilde{x}_2$.*

(i) There exists a unique subsidy scheme $\tilde{s} = (\tilde{s}_i)$ that implements $\left(p_1^{\tilde{x}_1}, p_2^{\tilde{x}_2}\right)$.

(ii) For each $i \in \mathscr{N}$, the subsidy $\tilde{s}_i$ pursuant to the scheme is given by (10).

The characterization of optimal subsidies for asymmetric policy targets in (10) underlines the importance of strategic uncertainty for policy design. Compared to the symmetric equilibrium case in which all players coordinate on the same strategy vector, under an asymmetric equilibrium target players in $\mathscr{N}_1$ receive higher subsidies while players in $\mathscr{N}_2$

receive lower subsidies. The reason is the variation in players' strategic equilibrium beliefs between subsets. In equilibrium, a player i in $\mathscr{N}_1$ who observes signal $x_i^\varepsilon = \tilde{x}_1$ knows that no other player will have received a signal greater than $\tilde{x}_2$. Hence, player i also knows that all players who play strategy $p^{\tilde{x}_2}$ will play 0. Upon observing the signal $\tilde{x}_1$, there consequently is no uncertainty about the actions played by all players in $\mathscr{N}_2$; the only remaining strategic uncertainty pertains to i's opponents in $\mathscr{N}_1$. This is reflected in $\tilde{s}_i$ in (10). Similarly, a player $j \in \mathscr{N}_2$ who observes signal $x_j^\varepsilon = \tilde{x}_2$ knows that no player has received a signal below $\tilde{x}_1$. Hence, in equilibrium player j takes as given that all players $i \in \mathscr{N}_1$ play 1; the only remaining strategic uncertainty pertains to the actions of his opponents in $\mathscr{N}_2$. This is reflected in $\tilde{s}_j$ in (10).

5.5 Induced Coordination Failure

Because the planner must commit to its policy before Nature draws the parameter x, subsidies can end up stimulating an action that is ex post inefficient. This possibility warrants policy moderation, as is most simply illustrated in a symmetric game.

Consider a coordination problem among symmetric players such that $c_i = c$ and $w_i(n) = w(n)$ for all $i \in \mathscr{N}$. In symmetric games, $\overline{a}$ is the efficient Nash equilibrium of the complete information game $\Gamma(x)$ for all $x > \underline{x}$ where $\underline{x} = c - w(N-1)$. Let s^* denote the *optimal subsidy* in a symmetric game in the sense that s^* induces coordination on $p^{\underline{x}}$ as the unique Bayesian Nash equilibrium of $\Gamma^\varepsilon(s^*)$. Per the characterization in Theorem 1, we have

$$s^* = w(N-1) - \sum_{n=0}^{N-1} \frac{w(n)}{N}.$$

The following corollary says that s^* is not only sufficient to induce coordination on an efficient equilibrium; subsidization in excess of s^* causes equilibrium inefficiency.

Corollary 1. *In symmetric games, subsidies $\hat{s}$ such that $\hat{s} > s^*$ are inefficient. Specifically, the subsidy $\hat{s}$ induces coordination on $p^{\hat{x}}$ where $\hat{x} < \underline{x}$. In the limit as $\varepsilon \to 0$, players thus coordinate on an inefficient outcome of $\Gamma(x)$ for all $x \in (\hat{x}, \underline{x})$ with probability 1.*

The possibility of policy-induced coordination failure due to excessive subsidization is illustrated in Figure 2. Though Corollary 1 follows trivially from Theorem 1, we single it out to emphasize an important economic implication of our results. Prior uncertainty about x implies prior uncertainty about the efficient outcome of $\Gamma(x)$. If the planner must commit to its policy s before the random variable x is drawn, such uncertainty warrants policy moderation as high subsidies risk stimulating coordination on $\overline{a}$ even when $\underline{a}$ turns out to be the efficient outcome. Intuitively, under prior uncertainty about payoff functions the planner should subsidize conservatively to avoid picking inefficient winners; the planner should not be wedded to the idea that coordination on $\overline{a}$ must always be achieved.

