Gordon: Privacy Budget Manager for W3C's Privacy-Preserving Attribution API

Abstract

Privacy-preserving advertising APIs offer a path to improving online privacy, but their design requires a strong theoretical foundation. The Cookie Monster paper (SOSP 2024) introduced an individual differential privacy framework for on-device attribution measurement APIs, forming the basis of W3C's Privacy-Preserving Attribution (PPA) draft standard. However, it left open key challenges, particularly in managing granular per-site privacy budgets to balance user privacy with utility in the adversarial advertising ecosystem. This gap requires new foundational design to avoid ad-hoc, unprincipled solutions that might otherwise be adopted.

We introduce *Gordon*, an extension of Cookie Monster that provides this new foundation through a structured privacy budget management approach. In addition to per-site budgets, Gordon introduces several additional budgets to protect privacy against adversarial collusion and ensure utility isolation among advertisers. Implemented as a general on-device differential privacy library, and integrated into Mozilla Firefox's Private Attribution system, Gordon provides a reference implementation for PPA. Evaluations on real-world datasets from Criteo and the W3C show that Gordon enforces strong privacy guarantees while preserving utility, outperforming existing ad-hoc approaches and offering a rigorous foundation for further advancing the PPA standard.

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1 Introduction

Privacy-preserving advertising APIs, now under development and standardization in major browsers via the W3C, offer a rare opportunity to enhance online privacy while sustaining the web's primary funding model. Historically, browsers have lacked structured support for ad-related tasks like *conversion attribution measurement*, which requires linking ads viewed on content sites to purchases made on seller sites– a cross-origin function that is fundamentally at odds with the same-origin principle that underpins browser designs. This lack of support for the advertising workload has fueled widespread cross-site tracking through third-party cookies, fingerprinting, and other workarounds. The goal of the new APIs is to provide a well-structured, privacy-preserving alternative that aligns with browser principles while meeting advertising needs. However, these APIs remain in their early stages, with significant technical challenges still unresolved – creating a big opportunity for the academic community to contribute.

Such collaborations have already had an immediate impact, showing that the space is ripe for foundational contributions. The *Cookie Monster* paper, presented at SOSP last year [**cm-paper**], introduced the first formal framework based on individual differential privacy (individual DP) to systematically analyze and optimize these APIs—a framework later adopted by Google in the privacy analysis of its own API, called ARA [**ara-paper**]. This framework now underpins Privacy-Preserving Attribution (PPA), the W3C's unified draft API undergoing standardization since October 2024 [**ppa**]. Thus, the API and privacy framework ecosystem is now largely consolidated, with PPA and Cookie Monster now serving as the main focus of ongoing design and standardization in W3C.

This paper builds on PPA by addressing a key open challenge: *privacy budget management*, providing further foundational support for the standard. PPA replaces cross-site tracking with a system where content sites register ads with the browser, seller sites request encrypted reports, and reports are only accessible through differentially private (DP) aggregation via secure multi-party computation or a trusted execution environment. Before sending an encrypted report, the browser deducts privacy loss from a *per-site privacy budget*, limiting how much new information a site can infer about a user. While PPA, through Cookie Monster's algorithm, optimizes privacy loss accounting within each per-site budget using individual DP, it does not address how to manage these highly granular budgets to balance strong privacy with practical utility in an inherently adversarial advertising ecosystem.

The absence of a principled approach to privacy budget management has led to unresolved questions within the W3C PPA working group,[1] creating uncertainty in key design decisions. For instance, should some sites get budget while others do not—and if so, based on what criteria?[2] Should there be a cap on the number of sites allocated budget, and if so, how can we prevent a denial-of-service attack where one entity exhausts it?[3] Should API invocations be rate-limited to prevent privacy or DoS attacks, and how should such limits be configured? To date, there is no consensus, largely due to the lack of a solid foundation to drive solutions.

We propose *Gordon*, a *privacy budget manager for PPA* that addresses semantic gaps in per-site privacy loss accounting and challenges arising from the introduction of a coarse-grained global budget to protect user privacy against adversaries controlling multiple sites. To clarify the semantics of

[1] For brevity, we refer to the PPA W3C working group as simply *W3C*.
[2] Live discussion in W3C's PAT community group, April 2024.
[3] https://github.com/w3c/ppa/issues/69

PPA's ambiguous per-site budgeting—often affected by the shifting roles of third parties—we propose changes to its interface, protocol, and terms of use, some of which have already been acknowledged by W3C.

For the global budget, the challenge is configuring and managing it to support benign workloads while resisting malicious attempts to deplete it. Our insight is to treat the global privacy budget as a *shared resource*—analogous to traditional computing resources but governed by privacy constraints—and to apply classic resource isolation techniques, such as quotas and fair-share scheduling, to this new domain. First, beyond per-site and global budgets, Gordon introduces *quota budgets* that regulate global filter consumption, ensuring graceful utility degradation for compliant sites under attack by forcing adversaries to operate within the bounds of expected workloads—bounds they can currently evade to wreak havoc on PPA's global budget. Second, recognizing that quotas can underutilize the global filter under benign conditions, we explore *batched operation*—collecting and scheduling report requests periodically—to improve utilization while maintaining DoS protection. We use a fair-share-inspired algorithm, illustrating how OS scheduling research can address foundational gaps in emerging privacy solutions. Together, these mechanisms establish a principled and practical foundation for PPA, offering browsers a robust basis for enforceable defenses.

