Dynamic Time Warping for Gait Analysis in Rehabilitation: A Dynamic Programming Approach

Team 11

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1 Problem Description

1.1 Background and Clinical Motivation

Gait analysis plays a critical role in physical therapy and rehabilitation, particularly for patients recovering from neurological conditions such as stroke. One of the main objectives of rehabilitation is to help patients regain a walking pattern that is as close as possible to a healthy, normal gait. Clinicians often rely on visual observation as shown in fig(1) or simple numerical metrics to assess a patient's walking performance, which can be subjective and inaccurate.

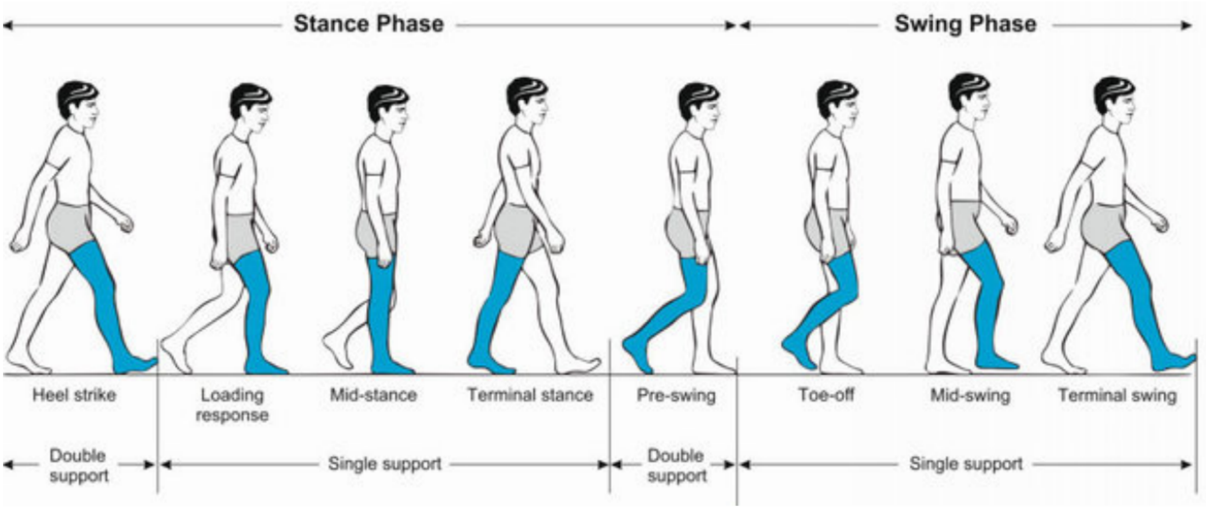

Figure 1: Phases of the normal gait cycle

With the availability of wearable sensors and motion capture systems, gait can now be represented as time-series signals, such as acceleration, joint angles, or limb displacement over time. These signals provide an objective way to analyze walking patterns and track patient progress during rehabilitation.

1.2 Problem Description

A major challenge in gait analysis is that patients typically walk slower and less consistently than healthy individuals. Even if a patient performs the correct movement pattern, the timing of the gait cycle may be stretched or compressed due to weakness, fatigue, or hesitation. As a result, direct point-to-point comparison methods, such as Euclidean distance, fail to accurately measure similarity between a patient's gait signal and a healthy reference signal.

This mismatch occurs because traditional comparison techniques assume that corresponding features (such as peaks and valleys in the signal) occur at the same time indices. When the patient's gait is slower, these features are temporally misaligned, leading to an overestimation of the difference between the two signals.

1.3 Input and Output Definition

Inputs:

- A reference time-series signal representing healthy gait (e.g., vertical leg acceleration during normal walking).
- A patient time-series signal representing impaired gait (e.g., slower or irregular walking).

Outputs:

- A numerical similarity score that quantifies how closely the patient's gait matches the healthy reference.
- An alignment between the two signals that shows which parts of the patient's gait correspond to parts of the healthy gait.

1.4 Why Dynamic Programming is Appropriate

The core difficulty of this problem lies in handling temporal variability between gait signals. The alignment between the patient and healthy signals is non-linear, meaning that one signal may need to be stretched or compressed in time to achieve a meaningful comparison.

Dynamic Programming is well-suited for this problem because it:

- Breaks the alignment task into smaller overlapping subproblems.
- Considers all valid temporal alignments while preserving the order of motion.
- Guarantees an optimal solution by minimizing the total alignment cost.