In our view, the notion that policy intervention can itself be a source of inefficiency is realistic and consistent with historical evidence (*cf.* Cowan, 1990). The literature on coordination games has not always emphasized this possibility. In the coordination problems studied by Sakovics and Steiner (2012), Bernstein and Winter (2012), and Halac et al. (2020, 2021) for example, coordination on 1 is always the strictly Pareto-dominant outcome of

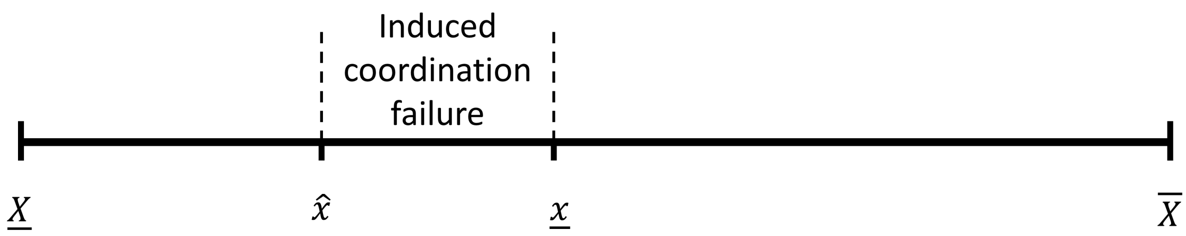

Figure 2: If $\hat{s} > s^*$, the unique equilibrium of $\Gamma^\varepsilon(\hat{s})$ has players coordinate on $p^{\hat{x}}$ with $\hat{x} < \underline{x}$. For $\varepsilon \to 0$, the policy $\hat{s}$ thus induces coordination on an inefficient outcome of the game – a policy-induced coordination failure – for all $x \in (\hat{x}, \underline{x})$.

the game. The inefficiency associated with excessive subsidization in those models is that the planner ends up spending more on subsidies than would be strictly necessary. Our analyis complements those results by making explicit another, non-budgetary source of policy inefficiency: the possibility that excessive intervention induces coordination on an inefficient outcome.

6 Concluding Remarks

The results in this paper suggest that a planner needs only modest policy intervention to solve coordination problems. The core mechanism driving this result is an *unraveling* effect of policies in coordination games. A subsidy raises a player's incentive to play the subsidized action. Her opponents know this and, given their desire to match actions, experience an increase in their own incentives, which in turn raises the player's incentive to pursue the subsidized action even further, and so on. Under common knowledge of the policy, this positive feedback loop compounds indefinitely. Compounding means that seemingly modest policy intervention can provide enough stimulus to steer rational agents toward efficiency. This unraveling effect is illustrated in Figure 1.

We derive our results in a global coordination game. We use a global games approach to address equilibrium selection explicitly. Coordination games tend to have multiple strict Nash equilibria. Policies need to work conditional on players' beliefs about their opponents' equilibrium strategies, which depend upon the equilibrium they believe others will be playing. With multiple equilibria, strategic beliefs therefore are not uniquely defined and this complicates policy design. It also motivates invasive policymaking, as policies must work even against the most pessimistic (strategic) beliefs. However, certain expectations about equilibrium play would seem highly counterintuitive; for example, one would not expect a subsidy to decrease incentives to play the subsidizes action. That such outcomes are consistent with equilibrium play is an artefact of the equilibrium multiplicity in complete information coordination games. We therefore address equilibrium selection explicitly, and we do so in a global game. We provide conditions under which rational agents will be able to select a unique equilibrium of the coordination game they are playing. We characterize the unique equilibrium as a function of the subsidy in place and use this characterization to derive the optimal subsidy, that is, the subsidy that makes the Pareto efficient strategy the unique Bayesian Nash equilibrium of the global game.

A useful feature of global games is that these games are solved by iterated elimination of strictly dominated strategies. Iterated dominance depends upon players' ability to form higher-order beliefs about the strategies pursued by their opponents and, consequently, their strategic beliefs. This is useful because our analysis of subsidy design and the unraveling effect presupposes the same ability with respect to the effect of subsidies on strategies. For this reason, and though other theories of equilibrium selection exist, we think of global games as the best suited framework to study our problem.

The unraveling effect makes it *possible* to solve coordination problems through modest policy intervention. Prior uncertainty about the efficient outcome of the game makes modest intervention necessary. Subsidies that are too high risk stimulating players to pursue a particular action even if coordination on that action is an inefficient, or Pareto-dominated, outcome of the game. This warrants a conservative approach toward policy as the planner should avoid picking possibly Pareto-inferior winners. This possibility naturally complicates policy design compared to cases in which the efficient outcome is assumed to be known a priori (*cf.* Bernstein and Winter, 2012; Sakovics and Steiner, 2012; Halac et al., 2020) and favors an noninvasive planner who subsidizes modestly.