We implement Gordon in two components: (1) `pdslib`, a generic on-device individual DP library that subsumes Cookie Monster and extends it with Gordon's budget management, and (2) integration into Mozilla Firefox's Private Attribution, a minimal PPA implementation. Upon release, these prototypes will serve as reference implementations for PPA, a service the W3C has acknowledged as valuable.

We evaluate Gordon on real-world datasets from the Criteo ad-tech company and a W3C-released dataset. Our results show that ... TODO(Pierre).

2 PPA Overview and Gaps

2.1 PPA architecture

Fig. 1(a) illustrates the architecture of *Privacy-Preserving Attribution (PPA)*, W3C's browser-based API that enables *conversion attribution measurement* while preserving user privacy. Traditionally, browsers enforce the *same-origin policy*, which allows a site to access only its own stored data (e.g., cookies) and prevents cross-site data access. However, conversion attribution–the process of determining whether users who see an ad later make a purchase–is inherently *cross-origin*. It requires linking ad impressions shown on content sites (e.g., *news.ex*, *social.ex*) to conversions occurring on advertiser sites (e.g., *shoes.ex*). Ben: In fig 1, we could have the impressions stored under the conversion domain, curious @Martin what PPA has decided to do here as I know we'd discuss a couple options.

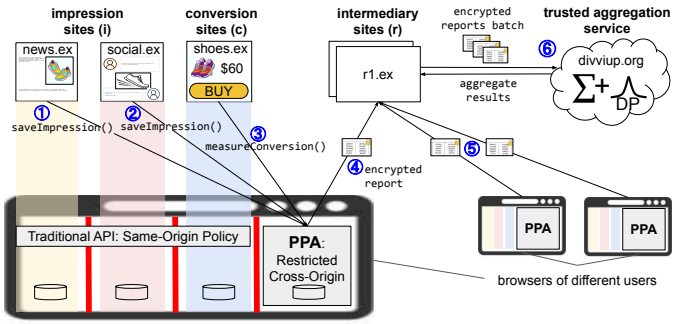
Because no structured browser API exists for this purpose, advertisers and adtech platforms have relied on workarounds such as third-party cookies, fingerprinting, and backend data exchanges. While essential to the web economy, these methods effectively bypass browser restrictions and expose user activity across sites. PPA addresses this gap by introducing a privacy-preserving attribution framework that permits controlled cross-origin measurement while ensuring that user activity remains inaccessible across domains. It does so using differential privacy (DP) and secure aggregation, implemented via secure multi-party computation (MPC) or a trusted execution environment (TEE). This approach bounds the amount of information leaked by the API while still enabling effective ad measurement.

PPA defines four key types of participants, each interacting with the API through specific functions. **Impression sites** (*news.ex*, *social.ex*) are content sites where ads are displayed. These sites register ad impressions with the browser using the `saveImpression()` function. **Conversion sites**, a.k.a. **advertiser sites** (*shoes.ex*), are sites where purchases or other conversions occur. When a user completes a conversion, these sites invoke `measureConversion()` to link the event to any relevant prior ad impressions. **Intermediary sites** (*r1.ex*, *r2.ex*) are adtech platforms that are embedded as frames within impression and conversion sites to facilitate ad delivery and measurement. Unlike traditional tracking-based adtechs, they do not collect cross-site data directly but instead receive encrypted reports from `measureConversion()`, which they submit for secure aggregation. **Aggregation services** (e.g., *divviup.org*) are trusted entities that process encrypted reports, applying DP to produce aggregated conversion metrics while ensuring no single entity can reconstruct individual user data.

2.2 Example workflow

Fig. 1(b) shows an example workflow for PPA, consisting of six steps (the same steps are also marked in the Fig. 1(a) architecture). The example entails an advertiser, *shoes.ex*, that launches an ad campaign to promote a new product. To compare the effectiveness of two ad creatives–a colorful ad highlighting the shoe's design and a black-and-white ad emphasizing materials and comfort–*shoes.ex* partners with two placement adtechs, *r1.ex* and *r2.ex*. Each adtech places the ads on content sites, *e.g., r1.ex* on *blog.ex* and *r2.ex* on *news.ex*. In addition to placing ads, these adtechs provide a *measurement service* that allows *shoes.ex* to compare the performance of its creatives within their respective networks.

