Selection and parallel trends*

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Abstract

We study the connection between selection into treatment and the parallel trends assumptions underlying difference-in-differences (DiD) designs. Our framework accommodates general selection mechanisms, including selection on fixed effects, potential outcomes, treatment effects, and other economic models of selection. First, we derive necessary and sufficient conditions for the parallel trends assumption. These conditions theoretically clarify the empirical content of the parallel trends assumption and demonstrate trade-offs between restrictions on selection and the distribution of time-varying unobservables. Second, we provide a menu of interpretable primitive sufficient conditions, which constitute a formal framework for justifying DiD in practice. Third, we propose novel tools for selection-based sensitivity analyses when our necessary conditions for parallel trends are questionable. Finally, we show that for settings with time-varying covariates, typical conditional parallel trends assumptions imply combinations of time homogeneity and separability restrictions. We therefore provide sufficient conditions for a weaker conditional parallel trends assumption that accommodates a rich class of nonseparable models.

Keywords: causal inference, conditional parallel trends, covariates, difference-in-differences, selection mechanism, time-invariant and time-varying unobservables, treatment effects

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...while the new papers [in the DiD literature] clarify very well the statistical assumptions needed for estimation, effective use of these methods also requires being able to understand what the threats to these assumptions are in different contexts, and to make a plausible rhetorical argument as to why we should think the assumptions hold.
— David McKenzie, *World Bank Development Impact Blog* (McKenzie, 2022)

1 Introduction

Difference-in-differences (DiD) designs are widely used in practice to estimate causal effects. One of the perceived advantages of DiD is that it does not require explicit assumptions on how units select into treatment but instead relies on parallel trends assumptions. However, when justifying DiD in empirical applications, researchers often argue that the treatment is "quasi-randomly" assigned. Although these discussions allude to selection mechanisms, they are often not explicit about what constitutes "quasi-random" assignment, arguably due to the lack of formal guidance.

In this paper, we study parallel trends assumptions through the lens of selection mechanisms. We have four goals: (i) explicitly examine the role of selection into treatment in the context of the parallel trends assumption, (ii) offer practitioners a menu of theory-based templates for justifying parallel trends in applications, (iii) provide tools for sensitivity analyses that exploit contextual knowledge about selection, and (iv) study the role of time-varying covariates in DiD designs through the lens of selection. As a byproduct, our analysis provides foundations for comparing DiD to competing methods and developing new ones.

Consider the classical 2×2 DiD setup, where we observe N units over two time periods. In the first period, none of the units is treated; in the second period, some units select into treatment (treatment group), while others remain untreated (control group). Let $Y_{it}(0)$ denote the untreated potential outcome for unit $i = 1, \ldots, N$ in time period $t = 1, 2$. The identifying assumption of DiD is the parallel trends assumption. This assumption requires that the expected change across time in the untreated potential outcome, $Y_{it}(0)$, is identical in the treatment and control group,

$$E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1] = E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0],$$

where $G_i = 1$ indicates the treatment group, and $G_i = 0$ indicates the control group. We focus on the 2×2 case for expositional simplicity. Our results directly extend to general DiD designs with multiple groups and multiple periods; see Appendix C.2.

We consider a general nonseparable model for the untreated potential outcome,

$$Y_{it}(0) = \xi_t(\alpha_i, \varepsilon_{it}), \tag{1}$$

where $\xi_t(\cdot)$ is an arbitrary time-varying function, and α_i and ε_{it} are random vectors. The outcome model (1) does not impose any restrictions on $Y_{it}(0)$, but allows us to explicitly distinguish between selection on time-invariant unobservables, α_i, and time-varying unobservables, ε_{it}.

A unit's decision to select into the treatment can depend on the unobservable determinants of $Y_{it}(0)$ as well as additional time-invariant and time-varying unobservables $(\nu_i, \eta_{i1}, \eta_{i2})$,

$$G_i = g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}). \tag{2}$$

The selection mechanism in Equation (2) accommodates selection on time-invariant unobservables ("fixed effects"), selection on untreated potential outcomes, selection on treatment effects (Roy-style selection), and other economic models of selection.

Our first contribution is to provide necessary and sufficient conditions for parallel trends, which clarify the empirical content of this assumption. We first consider a scenario where researchers are not willing to restrict the selection mechanism.[1] We show that absent any restrictions on selection, parallel trends holds if and only if $Y_{i2}(0) - E[Y_{i2}(0)] = Y_{i1}(0) - E[Y_{i1}(0)]$. If the outcome model is separable, $Y_{it}(0) = \alpha_i + \lambda_t + \varepsilon_{it}$, the necessary and sufficient condition becomes $\varepsilon_{i2} = \varepsilon_{i1}$. Thus, absent restrictions on selection, parallel trends rules out time-varying shocks, requiring that the untreated potential outcomes are constant across time up to location shifts, $\lambda_2 - \lambda_1$, which is not plausible in most applications.

This negative result motivates restricting the selection mechanism. We derive necessary conditions for parallel trends under two types of restrictions that can be motivated based on classical examples of selection as well as the information sets available to units at the time of the decision. First, if the units only have information from the pre-treatment period, including α_i and ε_{i1}, and selection therefore cannot depend on post-treatment shocks (*imperfect foresight*), parallel trends implies a martingale-type property on $Y_{it}(0) - E[Y_{it}(0)]$. Second, if the units select into treatment based on *fixed effects* so that selection does not depend on time-varying unobservables, parallel trends implies time homogeneity of $E[Y_{it}(0)|\alpha_i] - E[Y_{it}(0)]$. Under additional assumptions, these two necessary conditions are also sufficient for parallel trends.

Taken together, the necessary and sufficient conditions imply that researchers relying on

[1] In Appendix C.1, we provide necessary and sufficient conditions under an alternative scenario where researchers are not willing to impose any restrictions on the distribution of unobservables.

parallel trends assumptions face a trade-off between restrictions on selection into treatment and restrictions on the time-series properties of the outcomes. Given this trade-off, we offer a menu of primitive sufficient conditions that provide practitioners with theory-based templates for justifying parallel trends assumptions based on contextual knowledge about selection, such as what units select on and what information sets are available to them at the time of the selection decision.[2]

Our necessary and sufficient conditions motivate novel selection-based sensitivity analyses. Suppose, for example, that the units have imperfect foresight. In this case, as we show, martingale assumptions on the untreated potential outcomes are necessary for the parallel trends assumption. Such assumptions may be restrictive in applications, and if they are violated, DiD is biased for the average treatment effect on the treated (ATT). We derive bounds on the ATT under violations of these martingale assumptions for settings with *and* without additional pre-treatment periods. These bounds allow researchers to leverage contextual knowledge about selection when performing sensitivity analyses and constructing confidence intervals that are robust to violations of the relevant necessary conditions for parallel trends.

We then examine the role of (time-varying) covariates in DiD analyses.[3] Our necessary and sufficient conditions generalize directly to settings with covariates. They demonstrate that parallel trends assumptions that condition on the trajectory of covariates imply combinations of time homogeneity and separability restrictions on how the covariates enter the outcome model, even when selection only depends on time-invariant unobservables. We therefore consider a weaker conditional parallel trends assumption, designed specifically to accommodate nonseparability between observables and unobservables in the outcome model. We provide a menu of sufficient conditions for this weaker conditional parallel trends assumption and establish connections between these selection-based conditions and identification assumptions in the literature on nonseparable panel data models.

This paper contributes to several branches of the literature on causal inference using panel data. Our first contribution is to the classical literature on canonical DiD setups. See, e.g., Ashenfelter (1978), Ashenfelter and Card (1985), Heckman and Robb (1985), Card (1990), Card and Krueger (1994), Meyer et al. (1995), and Angrist and Krueger (1999) for early developments, and Section 2 of Lechner (2010) for a historical perspective. Our contribution is to provide foundations for the parallel trends assumption to hold in non-experimental settings, where selection into treatment may depend on time-invariant and time-varying unobservables.

[2]For example, Arellano et al. (2022) document heterogeneity in the information available to individuals regarding their future incomes.

[3]We assume that covariates are not affected by the treatment. See Caetano et al. (2022) for some recent results relaxing this assumption.

Our second contribution is to the more recent literature on DiD methods. See, e.g., de Chaisemartin and D'Haultfœuille (2021) and Roth et al. (2022) for surveys. Our paper is most closely related to Roth and Sant'Anna (2021), Arkhangelsky et al. (2021), and Arkhangelsky and Imbens (2022), though our focus greatly differs from theirs. Roth and Sant'Anna (2021) discuss necessary and sufficient conditions under which the parallel trends assumption is satisfied for all (monotonic) transformations of the untreated potential outcome. We, on the other hand, take the outcome model (and thus the specific transformation) as given and study the connection between parallel trends and selection into treatment. Arkhangelsky et al. (2021) and Arkhangelsky and Imbens (2022) propose doubly robust estimation methods that leverage restrictions on outcome models and/or selection models with unconfoundedness-type restrictions; see also Athey et al. (2021). Our results complement theirs as we maintain the parallel trends assumption and discuss the types of restrictions on selection compatible with it. Moreover, our analysis shows that parallel trends is compatible with various types of selection on unobservables, unlike standard unconfoundedness assumptions (e.g., Imbens, 2004; Imbens and Wooldridge, 2009).

Our third contribution is to the literature proposing bounds and robust inference procedures under violations of parallel trends. Our approach differs from the methods in Manski and Pepper (2018), Ban and Kédagni (2023), and Rambachan and Roth (2023) in that we explicitly exploit assumptions on selection into treatment to obtain informative bounds on the ATT with or without data from additional pre-treatment periods. Moreover, even when such data are available, the way we exploit this additional information departs from these papers. We specifically use the additional pre-treatment periods to learn about the time-series properties of the outcomes, instead of directly making assumptions about how the parallel trends violation changes over time. For these reasons, our bounding strategy complements these existing approaches. Our selection-based approach to sensitivity analysis and robust inference also differs from the analysis by Marx et al. (2022). They derive partial identification results under monotone treatment selection assumptions on the untreated potential outcome, which they motivate using an economic model of learning with binary outcomes. By contrast, we directly exploit necessary conditions for parallel trends under restrictions on the selection mechanism.

Our fourth contribution is to the literature imposing explicit selection and/or outcome models to develop and compare different methods for estimating treatment effects, including DiD (e.g., Ashenfelter and Card, 1985; Heckman and Robb, 1985; Card and Hyslop, 2005; Chabé-Ferret, 2015; Blundell and Costa Dias, 2009; de Chaisemartin and D'Haultfœuille, 2018; Verdier, 2020; Marx et al., 2022). These selection mechanisms were developed for economic models, some of which are tailored to applications such as job training and technology adoption. Our results complement this strand of the literature in several ways. First, our necessary and sufficient conditions are derived for general selection and outcome models that nest models considered in this literature. Our conditions thus clarify trade-offs between assumptions on selection and time-varying unobservables that are relevant for those models. Second, our primitive sufficient conditions nest several of the existing application-specific restrictions. Third, we provide results for general nonseparable models and clarify the role of covariates in the context of parallel trends assumptions. It is worth noting that while most papers in this literature examine sharp DiD designs, as we do, de Chaisemartin and D'Haultfœuille (2018) and Marx et al. (2022) also consider fuzzy DiD designs.

Finally, we establish an explicit connection between DiD and the literature on nonseparable panel models.[4] A strand of this literature has analyzed the identification of average effects either by allowing for fixed effects and imposing time homogeneity (e.g. Hoderlein and White, 2012; Chernozhukov et al., 2013) or restricting individual heterogeneity via nonparametric correlated random effects assumptions (e.g. Altonji and Matzkin, 2005; Bester and Hansen, 2009). We show that our sufficient conditions for parallel trends imply combinations of time homogeneity and (correlated) random effects restrictions. Our results demonstrate how restrictions on the selection mechanism can be used to justify identification assumptions in the nonseparable panel literature.

Notation. For a random vector W_{it}, where $i = 1, \dots, N$ and $t = 1, 2$, we denote its time series by $W_i \equiv (W_{i1}, W_{i2})$.[5] We use F_W to denote the distribution of the random vector W. Let $f(z, w)$ be a function defined on $\mathcal{Z} \times \mathcal{W}$. We say that $f(z, w)$ is a trivial function of w if $f(z, w) = f(z, w') = h(z)$ for all $z \in \mathcal{Z}$, $w \neq w'$, and $(w, w') \in \mathcal{W}^2$. We say that $f(z, w)$ is a symmetric function in z and w if $f(z, w) = f(w, z)$ for all $(z, w) \in \mathcal{Z} \times \mathcal{W}$. For a vector W_i, W_i^j is the j^{th} element of W_i. We use the notation $\overset{d}{=}$ to denote equality of distribution. For random variables, X_i, Z_i, and W_i, $Z_i|W_i, X_i \overset{d}{=} Z_i|X_i, W_i$ denotes that $F_{Z_i|W_i,X_i}(z|w, x) = F_{Z_i|X_i,W_i}(z|w, x)$ for $(z, w, x) \in \mathcal{Z} \times \mathcal{W} \times \mathcal{X}$.

[4]See, e.g., Altonji and Matzkin (2005); Athey and Imbens (2006); Bester and Hansen (2009); Hoderlein and White (2012); Chernozhukov et al. (2013); Arellano and Bonhomme (2016); Ghanem (2017). This work extends notions of fixed effects and correlated random effects that originated in the linear model (Mundlak, 1961, 1978; Chamberlain, 1982, 1984). Recent surveys (Arellano and Honoré, 2001; Arellano and Bonhomme, 2011) and textbook treatments (Arellano, 2003; Wooldridge, 2010) further describe the role of restrictions on time and individual heterogeneity in linear and nonlinear models. Such restrictions have been imposed in the context of identification in limited dependent variable models (e.g. Manski, 1987; Honoré, 1993; Kyriazidou, 1997; Honoré and Kyriazidou, 2000a,b) and random coefficient models (e.g. Chamberlain, 1992; Graham and Powell, 2012; Arellano and Bonhomme, 2012). Nonparametric identification of panel models with additivity restrictions has been examined, e.g., in Evdokimov (2010) and Freyberger (2017).

[5]We define all vectors in this paper as row vectors.

2 Setup, selection mechanism, and examples

We consider the classical DiD setup with two groups and two periods and abstract from covariates. We discuss the role of covariates in Section 5 and generalize our results to DiD designs with multiple groups and multiple periods in Appendix C.2. Let D_{it} and Y_{it} denote the treatment status and outcome for unit i in period t. Here the index i refers to the unit making the decision to select into treatment. This could be an individual or a more aggregate administrative unit, such as county or state. The treatment group ($G_i = 1$) selects the treatment path $D_i = (0, 1)$; the control group ($G_i = 0$) selects $D_i = (0, 0)$. The potential outcomes with and without the treatment are $Y_{it}(1)$ and $Y_{it}(0)$, respectively.[6]

We consider the standard parallel trends assumption. Throughout the paper, we assume that all relevant moments exist.

Assumption PT. *The (unconditional) parallel trends assumption holds:*

$$E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1] = E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0].$$

Under Assumption PT, the average treatment effect on the treated group in period $t = 2$, $\text{ATT} \equiv E[Y_{i2}(1) - Y_{i2}(0)|G_i = 1]$, is identified from the "difference-in-differences" as follows

$$\text{ATT} = E[Y_{i2} - Y_{i1}|G_i = 1] - E[Y_{i2} - Y_{i1}|G_i = 0] \equiv \text{DiD}\,.$$

We work with a general nonseparable model for $Y_{it}(0)$,

$$Y_{it}(0) = \xi_t(\alpha_i, \varepsilon_{it}), \quad i = 1, \ldots, N, \quad t = 1, 2, \tag{3}$$

where α_i, ε_{i1}, and ε_{i2} are finite-dimensional vector-valued random variables, and $\xi_t(\cdot)$ is an unrestricted time-varying function. The outcome model (3), while not imposing any restrictions on $Y_{it}(0)$, allows us to distinguish between time-invariant and time-varying unobservables. This is necessary to define selection mechanisms that can directly depend on these unobservables. If, instead, we were to work directly with potential outcomes, this would rule out important examples of selection mechanisms such as selection on time-invariant unobservables (e.g., Ashenfelter and Card, 1985).

We consider a general class of selection mechanisms in which units select into treatment based on $(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$ as well as an additional vector of time-invariant and time-varying

[6] We assume that the units do not anticipate their treatment. As a result, at $t = 1$ we observe $Y_{i1}(0)$ for all units, while at $t = 2$ we observe $Y_{i2}(1)$ for treated and $Y_{i2}(0)$ for untreated units.

random variables, $(\nu_i, \eta_{i1}, \eta_{i2})$,

$$G_i = g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}). \tag{4}$$

This selection mechanism accommodates many different types of selection, including random assignment, selection on fixed effects, selection on untreated potential outcomes, selection on treatment effects, and other economic models of selection (e.g. Heckman and Robb, 1985; Chabé-Ferret, 2015; Marx et al., 2022). Note that since $G_i = D_{i2}$, $g(\cdot)$ can equivalently be viewed as the selection mechanism for D_{i2}. Let $\mathcal{G}_{\text{all}}$ denote the set of all selection mechanisms $g(\cdot)$, mapping from the support of the unobservables to $\{0, 1\}$.

Throughout the paper, we will come back to the following three leading examples of selection, specifically selection on outcomes, on treatment effects, and on fixed effects.