A Proofs

PROOF OF LEMMA 1

Proof. First, observe that

$$\begin{aligned}
u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) &= \int u_i\left(p_{-i}\left(x_{-i}^\varepsilon\right) \mid x\right) \, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) \\
&= \int w_i\left(p_{-i}\left(x_{-i}^\varepsilon\right)\right) + x \, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon) - c_i \\
&= \int w_i\left(p_{-i}\left(x_{-i}^\varepsilon\right)\right) \, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon) + x_i^\varepsilon - c_i,
\end{aligned}$$

for any strategy vector p_{-i}.

To prove part (i), it suffices to show that $\int w_i\left(p_{-i}^y\left(x_{-i}^\varepsilon\right)\right) \, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon)$ is increasing in x_i^ε. First we introduce a random variable $v_i(x_{-i}) = w_i(p_{-i}^y(x_{-i}^\varepsilon))$ and observe that, since $w_i(p_{-i}^y(x_{-i}^\varepsilon))$ is increasing in $p_{-i}^y(x_{-i}^\varepsilon)$ and $p_{-i}^y(x_{-i}^\varepsilon)$ is increasing in x_{-i}^ε, v_i is increasing in x_{-i}^ε. Next, we note that the distribution $\Phi^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon)$ is first-order stochastic dominant over the distribution $\Phi^\varepsilon(x_{-i}^\varepsilon \mid \hat{x}_i^\varepsilon)$ iff $x_i^\varepsilon > \hat{x}_i^\varepsilon$; this follows from Bayes' theorem upon application of the two facts that (a) each ε_j (and indeed ε_i) is drawn independently of x, and (b) player i's conditional distribution on x given x_i^ε first-order stochastic dominates his conditional distribution on x given $\hat{x}_i^\varepsilon$ iff $x_i^\varepsilon > \hat{x}_i^\varepsilon$. Hence, because v_i is increasing we have $\int v_i(x_{-i}^\varepsilon)\mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon) > \int v_i(x_{-i}^\varepsilon)\mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid \hat{x}_i^\varepsilon)$ and the result follows.

To prove part (ii), we reiterate the observation from the proof of part (i) that the distribution $\Phi^\varepsilon(x_{-i}^\varepsilon \mid x_i^\varepsilon)$ is first-order stochastic dominant over the distribution $\Phi^\varepsilon(x_{-i}^\varepsilon \mid \hat{x}_i^\varepsilon)$ iff $x_i^\varepsilon > \hat{x}_i^\varepsilon$. Next, we note that $p_{-i}^y(x_{-i}^\varepsilon)$ is (weakly) decreasing in $y_j \in y$, all $j \neq i$ (and, therefore, the random variable $v_i(x_{-i}^\varepsilon)$ we introduced in the proof of part (i) is also decreasing in y_j). Therefore $\int w_i\left(p^{y}_{-i}\left(x^{\varepsilon}_{-i}\right)\right)\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_{-i}\mid x^{\varepsilon}_i)$ is decreasing in y_j, all $j\neq i$, and the result follows. □

PROOF OF LEMMA 2

Proof. When $x^{\varepsilon}_i > \overline{x} - s + \varepsilon$, the entire support of player i's conditional distribution on x lies in a region where $x > \overline{x}$. For any such x, playing 1 is strictly dominant. (i) follows. Part (ii) is proven in a similar way. □

PROOF OF LEMMA 4

Proof. By construction, $l_i \leq r_i$. Define $\epsilon_i := r_i - l_i$, so $\epsilon_i \geq 0$. We first establish a useful claim.