Dynamic Time Warping (DTW), a Dynamic Programming-based algorithm, enables fair comparison of gait patterns by aligning signals based on movement shape rather than speed. This makes it particularly suitable for rehabilitation assessment, where timing differences are expected and clinically relevant.

1.5 Clinical Relevance

By using Dynamic Time Warping for gait analysis, clinicians can obtain an objective and reliable measure of patient recovery. The resulting similarity score can be used to track rehabilitation progress over time, identify abnormal gait phases, and support data-driven clinical decision-making.

2 Related Work

Gait analysis has been widely studied in the fields of biomechanics, rehabilitation engineering, and physical therapy. Traditional approaches for comparing gait patterns often rely on direct point-to-point distance measures, such as Euclidean distance or correlation-based methods. While these techniques are simple to implement, they assume that gait signals are temporally aligned, which is rarely the case in rehabilitation scenarios where patients tend to walk slower or exhibit irregular timing.

To address temporal variability, Dynamic Time Warping (DTW) has been introduced as an effective method for aligning time-series signals with different speeds or durations.

DTW was originally developed for speech recognition but has since been successfully applied to various biomedical signal analysis tasks, including electrocardiogram (ECG) comparison, electromyography (EMG) analysis, and gait pattern recognition.

Several studies have demonstrated that DTW provides a more robust similarity measure for gait signals compared to traditional distance metrics, particularly in clinical settings involving neurological disorders such as stroke or Parkinson's disease. By allowing non-linear alignment along the time axis, DTW enables meaningful comparison of movement patterns even when patients exhibit pauses, slow movements, or inconsistent gait cycles.

More recently, machine learning and deep learning approaches have been proposed for gait analysis and rehabilitation assessment. Although these methods can achieve high accuracy, they often require large labeled datasets, extensive training, and lack interpretability. In contrast, DTW offers a deterministic, explainable, and data-efficient alternative, making it suitable for small-scale clinical studies and educational biomedical projects.

3 Formal Problem Definition

Let the healthy gait signal be represented as a discrete time-series:

$$H = \{h_1, h_2, \ldots, h_n\} \tag{1}$$

where h_j denotes a gait-related measurement (such as vertical leg acceleration or joint angle) recorded at time index j for a healthy individual.

Similarly, let the patient gait signal be represented as:

$$P = \{p_1, p_2, \ldots, p_m\} \tag{2}$$

where p_i denotes the corresponding gait measurement recorded at time index i for a patient undergoing rehabilitation.

In general, the lengths of the two signals are not equal ($m \neq n$) due to differences in walking speed and gait duration.

3.1 Objective

The objective of this problem is to compute an optimal temporal alignment between the patient gait signal P and the healthy reference signal H that minimizes the total cost of alignment. The alignment must preserve the chronological order of both signals while allowing non-linear stretching or compression of the time axis to compensate for speed variations.

3.2 Local Cost Function

A local cost function is defined to measure the dissimilarity between individual samples of the two signals:

$$\text{cost}(i, j) = |p_i - h_j| \tag{3}$$

This cost quantifies the instantaneous difference between the patient's gait measurement at time index i and the healthy reference measurement at time index j. Smaller values indicate higher similarity between the two gait signals at the corresponding points.

3.3 Warping Path and Constraints

An alignment between the two signals is defined by a warping path that maps elements of P to elements of H. The warping path must satisfy the following constraints:

1. **Boundary Constraint:** The alignment starts at the first samples of both signals and ends at their last samples.
2. **Monotonicity Constraint:** The time indices of the alignment must be non-decreasing, ensuring that the temporal order of the gait signals is preserved.
3. **Continuity Constraint:** The alignment progresses step-by-step without skipping samples, allowing only small temporal adjustments between consecutive points.

Boundary condition:

$$p_1 = (1,1) = c(x_1, y_1) \text{ and } p_L = (m,m) = c(x_i, y_i)$$

Monotonicity condition:

$$m_1 \leq m_2 \leq \cdots \leq m_L \text{ and } n_1 \leq n_2 \leq \cdots \leq n_L$$

Continuity condition:

$$p_{L+1} - p_L \in \{(1,0), (0,1), (1,1)\}$$

Figure 2: conditions of Warping path alignment

3.4 Problem Statement

Given two time-series gait signals of potentially different lengths, the problem is to determine a non-linear temporal alignment that minimizes the total accumulated cost defined by the local cost function while satisfying the boundary, monotonicity, and continuity constraints. The resulting alignment provides both a numerical similarity score and a mapping between the patient's gait phases and the healthy reference gait phases.