(1) When a user visits *blog.ex*, *r1.ex* displays the colorful ad and registers the impression by calling: `saveImpression()` with the parameters shown in the figure. (2) Later, the user visits *news.ex*, where *r2.ex* displays the black-and-white ad and registers it by also calling `saveImpression()`. These impressions are stored *locally in the browser* within an Impression Store, along with metadata such as timestamps, campaign



(a) PPA architecture

① **Epoch e1:** User visits *blog.ex*. *r1.ex* script embedded within *blog.ex* displays colorful ad (*id 1*) for *shoes.ex*'s campaign (*0x22*) and invokes:
`saveImpression(conversionSites=[shoes.ex], filterData=0x22, histogramBucket=1)`

② **Epoch e2:** User visits *news.ex*. *r2.ex* script displays a black-and-white ad (*id 2*) and invokes:
`saveImpression(conversionSites=[shoes.ex], filterData=0x22, histogramBucket=2)`

③ **Epoch e3:** User visits *shoes.ex* and purchases $60 of advertised shoes. *Shoes.ex* invokes:
`measureConversion(impressionSites=[blog.ex,news.ex], intermediarySites=[r1.ex,r2.ex], filterData=0x22, epsilon=0.5, value=60, maxValue=100) → attributionObject`

AttributionObject contains the full attribution histogram `{r1.ex:1=30, r2.ex:2=30}` and privacy loss tracking. Privacy loss is applied when `getReport(.)` is called.

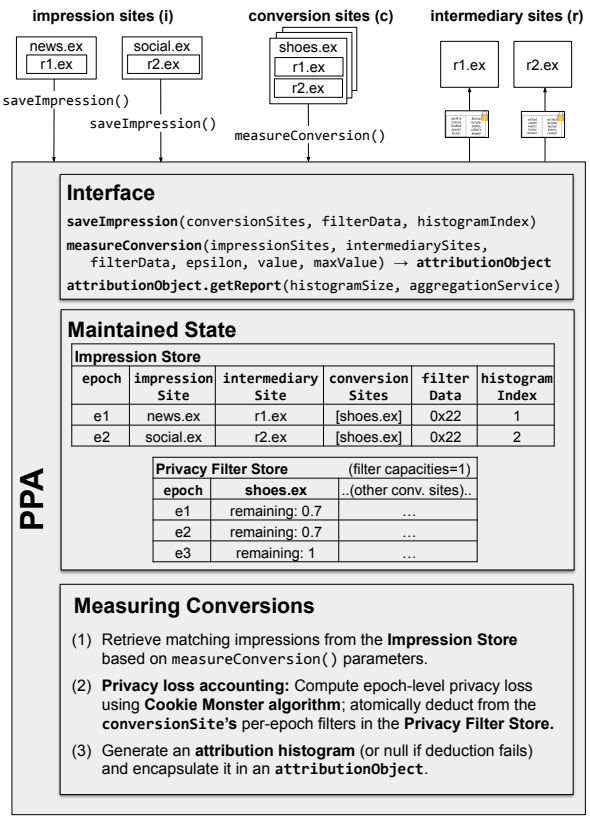
④ *Shoes.ex* hands attributionObject to *r1.ex* and *r2.ex* embedded scripts, which invoke:
`attributionObject.getReport(histogramSize=2, aggregationService=divviup.org)`

to retrieve **encrypted reports**. *r1.ex* gets encryption of padded `{1:30}`, *r2.ex* gets encryption of padded `{2:30}`, both with authenticated data: `{epsilon=0.5, value=60, maxValue=100}`.

⑤ As other users convert on *shoes.ex*, *r1.ex* and *r2.ex* get encrypted reports from them and batch them.

⑥ At some point, *r1.ex* sends the batch to the aggregation service, which uses MPC or TEE to check consistency of authenticated data and uniqueness of report use. Then sums all the histograms and adds noise, e.g. from `Laplace(maxValue/epsilon)`, to ensure `epsilon`-DP.

(b) Example workflow



(c) Privacy loss accounting with Cookie Monster

Fig. 1. PPA overview.

identifiers, and the adtech responsible for each impression. This store, shown in Fig. 1(c), is write-only until a conversion.

③ Subsequently, if the user visits *shoes.ex* and purchases the shoes for $60, the site invokes `measureConversion()` with the parameters shown in Fig. 1(b). This function searches the Impression Store in the browser for *relevant impressions*, matching the `impressionSite` and `conversionSite` fields of the impressions to the parameters of `measureConversion()`. It then generates an *attributionObject*, which encapsulates the attribution histogram and manages privacy loss accounting. Assuming that PPA applies *uniform attribution*, it will assign the $60 conversion value is equally between the two registered impressions, assigning $30 to each and resulting in the following attribution histogram: `{1:30, 2:30}`.

④ The attributionObject is *lazy*, i.e., no privacy loss occurs until it is used to request a report. To support DP queries, *shoes.ex* hands over the attributionObject to the *r1.ex* and *r2.ex* contexts within the browser, which invoke `attribution-Object.getReport()`, specifying the histogram size they expect and the aggregation service they intend to use (from a list of such services supported and trusted by the browser). The browser processes these invocations by: (1) filtering the attribution histogram so that each intermediary only sees its own contributions (*r1.ex* gets `{1:30}`, *r2.ex* gets `{2:30}`); (2) trimming or padding the histogram to match the intermediary's expected size; (3) encrypting the report and secret-sharing it (if MPC is used), while attaching some critical parameters as authenticated data, such as `epsilon` and `maxValue`; and (4) performing privacy loss accounting before sending the encrypted reports over to the intermediaries.