Example 1 (Selection on outcomes)**.** *We consider a class of threshold-crossing selection mechanisms, generalizing the selection mechanisms analyzed in Ashenfelter and Card (1985), who study the effect of training programs on earnings. Let ω_i denote the information set available to the units when deciding whether to participate in the training program and consider the following mechanism,*

$$G_i = 1\left\{E[Y_{i1}(0) + \beta Y_{i2}(0)|\omega_i] \leq E[C_{i2}|\omega_i]\right\}, \tag{5}$$

where $\beta \in [0, 1]$ is a discount factor, G_i indicates participation in a job training program, $Y_{it}(0)$ denotes untreated potential earnings, C_{i2} is the individual-specific cost of participation, which is assumed to be an element of η_{i2}. The selection mechanism (5) can be expressed as $G_i = \check{g}(\omega_i)$ and is therefore a special case of the mechanism (4) if ω_i is a subvector of $(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2})$. □

Example 2 (Selection on treatment effects (Roy-style selection))**.** *Suppose that units select into the treatment if the expected gains from treatment given the information set ω_i, $E[Y_{i2}(1) - Y_{i2}(0)|\omega_i]$, exceed the expected cost of treatment, $E[C_{i2}|\omega_i]$,*

$$G_i = 1\{E[Y_{i2}(1) - Y_{i2}(0)|\omega_i] \geq E[C_{i2}|\omega_i]\}. \tag{6}$$

The selection mechanism (6) is again a special case of mechanism (4) if ω_i is a subvector of $(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2})$. This example shows that it is important to allow $g(\cdot)$ to depend on a vector of additional unobservables, such that we can allow η_{i2} (and thereby the information set) to include $(Y_{i2}(1), C_{i2})$. □

Example 3 (Selection on fixed effects)**.** *DiD methods have traditionally been motivated using two-way fixed effects models. Fixed effects assumptions allow for unrestricted dependence between time-invariant unobservables and the regressors, thereby implicitly allowing for selection on time-invariant unobservables.*[7] *The general selection mechanism (4) accommodates this classical type of selection if* $g(\cdot)$ *is a trivial function of* $(\varepsilon_{i1}, \varepsilon_{i2}, \eta_{i1}, \eta_{i2})$*. A simple example is* $G_i = 1\{\alpha_i \leq c\}$*, which corresponds to the selection mechanism on p.650 in Ashenfelter and Card (1985).* □

Remark 1 (Parallel trends and functional form)**.** *Throughout this paper, we take the functional form of the outcome as given. We thereby abstract from the issues arising from the sensitivity of DiD to functional form specification; see Roth and Sant'Anna (2021) for a discussion.* □

3 Necessary and sufficient conditions for parallel trends

3.1 No restrictions on selection

To better understand the implications of parallel trends, we derive necessary and sufficient conditions for this assumption. We start by analyzing a scenario where researchers are not willing to make any assumptions on the selection mechanism so that parallel trends needs to hold for all selection mechanisms.

To ensure non-degeneracy of the selection mechanisms we use to derive necessary and sufficient conditions for parallel trends, we impose the following weak regularity condition.

Assumption SEL. *There exists a component of* ν_i*, labeled* ν_i^1 *(w.l.o.g.), such that* $\nu_i^1 \perp (\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$ *and* $P(\nu_i^1 > c) \in (0, 1)$ *for some* $c \in \mathbb{R}$*.*

The following proposition presents a necessary and sufficient condition for parallel trends holding for all selection mechanisms. To simplify exposition, we use $\dot{Y}_{it}(0)$ to denote the centered potential outcome without the treatment, $\dot{Y}_{it}(0) \equiv Y_{it}(0) - E[Y_{it}(0)]$ for $t = 1, 2$.

Proposition 3.1 (Necessary and sufficient condition for $g \in \mathcal{G}_{\text{all}}$)**.** *Suppose that Assumption SEL holds and either* $P(\dot{Y}_{i2}(0) > \dot{Y}_{i1}(0)) < 1$ *or* $P(\dot{Y}_{i2}(0) < \dot{Y}_{i1}(0)) < 1$*. Then, Assumption PT holds for all* $g \in \mathcal{G}_{all}$ *satisfying* $P(G_i = 1) \in (0, 1)$ *if and only if* $\dot{Y}_{i1}(0) = \dot{Y}_{i2}(0)$ *a.s.*

It is helpful to interpret the necessary and sufficient condition in Proposition 3.1 under a simple linear two-way model for $Y_{it}(0)$,

$$Y_{it}(0) = \alpha_i + \lambda_t + \varepsilon_{it}, \quad E[\varepsilon_{it}] = 0. \tag{7}$$

[7] See, e.g., Chamberlain (1984); Arellano (2003); Evdokimov (2010); Wooldridge (2010); Hoderlein and White (2012); Chernozhukov et al. (2013).

Under model (7), the necessary condition simplifies to $\varepsilon_{i1} = \varepsilon_{i2}$. That is, Assumption PT holds for all selection mechanisms $g(\cdot)$ if and only if the time-varying unobservables are in fact time-invariant. Put simply, if one were to allow for an *unrestricted* selection mechanism, one would need to rule out time-varying shocks. Given that this condition is implausible in many applications, we consider restricted versions of the selection mechanism in Section 3.2.

The above discussion raises the question: Can we allow for selection to depend on both ε_{i1} and ε_{i2}, albeit in a restricted way, while allowing $Y_{it}(0)$ to vary across time beyond location shifts? In Section 3.3, we show that the answer to this question is yes. Specifically, we provide a set of sufficient conditions that allows selection to symmetrically depend on both ε_{i1} and ε_{i2} (see Assumption SC1).

3.2 Necessary conditions for restricted selection mechanisms

Motivated by Proposition 3.1, we consider two restricted classes of selection mechanisms. These classes of mechanisms are directly related to and motivated by the information sets available to the units when making the decision to select into the treatment.

First, we examine a class of selection mechanisms in which individuals have *imperfect foresight* so that selection depends on the time-invariant and pre-treatment unobservables,

$$\mathcal{G}_{\text{if}} = \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, e_2, v, t_1, t_2) \text{ is a trivial function of } (e_2, t_2)\}.$$

In Example 1, $\mathcal{G}_{\text{if}}$ captures settings where individuals know their permanent income component, α_i, and the pre-treatment idiosyncratic earnings shock, ε_{i1}, but not unobservables from the post-treatment period, specifically ε_{i2} and C_{i2}. For empirical evidence on the heterogeneity in income uncertainty faced by different individuals, see, e.g., Arellano et al. (2022). In Example 2, assuming that $g \in \mathcal{G}_{\text{if}}$ requires that individuals do not know their treatment effects, $Y_{i2}(1) - Y_{i2}(0)$, and costs, C_{i2}, while their information set can contain all time-invariant and pre-treatment unobservables $(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})$.

Second, we consider a class of mechanisms where selection only depends on the *fixed effects* (α_i, ν_i),

$$\mathcal{G}_{\text{fe}} = \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, e_2, v, t_1, t_2) \text{ is a trivial function of } (e_1, e_2, t_1, t_2)\}.$$

The class of selection mechanisms $\mathcal{G}_{\text{fe}}$ captures the classical selection on fixed effects scenario. Assuming that $g \in \mathcal{G}_{\text{fe}}$ is plausible if either the units' information set only contains the time-invariant unobservables in Examples 1 and 2, so that $\omega_i = (\alpha_i, \nu_i)$, or if selection is directly based on fixed effects as in Example 3.

The next two propositions provide necessary conditions for parallel trends when the selection mechanism belongs to $\mathcal{G}_{\text{if}}$ and $\mathcal{G}_{\text{fe}}$, respectively.

Proposition 3.2 (Necessary condition for $g \in \mathcal{G}_{\text{if}}$)**.** *Suppose that Assumption SEL holds and either* $P(E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] > \dot{Y}_{i1}(0)) < 1$ *or* $P(E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] < \dot{Y}_{i1}(0)) < 1$*. If Assumption PT holds for all* $g \in \mathcal{G}_{if}$ *satisfying* $P(G_i = 1) \in (0, 1)$*, then* $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] = \dot{Y}_{i1}(0)$ *a.s.*

Proposition 3.3 (Necessary condition for $g \in \mathcal{G}_{\text{fe}}$)**.** *Suppose that Assumption SEL holds and either* $P(E[\dot{Y}_{i2}(0)|\alpha_i] > E[\dot{Y}_{i1}(0)|\alpha_i]) < 1$ *or* $P(E[\dot{Y}_{i2}(0)|\alpha_i] < E[\dot{Y}_{i1}(0)|\alpha_i]) < 1$*. If Assumption PT holds for all* $g \in \mathcal{G}_{fe}$ *satisfying* $P(G_i = 1) \in (0, 1)$*, then* $E[\dot{Y}_{i1}(0)|\alpha_i] = E[\dot{Y}_{i2}(0)|\alpha_i]$ *a.s.*

The two propositions demonstrate that while parallel trends is compatible with the presence of time-varying unobservables under the restricted classes of selection mechanisms, it implies time series restrictions on $\dot{Y}_{it}(0)$. Under the separable model (7), the necessary condition in Proposition 3.2 becomes $E[\varepsilon_{i2}|\alpha_i, \varepsilon_{i1}] = \varepsilon_{i1}$, a martingale-type property that implies $\varepsilon_{i2} = \varepsilon_{i1} + \zeta_{i2}$, where ζ_{i2} is an innovation satisfying $E[\zeta_{i2}|\varepsilon_{i1}] = 0$.[8] The necessary condition in Proposition 3.3 simplifies to $E[\varepsilon_{i2}|\alpha_i] = E[\varepsilon_{i1}|\alpha_i]$, a time homogeneity assumption on the conditional mean. In general, the stability of the conditional mean, is implied by (and weaker than) the textbook strict exogeneity assumption, $E[\varepsilon_{it}|G_i, \alpha_i] = 0$, since in our framework selection may depend on additional unobservables $(\nu_i, \eta_{i1}, \eta_{i2})$.

The necessary conditions in Propositions 3.2 and 3.3 do not imply parallel trends in general due to the presence of the additional unobservables $(\nu_i, \eta_{i1}, \eta_{i2})$. The following proposition provides simple sufficient conditions in terms of the conditional distribution of $(\nu_i, \eta_{i1}, \eta_{i2})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}$, under which these necessary conditions are also sufficient.

Proposition 3.4 (Sufficient conditions)**.** *Suppose that* $P(G_i = 1) \in (0, 1)$*.*

(i) Suppose that $g \in \mathcal{G}_{if}$*. If* $(\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \overset{d}{=} (\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}$*, then* $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] = \dot{Y}_{i1}(0)$ *implies Assumption PT.*

(ii) Suppose that $g \in \mathcal{G}_{fe}$*. If* $\nu_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \overset{d}{=} \nu_i|\alpha_i$*, then* $E[\dot{Y}_{i1}(0)|\alpha_i] = E[\dot{Y}_{i2}(0)|\alpha_i]$ *implies Assumption PT.*

Taken together, our necessary conditions show that Assumption PT cannot hold absent additional restrictions on the selection mechanism and/or the distribution of unobservables.

[8] The result in Proposition 3.2 relates to the consistency of the first-differences estimator under violations of strict exogeneity when the idiosyncratic shocks follow a unit root. In fact, under sequential exogeneity, selection into treatment depends on the lagged outcome and the time-invariant unobservable such that $G_i = g(\alpha_i, \varepsilon_{i1})$ (Chamberlain, 2022) and, thus, $E[\varepsilon_{i2}|\alpha_i, \varepsilon_{i1}, G_i] = E[\varepsilon_{i2}|\alpha_i, \varepsilon_{i1}]$. If, in addition, $E[\varepsilon_{i2}|\alpha_i, \varepsilon_{i1}] = \varepsilon_{i1}$, then it follows that $E[\varepsilon_{i2} - \varepsilon_{i1}|\alpha_i, \varepsilon_{i1}, G_i] = 0$, which implies that $E[\varepsilon_{i2} - \varepsilon_{i1}|G_i] = E[\varepsilon_{i2} - \varepsilon_{i1}]$ and thus Assumption PT in the separable model (7). We thank Stéphane Bonhomme for pointing out this connection.

In particular, these results highlight the role of restrictions on time-varying unobservables, either in terms of how they vary over time or how they determine selection. As a result, researchers using DiD approaches cannot avoid making meaningful and nontrivial assumptions on selection and time-varying unobservables.

3.3 Justifying parallel trends using contextual knowledge about selection

The results in the previous sections illustrate that restrictions on time-varying unobservables are necessary for parallel trends to hold. Here we discuss three sets of sufficient conditions that practitioners can use to justify parallel trends in empirical applications, depending on the assumptions they are willing to impose on the selection mechanism. The exact form of these sufficient conditions depends on the model for the potential outcome in the absence of the treatment. Here, we present the conditions for the separable two-way model (7). See Section 5.2 for sufficient conditions for general nonseparable models.

The first sufficient condition demonstrates a case where selection can depend on both ε_{i1} and ε_{i2}, *and* the untreated potential outcomes can vary across time beyond location shifts. Define the class of symmetric selection mechanisms as

$$\mathcal{G}_{\text{sym}} = \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, e_2, v, t_1, t_2) \text{ is a symmetric function in } e_1 \text{ and } e_2\}.$$

Assumption SC1. *The following conditions hold: (i)* $g \in \mathcal{G}_{sym}$*, (ii)* $\varepsilon_{i1}, \varepsilon_{i2}|\alpha_i \overset{d}{=} \varepsilon_{i2}, \varepsilon_{i1}|\alpha_i$*, and (iii)* $(\nu_i, \eta_{i1}, \eta_{i2})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \overset{d}{=} (\nu_i, \eta_{i1}, \eta_{i2})|\alpha_i, \varepsilon_{i2}, \varepsilon_{i1}$.

In addition to symmetry of the selection mechanism, Assumption SC1 imposes two different types of exchangeability restrictions. First, it requires that the conditional distribution of $(\nu_i, \eta_{i1}, \eta_{i2})$ is exchangeable in ε_{i1} and ε_{i2} after conditioning on α_i. This notion of exchangeability has been employed, for example, in Altonji and Matzkin (2005). Second, it requires the distribution of $(\varepsilon_{i1}, \varepsilon_{i2})$ to be exchangeable conditional on α_i.

The next two sufficient conditions directly build on Propositions 3.2, 3.3, and 3.4.

Assumption SC2. *The following conditions hold: (i)* $g \in \mathcal{G}_{if}$*, (ii)* $E[\varepsilon_{i2}|\alpha_i, \varepsilon_{i1}] = \varepsilon_{i1}$*, and (iii)* $(\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \overset{d}{=} (\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}$.

Assumption SC3. *The following conditions hold: (i)* $g \in \mathcal{G}_{fe}$*, (ii)* $E[\varepsilon_{i1}|\alpha_i] = E[\varepsilon_{i2}|\alpha_i]$*, and (iii)* $\nu_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \overset{d}{=} \nu_i|\alpha_i$.

The following proposition formally establishes the sufficiency of Assumptions SC1, SC2, and SC3.

Proposition 3.5 (Sufficient conditions for parallel trends). *Suppose that the model for $Y_{it}(0)$ is (7) and $P(G_i = 1) \in (0, 1)$. Then (i) Assumption SC1 implies Assumption PT, (ii) Assumption SC2 implies Assumption PT, and (iii) Assumption SC3 implies Assumption PT.*

The sufficient conditions SC1, SC2, and SC3 provide practitioners with explicit theory-based templates for justifying parallel trends assumptions. These conditions can be used, for example, in conjunction with the selection mechanisms in Examples 1, 2, and 3. In Section 6, we discuss their practical implications.

3.4 Extensions

In this section, we discuss two extensions. We summarize the main results here and refer to Appendix C for details.

3.4.1 Parallel trends for any distribution

In Appendix C.1, we provide necessary and sufficient conditions for an alternative scenario where researchers are not willing to restrict the distribution of unobservables. Specifically, suppose researchers want parallel trends to hold for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}$, where $\mathcal{F}$ is a class of distributions.

We show that if $\mathcal{F}$ is a complete class of distributions, then Assumption PT holds for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}$ if and only if

$$P(G_i = 1|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) = P(G_i = 1) \text{ a.s. for all } F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}.$$

That is, parallel trends (holding for any distribution of unobservables) is equivalent to selection being independent of all the unobservable determinants of the untreated potential outcome. Intuitively, completeness of $\mathcal{F}$, which is formally defined in Definition C.1, requires that the class of possible distributions of unobservables is "rich enough." This condition is trivially satisfied if $\mathcal{F}$ is completely unrestricted.

3.4.2 Multiple periods and multiple groups

In Appendix C.2, we extend our results to DiD designs with multiple periods and multiple groups.[9] Specifically, we consider a staggered adoption setting with T periods, where no units are treated at $t = 1$ and some units remain untreated at $t = T$. The group indicator

[9] Our setup and notation build on Callaway and Sant'Anna (2021), Sun and Abraham (2021), and Roth et al. (2022).

G_i denotes the first period in which units select into the treatment. We set $G_i = \infty$ for the never-treated units so that $G_i \in \{2, \dots, T, \infty\}$.