3.5 Definition of Dynamic Time Warping Distance

The Dynamic Time Warping (DTW) distance between the patient gait signal P and the healthy reference signal H is defined as the minimum possible total alignment cost over all valid warping paths. This distance serves as an objective measure of gait similarity and is used to assess rehabilitation progress.

4 Dynamic Programming Formulation

Dynamic Time Warping (DTW) is formulated as a Dynamic Programming problem in order to efficiently compute the optimal non-linear alignment between the patient gait signal and the healthy reference signal.

4.1 Subproblem Definition

Let $DTW[i][j]$ denote the minimum cumulative alignment cost for aligning the first i samples of the patient gait signal $P = \{p_1, p_2, \ldots, p_m\}$ with the first j samples of the healthy gait signal $H = \{h_1, h_2, \ldots, h_n\}$.

Each subproblem represents the optimal alignment cost up to the corresponding time indices i and j.

4.2 Recurrence Relation

To compute $DTW[i][j]$, a local cost is first calculated between the two signal samples:

$$\text{cost}(i, j) = |p_i - h_j| \tag{4}$$

The cumulative cost is then computed by adding the local cost to the minimum of the three possible previous alignment states:

$$DTW[i, j] = \text{cost}(i, j) + \min \begin{cases} DTW[i-1, j] & \text{(Insertion)} \\ DTW[i, j-1] & \text{(Deletion)} \\ DTW[i-1, j-1] & \text{(Match)} \end{cases} \tag{5}$$

The three transitions correspond to different temporal alignment scenarios:

- **Insertion:** The patient signal advances while the healthy signal remains at the same index, modeling slower movement or pauses.
- **Deletion:** The healthy signal advances while the patient signal remains at the same index.
- **Match:** Both signals advance simultaneously, representing synchronized movement.

4.3 Base Cases

The Dynamic Programming table is initialized as follows:

$$DTW[0][0] = 0 \tag{6}$$

$$DTW[i][0] = \infty \quad \text{for all } i > 0 \tag{7}$$

$$DTW[0][j] = \infty \quad \text{for all } j > 0 \tag{8}$$

These base cases ensure that the alignment starts at the first samples of both signals and that invalid alignments are excluded.

	0	H1	H2	H3
		1	2	4
0	0	∞	∞	∞
P1=1	∞	?	?	?
P2=3	∞	?	?	?
P3=4	∞	?	?	?

Figure 3: the initial form of DP matrix

4.4 DP Table Construction

The DP table is constructed using a bottom-up approach. The values of $DTW[i][j]$ are computed iteratively for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, ensuring that all required subproblems are solved before computing larger ones.

The final DTW distance is obtained from the value stored in $DTW[m][n]$, which represents the minimum cumulative cost of aligning the complete patient gait signal with the healthy reference signal.

DP Table Construction (First 10x10)

Ref	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
0.0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
∞	[0.0]	0.1	0.4	0.8	1.3	2.0	2.7	3.4	4.2	5.2
∞	0.2	[0.1]	0.2	0.5	0.9	1.4	2.0	2.6	3.3	4.1
∞	0.5	0.2	[0.1]	0.3	0.6	1.0	1.4	1.8	2.4	3.1
∞	0.9	0.5	0.2	[0.2]	0.3	0.6	0.9	1.2	1.7	2.3
∞	1.4	0.9	0.5	0.2	[0.2]	0.4	0.6	0.8	1.1	1.6
∞	2.0	1.4	0.8	0.3	0.3	[0.3]	0.4	0.5	0.7	1.1
∞	2.7	2.0	1.3	0.6	0.4	0.3	[0.3]	[0.3]	0.4	0.7
∞	3.5	2.7	1.8	0.9	0.6	0.4	0.4	0.4	[0.3]	0.5
∞	4.4	3.5	2.4	1.3	1.0	0.6	0.5	0.5	[0.3]	0.4
∞	5.3	4.3	3.1	1.8	1.3	0.9	0.7	0.7	0.4	[0.4]

* Values in [Brackets] are part of the optimal backtracking path.
* Showing first 10x10 cells only.

Figure 4: Example of Healthy case Backtracking matrix

4.5 Backtracking and Alignment Reconstruction

To recover the optimal warping path, backtracking is performed starting from the cell $DTW[m][n]$ and moving backward to $DTW[0][0]$. At each step, the predecessor cell that contributed to the minimum value is selected.

This backtracking process produces a sequence of index pairs that define the optimal alignment between the patient and healthy gait signals. The resulting warping path can be visualized by drawing correspondence lines between the two signals, providing insight into temporal distortions such as slowed or paused gait phases.