Claim 1. *If $\epsilon_i = \epsilon$ for all $i \in \mathscr{N}$, then $\epsilon = 0$.*

Proof of the claim. With a slight abuse of notation, we write $z_{-i} + \epsilon = (z_j + \epsilon)_{j\neq i}$. Observe that $\Phi(z_{-i}\mid x^{\varepsilon}_i) = \Phi(z_{-i}+\epsilon \mid x^{\varepsilon}_i + \epsilon)$ for all z_{-i} and x^{ε}_i. If $\epsilon_i = \epsilon$ for all $i\in\mathscr{N}$, this implies

$$\int w_i(p^{l_{-i}}_{-i})\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_{-i}\mid l_i) = \int w_i(p^{l_{-i}+\epsilon}_{-i})\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_{-i}\mid l_i+\epsilon) = \int w_i(p^{r_{-i}}_{-i})\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_{-i}\mid r_i).$$

Hence,

$$\begin{aligned} u^{\varepsilon}_i(p^{r_{-i}}_{-i}\mid r_i, s_i) &= r^*_i + s_i - c_i + \int w_i(p^{r_{-i}}_{-i})\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_{-i}\mid r_i) \\ &= r_i + s_i - c_i + \int w_i(p^{l^*_{-i}}_{-i})\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_{-i}\mid l_i) \\ &= l_i + \epsilon + s_i - c_i \int w_i(p^{l_{-i}}_{-i})\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_{-i}\mid l_i) \\ &= u^{\varepsilon}_i(p^{l_{-i}}_{-i}\mid l_i, s_i) + \epsilon. \end{aligned}$$

By definition, $u^{\varepsilon}_i(p^{r_{-i}}_{-i}\mid r_i, s_i) = u^{\varepsilon}_i(p^{l_{-i}}_{-i}\mid l_i, s_i)$, and it follows that $\epsilon = 0$. □

Now suppose that $\epsilon_i \neq \epsilon_j$ for at least one pair of players $i, j\in\mathscr{N}$. Let $\epsilon_i = \sup\{\epsilon_j \mid j\in\mathscr{N}\}$. Because $\epsilon_i \geq \epsilon_j$ for all $j\neq i$, we have

$$\int_{l_i-2\varepsilon}^{z} x^{\varepsilon}_j\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_j\mid l_i) \leq \int_{r_i-2\varepsilon}^{z+\epsilon_i} x^{\varepsilon}_j\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_j\mid r_i)$$

for all $z\in[l_i-2\varepsilon, l_i+2\varepsilon]$. The inequality is strict if $\epsilon_j < \epsilon_i$ (and by assumption there is at least one such j). The immediate implication is that

$$\int p^{l_{-i}}_{-i}(x^{\varepsilon}_{-i})\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_{-i}\mid l_i) < \int p^{r_{-i}}_{-i}(x^{\varepsilon}_{-i})\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_{-i}\mid r_i)$$

and, because $w_i(n)$ is increasing in n, we have

$$\int w_i\left(p^{l_{-i}}_{-i}(x^{\varepsilon}_{-i})\right)\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_{-i}\mid l_i) < \int w_i\left(p^{r_{-i}}_{-i}(x^{\varepsilon}_{-i})\right)\,\mathrm{d}\Phi^{\varepsilon}(x^{\varepsilon}_{-i}\mid r_i). \tag{I}$$

By definition, l satisfies $u_i^\varepsilon(p_{-i}^{l_{-i}} \mid l_i, s_i) = u_i^\varepsilon(p_{-i}^{r_{-i}} \mid r_i) = 0$ and

$$\begin{aligned} u_i^\varepsilon(p_{-i}^{l_{-i}} \mid l_i) &= l_i + s_i - c_i + \int w_i\left(p_{-i}^{l_{-i}}(x_{-i}^\varepsilon)\right) \, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid l_i) \\ &< r_i + s_i - c_i + \int w_i\left(p_{-i}^{l_{-i}}(x_{-i}^\varepsilon)\right) \, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid l_i) \\ &< r_i + s_i - c_i + \int w_i\left(p_{-i}^{r_{-i}}(x_{-i}^\varepsilon)\right) \, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid r_i) \\ &= u_i^\varepsilon(p_{-i}^{r_{-i}} \mid r_i). \end{aligned}$$

The first inequality follows from the assumption that $r_i > l_i$, the second from (I). We have now derived that $u_i^\varepsilon(p_{-i}^{l_{-i}} \mid l_i, s_i) \neq u_i^\varepsilon(p_{-i}^{r_{-i}} \mid r_i)$, contradicting at least one of the definitions of r and l that $u_i^\varepsilon(p_{-i}^{l_{-i}} \mid l_i, s_i) = u_i^\varepsilon(p_{-i}^{r_{-i}} \mid r_i) = 0$. It follows there can be no player i such that $\epsilon_i \geq \epsilon_j$ for all $j \neq i$ and with a strict inequality for at least one j. By implication, $\epsilon_i = \epsilon$ for all $i \in \mathscr{N}$. The claim at the beginning of the proof established that in this case $\epsilon = 0$. We therefore have $l_i = r_i$ for all $i \in \mathscr{N}$. □

