Two-Way Fixed Effects Difference-in-Differences with Time Differentials
Econ 672

The Takeaway

- The Two-Way Fixed Effects Difference-in-Differences (TWFEDD) estimator is a weighted group average of all potential 2x2 DD ATT estimates.
- When there is variation in treatment implementation or time differentials, the TWFEDD will provide a variance-weighted ATT, which is the weighted average of 2-by-2 ATT estimates based on sample size and variance in treatment.
- The Bacon Decomposition Theorem shows that you can get a biased estimate of the ATT when comparing late-treated to earlier-treated units.

Pros

- The TWFEDD can utilize variation in timing in the implementation of treatment, which is common for states adopting similar laws.
- The main assumption of TWFEDD with time differentials is called the variance-weighted common trends, which is not as strict as the parallel trends assumption with canonical 2-by-2 DD

Cons

- Like the 2-by-2 DD estimator, we can only estimate the ATT and not the ATE
- Like the 2-by-2 DD estimator, there are thorny issues
  - Heterogeneity treatment bias in time can still contaminate the estimate of ATT even if the main assumption of variance-weighted common trends is satisfied
  - Late-treated units compared to early-treated units can potentially create heterogeneity treatment bias in time

Assumptions

- Variance-Weighted Common Trends
  - VWCT is the main assumption of TWFEDD with time differentials. It is similar to parallel trends, but not as strict since weights can adjust the trends to zero
- Stable Unit Treatment Value Assumption (SUTVA)
  - There are no spillovers into the control group
- Treatment Effects are Homogenous in Time
  - The treatment effects of the early-treated units do not change over time

Testable Assumption

- We can indirectly test the variance-weighted common trends

The Estimator

- It is called a two-way fixed effects estimator because we utilize unit and time fixed effects in the model.
- $Y_{it} = \alpha_0 + \delta D_{it} + X_{it} + \alpha_i + \alpha_t + \epsilon_{it}$
  - α_i are unit fixed effects for unit i
  - α_t are time fixed effects for time t
  - δ is our parameter of interest

- D_{it} is out treatment variable for unit i at time t

Bacon Decomposition Theorem

- Assuming a balanced panel with T time periods and N cross-sectional Units
  - $\hat{\delta}^{DD} = \sum_{k \neq U} s_{kU} \hat{\delta}_{kU}^{2x2} + \sum_{k \neq U} \sum_{l>k} \left[s_{kl} \hat{\delta}_{kl}^{2x2,k} + s_{lk} \hat{\delta}_{kl}^{2x2,l} \right]$
  - Where weights
    - $s_{kU}, s_{kl}, and\ s_{lk}$ are the variance weights
    - The sum of the weights equals 1: $\sum_{k \neq U} s_{kU} + \sum_{k \neq U} \sum_{l>k} s_{kl} = 1$
    - Where the weights are a function of sample size and variance in treatment
    - $s_{kU} = \frac{(n_k + n_U)^2 n_{kU}(1 - n_{kU}) \bar{D}_k (1 - \bar{D}_k)}{\hat{V}^D}$
      - Where $\hat{V}_{kU}^D = n_{kU}(1 - n_{kU}) \bar{D}_k (1 - \bar{D}_k)$
    - $s_{kl}^k = \frac{\left((n_k + n_l)(1 - \bar{D}_l)\right)^2 n_{kl}(1 - n_{kl}) \frac{\bar{D}_k - \bar{D}_l}{1 - \bar{D}_l} \frac{1 - \bar{D}_k}{1 - \bar{D}_l}}{\hat{V}^D}$
      - Where $\hat{V}_{kl}^{D,k} = n_{kl}(1 - n_{kl}) \frac{\bar{D}_k - \bar{D}_l}{1 - \bar{D}_l} \frac{1 - \bar{D}_k}{1 - \bar{D}_l}$
    - $s_{kl}^l = \frac{\left((n_k + n_l)\bar{D}_k\right)^2 n_{kl}(1 - n_{kl}) \frac{\bar{D}_l}{\bar{D}_k} \frac{\bar{D}_k - \bar{D}_l}{\bar{D}_k}}{\hat{V}^D}$
      - Where $\hat{V}_{kl}^{D,l} = n_{kl}(1 - n_{kl}) \frac{\bar{D}_l}{\bar{D}_k} \frac{\bar{D}_k - \bar{D}_l}{\bar{D}_k}$
    - Where
      - n is the **sample share** for each timing group (n_k, n_l, n_U)
        - $n_k = \sum_i 1\{t_i = k\} / N$
      - $\bar{D}_k$ is the **time share** spent in treatment
        - $\bar{D}_k = \sum_t 1\{t \geq k\} / T$
  - Where ATTs
    - $\hat{\delta}_{kU}^{2x2}$ is the ATT between treated and never-treated units
    - $\hat{\delta}_{kl}^{2x2,k}$ is the ATT between early-treated and late-treated units
    - $\hat{\delta}_{kl}^{2x2,l}$ is the ATT between late-treated to already treated units
- As the sample size grows larger and with time fixed
  - $\operatorname{plim}_{N \to \infty} \hat{\beta}^{DD} = \beta^{DD} = VWATT + VWCT + \Delta ATT$
  - Where
    - VWATT is the variance-weighted ATT
      - $VWATT = \sum_{k \neq U} \sigma_{kU} ATT_k\big(Post(k)\big) + \sum_{k \neq U} \sum_{l>k} \left[\sigma_{kl}^k ATT_k\big(Mid(k,l)\big) + \sigma_{kl}^l ATT_l\big(Post(l)\big) \right]$
    - VWCT is the variance-weighted common trends (Selection Bias)
    - ΔATT is the weighted sum of the change in treatment effects within each time group's before and after treatment time (Heterogeneity Treatment Bias)
      - $\hat{\delta}_{kl}^{2x2,l}$ can have heterogeneity treatment bias due to changes in treatment effect for already-treated k is compared to l