5 Correctness Argument

The Dynamic Time Warping algorithm correctly computes the optimal alignment between the patient gait signal and the healthy reference signal by relying on the principle of optimality. Each entry $DTW[i][j]$ represents the minimum possible alignment cost between the first i samples of the patient signal and the first j samples of the healthy signal.

The recurrence relation considers all valid ways to extend a partial alignment by aligning the current samples p_i and h_j. By selecting the minimum among insertion, deletion, and match transitions, the algorithm ensures that every possible valid warping path is examined. Since the alignment cost is accumulated from optimal sub-alignments, the solution stored in $DTW[i][j]$ is optimal.

By computing the DP table in a bottom-up manner, all subproblems are solved before they are needed. Therefore, the final value $DTW[m][n]$ represents the minimum total cost over all valid warping paths that align the entire patient gait signal with the healthy reference signal. Consequently, the algorithm guarantees that the computed DTW distance corresponds to the globally optimal alignment.

6 Time and Space Complexity

Let m and n denote the lengths of the patient and healthy gait signals, respectively.

- **Time Complexity:** $O(m \times n)$
- **Space Complexity:** $O(m \times n)$

This complexity is acceptable for short to medium-length gait signals commonly used in rehabilitation analysis.

7 Experiments

7.1 Test Case 1: Healthy Reference vs. Healthy Patient (Baseline)

In this test, we generated a patient signal that closely mimics the healthy reference with minor random noise.

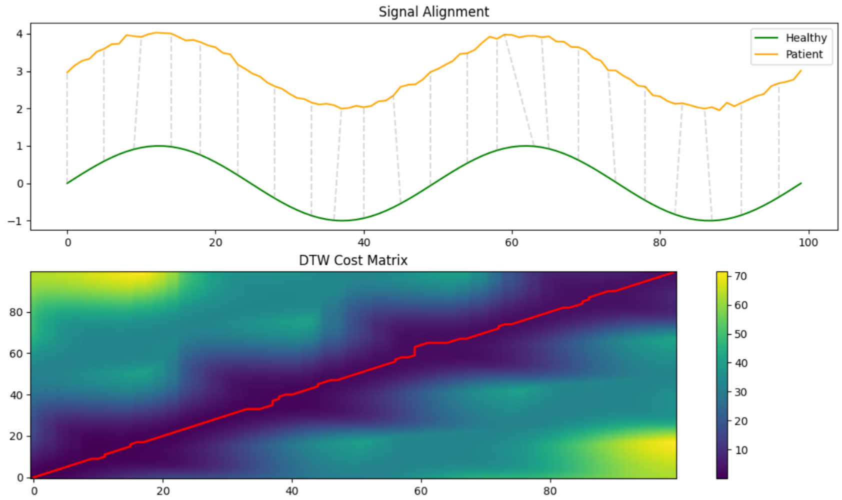

Figure 5: The alignment between the two signals for test 1

Visual Observation (Figure 5): The gray alignment lines are nearly vertical, and the optimal path (red line) on the cost matrix follows a perfect diagonal. **Quantitative Result:** The system calculated a final alignment score of 3.55.

Interpretation: This low score indicates a high degree of similarity. The algorithm correctly identified that no significant time-warping was required, validating the base case. [Continue with more text here...]

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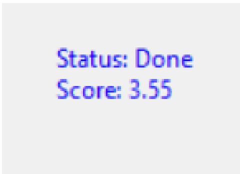

Figure 6: Numerical value for test 1

7.2 Test Case 2: The "Slow Walker" Scenario

We simulated a patient walking 40% slower than the healthy reference. A standard Euclidean distance check would fail here because the peaks do not line up in time.