PROOF OF LEMMA 5

Proof. Let $p = (p_i)$ be a BNE of $\Gamma^\varepsilon(s)$. For any player i, define

$$\underline{\underline{x}}_i = \inf\{x_i^\varepsilon \mid p_i(x_i^\varepsilon) > 0\}, \tag{11}$$

and

$$\overline{\overline{x}}_i = \sup\{x_i^\varepsilon \mid p_i(x_i^\varepsilon) < 1\}. \tag{12}$$

Observe that $\underline{\underline{x}}_i \leq \overline{\overline{x}}_i$. Now define

$$\underline{\underline{x}} = \min\{\underline{\underline{x}}_i\}, \tag{13}$$

and

$$\overline{\overline{x}} = \max\{\overline{\overline{x}}_i\}. \tag{14}$$

By construction, $\overline{\overline{x}} \geq \overline{\overline{x}}_i \geq \underline{\underline{x}}_i \geq \underline{\underline{x}}$. Observe that p is a BNE of $\Gamma^\varepsilon(s)$ only if, for each i, it holds that $u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i) \geq 0$. Consider then the expected incentive $u_i^\varepsilon(p_{-i}^{\underline{\underline{x}}}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i)$. It follows from the definition of $\underline{\underline{x}}$ that $p^{\underline{\underline{x}}}(x^\varepsilon) \geq p(x^\varepsilon)$ for all x^ε. The implication is that, for each i, $u_i^\varepsilon(p_{-i}^{\underline{\underline{x}}}(x_i{-}i^\varepsilon) \mid \underline{\underline{x}}_i) \geq u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \underline{\underline{x}}_i) \geq 0$. From Proposition 7 then follows that $\underline{\underline{x}} \geq x$.

Similarly, if p is a BNE of $\Gamma^\varepsilon(s)$ then, for each i, it must hold that $u_i^\varepsilon(p_{-i}(x_{-i}^\varepsilon) \mid \overline{x}_i) \leq 0$. Consider the expected incentive $u_i^\varepsilon(p_{-i}^{\overline{\overline{x}}}(x_{-i}^\varepsilon) \mid \overline{\overline{x}}_i)$. It follows from the definition of $\overline{\overline{x}}$ that $p^{\overline{\overline{x}}}(x^\varepsilon) \leq p(x^\varepsilon)$ for all x^ε. For each i it therefore holds that $u_i^\varepsilon(p_{-i}^{\overline{\overline{x}}}(x_{-i}^\varepsilon) \mid \overline{\overline{x}}_i) \leq u_i^\varepsilon(p_{-i}(x_i{-}i^\varepsilon) \mid \overline{\overline{x}}_i) \leq 0$. Hence $\overline{\overline{x}} \leq x$.

Since $\underline{\underline{x}} \leq \overline{\overline{x}}$ while also $\underline{\underline{x}} \geq x$ and $\overline{\overline{x}} \leq x$ it must hold that $\underline{\underline{x}} = \overline{\overline{x}} = x$. Moreover, since $p^{\underline{\underline{x}}} \geq p$ while also $p^{\overline{\overline{x}}} \leq p$, given $\underline{\underline{x}} = \overline{\overline{x}} = x$, it follows that $p_i(s_i^\varepsilon) = p_i^x(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x$ and all i (recall that for each player i one has $u_i^\varepsilon(p_{-i}^x \mid x) = 0$, explaining the singleton exeption at $x_i^\varepsilon = x$). Thus, if $p = (p_i)$ is a BNE of $\Gamma^\varepsilon(s)$ then it must hold that $p_i(x_i^\varepsilon) = p_i^x(x_i^\varepsilon)$ for all $x_i^\varepsilon \neq x$ and all i, as we needed to prove. □