We consider the following standard parallel trends assumption on the never-treated potential outcome $Y_{it}(\infty)$,

$$E[Y_{it}(\infty) - Y_{i(t-1)}(\infty) \,|\, G_i = g] = E[Y_{it}(\infty) - Y_{i(t-1)}(\infty)|G_i = \infty] \quad \text{for all } (g, t). \tag{8}$$

Under this assumption and a no-anticipation condition, group-time ATTs are identified (e.g., Callaway and Sant'Anna, 2021).

Selection into treatment can depend on the unobservable determinants of $Y_{it}(\infty)$ as well as additional unobservables, $(\nu_i, \eta_{i1}, \dots, \eta_{iT})$,

$$G_i = g(\alpha_i, \varepsilon_{i1}, \dots, \varepsilon_{iT}, \nu_i, \eta_{i1}, \dots, \eta_{iT}).$$

We generalize Propositions 3.1, 3.2, and 3.3 to the multiple-group, multiple-period case. As in the 2×2 case, we denote the set of all selection mechanisms by $\mathcal{G}_{\text{all}}$ and consider the following restricted classes of selection mechanisms,

$$\begin{aligned}
\mathcal{G}_{\text{if}} &= \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, \dots, e_T, v, t_1, \dots, t_T) \text{ is a trivial function of } (e_T, t_T)\}, \\
\mathcal{G}_{\text{if'}} &= \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, \dots, e_T, v, t_1, \dots, t_T) \text{ is a trivial function of } (e_2, \dots, e_T, t_2, \dots, t_T)\}, \\
\mathcal{G}_{\text{fe}} &= \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, \dots, e_T, v, t_1, \dots, t_T) \text{ is a trivial function of } (e_1, \dots, e_T, t_1, \dots, t_T)\},
\end{aligned}$$

where $\mathcal{G}_{\text{if}}$ and $\mathcal{G}_{\text{if}'}$ are different extensions of $\mathcal{G}_{\text{if}}$ in the 2×2 case.

We give four necessary conditions for the parallel trends assumption in Equation (8). First, parallel trends holds for all $g \in \mathcal{G}_{\text{all}}$ if and only if $\dot{Y}_{i1}(\infty) = \cdots = \dot{Y}_{iT}(\infty)$. Second, parallel trends for all $g \in \mathcal{G}_{\text{if}}$ implies that $E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}, \dots, \varepsilon_{i(t-1)}] = \dot{Y}_{i(t-1)}(\infty)$ for $t \geq 2$. Third, parallel trends for all $g \in \mathcal{G}_{\text{if}'}$ implies that $E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}] = E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i, \varepsilon_{i1}]$ for $t \geq 2$. Finally, parallel trends for all $g \in \mathcal{G}_{\text{fe}}$ implies that $E[\dot{Y}_{it}(\infty)|\alpha_i] = E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i]$ for $t \geq 2$. All these conditions can be viewed as natural generalizations of the results for the 2×2 case.

4 Selection-based sensitivity analyses

Here we build on the theoretical results in Section 3 to derive bounds for the ATT in applications where parallel trends is questionable.

To motivate our approach, we decompose the DiD estimand as[10]

$$\begin{aligned}\text{DiD} &= \text{ATT} + E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|G_i = 1] - E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|G_i = 0] \\ &\equiv \text{ATT} + \Delta_{\text{post}} \qquad (9)\end{aligned}$$

This decomposition shows that the DiD estimand is equal to the sum of the ATT and the bias term Δ_{post}, which captures the bias due to violations of Assumption PT. We provide an approach for bounding Δ_{post} and thus the ATT under assumptions on selection. We consider a setting with an additional pre-treatment period $t = 0$ in which no units are treated so that $Y_{i0} = Y_{i0}(0)$ for $i = 1, \ldots, N$.

Selection on potential outcomes is a major concern in many empirical applications. For example, individuals might selection into job training programs if their earnings fall below a certain cutoff (Ashenfelter and Card, 1985). The necessary conditions in Section 3 show that parallel trends implies strong restrictions on the time series properties of $\dot{Y}_{it}(0)$ when selection is based on outcomes. To show how restrictions on selection can be used to bound the ATT, we focus on settings where it is plausible that units have imperfect foresight when selecting into treatment.

We allow selection to also depend on the shocks in period $t = 0$,

$$G_i = g(\alpha_i, \varepsilon_i^1, \varepsilon_{i2}, \nu_i, \eta_i^1, \eta_{i2}),$$

where $\varepsilon_i^1 \equiv (\varepsilon_{i0}, \varepsilon_{i1})$ and $\eta_i^1 \equiv (\eta_{i0}, \eta_{i1})$ and modify the definition of $\mathcal{G}_{\text{if}}$ accordingly,

$$\mathcal{G}_{\text{if}} = \{g \in \mathcal{G}_{\text{all}} : g(a, e^1, e_2, v, t^1, t_2\} \text{ is a trivial function of } (e_2, t_2)\}.$$

We derive bounds under the following imperfect foresight assumption.

Assumption IF. *The following conditions hold: (i)* $g \in \mathcal{G}_{if}$ *and (ii)* $(\nu_i, \eta_i^1)|\alpha_i, \varepsilon_i^1, \varepsilon_{i2} \overset{d}{=} (\nu_i, \eta_i^1)|\alpha_i, \varepsilon_i^1$.

Assumption IF embeds two conditions: (i) the selection mechanism does not directly depend on future shocks, and (ii) conditions on the distribution of the additional time-invariant and time-varying unobservables in periods $t \in \{0, 1\}$. Together, these two conditions imply that $E[G_i|\alpha_i, \varepsilon_i^1, \varepsilon_{i2}] = E[G_i|\alpha_i, \varepsilon_i^1]$.

The following direct corollary of Proposition 3.2 shows Assumption PT holding for all $g \in \mathcal{G}_{\text{if}}$ implies a martingale property on $\dot{Y}_{i2}(0)$ conditional on α_i and ε_i^1.

[10] Rambachan and Roth (2023, Section 2.2) use a similar decomposition: DiD = ATT $+E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1] - E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0]$. We work with demeaned outcomes here to relate the bias to the results in Section 3.

Corollary 4.1 (Necessary condition for $g \in \mathcal{G}_{\text{if}}$)**.** *Suppose that Assumption SEL holds with ε_{i1} replaced by ε_i^1 and either $P(E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_i^1] > \dot{Y}_{i1}(0)) < 1$ or $P(E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_i^1] < \dot{Y}_{i1}(0)) < 1$. If Assumption PT holds for all $g \in \mathcal{G}_{if}$ satisfying $P(G_i = 1) \in (0,1)$, then $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_i^1] = \dot{Y}_{i1}(0)$ a.s.*

Corollary 4.1 motivates bounding Δ_{post} by bounding deviations from the martingale condition. To this end, we consider the following general class of relaxations of this condition. Other choices are possible.

Assumption REL. *The following relaxation of the martingale condition holds:*

$$E[\dot{Y}_{it}(0)|\alpha_i, \varepsilon_{i0}, \ldots, \varepsilon_{i(t-1)}] = \sigma_t \rho(\dot{Y}_{i(t-1)}(0)), \quad i = 1, \ldots, N, \quad t = 1, 2,$$

where $\rho(\cdot)$ is an arbitrary nonparametric function, and σ_1 is normalized to one.

If $\rho(\cdot)$ is the identity function and $\sigma_t = 1$ for all t, Assumption REL reduces to the martingale assumption $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_i^1] = \dot{Y}_{i1}(0)$. Note that Assumption REL allows for location shifts because it is imposed on the demeaned outcomes $\dot{Y}_{it}(0)$.

In Assumption REL, the time-invariant function $\rho(\cdot)$ can be identified from the pre-treatment data. Thus, the key unknown quantity is σ_2.[11] The following proposition provides bounds on the ATT under Assumption REL for settings where researchers are willing to specify a range of plausible values for σ_2.

Proposition 4.1 (Bounds on the ATT)**.** *Suppose that Assumption IF holds. Suppose further that $P(G_i = 1) \in (0,1)$. If Assumption REL holds, then*

$$\text{ATT} \equiv \text{ATT}(\sigma_2) = \text{DiD} - \Delta_{post}(\sigma_2),$$

where

$$\Delta_{post}(\sigma_2) = \frac{E[G_i(\sigma_2 \rho(\dot{Y}_{i1}(0)) - \dot{Y}_{i1}(0))]}{P(G_i = 1)P(G_i = 0)}.$$

If Assumption REL holds for some $\sigma_2 \in [\underline{\sigma}_2, \overline{\sigma}_2]$, where $-\infty < \underline{\sigma}_2 \leq \overline{\sigma}_2 < \infty$, then

$$\text{ATT} \in \{\text{ATT}(\sigma_2) : \sigma_2 \in [\underline{\sigma}_2, \overline{\sigma}_2]\} \equiv \Theta_I.$$

[11] Assumption REL yields a linear autoregressive model with a time-varying parameter σ_t, when $\rho(\cdot)$ is the identity function. This class of models has been studied extensively in the time series literature under restrictions on the heterogeneity of the coefficient (e.g., Nicholls and Quinn, 1982; Regis et al., 2022). We do not impose any structure on σ_t here, but instead perform sensitivity analyses relying on bounds on this parameter.

Proposition 4.1 provides bounds for the ATT given a pre-specified range of deviations from the martingale property. If the deviation from the martingale assumption and thus σ_2 is known so that $\underline{\sigma}_2 = \overline{\sigma}_2$, the ATT is point-identified. More generally, given a range of a priori plausible values of σ_2, the ATT is partially identified, and the identified set Θ_I is a closed interval. If $\underline{\sigma}_2 \leq 1 \leq \overline{\sigma}_2$ and $\rho(\cdot)$ is the identity function, the DiD estimand is contained in Θ_I.

Proposition 4.1 can be used for constructing confidence intervals for the ATT that are robust to violations of the martingale property necessary for parallel trends under imperfect foresight. Such confidence intervals could be constructed, for example, using the approach proposed by Conley et al. (2012).

Remark 2 (Multiple pre-treatment periods)**.** *So far, we have assumed that there are only two pre-treatment periods. If more pre-treatment periods are available, the identification strategy can be refined. For example, one can estimate a parametric model for σ_t based on the pre-treatment data and use this model to impute or determine a range for σ_2.* □

5 Covariates

In many applications, parallel trends may only be plausible conditional on covariates (e.g., Heckman et al., 1997; Abadie, 2005; Sant'Anna and Zhao, 2020; Callaway and Sant'Anna, 2021). Therefore, we study the role of covariates in parallel trends assumptions. While many existing approaches focus on time-invariant covariates, we explicitly allow for a vector of both time-invariant and time-varying covariates, X_{it}, assuming that X_{it} is not affected by the treatment.[12]

We start by demonstrating that conditional parallel trends assumptions imply separability restrictions with respect to how the covariates can enter the outcome equation. We then provide a set of sufficient conditions for a weaker version of the parallel trends assumption that accommodates nonseparable models and discuss connections to the literature on nonseparable panel data models.

5.1 Conditional parallel trends implies separability

We consider a parallel trends assumption conditional on the time series of covariates.

Assumption PT-X. *The conditional parallel trends assumption holds:*

$$E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, X_i] = E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0, X_i] \text{ a.s.}$$

[12]See Caetano et al. (2022) for an analysis of settings where covariates can be affected by the treatment.

Under Assumption PT-X, the unconditional ATT is identified as

$$\begin{aligned}E[Y_{i2}(1)-Y_{i2}(0)|G_i=1] &= E\left[E[Y_{i2}(1)-Y_{i2}(0)|G_i=1,X_i]|G_i=1\right] \\ &= E\left[E[Y_{i2}-Y_{i1}|G_i=1,X_i]-E[Y_{i2}-Y_{i1}|G_i=0,X_i]|G_i=1\right].\end{aligned}$$

In the presence of covariates, potential outcomes and selection into treatment may naturally depend on them. We therefore consider the following outcome model and selection mechanism,

$$\begin{aligned}Y_{it}(0) &= \xi_t(X_{it},\alpha_i,\varepsilon_{it}), \\ G_i &= g(\alpha_i,X_{i1},X_{i2},\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}).\end{aligned}$$

Let $\mathcal{G}_{\text{all}}$ denote the class of all selection mechanisms.

All the necessary and sufficient conditions in Section 3 generalize straightforwardly to settings with covariates. The resulting conditions demonstrate that Assumption PT-X implies separability requirements on how the covariates can enter the outcome models. This is true, even when we allow selection to only depend on time-invariant unobservables and covariates. The following corollary to Proposition 3.3 provides the necessary condition for Assumption PT-X in this case. To state the result formally, define the following class of selection mechanisms,

$$\mathcal{G}_{\text{fe}}=\{g\in\mathcal{G}_{\text{all}}: g(a,x_1,x_2,e_1,e_2,v,t_1,t_2) \text{ is a trivial function of } (e_1,e_2,t_1,t_2)\}.$$

Corollary 5.1 (Necessary condition for $g\in\mathcal{G}_{\text{fe}}$)**.** *Suppose that $\nu_i^1\perp(\alpha_i,\varepsilon_{i1},\varepsilon_{i2},X_{i1},X_{i1})$, $P(\nu_i^1>c)\in(0,1)$ for some $c\in\mathbb{R}$, and either $P(E[\dot{Y}_{i2}(0)|\alpha_i,X_i]>E[\dot{Y}_{i1}|\alpha_i,X_i])<1$ and $P(E[\dot{Y}_{i2}(0)|\alpha_i,X_i]<E[\dot{Y}_{i1}|\alpha_i,X_i])<1$. If Assumption PT-X holds for all $g\in\mathcal{G}_{fe}$ satisfying $P(G_i=1|X_i)\in(0,1)$ a.s., then $E[Y_{i2}(0)|\alpha_i,X_i]-E[Y_{i1}(0)|\alpha_i,X_i]=E[Y_{i2}(0)|X_i]-E[Y_{i1}(0)|X_i]$ a.s.*

The necessary condition in Corollary 5.1 embeds separability and time-invariance restrictions on $E[Y_{it}(0)|\alpha_i,X_i]$. To illustrate those restrictions, consider a generalized random coefficient model (e.g. Chamberlain, 1992) where α_i interacts with X_{it},

$$\xi_t(X_{it},\alpha_i,\varepsilon_{it})=\alpha_i\gamma_t(X_{it})+\lambda_t+\varepsilon_{it}. \tag{10}$$

Here $\gamma_t(\cdot)$ is an arbitrary time-varying function. Even under the assumption that $E[\varepsilon_{it}|\alpha_i,X_i]=0$, this model generally violates the necessary condition due to the combination of nonseparability between α_i and X_{it} and the time variability in the structural function through

$\gamma_t(\cdot)$,

$$E[Y_{i2}(0)|\alpha_i, X_i] - E[Y_{it}(0)|\alpha_i, X_i] = \alpha_i(\gamma_2(X_{i2}) - \gamma_1(X_{i1})) + \lambda_2 - \lambda_1.$$

Allowing for interactions between the unobservable determinants of selection and some covariates is important in applications. Therefore, we consider a weaker conditional parallel trends assumption that allows for such interactions in Section 5.2.

Remark 3 (Sufficient conditions for separable models with covariates)**.** *The discussion in this section shows that Assumption PT-X requires separability between the observable and unobservable determinants of selection in the outcome model. In Appendix D, we provide three sets of primitive sufficient conditions for Assumption PT-X based on the following model,*

$$Y_{it}(0) \quad = \alpha_i + \gamma_t(X_{it}) + \lambda_t + \varepsilon_{it}. \tag{11}$$

In model (11), the covariates enter in an additively separable manner through the arbitrary and potentially time-varying function $\gamma_t(\cdot)$. These sufficient conditions are conditional versions of Assumptions SC1, SC2, and SC3. □

5.2 A parallel trends assumption for nonseparable models

Motivated by Section 5.1, we consider a weaker (than Assumption PT-X) conditional parallel trends assumption. To define this assumption, we explicitly differentiate between two types of covariates: (i) X_{it}^{μ} are covariates that interact with the unobservable determinants of selection in the outcome model; (ii) X_{it}^{λ} are covariates that do not interact with these unobservables in the outcome model. Both types of covariates can enter the selection mechanism in an arbitrary way. The conditional parallel trends assumption we introduce next holds for subpopulations that experience no change in X_{it}^{μ} and the same trajectory in X_{it}^{λ}.

Assumption PT-NSP. *The (modified) conditional parallel trends assumption holds:*

$$E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] = E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0, X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}] \ a.s.$$

Under Assumption PT-NSP, we can no longer identify the ATT, $E[Y_{i2}(1)-Y_{i2}(0)|G_i = 1]$, because we cannot identify the conditional ATT, $E[Y_{i2}(1) - Y_{i2}(0)|G_i = 1, X_i^{\lambda}, X_i^{\mu}]$. Instead, we can identify $E[Y_{i2}(1) - Y_{i2}(0)|G_i = 1, X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}]$. After integrating out with respect to the distribution of covariates, we can identify the ATT for subpopulations that do not experience changes in X_{it}^{μ},

$$E[Y_{i2}(1) - Y_{i2}(0)|G_i = 1, X_{i1}^{\mu} - X_{i2}^{\mu} = 0].$$

Note that if X_{it}^μ is time-invariant, then $X_{i1}^\mu = X_{i2}^\mu$ holds by definition such that Assumptions PT-X and PT-NSP are equivalent.

In view of Assumption PT-NSP, we consider the following nonseparable model which consists of a time-invariant and time-varying component.