⑤ As more users purchase *shoes.ex*'s advertised product, additional encrypted reports are generated, each containing zero, one, or two attributed ads. ⑥ The intermediaries batch these reports and submit them to an aggregation service, which performs the final step: (1) validating the reports, ensuring all parameters in authenticated data match and that no report is reused; (2) summing the attribution values; and (3) applying differential privacy, adding noise (such as from a Laplace distribution with scale `maxValue/epsilon`) to protect individual users. The resulting *noised, aggregated conversion metrics* are then provided to *r1.ex* and *r2.ex*, which relay the final effectiveness comparison back to *shoes.ex*. This helps it discern which of the colorful versus black-and-white ads leads to higher purchase revenue.

2.3 Privacy loss accounting with Cookie Monster

PPA adopts the privacy loss accounting algorithm from the Cookie Monster paper (§3.3 of [**cm-paper**]). Fig. 1(c) shows PPA's on-device components, including its detailed interface, maintained state, and the inner workings of conversion measurement which incorporates the Cookie Monster algorithm.

With it, PPA enforces *individual DP* [**<empty citation>**]: each browser maintains a separate privacy guarantee and accounts for privacy loss based on its own contribution to a query. Unlike traditional DP, which tracks a global guarantee across devices, individual DP allows finer-grained privacy control, optimizing privacy loss accounting.

Within each browser, PPA enforces individual DP at the *epoch* level, dividing the impression stream into time-based partitions, say of a week, where privacy budgets are managed separately. Each device maintains an *Impression Store* to log impressions per epoch and a *Privacy Filter Store* to track per-epoch privacy budgets. A *privacy filter* acts as a per-epoch budget manager, deducting privacy loss only when it has remaining budget and when data from that epoch contributes to a query; when out of budget, the filter denies access to the data from that epoch. This mechanism prevents excessive budget depletion and blocks any further use of impression data from an epoch once its budget is exhausted. Importantly, PPA maintains separate epoch-level privacy filters *per site*, a design choice, that as we shall see, raises substantial budget management questions.

The Cookie Monster paper defines the algorithm for computing *epoch-level individual privacy loss* for conversion attribution queries, which in PPA are summations over capped attribution histograms (`maxValue`). Instead of using global sensitivity, PPA applies *individual sensitivity*, scaling privacy loss based on actual epoch contributions to the query. For example, in Fig. 1(b), epochs $e1$ and $e2$ incur an individual privacy loss of `value/maxValue*epsilon=0.3`, whereas standard DP would apply the global privacy loss of `epsilon=0.5`. Even better, epoch $e3$, which lacks relevant impressions, incurs *zero* individual privacy loss.

Fig. 1(c) shows the three steps of `measureConversion()`, initially outlined by the Cookie Monster paper and subsequently refined by PPA to address, partially and not always satisfactorily, gaps left by that paper. We describe the PPA spec as-is here before describing its foundational gaps in §2.5. (1) The browser fetches relevant impressions from the Impression Store, filtering based on metadata and the `measureConversion()` parameters. (2) PPA applies the Cookie Monster algorithm to compute epoch-level privacy losses and attempts to deduct them atomically across all epochs from the *conversion site's privacy filters* (*shoes.ex*). If any epoch lacks sufficient budget, PPA generates a null attribution histogram. Otherwise, it will generate the real attribution histogram based on impressions. (3) PPA constructs an `attributionObject`, encapsulating the histogram and privacy loss accounting, and returns it to the conversion site (*shoes.ex*). The intermediaries then use it to obtain encrypted reports. Although privacy loss is described as being deducted in `measureConversion()`, implementations can apply lazy deduction, postponing privacy loss enforcement until `getReport()`; if no report is ever requested, no privacy is lost. For our example, Fig. 1(c) shows the filter state after *r1.ex* and *r2.ex* request their reports: assuming an initial filter capacity of 1, *shoes.ex* retains 0.7 privacy budget for epochs $e1$ and $e2$, and the full 1 for $e3$.

Stock-and-flow pattern of individual DP optimizations. A key informal argument for PPA's practicality, voiced in W3C discussions, is that under *intended use*, individual DP-based optimizations are expected to limit privacy consumption by tying it to *user actions on both content and purchase sites*. Under individual DP, non-zero privacy loss arises only when both an impression (signifying user visit to a content site) and a conversion (signifying visit to a purchase site) are present. This creates a *stock-and-flow pattern* of intended use, in which *privacy stock* is created on impression sites as impressions are stored, and *privacy flow* is triggered on conversion sites when reports are requested over those impressions—*both gated by user actions*. W3C discussions generally endorse the view that users who engage more with content and conversion sites should, naturally, incur more privacy loss—up to a limit, which we discuss next.

2.4 Global privacy filter

PPA acknowledges that relying solely on per-site filters risks exposing users to adversaries capable of correlating activity across sites [**ppa**]. According to the spec, such coordination increases the adversary's information gain from attribution, proportionally to the number of sites involved. Per-site filters alone impose no limit on this cross-site leakage. To mitigate this, PPA recommends "safety limits:" per-epoch global filters that ignore site boundaries. While the spec lacks details on managing these filters—a gap this paper addresses (see next section)—our input has influenced their design direction. Specifically: (1) global filter capacities are by necessity expected to be significantly larger than per-site budgets; and (2) these filters should "remain inactive during normal browsing and [trigger] only under high-intensity use or attack" [**ppa**]. We adopt this approach here, as contributed by us in W3C discussions and agreed upon by the group.