PROOF OF LEMMA 6

Proof. Suppose, in contrast, that there are two distinct vectors of subsidies $\hat{s}_1 = (\hat{s}_{1i})$ and $\hat{s}_2 = (\hat{s}_{2i})$ that both implement $p^{\hat{x}}$ such that $\hat{s}_1 \neq \hat{s}_2$. Per Lemmas 4 and 5, $\hat{s}_1$ and $\hat{s}_2$ must solve $x(\hat{s}_1) = x(\hat{s}_2) = \hat{x}$. By (4.1), this means that $\hat{s}_{1i}$ and $\hat{s}_{2i}$ are both solutions to

$$u_i^{\varepsilon}\left(p_{-i}^{\hat{x}} \mid \hat{x}_i, \hat{s}_{1i}\right) = u_i^{\varepsilon}\left(p_{-i}^{\hat{x}} \mid \hat{x}_i, \hat{s}_{2i}\right) = 0, \tag{15}$$

for each $i \in \mathscr{N}$. Using (3), we thus have

$$u_i^{\varepsilon}\left(p_{-i}^{\hat{x}} \mid \hat{x}_i\right) + s_{1i} = u_i^{\varepsilon}\left(p_{-i}^{\hat{x}} \mid \hat{x}_i\right) + s_{2i}, \tag{16}$$

which implies

$$\hat{s}_{1i} = \hat{s}_{2i} \tag{17}$$

for all $i \in \mathscr{N}$. This contradicts our assumption that $\hat{s}_1 \neq \hat{s}_2$ and thus proves the result. $\square$

PROOF OF LEMMA 7

Proof. First fix $x \in [\underline{X}+\varepsilon, \overline{X}-\varepsilon]$. Each player $j \neq i$ is assumed to play p_j^X, so the probability that $x_j = 1$ is given by

$$\Pr[x_j^{\varepsilon} > X \mid x] = \Pr[\varepsilon_j > X - x] = \frac{x + \varepsilon - X}{2\varepsilon}, \tag{18}$$

for all $X \in [x - \varepsilon, x + \varepsilon]$ while $\Pr[x_j^{\varepsilon} > X \mid x]$ is either 0 or 1 otherwise. Clearly, $a_j = 0$ is played with the complementary probability (given x and X). Since each ε_j is drawn independently from the same distribution, the probability that m given players $j \neq i$ play $a_j = 1$ while the remaining $N - m - 1$ players play $a_j = 0$ (given p_{-i}^X and x) is:

$$\left[\frac{x + \varepsilon - X}{2\varepsilon}\right]^m \left[\frac{X + \varepsilon - x}{2\varepsilon}\right]^{N-m-1}. \tag{19}$$

As there are $\binom{N-1}{m}$ unique ways in which m out of $N - 1$ players j can choose $a_j = 1$, the total probability of this happening, as a function of x, is:

$$\binom{N-1}{m} \left[\frac{x + \varepsilon - X}{2\varepsilon}\right]^m \left[\frac{X + \varepsilon - x}{2\varepsilon}\right]^{N-m-1}. \tag{20}$$

The derivation so far took x as known and given. We next take account of the fact that player i does not observe x directly but only the noisy signal x_i^{ε}. Given $p_{-i} = p_{-i}^X$ and $x_i^{\varepsilon} = X$, the expected incentive for player i to play $a_i = 1$ becomes:

$$\begin{aligned} u_i^{\varepsilon}(p_{-i}^X \mid X) = & \frac{1}{2\varepsilon} \int_{X-\varepsilon}^{X+\varepsilon} x \, \mathrm{d}x - c_i \\ & + \sum_{m=0}^{N-1} w_i(m) \binom{N-1}{m} \frac{1}{2\varepsilon} \int_{X-\varepsilon}^{X+\varepsilon} \left[\frac{x + \varepsilon - X}{2\varepsilon}\right]^m \left[\frac{X + \varepsilon - x}{2\varepsilon}\right]^{N-m-1} \mathrm{d}x \end{aligned} \tag{21}$$

$$=X - c_i + \sum_{m=0}^{N-1} w_i(m) \binom{N-1}{m} \int_0^1 q^m (1-q)^{N-m-1} \mathrm{d}q \tag{22}$$

$$=X - c_i + \sum_{m=0}^{N-1} w_i(m) \frac{(N-1)!}{m!\,(N-m-1)!} \frac{m!\,(N-m-1)!}{N!} \tag{23}$$

$$=X - c_i + \sum_{m=0}^{N-1} \frac{w_i(m)}{N}. \tag{24}$$

Equation (21) takes the expression for $u_i(m \mid x)$ given in (2) and integrates out x and m, given $x_i^\varepsilon = X$ and $p_{-i} = p_{-i}^X$. Equation (22) uses integration by substitution (using $q = 1/2 - (X-x)/2\varepsilon$) to rewrite the second integral in (21). Equation (23) rewrites both the integral in (22) and the binomial coefficient $\binom{N-1}{m}$ in terms of factorials. Equation (24) simplifies. Finally, we know that $u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s) = u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon) + s$. □