Assumption NSP-X.

$$Y_{it}(0) = \mu(X_{it}^\mu, \alpha_i^\mu, \varepsilon_{it}^\mu) + \lambda_t(X_{it}^\lambda, \alpha_i^\lambda, \varepsilon_{it}^\lambda), \quad i = 1, \ldots, N, \quad t = 1, 2,$$

where X_{it}^μ, X_{it}^λ, α_i^μ, α_i^λ, ε_{it}^μ, and ε_{it}^λ are finite-dimensional random vectors.

Without further restrictions on the unobservables, the additive structure in Assumption NSP-X is without loss of generality and the superscripts μ and λ are merely labels. Indeed, if $X_{it}^\mu = X_{it}^\lambda$, $\alpha_i^\mu = \alpha_i^\lambda$, and $\varepsilon_{it}^\mu = \varepsilon_{it}^\lambda$, the model is fully nonseparable and time-varying in an arbitrary way. In the following, we use $\mathcal{X}_\mu$, $\mathcal{X}_\lambda$, $\mathcal{A}$, and $\mathcal{E}$ to denote the supports of X_{it}^μ, X_{it}^λ, α_i^μ, and ε_{it}^μ, respectively.

In view of the necessary condition in Corollary 5.1, it is natural to consider selection based on the unobservables entering $\mu(\cdot)$. We therefore impose the following condition on the projected selection mechanism.

Assumption SEL-CI.

$$E[G_i|\alpha_i^\mu, \alpha_i^\lambda, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu, \varepsilon_{i1}^\lambda, \varepsilon_{i2}^\lambda] = E[G_i|\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu].$$

Assumption SEL-CI allows the projected selection mechanism to depend on all covariates, but only on the unobservables that enter the time-invariant component of the structural function. In view of Assumption SEL-CI, we define

$$\begin{aligned}&\bar{g}(a^\mu, x_1^\mu, x_2^\mu, x_1^\lambda, x_2^\lambda, e_1^\mu, e_2^\mu)\\ &\equiv E[G_i|\alpha_i^\mu = a^\mu, X_{i1}^\mu = x_1^\mu, X_{i2}^\mu = x_2^\mu, X_{i1}^\lambda = x_1^\lambda, X_{i2}^\lambda = x_2^\lambda, \varepsilon_{i1}^\mu = e_1^\mu, \varepsilon_{i2}^\mu = e_2^\mu].\end{aligned}$$

We present three sets of sufficient conditions for Assumption PT-NSP. Each set of conditions consists of assumptions on the projected selection mechanism as well as distributional restrictions on the unobservables. Our first sufficient condition allows selection to depend on all covariates as well as the unobservables that enter the time-invariant component of the structural function, while imposing a symmetry restriction on the projected selection mechanism similar to Assumption SC1.

Assumption SC1-NSP. *The following conditions hold:*

(i) $\bar{g}(a^{\mu}, x_1^{\mu}, x_2^{\mu}, x_1^{\lambda}, x_2^{\lambda}, e_1^{\mu}, e_2^{\mu})$ *is a symmetric function in* e_1^{μ} *and* e_2^{μ}.

(ii) $(\varepsilon_{i1}^{\mu}, \varepsilon_{i2}^{\mu})|\alpha_i^{\mu}, X_i^{\mu}, X_i^{\lambda} \stackrel{d}{=} (\varepsilon_{i2}^{\mu}, \varepsilon_{i1}^{\mu})|\alpha_i^{\mu}, X_i^{\mu}, X_i^{\lambda}$.

(iii) $(\alpha_i^{\mu}, \varepsilon_{i1}^{\mu}, \varepsilon_{i2}^{\mu}) \perp (\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda})|X_i^{\mu}, X_i^{\lambda}$.

Here we require the conditional distribution of $(\varepsilon_{i1}^{\mu}, \varepsilon_{i2}^{\mu})|\alpha_i^{\mu}, X_i^{\mu}, X_i^{\lambda}$ to be exchangeable. Since the projected selection mechanism depends on $(\alpha_i^{\mu}, \varepsilon_{i1}^{\mu}, \varepsilon_{i2}^{\mu})$, we require them to be independent of the unobservables entering $\lambda_t(\cdot)$ conditional on $(X_i^{\mu}, X_i^{\lambda})$.

The exchangeability restriction in Assumption SC1-NSP is different from the exchangeability assumption in Altonji and Matzkin (2005). The exchangeability assumption in Altonji and Matzkin (2005) requires the conditional distribution of all unobservables that enter $\mu(\cdot)$ and $\lambda_t(\cdot)$ to be invariant to permutations of covariates in the conditioning set, which is a nonparametric correlated random effects restriction (Ghanem, 2017). By contrast, we assume that the time-varying unobservables are exchangeable conditional on $(\alpha_i^{\mu}, X_i^{\mu}, X_i^{\lambda})$ without imposing any restrictions on the distribution of $\alpha_i^{\mu}|G_i, X_i^{\mu}, X_i^{\lambda}$.

Next, in the spirit of Assumption SC2, we consider a projected selection mechanism that is a trivial function of ε_{i2}^{μ} in the following sufficient condition.

Assumption SC2-NSP. *The following conditions hold:*

(i) $\bar{g}(a^{\mu}, x_1^{\mu}, x_2^{\mu}, x_1^{\lambda}, x_2^{\lambda}, e_1^{\mu}, e_2^{\mu})$ *is a trivial function of* e_2^{μ}.

(ii) $(\alpha_i^{\mu}, \varepsilon_{i1}^{\mu}) \perp \Delta_{\mu,i}|X_i^{\lambda}, X_{i1}^{\mu} = X_{i2}^{\mu}$, *where* $\Delta_{\mu,i} \equiv \mu(X_{i2}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i2}^{\mu}) - \mu(X_{i1}^{\mu}, \alpha_i^{\mu}, \varepsilon_{i1}^{\mu})$.

(iii) $(\alpha_i^{\mu}, \varepsilon_{i1}^{\mu}) \perp (\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda})|X_i^{\mu}, X_i^{\lambda}$.

Assumption SC2-NSP.ii implicitly imposes separability conditions on $\mu(\cdot)$ (but not on $\lambda_t(\cdot)$) and restrictions on time series dependence. The independence condition in Assumption SC2-NSP.iii requires that the unobservable determinants of selection are independent of the unobservables that enter $\lambda_t(\cdot)$, conditional on the times series of the covariates.

The last sufficient condition restricts the projected selection mechanism to only depend on covariates and the time-invariant unobservables.

Assumption SC3-NSP. *The following conditions hold:*

(i) $\bar{g}(a^{\mu}, x_1^{\mu}, x_2^{\mu}, x_1^{\lambda}, x_2^{\lambda}, e_1^{\mu}, e_2^{\mu})$ *is a trivial function of* e_1^{μ} *and* e_2^{μ}.

(ii) $\varepsilon_{i1}^{\mu}|\alpha_i^{\mu}, X_i^{\mu}, X_i^{\lambda} \stackrel{d}{=} \varepsilon_{i2}^{\mu}|\alpha_i^{\mu}, X_i^{\mu}, X_i^{\lambda}$.

(iii) $\alpha_i^{\mu} \perp (\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda})|X_i^{\mu}, X_i^{\lambda}$.

Assumption SC3-NSP requires the distribution of ε_{it}^{μ}, which enters $\mu(\cdot)$, to be time-invariant conditional on $(\alpha_i^{\mu}, X_i^{\mu}, X_i^{\lambda})$. The unobservables entering $\lambda_t(\cdot)$, $(\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda})$, are required to be independent of the unobservables that determine selection, α_i^μ, conditional on (X_i^μ, X_i^λ).

Each of the sufficient conditions consists of three components: (i) a restriction on how/which unobservables determine the projected selection mechanism, (ii) a restriction on the unobservables entering the time-invariant component of the structural function, and (iii) an independence assumption that ensures that the time-varying component of the structural function is independent of G_i conditional on the time series of the covariates.

The following proposition formally establishes sufficiency of each set of conditions.

Proposition 5.1 (Sufficient conditions). *Suppose that Assumptions NSP-X and SEL-CI hold and $P(G_i = 1 | X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu) \in (0,1)$ a.s. Then (i) Assumption SC1-NSP implies Assumption PT-NSP, (ii) Assumption SC2-NSP implies Assumption PT-NSP, and (iii) Assumption SC3-NSP implies Assumption PT-NSP.*

5.3 Connections to identifying assumptions in panel models

DiD methods have traditionally been motivated using two-way fixed effects models. Fixed effects assumptions allow for unrestricted dependence between time-invariant unobservables and the regressors in separable and nonseparable models, thereby implicitly allowing for selection on time-invariant unobservables.[13] In this paper, we explicitly analyze the connection between selection mechanisms and the parallel trends assumptions underlying DiD. Therefore, a natural question is how our sufficient conditions relate to the identification assumptions in the nonseparable panel literature.

The literature on nonseparable panel models has considered two broad categories of identification assumptions. First, time homogeneity conditions (e.g., Hoderlein and White, 2012; Chernozhukov et al., 2013) require the distribution of time-varying unobservables to be stationary across time while allowing for unrestricted individual heterogeneity (fixed effects). Second, nonparametric correlated random effects restrictions (e.g., Altonji and Matzkin, 2005; Bester and Hansen, 2009) allow for unrestricted time heterogeneity by imposing restrictions on individual heterogeneity, generalizing the classical notion of correlated random effects (e.g., Mundlak, 1978; Chamberlain, 1984). However, neither category of assumptions is explicit about the selection mechanism and, in particular, about how unobservables determine selection.

The existing identification results based on time homogeneity or correlated random effects assumptions suggest a trade-off between restrictions on time and individual heterogeneity.

[13] See, e.g., Chamberlain (1984); Arellano (2003); Evdokimov (2010); Wooldridge (2010); Hoderlein and White (2012); Chernozhukov et al. (2013).

Here we show that our sufficient conditions for parallel trends constitute interpretable primitive conditions on the selection mechanism that imply *combinations* of time homogeneity and correlated random effects restrictions from the nonseparable panel literature.

The following assumption is the time homogeneity assumption from Chernozhukov et al. (2013) imposed on ε_{it}^{μ} in Assumption NSP-X, conditional on the time series of all covariates that enter the outcome equation.

Assumption TH. $\varepsilon_{i1}^{\mu}|G_i, X_i^{\mu}, X_i^{\lambda}, \alpha_i^{\mu} \overset{d}{=} \varepsilon_{i2}^{\mu}|G_i, X_i^{\mu}, X_i^{\lambda}, \alpha_i^{\mu}$

Assumption TH requires the distribution of ε_{it}^{μ} to be homogeneous across time conditional on G_i, X_i^{μ}, X_i^{λ}, and α_i^{μ}. However, it does not impose any restrictions on the conditional distribution of ε_{it}^{μ}. Furthermore, there are no restrictions imposed on the distribution of $\alpha_i^{\mu}|G_i, X_i^{\mu}, X_i^{\lambda}$, consistent with the notion of fixed effects.

The next assumption is a nonparametric correlated random effects assumption (e.g., Altonji and Matzkin, 2005; Ghanem, 2017).

Assumption CRE. $(\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda})|G_i, X_i^{\mu}, X_i^{\lambda} \overset{d}{=} (\alpha_i^{\lambda}, \varepsilon_{i1}^{\lambda}, \varepsilon_{i2}^{\lambda})|X_i^{\mu}, X_i^{\lambda}$.

Assumption CRE is a conditional independence condition between G_i and the unobservables that enter the time-varying component of the structural function, $\lambda_t(\cdot)$. This assumption does not imply conditional random assignment, $(Y_{i1}(0), Y_{i2}(0)) \perp G_i|X_i^{\mu}, X_i^{\lambda}$, since selection into treatment can depend on the unobservables entering the time-invariant component $\mu(\cdot)$.

Together, Assumptions TH and CRE imply Assumption PT-NSP.

Proposition 5.2 (Time homogeneity and correlated random effects imply Assumption PT-NSP)**.** *Suppose that Assumption NSP-X holds and* $P(G_i = 1|X_{i1}^{\mu} = X_{i2}^{\mu}, X_i^{\lambda}) \in (0,1)$ *a.s. Then Assumptions TH and CRE imply Assumption PT-NSP.*

In view of Proposition 5.2 it is interesting to explore the connection between selection, time homogeneity, and correlated random effects in the nonseparable DiD framework. To this end, Proposition 5.3 shows that Assumptions SC1-NSP and SC3-NSP are primitive sufficient conditions on the selection mechanism for the nonseparable model satisfying Assumptions TH and CRE.[14]

Proposition 5.3 (Connection between selection, time homogeneity, and correlated random effects)**.** *Suppose that Assumption NSP-X holds and* $G_i = g(\alpha_i^{\mu}, X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda}, \varepsilon_{i1}^{\mu}, \varepsilon_{i2}^{\mu})$. *Then (i) Assumption SC1-NSP with* $g(\cdot)$ *in lieu of* $\bar{g}(\cdot)$ *implies Assumptions TH and CRE*

[14]In the context of correlated random coefficient models, Graham and Powell (2012) impose a similar structure on their model.

if $P(G_i = 1|\alpha_i^\mu, X_i^\mu, X_i^\lambda) \in (0,1)$ *a.s., (ii) Assumption SC3-NSP with* $g(\cdot)$ *in lieu of* $\bar{g}(\cdot)$ *implies Assumptions TH and CRE.*

Proposition 5.3 demonstrates how restrictions on selection can be used to justify combinations of Assumptions TH and CRE.

6 Implications for practice

Restrictions on selection are unavoidable in DiD designs. Our necessary and sufficient condition in Proposition 3.1 underscores that if researchers are not willing to impose any restrictions on selection, then parallel trends rules out time-varying unobservables. Therefore, in realistic settings, relying on parallel trends assumptions implicitly imposes restrictions on the time-varying unobservables and how selection depends on them.

Parallel trends can be compatible with selection on time-varying unobservables. It is well-understood that selection on time-invariant unobservables is compatible with parallel trends in the classical two-way fixed effects model under strict exogeneity (e.g., Blundell and Costa Dias, 2009). The primitive sufficient conditions in Section 3.3 provide cases where parallel trends could hold despite selection depending on time-invariant *and* time-varying unobservables. An important implication is that parallel trends can be compatible with selection on untreated potential outcomes (Example 1) and selection on treatment effects (Example 2).

Contextual knowledge about selection can be used to justify parallel trends assumptions. The menu of primitive sufficient conditions in Section 3.3 provides practitioners with explicit theory-based templates for justifying parallel trends assumptions. These conditions consist of different combinations of restrictions on (i) which/how unobservables determine selection and (ii) how their distribution varies over time. We recommend that applied researchers relying on these conditions use contextual information to assess and explicitly discuss which determinants of the untreated potential outcome affect selection. In doing so, it is crucial to consider the timing of the decision as well as the information set available to the units.[15] Once a suitable selection mechanism is identified, the next step is to discuss the plausibility of the corresponding assumption on the distribution of the unobservables. In this context, periodicity is crucial both to distinguish between time-invariant and time-varying factors and to justify the distributional assumptions.

[15] The importance of the information available to units is underscored by the results in Marx et al. (2022), who study economic models of selection including learning and optimal stopping.

Assumptions on selection are useful for sensitivity analyses. Assumptions on selection into treatment are useful for bounding the ATT in applications where the validity of the parallel trends assumption is questionable. To illustrate, we provide bounds under imperfect foresight for settings where the martingale assumptions necessary for parallel trends may be violated. For applications where contextual knowledge about selection is available, these bounds provide a new tool for performing sensitivity analyses and constructing confidence intervals that are robust to violations of the relevant necessary conditions for parallel trends.

How to condition on covariates depends on how they enter the outcome model. If the covariates and the unobservable determinants of selection enter the outcome model separably, researchers can condition on the entire time series of covariates and identify the overall ATT. If there are time-varying covariates that interact with the unobservable determinants of selection in the outcome model, researchers should condition on these covariates not changing over time and settle for identification of the ATT for a subpopulation.

Restrictions on nonseparable outcome models can also be used to justify parallel trends. An implication of Section 5.3 is that parallel trends is consistent with a nonseparable outcome model satisfying a combination of time homogeneity and correlated random effects. This provides researchers with an alternative avenue for justifying parallel trends based on restrictions on the untreated potential outcome and its unobservable determinants.

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Appendix

A Auxiliary lemmas

Lemma A.1. *Let ω_i denote a vector of random variables. Suppose that $P(G_i = 1|\omega_i) \in (0,1)$ a.s. Then $E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i] = E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0, \omega_i]$ if and only if $E[G_i(Y_{i2}(0) - Y_{i1}(0))|\omega_i] = E[G_i|\omega_i]E[Y_{i2}(0) - Y_{i1}(0)|\omega_i]$ a.s.*

Proof. In the following, all equalities involving conditional expectations are understood as a.s. equalities.

"$\Longrightarrow$": First, note that by the law of total probability, $E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i] = E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0, \omega_i]$ implies

$$E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i] = E[Y_{i2}(0) - Y_{i1}(0)|\omega_i].$$

The result follows from noting that $E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i] = \frac{E[G_i(Y_{i2}(0)-Y_{i1}(0))|\omega_i]}{P(G_i=1|\omega_i)}$ by definition.