2.5 Foundational gaps

We identify two key gaps in PPA related to managing its two filter types: per-site and global.

Gap 1: Unclear semantics of per-site filters. PPA adopts Cookie Monster's accounting model, which tracks privacy loss *per querier* but is unclear about who counts as a querier in real-world deployments. PPA maps queriers to conversion sites for **single-advertiser queries**, where intermediaries request reports on behalf of a specific advertiser (e.g., *shoes.ex*), and privacy loss is charged to that advertiser's budget. Yet intermediaries also receive the reports and may reuse them for their own purposes, raising questions about whether they too should be considered queriers. The ambiguity deepens with PPA's planned support for **cross-advertiser queries**, where intermediaries optimize across multiple advertisers (e.g., selecting which ad to show) and are the primary beneficiaries. W3C is considering letting intermediaries hold their

own budgets on user devices, but this blurs the line between client-serving and self-serving queries, undermining the semantics of per-site accounting and opening the door to report misuse. In W3C discussions, the global filter has often been invoked as a source of consolation, offering clear semantics even if per-site semantics break down—but this introduces its own configuration and management challenges, as we discuss below. This paper proposes changes to the PPA API, protocol, and terms of use to clarify the semantics of per-site filters and the assumptions under which they hold (§4.1).

Gap 2: Lack of mechanisms to manage the global filter. A critical yet under-specified aspect of PPA is the configuration and management of the global filter, a shared resource across all parties requesting reports from a browser. This raises two key challenges: (1) how to set its capacity to support benign workloads, and (2) how to prevent malicious actors from depleting it—either to boost their own utility or to deny service to others (e.g., competitors). While per-site budgets cap consumption per domain, they offer weak protection, as domain names are cheap and easily acquired. Proposed mitigations range from requiring sites to register with a trusted authority to browser-side heuristics for identifying legitimate use of the API. But site registration faces resistance from some industry participants for undermining the API's open nature—a core tenet of the web—while heuristics rely on hard-to-define notions of "legitimacy," especially for a nascent API with no deployment history and many possible valuable, unforeseen use cases. For instance, should the number of invocations be limited? Over what time window and to what value? Should access to device-side budgets be restricted? On what grounds? While discussion continues, we argue that W3C lacks a foundation—a minimal set of principled mechanisms with well-defined properties under clear assumptions—to guide browsers toward targeted, defense-in-depth strategies that are both protective and not over-constraining for the API. This paper contributes such a foundation for two settings: the current PPA design (§4.3) and an extended version that permits more efficient mechanisms (§4.4).

3 Gordon Overview

We address PPA's gaps by (1) clarifying the two distinct threat models that per-site and global guarantees address (§3.1), and (2) introducing Gordon to both restore the semantics of per-site filters and manage the global filter to support legitimate use while deterring abuse (§3.3). §3.2 introduces an example.

3.1 Threat model

Gordon adopts the same threat model as PPA. Users trust their browser, device, and browser-supported aggregation services. They also partially trust first-party sites they navigate to intentionally–i.e., through *explicit actions* such as via direct navigation or clicks–granting them access to first-party data and cookies. Moreover, embedded intermediaries are not trusted at all, and no site—first party or otherwise—is trusted with raw cross-site information. To enforce this, PPA seeks to ensure that cross-site measurements satisfy DP guarantees at two levels: per-site and global.

As API designers, we (and PPA) must account for two threat levels. First, we must support the **intended use** of the API—making it easy for well-intentioned actors to comply through the API's design, semantics, and terms of use. In idealized security models, such actors are often called *honest-but-curious*: they follow the protocol but attempt to learn as much as possible from the outputs. In practice, especially for new APIs like PPA, not all rules can be protocol-enforced. This makes it critical to define clear terms of use that bridge enforcement gaps. The resulting semantics should hold for adversaries that comply with both the protocol and the terms of use—our definition of honest-but-curious in this paper.

Per-site filters in PPA are intended to provide strong privacy guarantees against individual honest-but-curious sites. But the current API lacks key protocol elements, and its terms of use remain undefined—leaving per-site accounting semantically unclear. Gordon addresses these deficiencies.

Second, we must account for **adversarial use** of the API, where actors may attempt to subvert its limits—either to increase the utility of their DP queries by acquiring excess budget or to extract more information about individual users. While per-site budgets offer protection when queriers operate independently or with limited coordination—thanks to DP's compositionality—they break down under *large-scale Sybil attacks*, where an adversary registers numerous fake domains (*Sybils*) to bypass per-site privacy caps. Given the ease and low cost of domain registration, this attack poses a serious threat. For instance, a malicious conversion site X might use automatic redirection to cycle through Sybil conversion sites, each triggering a single-advertiser query and maxing out its own filter. This amplifies the user's privacy loss toward entity X by the number of Sybils.

Global filters in PPA are intended to provide coarser-grained privacy protection against large-scale Sybil attacks. But they also introduce a new vulnerability: *denial-of-service (DoS) depletion attacks* targeting the filter itself. A malicious actor can deplete the global budget, blocking legitimate queries—either to increase their own query utility or to sabotage competitors. These attacks can mirror the Sybil strategies used to defeat per-site filters, and in §4.3, we demonstrate concrete examples that PPA does not currently defend against.