PROOF OF PROPOSITION 1

Proof. Given a vector of investment subsidies s, the payoff to investor i is

$$\pi_i(a \mid x) = \begin{cases} b_i + s_i - c_i & \text{if} \quad a_i = 1 \text{ and } A \geq I \\ s_i - c_i & \text{if} \quad a_i = 1 \text{ and } A < I \\ -x & \text{if} \quad a_i = 0, \end{cases}$$

and an investor's incentive to invest is given by

$$u_i(a_{-i} \mid x) = x + b_i \cdot \mathbb{1}_I(A) + s_i - c_i,$$

where $\mathbb{1}_I$ is the indicator function such that $\mathbb{1}_I(A_{-i}) = 1$ if $A \geq I$ and $\mathbb{1}_I(A) = 0$ otherwise. Because investors do not observe x but only a signal x_i^ε, the expected incentive to investment (given the subsidies s) is

$$u_i^\varepsilon(p_{-i} \mid x_i^\varepsilon, s_i) = \int b_i \cdot \mathbb{1}_I(\| \, p_{-i}(x_{-i}^\varepsilon) \, \|) + s_i - c_i + x \, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid x_i^\varepsilon),$$

where $\| \cdot \|$ denotes the norm of a vector.

If $\tilde{s}$ implements $p^{\tilde{x}}$, then by construction $\tilde{s}_i$ must solve

$$u_i^\varepsilon(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}_i) = b_i \cdot \int \mathbb{1}_I(\| \, p_{-i}(x_{-i}^\varepsilon) \, \|) \, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid \tilde{x}) + \tilde{s}_i - c_i + \tilde{x} = 0.$$

By Lemma 7, we know that

$$\int \mathbb{1}_I(\| \, p_{-i}(x_{-i}^\varepsilon) \, \|) \, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid \tilde{x}) = \sum_{n=1}^{N} \frac{\mathbb{1}_I(n)}{N} = \frac{n^*}{N}$$

since, by definition, $N - n^*$ is the smallest integer greater than I. Plugging $\int \mathbb{1}_I(\| \, p_{-i}(x_{-i}^\varepsilon) \, \|$ $) \, \mathrm{d}\Phi^\varepsilon(x, x_{-i}^\varepsilon \mid \tilde{x}) = \frac{n^*}{N}$ back into the condition that $u_i^\varepsilon(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{s}_i) = 0$ we obtain

$$\frac{n^*}{N} \cdot b_i + \tilde{s}_i + \tilde{x} - c_i = 0,$$

giving

$$\tilde{s}_i = c_i - \tilde{x} - \frac{n^*}{N} \cdot b_i.$$

Setting $\tilde{x} = 0$ gives the Proposition. □

PROOF OF PROPOSITION 2

Proof. Given a_{-i} and the reward scheme v, the payoff to agent i is given by:

$$\pi_i(a_i, a_{-i} \mid x, v_i) = \begin{cases} v_i - c_i & \text{if the project succeeds and } a_i = 1 \\ v_i - x & \text{if the project succeeds and } a_i = 0 \\ -c_i & \text{if the project does not succeed and } a_i = 1 \\ -x & \text{if the project does not succeed and } a_i = 0 \end{cases} \tag{25}$$

Since project success is stochastic and agents do not observe x, their (conditional) expected payoff is:

$$\pi_i^\varepsilon(a_i, a_{-i} \mid x_i^\varepsilon, v_i) = \begin{cases} q(A_{-i}+1) \cdot v_i - c_i & \text{if } a_i = 1 \\ q(A_{-i}) \cdot v_i - x_i^\varepsilon & \text{if } a_i = 0, \end{cases} \tag{26}$$

yielding her expected incentive to work:

$$u_i^\varepsilon(a_{-i} \mid x_i^\varepsilon, v_i) = (q(A_{-i}+1) - q(A_{-i})) \cdot v_i - c_i + x_i^\varepsilon. \tag{27}$$