"$\Longleftarrow$": Since $P(G_i = 1|\omega_i) \in (0,1)$, it follows that $E[Y_{i2}(0) - Y_{i1}(0))|G_i = 1, \omega_i] = E[Y_{i2}(0) - Y_{i1}(0))|\omega_i]$. It then follows that

$$\begin{aligned} &E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i]P(G_i = 1|\omega_i) + E[Y_{i2}(0) - Y_{i1}(0)|G_i = 0, \omega_i]P(G_i = 0|\omega_i) \\ &= E[Y_{i2}(0) - Y_{i1}(0)|G_i = 1, \omega_i]. \end{aligned}$$

The result follows from subtracting the first term on the left-hand side and dividing by $P(G_i = 0|\omega_i)$. $\square$

Lemma A.2. *For a scalar random variable W_i, let $\dot{W}_i = W_i - E[W_i]$. If $E[\dot{W}_i 1\{\dot{W}_i \leq 0\}] = 0$ or $E[\dot{W}_i 1\{\dot{W}_i \geq 0\}] = 0$, then $W_i = E[W_i]$ a.s.*

Proof. We prove the results for the case where $E[\dot{W}_i 1\{\dot{W}_i \leq 0\}] = 0$, since the proof for the other case follows by identical arguments. First, note that by definition $E[\dot{W}_i] = 0$, which is equivalent to

$$E[\dot{W}_i^+] = E[\dot{W}_i^-], \tag{12}$$

where $\dot{W}_i^+ = |\dot{W}_i| 1\{\dot{W}_i > 0\}$ and $\dot{W}_i^- = |\dot{W}_i| 1\{\dot{W}_i < 0\}$.

Now suppose that $E[\dot{W}_i 1\{\dot{W}_i \leq 0\}] = 0$ holds, which is equivalent to

$$E[\dot{W}_i^+ 1\{\dot{W}_i \leq 0\}] = E[\dot{W}_i^- 1\{\dot{W}_i \leq 0\}], \tag{13}$$

since, by definition, $\dot{W}_i = \dot{W}_i^+ - \dot{W}_i^-$. Note that the left-hand side equals zero by the definition of $\dot{W}_i^+$. As a result, $E[\dot{W}_i^- 1\{\dot{W}_i \leq 0\}] = E[\dot{W}_i^-] = 0$. Since $\dot{W}_i^- \geq 0$, this implies that $P(\dot{W}_i^- = 0) = 1$. Now note that $P(\dot{W}_i^- = 0) = P(|\dot{W}_i| 1\{\dot{W}_i < 0\} = 0) = P(1\{\dot{W}_i < 0\} = 0) = 1$, which implies $P(\dot{W}_i < 0) = 0$.

Since $E[\dot{W}_i] = 0$, (12) further implies that $E[\dot{W}_i^-] = E[\dot{W}_i^+] = 0$. Since $\dot{W}_i^+ \geq 0$, it follows that $P(\dot{W}_i^+ = 0) = 1$. Now note that $P(\dot{W}_i^+ = 0) = P(|\dot{W}_i| 1\{\dot{W}_i > 0\} = 0) = P(1\{\dot{W}_i > 0\} = 0) = 1$, which implies $P(\dot{W}_i > 0) = 0$.

Together, $P(\dot{W}_i < 0) = 0$ and $P(\dot{W}_i > 0) = 0$ imply that $P(\dot{W}_i = 0) = 1 - (P(\dot{W}_i < 0) + P(\dot{W}_i > 0)) = 1$, which completes the proof. $\square$

Lemma A.3. *Let ω_i denote a subvector of $(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$. Suppose that $P(v_i^1 > c) \in (0,1)$ for some $c \in \mathbb{R}$, and $\nu_i^1 \perp (\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$.*

(i) If $P(E[\dot{Y}_{i2}(0)|\omega_i] > E[\dot{Y}_{i1}(0)|\omega_i]) < 1$ and Assumption PT holds for $G_i = 1\{\nu_i^1 > c\} 1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\}$, then $E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] = 0$ a.s.

(ii) If $P(E[\dot{Y}_{i2}(0)|\omega_i] < E[\dot{Y}_{i1}(0)|\omega_i]) < 1$ and Assumption PT holds for $G_i = 1\{\nu_i^1 > c\} 1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \geq 0\}$, then $E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] = 0$ a.s.

Proof. We only prove (i). The proof of (ii) follows from the same arguments. Under the maintained assumptions the selection mechanism is nondegenerate,

$$P(1\{\nu_i^1 > c\} 1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\}) \in (0,1).$$

Thus, by Lemma A.1, Assumption PT holding for $G_i = 1\{\nu_i^1 > c\} 1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\}$

is equivalent to

$$E[1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\}(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] = 0,$$

which, by $P(v_i^1 > c) \in (0,1)$ and $\nu_i^1 \perp (\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$, is equivalent to

$$E[1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\}(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] = 0.$$

By the LIE, this is further equivalent to

$$E[1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i] \leq 0\}E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i]] = 0$$

Since $E[E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\omega_i]] = 0$, the result follows by Lemma A.2. □

Lemma A.4. *Let $(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$ denote a vector of random variables. Suppose that $\varepsilon_{i1}, \varepsilon_{i2}|\alpha_i \overset{d}{=} \varepsilon_{i2}, \varepsilon_{i1}|\alpha_i$ holds. Then*

(i) $F_{\varepsilon_{i1}|\alpha_i}(e|a) = F_{\varepsilon_{i2}|\alpha_i}(e|a)$ a.e. $(a, e) \in \mathcal{A} \times \mathcal{E}$

(ii) $F_{\varepsilon_{i1}|\varepsilon_{i2},\alpha_i}(e_1|e_2, a) = F_{\varepsilon_{i2}|\varepsilon_{i1},\alpha_i}(e_1|e_2, a)$ a.e. $(a, e_1, e_2) \in \mathcal{A} \times \mathcal{E}^2$.

Proof. **(i)** By the definition of the marginal distribution, the conditional exchangeability restriction implies (i) by the following *a.e.*

$$F_{\varepsilon_{i1}|\alpha_i}(e_1|a) = \lim_{e_2 \to \infty} F_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_1, e_2|a) = \lim_{e_2 \to \infty} F_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_2, e_1|a) = F_{\varepsilon_{i2}|\alpha_i}(e_1|a). \tag{14}$$

(ii) By the definition of the conditional distribution and (i) of this lemma, the conditional exchangeability restriction implies (ii) by the following

$$F_{\varepsilon_{i1}|\varepsilon_{i2},\alpha_i}(e_1|e_2, a) = \frac{F_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_1, e_2|a)}{F_{\varepsilon_{i2}|\alpha_i}(e_2|a)} = \frac{F_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_2, e_1|a)}{F_{\varepsilon_{i1}|\alpha_i}(e_2|a)} = F_{\varepsilon_{i2}|\varepsilon_{i1},\alpha_i}(e_1|e_2, a), \tag{15}$$

a.e. $(a, e_1, e_2) \in \mathcal{A} \times \mathcal{E}^2$. □

B Proofs of main results

B.1 Proof of Proposition 3.1

"$\Longrightarrow$": We first consider the case where $P(\dot{Y}_{i2}(0) > \dot{Y}_{i1}(0)) < 1$. Note that if Assumption PT holds for all $g \in \mathcal{G}_{\text{all}}$, then it holds for $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0) \leq 0\}$. By Assumption SEL and $P(\dot{Y}_{i2}(0) > \dot{Y}_{i1}(0)) < 1$, we can invoke Lemma A.3.i with $\omega_i = (\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$, which implies $\dot{Y}_{i1}(0) = \dot{Y}_{i2}(0)$ a.s.

The proof for the case where $P(\dot{Y}_{i2}(0) < \dot{Y}_{i1}(0)) < 1$ follows symmetrically using the selection mechanism $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0) \geq 0\}$ and invoking Lemma A.3.ii.

"$\Longleftarrow$": This direction is immediate. $\square$

B.2 Proof of Proposition 3.2

We first consider the case where $P(E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] > \dot{Y}_{i1}(0)) < 1$. Note that if Assumption PT holds for all $g \in \mathcal{G}_{\text{if}}$, then it holds for $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i, \varepsilon_{i1}] \leq 0\}$. By Assumption SEL and $P(E[Y_{i2}(0)|\alpha_i, \varepsilon_{i1}] > \dot{Y}_{i1}(0)) < 1$, we can invoke Lemma A.3.i with $\omega_i = (\alpha_i, \varepsilon_{i1})$, which implies $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] = \dot{Y}_{i1}(0)$ a.s.

The proof for the case where $P(E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] < \dot{Y}_{i1}(0)) < 1$ follows symmetrically using the selection mechanism $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i, \varepsilon_{i1}] \geq 0\}$ and invoking Lemma A.3.ii. $\square$

B.3 Proof of Proposition 3.3

We first consider the case where $P(E[\dot{Y}_{i2}(0)|\alpha_i] > E[\dot{Y}_{i1}(0)|\alpha_i]) < 1$. Note that if Assumption PT holds for all $g \in \mathcal{G}_{\text{fe}}$, then it holds for $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i] \leq 0\}$. By Assumption SEL and $P(E[\dot{Y}_{i2}(0)|\alpha_i] > E[Y_{i1}(0)|\alpha_i]) < 1$, we can invoke Lemma A.3.i with $\omega_i = \alpha_i$, which implies $E[\dot{Y}_{i1}(0)|\alpha_i] = E[\dot{Y}_{i2}(0)|\alpha_i]$ a.s.

The proof for the case where $P(E[\dot{Y}_{i2}(0)|\alpha_i] < E[\dot{Y}_{i1}(0)|\alpha_i]) < 1$ follows symmetrically using the selection mechanism $g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1\{\nu_i^1 > c\}1\{E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i] \geq 0\}$ and invoking Lemma A.3.ii. $\square$

B.4 Proof of Proposition 3.4

(i) Since $g \in \mathcal{G}_{\text{if}}$, then we can simplify the following expression by the LIE as follows,

$$\begin{aligned}
E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] &= E[g(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] \\
&= E[E[E[g(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}](\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))|\alpha_i, \varepsilon_{i1}]] \\
&= E[E[g(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}]E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i, \varepsilon_{i1}]].
\end{aligned}$$

The third equality follows from $(\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \stackrel{d}{=} (\nu_i, \eta_{i1})|\alpha_i, \varepsilon_{i1}$.[16] If $E[\dot{Y}_{i2}(0)|\alpha_i, \varepsilon_{i1}] = \dot{Y}_{i1}(0)$ a.s., then the last term equals zero, which implies the result by Lemma A.1.

(ii) Similar to (i), since $g \in \mathcal{G}_{\text{fe}}$ and $\nu_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2} \stackrel{d}{=} \nu_i|\alpha_i$, then we can simplify the following expression by the LIE as follows,

$$\begin{aligned} E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] &= E[g(\alpha_i, \nu_i)(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] \\ &= E[E[E[g(\alpha_i, \nu_i)|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}](\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))|\alpha_i]] \\ &= E[E[g(\alpha_i, \nu_i)|\alpha_i]E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)|\alpha_i]]. \end{aligned}$$

If $E[\dot{Y}_{i1}(0)|\alpha_i] = E[\dot{Y}_{i2}(0)|\alpha_i]$ a.s., then the last term equals zero, which implies the result by Lemma A.1. ☐

B.5 Proof of Proposition 3.5

(i) We first show that (i) and (iii) of Assumption SC1 imply the symmetry of $\bar{g}(a, e_1, e_2) = E[G_i|\alpha_i = a, \varepsilon_{i1} = e_1, \varepsilon_{i2} = e_2]$ in e_1 and e_2. To do so, we note that these two conditions imply the following for $(a, e_1, e_2) \in \mathcal{A} \times \mathcal{E}^2$

$$\begin{aligned} \bar{g}(a, e_1, e_2) &= \int g(a, e_1, e_2, v, t_1, t_2) dF_{\nu_i, \eta_{i1}, \eta_{i2}|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}}(v, t_1, t_2|a, e_1, e_2) \\ &= \int g(a, e_2, e_1, v, t_1, t_2) dF_{\nu_i, \eta_{i1}, \eta_{i2}|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}}(v, t_1, t_2|a, e_2, e_1) = \bar{g}(a, e_2, e_1), \end{aligned} \tag{16}$$

where the penultimate equality follows by the symmetry of $g(\cdot)$ and $F_{\nu_i, \eta_{i1}, \eta_{i2}|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}}$ in e_1 and e_2 imposed in (i) and (iii) in Assumption SC1, respectively.

Next, by the LIE, we can decompose $E[G_i(\varepsilon_{i2} - \varepsilon_{i1})]$ and then invoke the symmetry restrictions on $\bar{g}(\cdot)$ and $F_{\varepsilon_{i1}, \varepsilon_{i2}|\alpha_i}$ implied by (i) and (iii) of Assumption SC1 as well as (ii) of Assumption SC1, respectively:

[16] The conditional independence restriction specifically implies the following:

$$\begin{aligned} E[g(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})|\alpha_i = a, \varepsilon_{i1} = e_1, \varepsilon_{i2} = e_2] &= \int g(a, e_1, v, t_1) dF_{\nu_i, \eta_{i1}|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}}(v, t_1|a, e_1, e_2) \\ = \int g(\alpha_i, \varepsilon_{i1}, v, t_1) dF_{\nu_i, \eta_{i1}|\alpha_i, \varepsilon_{i1}}(v, t_1|a, e_1) &= E[g(\alpha_i, \varepsilon_{i1}, \nu_i, \eta_{i1})|\alpha_i = a, \varepsilon_{i1} = e_1]. \end{aligned}$$

$$
\begin{aligned}
&E[G_i(\varepsilon_{i2}-\varepsilon_{i1})] = E[E[\bar{g}(\alpha_i,\varepsilon_{i1},\varepsilon_{i2})\varepsilon_{i2}|\alpha_i] - E[\bar{g}(\alpha_i,\varepsilon_{i1},\varepsilon_{i2})\varepsilon_{i1}|\alpha_i]] \\
&= \int \left(\int \bar{g}(a,e_1,e_2)e_2 dF_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_1,e_2|a) - \int \bar{g}(a,e_1,e_2)e_1 dF_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_1,e_2|a) \right) dF_{\alpha_i}(a) \\
&= \int \left(\int \bar{g}(a,e_2,e_1)e_2 dF_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_2,e_1|a) - \int \bar{g}(a,e_1,e_2)e_1 dF_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}(e_1,e_2|a) \right) dF_{\alpha_i}(a) = 0.
\end{aligned}
$$

The second equality follows from the symmetry restrictions on $\bar{g}(\cdot)$ and $F_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i}$. Together, they imply that both conditional expectations in the parentheses equal $E[\bar{g}(\alpha_i,\varepsilon_{i1},\varepsilon_{i2})\varepsilon_{i1}|\alpha_i]$, and therefore the difference between them is zero. As a result, Assumption SC1 implies Assumption PT.

(ii) This result follows from the proof of Proposition 3.4.i by plugging-in $Y_{it}(0) = \alpha_i + \lambda_t + \varepsilon_{it}$ for $t = 1, 2$.

(iii) This result follows from the proof of Proposition 3.4.ii by plugging-in $Y_{it}(0) = \alpha_i + \lambda_t + \varepsilon_{it}$ for $t = 1, 2$. □

B.6 Proof of Proposition 4.1

First, we simplify Δ_{post} as follows

$$
\begin{aligned}
\Delta_{\text{post}} &= \frac{E[G_i(\dot{Y}_{i2}(0)-\dot{Y}_{i1}(0))]}{P(G_i=1)} - \frac{E[(1-G_i)(\dot{Y}_{i2}(0)-\dot{Y}_{i1}(0))]}{P(G_i=0)} \\
&= \frac{E[G_i(\dot{Y}_{i2}(0)-\dot{Y}_{i1}(0))]}{P(G_i=1)} + \frac{E[G_i(\dot{Y}_{i2}(0)-\dot{Y}_{i1}(0))]}{P(G_i=0)} \\
&= \frac{E[G_i(\dot{Y}_{i2}(0)-\dot{Y}_{i1}(0))]}{P(G_i=1)P(G_i=0)} \qquad (17)
\end{aligned}
$$

By the law of iterated expectations, it then follows that

$$
\begin{aligned}
\Delta_{\text{post}} &= \frac{E[E[E[G_i|\alpha_i,\varepsilon_i^1,\varepsilon_{i2}](\dot{Y}_{i2}(0)-\dot{Y}_{i1}(0))|\alpha_i,\varepsilon_i^1]]}{P(G_i=0)P(G_i=1)} \\
&= \frac{E[E[G_i|\alpha_i,\varepsilon_i^1]E[\dot{Y}_{i2}(0)-\dot{Y}_{i1}(0)|\alpha_i,\varepsilon_i^1]]}{P(G_i=0)P(G_i=1)} \\
&= \frac{E[E[G_i|\alpha_i,\varepsilon_i^1](\sigma_2\rho(\dot{Y}_{i1}(0))-\dot{Y}_{i1}(0))]}{P(G_i=0)P(G_i=1)} \\
&= \frac{E[G_i(\sigma_2\rho(\dot{Y}_{i1}(0))-\dot{Y}_{i1}(0))]}{P(G_i=0)P(G_i=1)} \qquad (18)
\end{aligned}
$$

where the second equality follows from $E[G_i|\alpha_i, \varepsilon_i^1, \varepsilon_{i2}] = E[G_i|\alpha_i, \varepsilon_i^1]$, which is an implication of Assumption IF. The third equality follows from Assumption REL. The last equality follows by the law of iterated expectations. The result follows from the definition of the ATT. $\square$

B.7 Proof of Proposition 5.1

In this proof, all equalities involving random variables are understood to hold a.s.