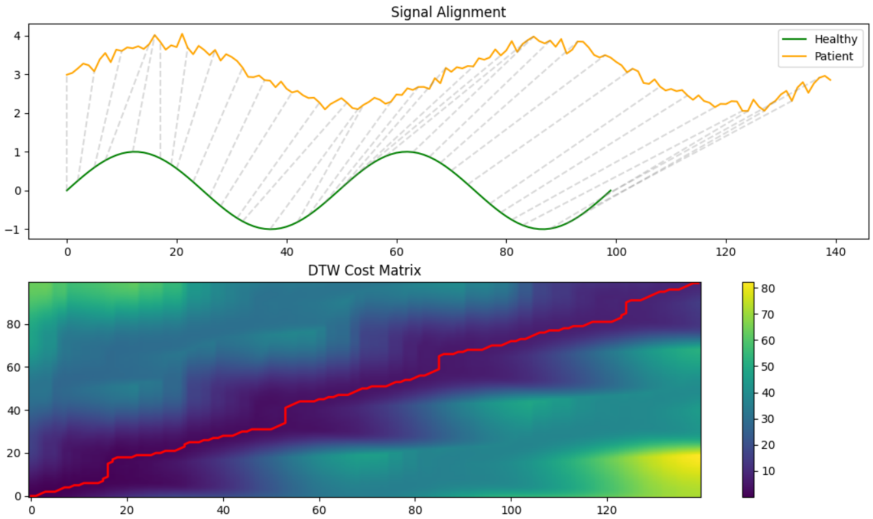

Figure 7: The alignment between the two signals for test 2

Visual Observation (Figure 7): As seen in the top graph, the gray dashed lines are diagonal, connecting the delayed peaks of the patient to the earlier peaks of the healthy signal. The red path in the heatmap curves significantly off the main diagonal.

Quantitative Result: The alignment score is 9.10.

Interpretation: Although the score is higher than the baseline, it is relatively low compared to the severe case. This proves the DTW algorithm successfully "warped" the time axis to match the gait cycles despite the speed difference.

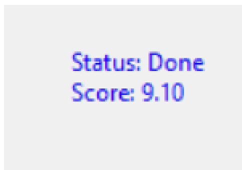

Figure 8: Numerical value for test 2

7.3 Test Case 3: Severe Asymmetry

We introduced significant distortion, phase shifting, and random noise to simulate a patient with severe gait pathology.

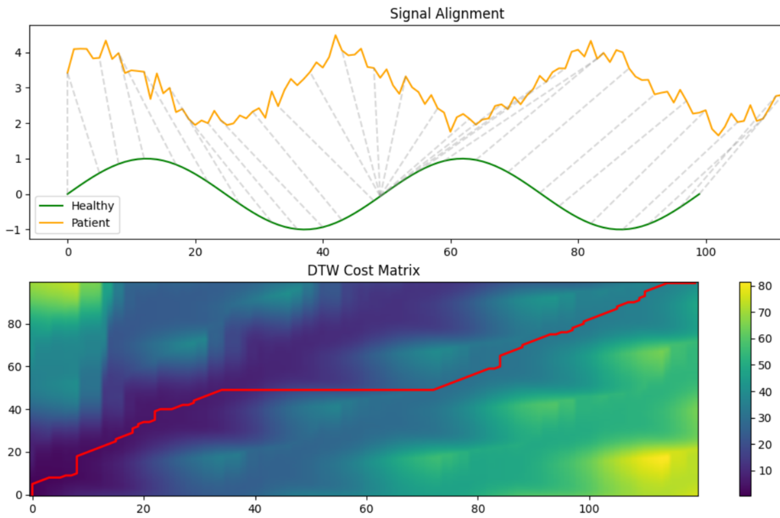

Figure 9: The alignment between the two signals for test 3

Visual Observation (Figure 9): The signal alignment is chaotic, and the optimal path (red line) is jagged and irregular as the algorithm struggles to find matching features.
Quantitative Result: The alignment score spiked to 39.76.

Interpretation: This high score serves as a quantitative metric for the severity of the gait abnormality. In a clinical setting, this high value would trigger an alert for physician intervention.

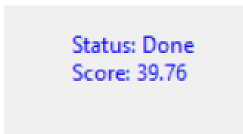

Figure 10: Numerical value for test 3

8 Discussion and Limitations

8.1 Strengths of DTW

- Uses dynamic programming to divide the alignment problem into small subproblems.
- Each DP cell stores the optimal alignment cost up to that point.
- Guarantees a globally optimal solution.
- Allows non-linear alignment between signals with different speeds.

8.2 Sensitivity to Noise

- Local DP cost depends on raw signal values.
- Noise increases local error and propagates through the DP table.
- Leads to incorrect warping paths and higher DTW distance.

8.3 Slow for Long Signals

The dynamic programming table has size $m \times n$, where m and n are the signal lengths. This means that DTW must fill a large table and compute all subproblems, which becomes very slow for long recordings.

Possible Solutions:

- Reduce signal length by splitting the signal into smaller segments.
- Use approximate DTW methods that compute only part of the DP table.

8.4 Suggested Solutions for the Limitations

- Smooth signals before DTW using filters.
- Normalize signals to remove amplitude variations.
- Use averaged samples instead of raw points.

9 References

References

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