While DoS depletion defenses might focus on identifying "illegitimate" domains or anomalous query patterns, Gordon builds *resilience directly into privacy budget management*. Our approach builds directly on the stock-and-flow model outlined in §2.3, relying on the same key assumption—frequently raised by W3C participants—that *privacy consumption in PPA should be driven by explicit user actions, such as navigations or clicks, separately on content and conversion first-party sites*. As long as legitimate API usage continues to follow this intended pattern—where both stock and flow are

moderated by real user behavior—our defenses can fulfill the global filter's design goal: to support normal workloads under benign conditions and to degrade utility gracefully under attack (§2.4). We assume that browsers can distinguish, through external means, between intentional user actions and automatic navigations, and that intentional actions are hard for malicious advertisers to fake—challenges that browsers have long addressed. If these assumptions break down, our privacy guarantees still hold, though our protections against DoS depletion may weaken. We do not attempt to defend against bot-controlled browsers, which would only consume their own budgets, not those of real users.

3.2 Running example

We update the *shoes.ex* example to support *cross-advertiser queries*, a feature PPA plans to add soon. Our Gordon design anticipates this shift, which significantly impacts privacy budget management. To reflect this, we modify the example: *shoes.ex* contracts with *r1.ex* and *r2.ex* for ad placement and evaluation as before, but now *r1.ex* and *r2.ex* also optimize placements across advertisers and content sites. They will each therefore be interested in obtaining two encrypted reports for each conversion: one for single-advertiser measurement on behalf of *shoes.ex* and one for cross-advertiser optimization on their own behalf. Additionally, we introduce *r3.ex*, which focuses solely on single-advertiser measurements and specializes in cross-intermediary reporting, providing a complete view of *shoes.ex*'s ad performance across the two placement intermediaries *r1.ex* and *r2.ex*. *r3.ex* will require only one encrypted report for the single-advertiser measurement on *shoes.ex*'s behalf.

3.3 Gordon architecture

Fig. 2 shows Gordon's architecture, with proposed changes to PPA highlighted in yellow (relative to Fig. 1(c)). Gordon modifies all three layers of PPA: the interface, the privacy filter architecture, and how privacy loss is accounted for during conversion measurement and report requests. These changes span four major conceptual shifts (yellow boxes on the left).

API changes for per-site semantics (Gap 1): We introduce changes in the API, protocol, and terms of use to resolve ambiguity in budget attribution. We add a *beneficiary site* parameter to the API, authenticate it toward the aggregation service, and restrict report usage—both at the aggregator and via terms of use—to ensure reports serve only the intended site's DP queries. These changes prevent intermediaries from re-purposing reports funded by conversion sites, restoring semantic clarity to per-site accounting for parties who comply with the protocol and its terms. This resolves PPA's Gap 1 (§2.5). Though not targeted at global filter defense, these changes outlaw Sybil and DoS attacks against it.

Online global filter management (Gap 2): Without further altering the API or protocol, we rework PPA's internal state and privacy loss accounting to defend the global filter from

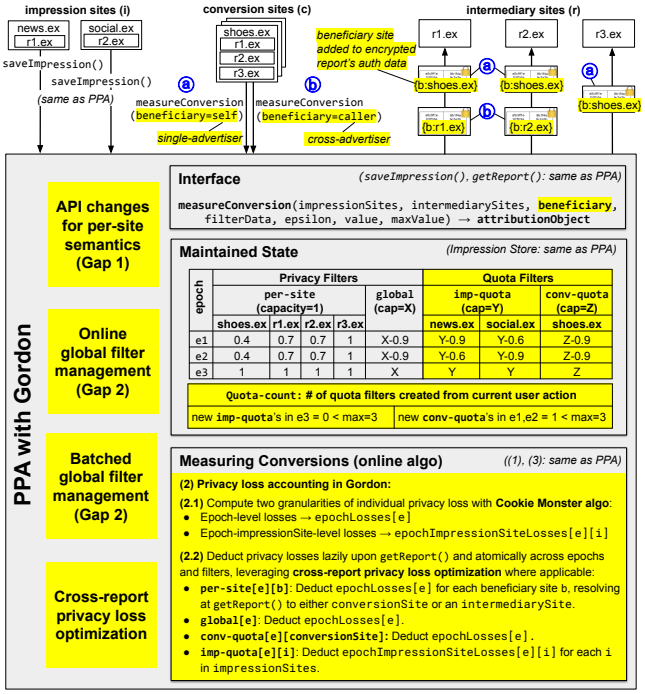


Fig. 2. Gordon architecture. Changes vs. PPA (Fig. 1(c)) in yellow. Roxana: Need to replace X, Y, Z with concrete values based on evaluation.