First, by Lemma A.1, Assumption PT-NSP under Assumption NSP-X holds if and only if

$$\begin{aligned}
&E[G_i(Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[G_i|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu].
\end{aligned} \tag{19}$$

Next, we state some preliminary observations and then proceed to show each statement separately.

Note that, by the LIE, Assumption SEL-CI and the definition of $\bar{g}(\cdot)$, the LHS of (19) equals the following

$$\begin{aligned}
&E[G_i(Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[E[G_i|\alpha_i^\mu, \alpha_i^\lambda, X_i^\mu, X_i^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu, \varepsilon_{i1}^\lambda, \varepsilon_{i2}^\lambda](Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)(Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu].
\end{aligned} \tag{20}$$

Similarly, by the LIE, the RHS of (19) equals the following,

$$\begin{aligned}
&E[G_i|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]
\end{aligned} \tag{21}$$

As a result, in the following, to show that Assumptions SC1-NSP, SC2-NSP, and SC3-NSP are sufficient for Assumption PT-NSP, it suffices to show that each assumption implies the following equality,

$$\begin{aligned}
&E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)(Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]
\end{aligned}$$

(i) By Assumption NSP-X, it follows that

$$
\begin{aligned}
&E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)(Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)(\mu(X_{i2}^\mu, \alpha_i^\mu, \varepsilon_{i2}^\mu) - \mu(X_{i1}^\mu, \alpha_i^\mu, \varepsilon_{i1}^\mu))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
&+ E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)(\lambda_2(X_{i2}^\lambda, \alpha_i^\lambda, \varepsilon_{i2}^\lambda) - \lambda_1(X_{i1}^\lambda, \alpha_i^\lambda, \varepsilon_{i1}^\lambda))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu], \qquad (22)
\end{aligned}
$$

We first examine the first term on the RHS of the above equality. Note that by the symmetry restrictions in Assumptions SC1-NSP.i and SC1-NSP.ii, it follows that a.e. $(a, x^\mu, x_1^\lambda, x_2^\lambda) \in \mathcal{A} \times \mathcal{X}_\mu \times \mathcal{X}_\lambda^2$

$$
\begin{aligned}
&E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)\mu(X_{i1}^\mu, \alpha_i^\mu, \varepsilon_{i1}^\mu)|X_i^\lambda = (x_1^\lambda, x_2^\lambda), X_{i1}^\mu = X_{i2}^\mu = x^\mu, \alpha_i^\mu = a] \\
=&\int \bar{g}(a, x^\mu, x^\mu, x_1^\lambda, x_2^\lambda, e_1, e_2)\mu(x^\mu, a, e_1)dF_{\varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu, \alpha_i^\mu}(e_1, e_2|(x_1^\lambda, x_2^\lambda), x^\mu, a) \\
=&\int \bar{g}(a, x^\mu, x^\mu, x_1^\lambda, x_2^\lambda, e_2, e_1)\mu(x^\mu, a, e_1)dF_{\varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu, \alpha_i^\mu}(e_2, e_1|(x_1^\lambda, x_2^\lambda), x^\mu, a) \\
=&E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)\mu(X_{i2}^\mu, \alpha_i^\mu, \varepsilon_{i2}^\mu)|X_i^\lambda = (x_1^\lambda, x_2^\lambda), X_{i1}^\mu = X_{i2}^\mu = x^\mu, \alpha_i^\mu = a]. \qquad (23)
\end{aligned}
$$

As a result, the first summand in (22) equals zero by (23) and the LIE.

Next, we consider the second summand in (22),

$$
\begin{aligned}
&E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)(\lambda_2(X_{i2}^\lambda, \alpha_i^\lambda, \varepsilon_{i2}^\lambda) - \lambda_1(X_{i1}^\lambda, \alpha_i^\lambda, \varepsilon_{i1}^\lambda))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[\lambda_2(X_{i2}^\lambda, \alpha_i^\lambda, \varepsilon_{i2}^\lambda) - \lambda_1(X_{i1}^\lambda, \alpha_i^\lambda, \varepsilon_{i1}^\lambda)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\bar{g}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu, \varepsilon_{i2}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]. \qquad (24)
\end{aligned}
$$

The first equality follows from the conditional independence assumption in Assumption SC1-NSP.iii. The last equality follows from the time homogeneity of $F_{\varepsilon_{it}^\mu|X_i^\mu, X_i^\lambda, \alpha_i^\mu}$, which follows from the exchangeabilty restriction in Assumption SC1-NSP.ii by Lemma A.4, and implies that $E[\mu(X_{i2}^\mu, \alpha_i^\mu, \varepsilon_{i2}^\mu) - \mu(X_{i1}^\mu, \alpha_i^\mu, \varepsilon_{i1}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu, \alpha_i^\mu] = 0$ and

$$
E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] = E[\lambda_2(X_{i2}^\lambda, \alpha_i^\lambda, \varepsilon_{i2}^\lambda) - \lambda_1(X_{i1}^\lambda, \alpha_i^\lambda, \varepsilon_{i1}^\lambda))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]
$$

by the LIE. As a result, the above implies that Assumption PT-NSP holds.

(ii) By Assumption SC2-NSP.i, we can define $\check{\bar{g}}(a^\mu, x_1^\mu, x_2^\mu, x_1^\lambda, x_2^\lambda, e_1^\lambda) = \bar{g}(a^\mu, x_1^\mu, x_2^\mu, x_1^\lambda, x_2^\lambda, e_1^\lambda, e_2^\lambda)$. By Assumption NSP-X, it follows that

$$
\begin{aligned}
&E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu)(Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu)\Delta_{\mu,i}|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
&+ E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu)(\lambda_2(X_{i2}^\lambda, \alpha_i^\lambda, \varepsilon_{i2}^\lambda) - \lambda_1(X_{i1}^\lambda, \alpha_i^\lambda, \varepsilon_{i1}^\lambda))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[\Delta_{\mu,i}|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
&+ E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[\lambda_2(X_{i2}^\lambda, \alpha_i^\lambda, \varepsilon_{i2}^\lambda) - \lambda_1(X_{i1}^\lambda, \alpha_i^\lambda, \varepsilon_{i1}^\lambda)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda, \varepsilon_{i1}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \qquad (25)
\end{aligned}
$$

The second equality follows from the conditional independence conditions in Assumptions SC2-NSP.ii and SC2-NSP.iii. The last equality follows from Assumption NSP-X. Equation (25) then implies Assumption PT-NSP.

(iii) By Assumption SC3-NSP.i, we can define $\breve{\bar{g}}(a^\mu, x_1^\mu, x_2^\mu, x_1^\lambda, x_2^\lambda) = \bar{g}(a^\mu, x_1^\mu, x_2^\mu, x_1^\lambda, x_2^\lambda, e_1^\lambda, e_2^\lambda)$. Now by the Assumption NSP-X and SC3-NSP.i, it follows that

$$
\begin{aligned}
&E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda)(Y_{i2}(0) - Y_{i1}(0))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda)(\mu(X_{i2}^\mu, \alpha_i^\mu, \varepsilon_{i2}^\mu) - \mu(X_{i1}^\mu, \alpha_i^\mu, \varepsilon_{i1}^\mu))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
&+ E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda)(\lambda_2(X_{i2}^\lambda, \alpha_i^\lambda, \varepsilon_{i2}^\lambda) - \lambda_1(X_{i1}^\lambda, \alpha_i^\lambda, \varepsilon_{i1}^\lambda))|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda)E[\mu(X_{i2}^\mu, \alpha_i^\mu, \varepsilon_{i2}^\mu) - \mu(X_{i1}^\mu, \alpha_i^\mu, \varepsilon_{i1}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu, \alpha_i^\mu]|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
&+ E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[\lambda_2(X_{i2}^\lambda, \alpha_i^\lambda, \varepsilon_{i2}^\lambda) - \lambda_1(X_{i1}^\lambda, \alpha_i^\lambda, \varepsilon_{i1}^\lambda)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[\lambda_2(X_{i2}^\lambda, \alpha_i^\lambda, \varepsilon_{i2}^\lambda) - \lambda_1(X_{i1}^\lambda, \alpha_i^\lambda, \varepsilon_{i1}^\lambda)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] \\
=&E[\breve{\bar{g}}(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu],
\end{aligned}
$$

where the first equality follows from Assumption NSP-X. The second equality follows by applying the LIE to the first term and the conditional independence imposed in Assumption SC3-NSP.iii to the second term. The first term on the RHS of the second equality equals zero by the conditioning on $X_{i1}^\mu = X_{i2}^\mu$ and the time homogeneity condition in Assumption SC3-NSP.ii. The last equality follows from noting, similar as in the proof of (i), that since $E[\mu(X_{i2}^\mu, \alpha_i^\mu, \varepsilon_{i2}^\mu) - \mu(X_{i1}^\mu, \alpha_i^\mu, \varepsilon_{i1}^\mu)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu, \alpha_i^\mu] = 0$,

$$
E[Y_{i2}(0) - Y_{i1}(0)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu] = E[\lambda_2(X_{i2}^\lambda, \alpha_i^\lambda, \varepsilon_{i2}^\lambda) - \lambda_1(X_{i1}^\lambda, \alpha_i^\lambda, \varepsilon_{i1}^\lambda)|X_i^\lambda, X_{i1}^\mu = X_{i2}^\mu]
$$

by the LIE. This completes the proof. □

B.8 Proof of Proposition 5.2

Under Assumption NSP-X,

$$
\begin{aligned}
&E[Y_{i2}(0)-Y_{i1}(0)|G_i,X_i^{\lambda},X_{i1}^{\mu}=X_{i2}^{\mu}]\\
=&E[\mu(X_{i2}^{\mu},\alpha_i^{\mu},\varepsilon_{i2}^{\mu})-\mu(X_{i1}^{\mu},\alpha_i^{\mu},\varepsilon_{i1}^{\mu})|G_i,X_i^{\lambda},X_{i1}^{\mu}=X_{i2}^{\mu}] \qquad (26)\\
&+E[\lambda_2(X_{i2}^{\lambda},\alpha_i^{\lambda},\varepsilon_{i2}^{\lambda})-\lambda_1(X_{i1}^{\lambda},\alpha_i^{\lambda},\varepsilon_{i1}^{\lambda})|G_i,X_i^{\lambda},X_{i1}^{\mu}=X_{i2}^{\mu}]. \qquad (27)
\end{aligned}
$$

The remainder of the proof follows in two steps. First, we show that the term in (26) equals zero under our assumptions. Second, we show that the second term is conditionally mean independent of G_i, which implies Assumption PT-NSP.

We proceed to show that under Assumption TH the term in (26) equals zero by the following,

$$
\begin{aligned}
&E[\mu(X_{i1}^{\mu},\alpha_i^{\mu},\varepsilon_{i1}^{\mu})|G_i=g,X_i^{\lambda}=(x_1^{\lambda},x_2^{\lambda}),X_{i1}^{\mu}=X_{i2}^{\mu}=x^{\mu}]\\
=&\int \mu(x^{\mu},a^{\mu},e^{\mu})dF_{\alpha_i^{\mu},\varepsilon_{i1}^{\mu}|G_i,X_i^{\mu},X_i^{\lambda}}(a^{\mu},e^{\mu}|g,(x^{\mu},x^{\mu}),(x_1^{\lambda},x_2^{\lambda}))\\
=&\int \mu(x^{\mu},a^{\mu},e^{\mu})dF_{\alpha_i^{\mu},\varepsilon_{i2}^{\mu}|G_i,X_i^{\mu},X_i^{\lambda}}(a^{\mu},e^{\mu}|g,(x^{\mu},x^{\mu}),(x_1^{\lambda},x_2^{\lambda}))\\
=&E[\mu(X_{i2}^{\mu},\alpha_i^{\mu},\varepsilon_{i2}^{\mu})|G_i=g,X_i^{\lambda}=(x_1^{\lambda},x_2^{\lambda}),X_{i1}^{\mu}=X_{i2}^{\mu}=x^{\mu}], \qquad (28)
\end{aligned}
$$

where the first and last equalities follow by definition, whereas the penultimate equality follows from Assumption TH noting that it implies $\alpha_i^{\mu},\varepsilon_{i1}^{\mu}|G_i,X_i^{\mu},X_i^{\lambda}\overset{d}{=}\alpha_i^{\mu},\varepsilon_{i2}^{\mu}|G_i,X_i^{\mu},X_i^{\lambda}$.

Finally, we show that Assumption CRE implies the following for (27)

$$
\begin{aligned}
&E[\lambda_2(X_{i2}^{\lambda},\alpha_i^{\lambda},\varepsilon_{i2}^{\lambda})-\lambda_1(X_{i1}^{\lambda},\alpha_i^{\lambda},\varepsilon_{i1}^{\lambda})|G_i=g,X_i^{\lambda}=(x_1^{\lambda},x_2^{\lambda}),X_{i1}^{\mu}=X_{i2}^{\mu}=x^{\mu}]\\
=&\int(\lambda_2(x_2^{\lambda},a^{\lambda},e_2^{\lambda})-\lambda_1(x_1^{\lambda},a^{\lambda},e_1^{\lambda}))dF_{\alpha_i^{\lambda},\varepsilon_{i1}^{\lambda},\varepsilon_{i2}^{\lambda}|G_i,X_i^{\mu},X_i^{\lambda}}(a^{\lambda},(e_1^{\lambda},e_2^{\lambda})|g,(x^{\mu},x^{\mu}),(x_1^{\lambda},x_2^{\lambda}))\\
=&\int(\lambda_2(x_2^{\lambda},a^{\lambda},e_2^{\lambda})-\lambda_1(x_1^{\lambda},a^{\lambda},e_1^{\lambda}))dF_{\alpha_i^{\lambda},\varepsilon_{i1}^{\lambda},\varepsilon_{i2}^{\lambda}|X_i^{\mu},X_i^{\lambda}}(a^{\lambda},(e_1^{\lambda},e_2^{\lambda})|(x^{\mu},x^{\mu}),(x_1^{\lambda},x_2^{\lambda}))\\
=&E[\lambda_2(X_{i2}^{\lambda},\alpha_i^{\lambda},\varepsilon_{i2}^{\lambda})-\lambda_1(X_{i1}^{\lambda},\alpha_i^{\lambda},\varepsilon_{i1}^{\lambda})|X_i^{\lambda}=(x_1^{\lambda},x_2^{\lambda}),X_{i1}^{\mu}=X_{i2}^{\mu}=x^{\mu}], \qquad (29)
\end{aligned}
$$

where the penultimate equality follows by Assumption CRE. This completes the proof. □

B.9 Proof of Proposition 5.3

Throughout this proof, equalities involving conditioning statements are understood to hold *a.e.* We proceed to show each result separately.

(i) It suffices to show (i.a) Assumptions SC1-NSP.i and SC1-NSP.ii imply Assumption TH and (i.b) Assumptions SC1-NSP.i and SC1-NSP.iii imply Assumption CRE.

(i.a) Consider

$$F_{\varepsilon_{i1}^{\mu},G_i|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e_1,g|x^{\mu},x^{\lambda},a) = F_{G_i|\varepsilon_{i1}^{\mu},X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(g|e_1,x^{\mu},x^{\lambda},a)F_{\varepsilon_{i1}^{\mu}|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e_1|x^{\mu},x^{\mu},a),$$

where $x^{\mu} = (x_1^{\mu}, x_2^{\mu})$ and $x^{\lambda} = (x_1^{\lambda}, x_2^{\lambda})$. Assumption SC1-NSP.ii implies $F_{\varepsilon_{i1}^{\mu}|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e|x^{\mu},x^{\lambda},a) = F_{\varepsilon_{i2}^{\mu}|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e|x^{\mu},x^{\lambda},a)$ as well as $F_{\varepsilon_{i1}^{\mu}|\varepsilon_{i2}^{\mu},X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e_1|e_2,x^{\mu},x^{\lambda},a) = F_{\varepsilon_{i2}^{\mu}|\varepsilon_{i1}^{\mu},X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e_1|e_2,x^{\mu},x^{\lambda},a)$ by Lemma A.4, which implies

$$\begin{aligned}
&F_{G_i|\varepsilon_{i1}^{\mu},X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(g|e_1,x^{\mu},x^{\lambda},a)\\
=&\int 1\{g(a,x_1^{\mu},x_2^{\mu},x_1^{\lambda},x_2^{\lambda},e_1,e_2)\le g\}dF_{\varepsilon_{i2}^{\mu}|\varepsilon_{i1}^{\mu},X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e_2|e_1,x^{\mu},x^{\lambda},a)\\
=&\int 1\{g(a,x_1^{\mu},x_2^{\mu},x_1^{\lambda},x_2^{\lambda},e_2,e_1)\le g\}dF_{\varepsilon_{i1}^{\mu}|\varepsilon_{i2}^{\mu},X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e_2|e_1,x^{\mu},x^{\lambda},a)\\
=&F_{G_i|\varepsilon_{i2}^{\mu},X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(g|e_1,x^{\mu},x^{\lambda},a).
\end{aligned}\tag{30}$$

As a result,

$$\begin{aligned}
&F_{\varepsilon_{i1}^{\mu},G_i|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e_1,g|x^{\mu},x^{\lambda},a) = F_{G_i|\varepsilon_{i1}^{\mu},X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(g|e_1,x^{\mu},x^{\lambda},a)F_{\varepsilon_{i1}^{\mu}|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e_1|x^{\mu},x^{\lambda},a)\\
=&F_{G_i|\varepsilon_{i2}^{\mu},X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(g|e_1,x^{\mu},x^{\lambda},a)F_{\varepsilon_{i2}^{\mu}|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e_1|x^{\mu},x^{\lambda},a)\\
=&F_{\varepsilon_{i2}^{\mu},G_i|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e_1,g|x^{\mu},x^{\lambda},a).
\end{aligned}\tag{31}$$

This implies Assumption TH by the definition of a conditional distribution,

$$F_{\varepsilon_{it}^{\mu}|G_i,X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(e|g,x^{\mu},x^{\lambda},a) = \frac{F_{\varepsilon_{it}^{\mu},G_i|X_i^{\mu},X_i^{\lambda}\alpha_i^{\mu}}(e,g|x^{\mu},x^{\lambda})}{F_{G_i|X_i^{\mu},X_i^{\lambda}\alpha_i^{\mu}}(g|x^{\mu},x^{\lambda},a)},$$

where $F_{G_i|X_i^{\mu},X_i^{\lambda},\alpha_i^{\mu}}(g|x^{\mu},x^{\lambda},a) > 0$ a.s. for $g = 0, 1$ by assumption.