The planner seeks to implement $p^{\tilde{x}}$. The bonus scheme $\tilde{v}$ implements $p^{\tilde{x}}$ iff

$$u_i^\varepsilon(p_{-i}^{\tilde{x}} \mid \tilde{x}, \tilde{v}_i) = \tilde{v}_i \int \left(q\left(p_{-i}^{\tilde{x}}(x_{-i}^\varepsilon) + 1\right) - q\left(p_{-i}^{\tilde{x}}(x_{-i}^\varepsilon)\right)\right) \, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid \tilde{x}) - c_i + \tilde{x} = 0, \tag{28}$$

for all $i \in \mathscr{N}$. Invoking Lemma 7, we know that

$$\int \left(q\left(p_{-i}^{\tilde{x}}(x_{-i}^\varepsilon) + 1\right) - q\left(p_{-i}^{\tilde{x}}(x_{-i}^\varepsilon)\right)\right) \, \mathrm{d}\Phi^\varepsilon(x_{-i}^\varepsilon \mid \tilde{x}) = \sum_{n=0}^{N-1} \frac{q(n+1) - q(n)}{N} := \overline{q}.$$

Therefore, $\tilde{v}_i$ solves

$$\overline{q} \cdot \tilde{v}_i - c_i + \tilde{x} = 0 \implies \overline{q} \cdot \tilde{v}_i = c_i - \tilde{x},$$

as given. □

PROOF OF PROPOSITION 3

Proof. Proof here. □

PROOF OF PROPOSITION 4

Proof. Consider player $i \in \mathscr{N}_1$ who receives a signal $x_i^\varepsilon = \tilde{x}_1$. It is known from Lemma 5 that $\tilde{s}$ implements $p^* = (p_1^{\tilde{x}_1}, p_2^{\tilde{x}_2})$ iff $\tilde{s}_i$ solves

$$u_i^\varepsilon(p_{-i}^* \mid \tilde{x}_1, \tilde{s}_i) = u_i^\varepsilon(p_{-i}^* \mid \tilde{x}_1) + \tilde{s}_i = 0.$$

Because, by assumption, $\tilde{x}_2 > \tilde{x}_1 + 2\varepsilon$, we also know that $x_j^\varepsilon < \tilde{x}_2$ for all $j \in \mathscr{N}$ and, in particular, for all $j \in \mathscr{N}_2$, the probability that $n > N_1 - 1$ players $j \neq i$ play 1 is 0 – i.e. there is no strategic uncertainty about the actions of players in $\mathscr{N}_2$.

Given $x \in [\tilde{x}_1 - \varepsilon, \tilde{x}_1 + \varepsilon]$, the probability that each player $j \in \mathscr{N}_1 \setminus i$ (who plays strategy $p_j = p^{\tilde{x}_1}$) plays $a_j = 1$ is given by

$$\Pr[x_j^\varepsilon > \tilde{x}_1 \mid x] = \frac{x + \varepsilon - \tilde{x}_1}{2\varepsilon},$$

and the total probability that exactly $n \in \{0, 1, ..., N_1 - 1\}$ players play 1 is:

$$\binom{N_1 - 1}{n} \left[\frac{x + \varepsilon - \tilde{x}_1}{2\varepsilon}\right]^n \left[\frac{\tilde{x}_1 + \varepsilon - x}{2\varepsilon}\right]^{N_1 - n - 1}.$$

Hence, following the exact steps as in the proof of Lemma 7 but replacing N for N_1, one has

$$u_i^\varepsilon(p_{-i}^* \mid \tilde{x}_1, \tilde{s}_i) = \tilde{x}_1 + \sum_{n=0}^{N_1 - 1} \frac{w_i(n)}{N_1} - c_i + \tilde{s}_i = 0.$$

Solving for $\tilde{s}_i$ yields the result for all $i \in \mathscr{N}_1$. A similar procedure can be carried our for players $i \in \mathscr{N}_2$. In their case, we note that upon observing signal $x_i^\varepsilon = \tilde{x}_2$ the probability that $x_j^\varepsilon \leq \tilde{x}_1$ for any $j \neq i$ is 0. In particular, therefore, $x_j^\varepsilon > \tilde{x}_1$ for all $j \in \mathscr{N}_1$ and, as all those j play $p_j = p^{\tilde{x}_1}$, it can be taken as given that they all play 1; the only remaining strategic uncertainty pertains to i's opponents in $\mathscr{N}_2$. $\square$

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