DoS depletion by malicious sites. We introduce *quota filters*, which regulate consumption to either isolate compliant sites from attack or ensure their utility degrades gracefully. Unlike per-site and global filters, which enforce privacy semantics, quota filters are solely for utility preservation—a novel use in the DP literature. We bound three quantities: (1) *how much* each impression site can contribute to global budget consumption via impressions registered for an adversary; (2) *how much* each conversion site can trigger consumption through report requests from an embedded adversary; and (3) *how many* unique impression and conversion site domains can interact with the API following a single user action. The first two are DP filters that consume budget in sync with the global filter and are tuned to support benign workloads. The third is a lightweight counter that anchors quota instantiation to a user-driven trigger. Together, these address PPA's Gap 2 under its current operation.

Batched global filter management (Gap 2): While effective, quotas may underutilize the global filter under benign conditions. This is analogous to the classic tradeoff in scheduling, where static partitioning of resources across participants, while being fair, can lead to underutilization. In PPA's current online model, where each report request must be resolved immediately, quotas are the main tool for ensuring isolation. But relaxing them (e.g., redistributing unused quota) risks reintroducing attack vectors. To address this, we explore *batched operation*, where report requests are collected and scheduled periodically in fixed *scheduling intervals* (e.g., daily). In this setting, a scheduler can schedule all remaining unscheduled

requests fairly, improving utilization while maintaining comparable DoS protection. This also addresses Gap 2, but under a modified operation.

Cross-report privacy loss optimization: We formalize a new budget optimization across Gordon's filters. Building on insights first surfaced by W3C participants, we observe that multiple reports tied to the same conversion—e.g., issued by different intermediaries—may not incur additional privacy loss against a shared filter if their impression sets are disjoint, since they reveal no more than the original attribution histogram to a common observer. We prove this property formally and leverage it in Gordon to optimize accounting across multiple privacy and quota filters.

4 Detailed Design

We detail Gordon's four core components, grounding each in our running example and the right side of Fig. 2.

4.1 API changes for per-site semantic (Gap 1)

We begin by detailing our proposed changes to clarify the semantics of PPA's per-site filters, which are intended to guarantee privacy against honest-but-curious sites. Currently, they do not—due to ambiguity in how privacy loss is charged to one site's budget while reports are delivered to another. In our *shoes.ex* example, intermediaries *r1.ex* and *r2.ex* each request a report on a conversion event, nominally on behalf of *shoes.ex*. PPA treats this as a single-advertiser query and deducts privacy loss from *shoes.ex* 's budget, since the reports are meant to help it measure ad performance. However, both *r1.ex* and *r2.ex* receive the reports and may also use them for their own analytics. Now suppose *r1.ex* and *r2.ex* want to run cross-advertiser queries—e.g., training a model to choose between showing an ad from *shoes.ex*, *hats.ex*, or *bags.ex*, based on content-site context. In that setting, PPA plans to charge privacy loss to the intermediaries themselves, recognizing them as the primary beneficiaries. But when the same entities receive reports in both contexts, the line between client-serving and self-serving queries becomes blurred. Without clear guardrails, even well-meaning intermediaries might be tempted to reuse reports charged to *shoes.ex* 's budget to improve their own cross-advertiser DP query.Poor incentives also arise for conversion sites themselves, who may shard into domains like *shoes.ex*, *shoes-cart.ex*, and *shoes-purchase.ex* to stretch their privacy budgets for more accurate measurements. Without clear constraints on report usage, per-site privacy loss accounting becomes semantically incoherent even for honest-but-curious actors.

Proposed changes. Our approach in Gordon is to explicitly declare the *beneficiary site*—the site on whose behalf the DP query will be run. This requires modifications to the API, the protocol with the aggregation service, and the API's terms of use. For the **API**, we introduce a `beneficiary` parameter in `measureConversion()`, which resolves to the conversion site for single-advertiser measurements and to the requesting intermediary site for cross-advertiser optimizations. This resolution happens upon the `getReport()` invocation. The browser deducts privacy loss from the epoch-level filters of the beneficiary site, creating these filters if necessary. We place no restrictions on per-site filter creation, as for these filters we operate under the honest-but-curious model.

For the **protocol**, we include the query beneficiary site in the report's authenticated data and require the trusted aggregator to verify its consistency across all reports in a batch. If any two reports diverge on the beneficiary, the query will be rejected. This prevents intermediaries from re-purposing reports obtained for different conversion site clients—funded by the clients' distinct privacy budgets—for their own DP cross-advertiser queries.

For the API's **terms of use**, we mandate that: *DP-query results tied to a specific* `beneficiarySite` *shall not be used to improve queries for other beneficiaries*. Thus, not only can reports obtained for a given `beneficiarySite` solely be used in DP queries on its behalf, but also results obtained from them will only be shared with that site, regardless of who executes the queries. The preceding terms of use are more powerful than first meets the eye. In particular, it prohibits combining reports and DP results that a company might obtain by assuming multiple identities (e.g., *shoes.ex*, *shoes-cart.ex*, *shoes-purchase.ex*). It also makes Sybil behavior as described in §3.1 unlawful, hence we can expect honest-but-curious sites to not engage in it.

These measures remove ambiguity and make privacy loss accounting against a single site's budget semantically meaningful under well-defined assumptions clearly stated in the terms of use. PPA has already incorporated the protocol change following our recommendation and we plan to propose the other two changes shortly.