(i.b) This statement follows in a straightforward manner from the definition of G_i in Assumption SC1-NSP.i and the conditional independence condition in Assumption SC1-NSP.iii which together imply Assumption CRE. This completes the proof of (i).

(ii) To show the result, it suffices to show that (ii.a) Assumptions SC3-NSP.i and SC3-NSP.ii imply Assumption TH and (ii.b) Assumptions SC3-NSP.i and SC3-NSP.iii imply Assumption CRE.

(ii.a) Under Assumptions SC3-NSP.i and SC3-NSP.ii, $G_i = g(\alpha_i^{\mu}, X_{i1}^{\mu}, X_{i2}^{\mu}, X_{i1}^{\lambda}, X_{i2}^{\lambda})$ is a degenerate random variable equaling either zero or one with probability one conditional on

X_i^μ, X_i^λ and α_i^μ. As a result,

$$
\begin{aligned}
& F_{\varepsilon_{it}^\mu|G_i,X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e|g,x^\mu,x^\lambda,a) \\
&= \sum_{g=0,1} P(\varepsilon_{it}^\mu \le e|G_i = g(a,x_1^\mu,x_2^\mu,x_1^\lambda,x_2^\lambda), X_i^\mu = x^\mu, X_i^\lambda = x^\lambda, \alpha_i^\mu = a) 1\{g(a,x_1^\mu,x_2^\mu,x_1^\lambda,x_2^\lambda) = g\} \\
&= \sum_{g=0,1} P(\varepsilon_{it}^\mu \le e|X_i^\mu = x^\mu, X_i^\lambda = x^\lambda, \alpha_i^\mu = a) 1\{g(a,x_1^\mu,x_2^\mu,x_1^\lambda,x_2^\lambda) = g\} \\
&= \sum_{g=0,1} F_{\varepsilon_{it}^\mu|X_i^\mu,X_i^\lambda,\alpha_i^\mu}(e|a,x^\mu,x^\lambda) 1\{g(a,x_1^\mu,x_2^\mu,x_1^\lambda,x_2^\lambda) = g\}. \qquad (32)
\end{aligned}
$$

As a result, Assumption SC3-NSP.i together with the time homogeneity of $F_{\varepsilon_{it}^\mu|X_i^\mu,X_i^\lambda,\alpha_i^\mu}$ in Assumption SC3-NSP.ii is sufficient for the time homogeneity of $F_{\varepsilon_{it}^\mu|G_i,X_i^\mu,X_i^\lambda,\alpha_i^\mu}$, which yields Assumption TH.

(ii.b) The statement (ii.b) is immediate from noting that Assumption SC3-NSP.iii together with $G_i = g(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda)$ imply that $g(\alpha_i^\mu, X_{i1}^\mu, X_{i2}^\mu, X_{i1}^\lambda, X_{i2}^\lambda) \perp (\alpha_i^\lambda, \varepsilon_{i1}^\lambda, \varepsilon_{i2}^\lambda)|X_i^\mu, X_i^\lambda$, which is equivalent to Assumption CRE. This completes the proof of (ii). $\square$

Supplemental Online Appendix

C Extensions

C.1 Parallel trends for any distribution

In the main text, we derive necessary and sufficient conditions for a scenario where researchers are not willing to choose a specific selection mechanism. Here we consider an alternative scenario where researchers are not willing to impose any restrictions on the distribution of unobservables, $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}}$, and require parallel trends to hold for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}}$.

The following proposition shows that Assumption PT holds for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}}$ in a complete class if and only if selection is independent of the time-varying unobservables $(\varepsilon_{i1},\varepsilon_{i2})$. Before we state the proposition, we recall the definition of a complete class of distributions (Equations (4.8)–(4.9) on p.115 in Lehmann and Romano, 2005).

Definition C.1 (Completeness of a class of distributions)**.** *Let W be a vector of random variables. A family of distributions $\mathcal{F}$ is complete if*

$$E[f(W)] = 0 \quad \textit{for all} \;\; F_W \in \mathcal{F}$$

implies

$$f(w) = 0 \quad \textit{almost everywhere (a.e.)} \;\; \mathcal{F}.$$

Proposition C.1 (Necessary and sufficient condition for parallel trends for any distribution of unobservables)**.** *Suppose that $g \in \mathcal{G}_{all}$ and $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}$, where $\mathcal{F}$ is a complete family of probability distributions satisfying $P(\dot{Y}_{i1}(0) \neq \dot{Y}_{i2}(0)) = 1$ and $P(g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) = 1) \in (0,1)$. Assumption PT holds for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}$ if and only if $P(G_i = 1|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) = P(G_i = 1)$ a.s. for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}} \in \mathcal{F}$.*

In Proposition C.1, we require $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\nu_i,\eta_{i1},\eta_{i2}}$ to belong to a complete family of distributions, $\mathcal{F}$. Completeness requires that the class of possible distributions of unobservables is rich enough. This condition is key for showing that parallel trends implies that selection is independent of all unobservable determinants of $Y_{i1}(0)$ and $Y_{i2}(0)$.

C.2 Multiple periods and multiple groups

Here we generalize our results to DiD designs with multiple periods and multiple groups. The setup and notation are based on Callaway and Sant'Anna (2021), Sun and Abraham (2021), and Roth et al. (2022). We focus on necessary conditions.

Let $t \in \{1, 2, \dots, T\}$ index the periods. Suppose that at time $t = 1$, no units are treated, at $t = 2$, some units become treated, while others remain untreated, and so on. Previously treated units remain treated for all periods. Units can be categorized based on their treatment adoption pattern $D_i = (D_{i1}, \dots, D_{iT})$. We define the group indicator G_i as the first period in which units are treated, $G_i = \min\{t \in \{1, \dots, T\} : D_{it} = 1\}$, and set $G_i = \infty$ for the never-treated units so that $G_i \in \{2, \dots, T, \infty\}$.[17]

Potential outcomes are indexed by the entire treatment sequence $(d_1, \dots, d_T) \in \{0,1\}^T$, $Y_{it}(d_1, \dots, d_T)$. Since treatment is an absorbing state, the potential outcomes can be indexed by the first treatment period only. Define $Y_{it}(g) = Y_{it}(\mathbf{0}_{g-1}, \mathbf{1}_{T-g+1})$ for $g \in \{2, \dots, T\}$ and $Y_{it}(\infty) = Y_{it}(\mathbf{0}_T)$, where $\mathbf{0}_s \equiv (0, \dots, 0) \in \mathbb{R}^s$ and $\mathbf{1}_s \equiv (1, \dots, 1) \in \mathbb{R}^s$. Observed outcomes are given by $Y_{it} = \sum_{g\in\{2,\dots,T,\infty\}} 1\{G_i = g\}Y_{it}(g)$. We maintain a standard no-anticipation assumption (e.g., Roth et al., 2022).

Assumption NA. *For $g \in \{2, \dots, T, \infty\}$ and $t < g$, $Y_{it}(g) = Y_{it}(\infty)$.*

Our objects of interest are the group-time ATTs,

$$\text{ATT}(g, t) = E[Y_{it}(g) - Y_{it}(\infty)|G = g]. \tag{33}$$

We impose the following parallel trends assumption to identify the ATT(g, t).[18]

[17] Since G_i is a random variable with finite support, we emphasize that $\{\infty\}$ is merely a label.

[18] In our setting, this parallel trends assumption corresponds to the ones made by Borusyak et al. (2021), Callaway and Sant'Anna (2021), Gardner (2021), Sun and Abraham (2021), and Wooldridge (2021); see also de Chaisemartin and D'Haultfœuille (2020) and Marcus and Sant'Anna (2021) for related assumptions.

Assumption PT-MP. *For* $(g,t) \in \{2,\ldots,T\}^2$

$$E[Y_{it}(\infty) - Y_{i(t-1)}(\infty) \,|\, G_i = g] = E[Y_{it}(\infty) - Y_{i(t-1)}(\infty)|G_i = \infty] \tag{34}$$

We consider a general nonseparable outcome model.

$$Y_{it}(\infty) = \xi_t(\alpha_i, \varepsilon_{it}), \quad i = 1,\ldots,N, \quad t = 1,\ldots,T.$$

Selection into treatment can depend on the unobservable determinants of $Y_{it}(\infty)$ as well as additional unobservables,

$$G_i = g(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \nu_i, \eta_{i1}, \ldots, \eta_{iT}).$$

As before, let $\mathcal{G}_{\text{all}}$ denote the set of all selection mechanisms $g(\cdot)$ and define the following classes of restricted selection mechanisms, which are natural analogs of those considered in Section 3.

$$\begin{aligned}
\mathcal{G}_{\text{if}} &= \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, \ldots, e_T, v, t_1, \ldots, t_T) \text{ is a trivial function of } (e_T, t_T)\} \\
\mathcal{G}_{\text{if'}} &= \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, \ldots, e_T, v, t_1, \ldots, t_T) \text{ is a trivial function of } (e_2, \ldots, e_T, t_2, \ldots, t_T)\} \\
\mathcal{G}_{\text{fe}} &= \{g \in \mathcal{G}_{\text{all}} : g(a, e_1, \ldots, e_T, v, t_1, \ldots, t_T) \text{ is a trivial function of } (e_1, \ldots, e_T, t_1, \ldots, t_T)\}
\end{aligned}$$

The following assumption generalizes Assumption SEL to the multiple-period multiple-group setting. It ensures that the selection mechanisms used to establish the necessary and sufficient conditions for parallel trends are non-degenerate.

Assumption SEL-MP. *There exists a component of ν_i, labeled ν_i^1 (w.l.o.g.), such that $\nu_i^1 \perp (\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT})$. In addition, there exists a non-overlapping partition of the support of v_i^1, $\{B_g\}_{g=2}^T$, such that $P(v_i^1 \in B_g) \in (0,1)$ for $g \in \{2,\ldots,T\}$.*

The following four propositions extend the necessary and sufficient conditions in Propositions 3.1, 3.2, and 3.3 to the more general DiD setting in this section. All these conditions are natural generalizations of their counterparts in the 2×2 case.

Proposition C.2 (Necessary and sufficient condition for $g \in \mathcal{G}_{\text{all}}$)**.** *Suppose that Assumptions NA and SEL-MP hold. Suppose further that either $P(\dot{Y}_{it}(\infty) > \dot{Y}_{i(t-1)}(\infty)) < 1$ or $P(\dot{Y}_{it}(\infty) < \dot{Y}_{i(t-1)}(\infty)) < 1$ for each $t \in \{2,\ldots,T\}$. Then Assumption PT-MP holds for all $g \in \mathcal{G}_{all}$ satisfying $P(G_i = g) \in (0,1)$ for $g \in \{2,\ldots,T,\infty\}$ if and only if $\dot{Y}_{i1}(\infty) = \cdots = \dot{Y}_{iT}(\infty)$ a.s.*

Proposition C.3 (Necessary condition for $g \in \mathcal{G}_{\text{if}}$)**.** *Suppose that Assumptions NA and SEL-MP hold. Suppose further that either* $P(E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{i(t-1)}] > \dot{Y}_{i(t-1)}(\infty)) < 1$ *or* $P(E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{i(t-1)}] < \dot{Y}_{i(t-1)}(\infty)) < 1$ *for each* $t \in \{2, \ldots, T\}$*. If Assumption PT-MP holds for all* $g \in \mathcal{G}_{if}$ *satisfying* $P(G_i = g) \in (0,1)$ *for* $g \in \{2, \ldots, T, \infty\}$*, then* $E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{i(t-1)}] = \dot{Y}_{i(t-1)}(\infty)$ *a.s. for* $t \in \{2, \ldots, T\}$*.*

Proposition C.4 (Necessary condition for $g \in \mathcal{G}_{\text{if'}}$)**.** *Suppose that Assumptions NA and SEL-MP hold. Suppose further that either* $P(E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}] > E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i, \varepsilon_{i1}]) < 1$ *or* $P(E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}] < E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i, \varepsilon_{i1}]) < 1$ *for each* $t \in \{2, \ldots, T\}$*. If Assumption PT-MP holds for all* $g \in \mathcal{G}_{if'}$ *satisfying* $P(G_i = g) \in (0,1)$ *for* $g \in \{2, \ldots, T, \infty\}$*, then* $E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}] = E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i, \varepsilon_{i1}]$ *a.s. for* $t \in \{2, \ldots, T\}$*.*

Proposition C.5 (Necessary condition for $g \in \mathcal{G}_{\text{fe}}$)**.** *Suppose that Assumptions NA and SEL-MP hold. Suppose further that either* $P(E[\dot{Y}_{it}(\infty)|\alpha_i] > E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i]) < 1$ *or* $P(E[\dot{Y}_{it}(\infty)|\alpha_i] < E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i]) < 1$ *for each* $t \in \{2, \ldots, T\}$*. If Assumption PT-MP holds for all* $g \in \mathcal{G}_{fe}$ *satisfying* $P(G_i = g) \in (0,1)$ *for* $g \in \{2, \ldots, T, \infty\}$*, then* $E[\dot{Y}_{it}(\infty)|\alpha_i] = E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i]$ *a.s. for* $t \in \{2, \ldots, T\}$*.*

The necessary conditions in Propositions C.3, C.4 and C.5 are sufficient for PT-MP under straightforward extensions of the conditions in Section 3.2.

D Sufficient conditions for Assumption SP-X under a separable model

Consider the following separable model with covariates.

Assumption SP-X.

$$Y_{it}(0) = \alpha_i + \lambda_t + \gamma_t(X_{it}) + \varepsilon_{it}, \quad E[\varepsilon_{it}] = 0, \quad i = 1, \ldots, N, \quad t = 1, 2. \tag{35}$$

Assumption SP-X allows for nonparametric covariate-specific trends, which is a key reason for incorporating covariates in DiD analyses. It nests commonly used parametric specifications such as $\gamma_t(X_{it}) = X_{it}'\beta_t$. Recall that we assume that the treatment does not affect X_{it}.

To focus the discussion on the different roles played by the time-varying observable and unobservable determinants of $Y_{it}(0)$, we state our sufficient conditions in terms of the projected selection mechanism,

$$\bar{g}(a, x_1, x_2, e_1, e_2) = E[G_i|\alpha_i = a, X_{i1} = x_1, X_{i2} = x_2, \varepsilon_{i1} = e_1, \varepsilon_{i2} = e_2].$$

Assumption SC1-X. *The following conditions hold:*

(i) $\bar{g}(a, x_1, x_2, e_1, e_2)$ *is a symmetric function in* e_1 *and* e_2*.*

(ii) $\varepsilon_{i1}, \varepsilon_{i2} | \alpha_i, X_i \overset{d}{=} \varepsilon_{i2}, \varepsilon_{i1} | \alpha_i, X_i$*.*

Assumption SC2-X. *The following conditions hold:*

(i) $\bar{g}(a, x_1, x_2, e_1, e_2)$ *is a trivial function of* e_2*.*

(ii) $E[\varepsilon_{i2} - \varepsilon_{i1} | \alpha_i, \varepsilon_{i1}, X_i] = E[\varepsilon_{i2} - \varepsilon_{i1} | X_i]$*.*

Assumption SC3-X. *The following conditions hold:*

(i) $\bar{g}(a, x_1, x_2, e_1, e_2)$ *is a trivial function of* e_1 *and* e_2*.*

(ii) $E[\varepsilon_{i1} | \alpha_i, X_i] = E[\varepsilon_{i2} | \alpha_i, X_i]$*.*

Assumptions SC1-X, SC2-X, and SC3-X are conditional versions of Assumptions SC1, SC2, and SC3. They demonstrate that incorporating time-varying covariates makes the restrictions on the selection mechanism more plausible.

The following proposition shows that Assumptions SC1-X, SC2-X, and SC3-X are sufficient for Assumption PT-X.

Proposition D.1 (Sufficient conditions)**.** *Suppose that Assumption SP-X holds and* $P(G_i = 1 | X_i) \in (0, 1)$ *a.s. Then (i) Assumption SC1-X implies Assumption PT-X, (ii) Assumption SC2-X implies Assumption PT-X, and (iii) Assumption SC3-X implies Assumption PT-X.*

The results in this section have implications for the choice of covariates to be included in DiD analyses. Proposition D.1 provides several avenues for justifying the inclusion of covariates in DiD analyses. A key takeaway from Proposition D.1 is that time-invariant and time-varying covariates play different roles in ensuring that Assumption PT-X holds. Any (observable) time-varying factors that asymmetrically affect selection should be included as covariates. In addition, practitioners should include time-invariant and time-varying covariates that render the distributional restrictions plausible in their application.

All sufficient conditions in Proposition D.1 allow for selection on unobservable determinants of the untreated potential outcome. This is in contrast with the unconfoundedness assumptions commonly used in cross-sectional studies (e.g., Imbens, 2004; Imbens and Wooldridge, 2009). Therefore, these results elucidate the differences between conditional parallel trends and unconfoundedness-type assumptions.

E Proofs of results in Supplemental Online Appendix

E.1 Auxiliary lemma

Lemma E.1 (Equivalence with multiple periods)**.** *Suppose that Assumption NA holds and $P(G_i = g) \in (0,1)$ for $g \in \{2,\dots,T,\infty\}$. Then Assumption PT-MP is equivalent to $E[1\{G_i = g\}(\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty))] = 0$ for $g \in \{2,\dots,T,\infty\}$ and $t \in \{2,\dots,T\}$.*

Proof. Assumption PT-MP is equivalent to

$$E[\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty)|G_i = g] = E[\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty)|G_i = \infty] \quad \text{for } (g,t) \in \{2,\dots,T\}^2,$$

which, since $E[\dot{Y}_{it}(\infty)] = 0$, is also equivalent to

$$E[\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty)|G_i = g] = 0 \quad \text{for } (g,t) \in \{2,\dots,T,\infty\} \times \{2,\dots,T\}. \tag{36}$$

Thus, we need to show that (36) is equivalent to $E[1\{G_i = g\}(\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty))] = 0$ for $g \in \{2,\dots,T,\infty\}$ and $t \in \{2,\dots,T\}$. This follows because

$$E[\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty)|G_i = g] = \frac{E[1\{G_i = g\}(\dot{Y}_{it}(\infty) - \dot{Y}_{i(t-1)}(\infty))]}{P(G_i = g)}$$

for $(g,t) \in \{2,\dots,T,\infty\} \times \{2,\dots,T\}$, since $P(G_i = g) \in (0,1)$ for $g \in \{2,\dots,T,\infty\}$ by assumption. □

E.2 Proof of Proposition C.1

"$\Longrightarrow$": By Lemma A.1, Assumption PT is equivalent to $E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] = 0$, which in turn is equivalent to the following

$$E[\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1}))] = 0, \tag{37}$$

where $\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) = E[g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}) - E[G_i]|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}]$ and $\dot{\xi}_t(\alpha_i, \varepsilon_{it}) = \xi_t(\alpha_i, \varepsilon_{it}) - E[Y_{it}(0)]$ for $t = 1, 2$. The equivalence between $E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] = 0$ and (37) follows by the LIE and subtracting $E[G_i]E[\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0)]$, noting that it equals zero by construction.

It follows that Assumption PT holding for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\eta_{i1},\eta_{i2}} \in \mathcal{F}$ is equivalent to

$$E[\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1}))] = 0, \tag{38}$$

for all $F_{\alpha_i,\varepsilon_{i1},\varepsilon_{i2},\eta_{i1},\eta_{i2}} \in \mathcal{F}$. By completeness of $\mathcal{F}$, the last equality implies the following

(Lehmann and Romano, 2005, p.115)

$$P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1})) = 0) = 1 \quad \text{for all } F_{\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \eta_{i1}, \eta_{i2}} \in \mathcal{F}. \tag{39}$$

Now note that the left-hand side of (39) can be simplified as follows,

$$\begin{aligned}
&P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1}) = 0) \\
=&P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1})), \dot{\xi}_2(\alpha_i, \varepsilon_{i2}) = \dot{\xi}_1(\alpha_i, \varepsilon_{i1})) \\
&+ P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1})), \dot{\xi}_2(\alpha_i, \varepsilon_{i2}) \neq \dot{\xi}_1(\alpha_i, \varepsilon_{i1})) \\
=&P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) - \dot{\xi}_1(\alpha_i, \varepsilon_{i1}))|\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) \neq \dot{\xi}_1(\alpha_i, \varepsilon_{i1})) \\
=&P(\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) = 0) = 1,
\end{aligned} \tag{40}$$

where the penultimate equality follows since $P(\dot{\xi}_2(\alpha_i, \varepsilon_{i2}) \neq \dot{\xi}_1(\alpha_i, \varepsilon_{i1})) = P(\dot{Y}_{i2}(0) \neq \dot{Y}_{i1}(0)) = 1$ by assumption. As a result, by the definition of $\bar{g}(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2})$,

$$P(E[g(\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2})|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}] = E[G_i]) = 1 \quad \text{for all } F_{\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}} \in \mathcal{F}. \tag{41}$$

"$\Longleftarrow$": The if statement follows by the LIE. All following statements are understood to hold for all $F_{\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}, \nu_i, \eta_{i1}, \eta_{i2}} \in \mathcal{F}$. Note that $P(G_i = 1|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}) = P(G_i = 1)$ a.s. is equivalent to $E[G_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}] = E[G_i]$ a.s. Next, the LIE implies the following equality

$$\begin{aligned}
E[G_i(\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] &= E[E[G_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}](\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] \\
&= E[E[G_i](\dot{Y}_{i2}(0) - \dot{Y}_{i1}(0))] = 0.
\end{aligned} \tag{42}$$

The second equality follows from $E[G_i|\alpha_i, \varepsilon_{i1}, \varepsilon_{i2}] = E[G_i]$ a.s. The last equality follows from $E[\dot{Y}_{it}(0)] = 0$ for $t = 1, 2$ by definition. □

E.3 Proof of Proposition C.2

"$\Longrightarrow$": We first consider the case where $P(\dot{Y}_{it}(\infty) > \dot{Y}_{i(t-1)}(\infty)) < 1$ for $t \in \{2, \ldots, T\}$. Since Assumption PT-MP holds for all $g \in \mathcal{G}_{\text{all}}$, it holds for the following selection mechanism, where $\mathcal{G}_S = \{2, \ldots, T\}$ denotes the set of switcher groups,

$$\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = \begin{cases} g \text{ if } 1\{\dot{Y}_{ig}(\infty) \leq \dot{Y}_{i(g-1)}(\infty)\}1\{\nu_i^1 \in B_g\} = 1, g \in \mathcal{G}_S \\ \infty \text{ if } \breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) \notin \mathcal{G}_S, \end{cases}$$

where $\zeta_i = (\nu_i, \eta_{i1}, \ldots, \eta_{iT})$. By Lemma E.1, Assumption PT-MP implies that for any $g \in \mathcal{G}_S$,

$$\begin{aligned}
&E[1\{\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = g\}(\dot{Y}_{ig}(\infty) - \dot{Y}_{i(g-1)}(\infty)] \\
&= E[1\{\dot{Y}_{ig}(\infty) \leq \dot{Y}_{i(g-1)}(\infty)\}1\{\nu_i^1 \in B_g\}(\dot{Y}_{ig}(\infty) - \dot{Y}_{i(g-1)}(\infty))] = 0.
\end{aligned}$$

By Assumption SEL-MP and the additional regularity conditions in the proposition, we can invoke Lemma A.3.i while setting $\omega_i = (\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT})$ and replacing G_i $(t \in \{1, 2\})$ with $1\{G_i = g\}$ $(t \in \{g-1, g\})$ for each $g \in \mathcal{G}_S$. This implies that $\dot{Y}_{ig}(0) = \dot{Y}_{i(g-1)}(0)$ a.s. for each $g \in \mathcal{G}_S = \{2, \ldots, T\}$, which implies the result.

The proof for the case where $P(\dot{Y}_{it}(\infty) < \dot{Y}_{i(t-1)}(\infty)) < 1$ for $t \in \{2, \ldots, T\}$ follows symmetrically using the selection mechanism,

$$\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = \begin{cases} g \text{ if } 1\{\dot{Y}_{ig}(\infty) \geq \dot{Y}_{i(g-1)}(\infty)\}1\{\nu_i^1 \in B_g\} = 1, g \in \mathcal{G}_S \\ \infty \text{ if } \breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) \notin \mathcal{G}_S, \end{cases}$$

and invoking Lemma A.3.ii.

The proof for the case where $P(\dot{Y}_{it}(\infty) > \dot{Y}_{i(t-1)}(\infty)) < 1$ for $t \in \mathcal{G}_1 \subset \mathcal{G}_S$ and $P(\dot{Y}_{is}(\infty) < \dot{Y}_{i(s-1)}(\infty)) < 1$ for $s \in \mathcal{G}_2 = \mathcal{G}_1^c \cap \mathcal{G}_S$ follows from using the following selection mechanism

$$\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = \begin{cases} g \text{ if } 1\{\dot{Y}_{ig}(\infty) \leq \dot{Y}_{i(g-1)}(\infty)\}1\{\nu_i^1 \in B_g\} = 1, g \in \mathcal{G}_1, \\ g \text{ if } 1\{\dot{Y}_{ig}(\infty) \geq \dot{Y}_{i(g-1)}(\infty)\}1\{\nu_i^1 \in B_g\} = 1, g \in \mathcal{G}_2, \\ \infty \text{ if } \breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) \notin \mathcal{G}_S, \end{cases}$$

and invoking Lemma A.3.i for $g \in \mathcal{G}_1$ and Lemma A.3.ii for $g \in \mathcal{G}_2$.

"$\Longleftarrow$": This direction is immediate. $\square$

E.4 Proof of Proposition C.3

The proof follows from similar arguments as in Proposition C.2 using the following selection mechanism for the case where $P(E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{i(t-1)}] > \dot{Y}_{i(t-1)}(\infty)) < 1$ for $t \in \{2, \ldots, T\}$,

$$\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = \begin{cases} g \text{ if } 1\{E[\dot{Y}_{ig}(\infty)|\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{i(g-1)}] \leq \dot{Y}_{i(g-1)}(\infty)\}1\{\nu_i \in B_g\} = 1, g \in \mathcal{G}_S, \\ \infty \text{ if } \breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) \notin \mathcal{G}_S, \end{cases}$$

where $\mathcal{G}_S$ and ζ_i are defined in the proof of Proposition C.2. $\square$

E.5 Proof of Proposition C.4

The proof follows from similar arguments as in Proposition C.2 using the following selection mechanism for the case where $P(E[\dot{Y}_{it}(\infty)|\alpha_i, \varepsilon_{i1}] > E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i, \varepsilon_{i1}]) < 1$ for $t \in \{2, \ldots, T\}$,

$$\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = \begin{cases} g \text{ if } 1\{E[\dot{Y}_{ig}(\infty)|\alpha_i, \varepsilon_{i1}] \le E[\dot{Y}_{i(g-1)}(\infty)|\alpha_i, \varepsilon_{i1}]\}1\{\nu_i \in B_g\} = 1, g \in \mathcal{G}_S, \\ \infty \text{ if } \breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) \notin \mathcal{G}_S, \end{cases}$$

where $\mathcal{G}_S$ and ζ_i are defined in the proof of Proposition C.2. □

E.6 Proof of Proposition C.5

The proof follows from similar arguments as in Proposition C.2 using the following selection mechanism for the case where $P(E[\dot{Y}_{it}(\infty)|\alpha_i] > E[\dot{Y}_{i(t-1)}(\infty)|\alpha_i]) < 1$ for $t \in \{2, \ldots, T\}$,

$$\breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) = \begin{cases} g \text{ if } 1\{E[\dot{Y}_{ig}(\infty)|\alpha_i] \le E[\dot{Y}_{i(g-1)}(\infty)|\alpha_i]\}1\{\nu_i \in B_g\} = 1, g \in \mathcal{G}_S, \\ \infty \text{ if } \breve{g}(\alpha_i, \varepsilon_{i1}, \ldots, \varepsilon_{iT}, \zeta_i) \notin \mathcal{G}_S, \end{cases}$$

where $\mathcal{G}_S$ and ζ_i are defined in the proof of Proposition C.2.

E.7 Proof of Proposition D.1

In this proof, all equalities involving random variables are understood to hold a.s. By Lemma A.1, it suffices to show that each assumption implies $E[G_i(\varepsilon_{i2} - \varepsilon_{i1})|X_i] = E[G_i|X_i]E[\varepsilon_{i2} - \varepsilon_{i1}|X_i]$.

(i) The exchangeability restrictions in Assumption SC1-X imply the following:

$$\begin{aligned} &E[\bar{g}(\alpha_i, X_{i1}, X_{i2}, \varepsilon_{i1}, \varepsilon_{i2})\varepsilon_{i1}|\alpha_i = a, X_i = (x_1, x_2)] \\ &= \int \bar{g}(a, x_1, x_2, e_1, e_2)e_1 dF_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i,X_i}(e_1, e_2|a, x_1, x_2) \\ &= \int \bar{g}(a, x_1, x_2, e_2, e_1)e_1 dF_{\varepsilon_{i1},\varepsilon_{i2}|\alpha_i,X_i}(e_2, e_1|a, x_1, x_2) \\ &= E[\bar{g}(\alpha_i, X_{i1}, X_{i2}, \varepsilon_{i1}, \varepsilon_{i2})\varepsilon_{i2}|\alpha_i = a, X_i = (x_1, x_2)], \end{aligned} \tag{43}$$

a.e. $(a, x_1, x_2) \in \mathcal{A} \times \mathcal{X}^2$, where $\mathcal{X}$ denotes the support of X_{it}.

Integrating out $\alpha_i|X_i$ in the above yields the following a.e. equality:

$$\int E[\bar{g}(\alpha_i, X_{i1}, X_{i2}, \varepsilon_{i1}, \varepsilon_{i2})\varepsilon_{i1}|\alpha_i = a, X_i = (x_1, x_2)]dF_{\alpha_i|X_i}(a|(x_1, x_2))$$
$$= \int E[\bar{g}(\alpha_i, X_{i1}, X_{i2}, \varepsilon_{i1}, \varepsilon_{i2})\varepsilon_{i2}|\alpha_i = a, X_i = (x_1, x_2)]dF_{\alpha_i|X_i}(a|(x_1, x_2)). \tag{44}$$

As a result, by the LIE, we have that $E[G_i(\varepsilon_{i2} - \varepsilon_{i1})|X_i] = 0$. This completes the proof, since by Assumption SC1-X.ii $\varepsilon_{i1}|X_i \stackrel{d}{=} \varepsilon_{i2}|X_i$ by Lemma A.4 and therefore $E[\varepsilon_{i2}-\varepsilon_{i1}|X_i] = 0$.

(ii) Since under Assumption SC2-X, $\bar{g}(\cdot)$ is a trivial function of ε_{i2}, we can define $\breve{\bar{g}}(a, x_1, x_2, e_1) = \bar{g}(a, x_1, x_2, e_1, e_2)$. Note that

$$\begin{aligned} E[G_i(\varepsilon_{i2} - \varepsilon_{i1})|X_i] =& E[E[\bar{g}(\alpha_i, X_{i1}, X_{i2}, \varepsilon_{i1}, \varepsilon_{i2})(\varepsilon_{i2} - \varepsilon_{i1})|X_i, \alpha_i, \varepsilon_{i1}]|X_i] \\ =& E[\breve{\bar{g}}(\alpha_i, X_{i1}, X_{i2}, \varepsilon_{i1})E[\varepsilon_{i2} - \varepsilon_{i1}|X_i, \alpha_i, \varepsilon_{i1}]|X_i] \\ =& E[\breve{\bar{g}}(\alpha_i, X_{i1}, X_{i2}, \varepsilon_{i1})E[\varepsilon_{i2} - \varepsilon_{i1}|X_i]|X_i] \\ =& E[G_i|X_i]E[\varepsilon_{i2} - \varepsilon_{i1}|X_i], \end{aligned} \tag{45}$$

where the first equality follows by the LIE. The second equality follows from Assumption SC2-X.i. The third equality follows by Assumption SC2-X.ii, which implies the result in the last equality.

(iii) Since $\bar{g}(\cdot)$ is a trivial function of ε_{i1} and ε_{i2} under Assumption SC3-X, we can define $\breve{\breve{\bar{g}}}(a, x_1, x_2) = \bar{g}(a, x_1, x_2, e_1, e_2)$.

$$\begin{aligned} E[G_i(\varepsilon_{i2} - \varepsilon_{i1})|X_i] &= E[E[\bar{g}(\alpha_i, X_{i1}, X_{i2}, \varepsilon_{i1}, \varepsilon_{i2})(\varepsilon_{i2} - \varepsilon_{i1})|X_i, \alpha_i]|X_i] \\ &= E[\breve{\breve{\bar{g}}}(\alpha_i, X_{i1}, X_{i2})E[\varepsilon_{i2} - \varepsilon_{i1}|X_i, \alpha_i]|X_i] = 0. \end{aligned} \tag{46}$$

The first equality follows by the LIE. The second equality follows by Assumption SC3-X.i. The last equality follows from $E[\varepsilon_{i1}|X_i, \alpha_i] = E[\varepsilon_{i2}|X_i, \alpha_i]$ under Assumption SC3-X.ii. The result then follows from noting that $E[\varepsilon_{i2} - \varepsilon_{i1}|X_i] = 0$ under this assumption, which completes the proof. □