Example. In Fig. 2, *shoes.ex* invokes `measureConversion()` twice for a single $60 purchase: (a) once for its own measurement (`beneficiary = self`) and (b) once for intermediary optimization (`beneficiary = caller`), creating two separate `attributionObjects`. For the first `attributionObject` (a), any intermediary (*r1.ex*, *r2.ex*, *r3.ex*) can call `getReport()`, which resolves the beneficiary to *shoes.ex* and deducts from its per-epoch filters. Thanks to Cookie Monster's per-query accounting and Gordon's cross-report optimizations (§4.2), *shoes.ex* retains 0.4 budget in epochs $e1$ and $e2$, and a full 1 in $e3$. For the second `attributionObject` (b), *r1.ex* and *r2.ex* call `getReport()` on their own behalf, which resolves the beneficiary to *r1.ex* and *r2.ex*, respectively, triggering deduction from their own budgets and leaving 0.7 in $e1$ and $e2$ for each. Since *r3.ex* does not issue optimization queries, it receives no such object and preserves its budget. Each report returned by Gordon includes the authenticated beneficiary, shown in yellow over the ciphertext: `b:shoes.ex` marks single-advertiser

reports requested by all intermediaries; `b:r1.ex` and `b:r2.ex` label cross-advertiser reports for *r1.ex* and *r2.ex*, respectively.

4.2 Cross-report privacy loss optimization

We illustrate Gordon's *cross-report* privacy loss optimization using our running example, deferring a general treatment to Appendix A.2. This optimization is orthogonal to Cookie Monster's *per-report* individual-DP-based strategies (§2.3), and instead leverage structure *across* reports, often requested by different intermediaries for the same conversion.

In Fig. 2, *r1.ex*, *r2.ex*, and *r3.ex* request single-advertiser reports from `attributionObject` ⓐ, all on behalf of client *shoes.ex*; separately, *r1.ex* and *r2.ex* request cross-advertiser reports from ⓑ for their own purposes. All five reports operate on the same attribution histogram, assigning $30 to each of two impressions (epochs `e1`, `e2`). Cookie Monster computes a base epoch-level privacy loss of 0.3 per report (given `epsilon =1`, `maxValue=100`). Naïvely, one would expect a cumulative deduction of 0.9 from *shoes.ex*'s filters (three reports) and 1.5 from the global filter (five reports). Yet the Privacy Filters table shows only deductions of 0.6 and 0.9, respectively.

The discrepancy arises because some reports *shard* the histogram into non-overlapping pieces—enabling parallel-composition-like optimizations. *r1.ex* and *r2.ex*'s single-advertiser reports from ⓐ each include a disjoint portion: `{1:30}` and `{2:30}`, respectively. Since both are funded by the same per-site filter (of *shoes.ex*), their combined release leaks no more than a single full histogram toward *shoes.ex*, incurring only 0.3 privacy loss. They likewise count as one deduction against the shared global filter. In contrast, *r3.ex*'s report includes the full histogram (to give *shoes.ex* a complete view across intermediaries; see §3.2), overlapping with both *r1.ex* and *r2.ex* and adding another 0.3 of loss to both *shoes.ex*'s filter and the global filter. A similar optimization applies to cross-advertiser reports from ⓑ. These are funded from separate filters (those of *r1.ex* and *r2.ex*), so each incurs 0.3 loss. But against the global filter, they again count as one, bringing the total global filter deduction to 0.9 instead of the unoptimized 1.5.

Appendix A.2 formalizes the optimization, whose logic we encapsulate in the `attributionObject`. This object dynamically optimizes budget deduction across the per-site, global, and quota filters on each `getReport()` call, on the basis of prior invocations and deductions.

4.3 Online global filter management (Gap 2)

With per-site filters clarified and optimized, we can now hope for *strong privacy guarantees against individual honest-but-curious sites*, assuming tight configuration of these filters (e.g., capacity $\epsilon_{per-site} = 1$). Our terms of use also prohibit collusion, as DP results across identities must not be combined. Still, non-compliant behavior remains possible, making the global filter essential as a safeguard against worst-case privacy loss—i.e., an adversary accessing results from all sites. Reports are returned only if they can be funded by both per-site and global filters. Appendix **??** formalizes this multi-granularity filter accounting, which we have not seen articulated in prior work, despite its practical significance.

The key challenge lies in configuring and managing the global filter to support benign workloads while resisting depletion attacks. We begin by outlining specific attacks and limitations of existing defenses, motivating our defense.

DoS depletion attacks. An adversary X may aim to exhaust the global budget—either to increase their own utility or disrupt others'. The latter is especially potent in PPA via single-advertiser queries and becomes more dangerous if cross-advertiser support is added. To bypass per-site limits, X can register $s = \epsilon_{\text{global}}/\epsilon_{\text{per-site}}$ Sybil domains and distribute queries across them.

Attack 1: Cross-advertiser reports. X builds a site embedding the s Sybil domains as intermediaries. When user u visits, the site: (1) registers r impressions with X as the conversion site and a different Sybil as intermediary; and (2) has each intermediary request a cross-advertiser report, exhausting its per-site budget. This drains global budget for u. If many users visit X even once in an epoch, X can disrupt measurements by other sites for that epoch. If users continue arriving across epochs, X can sustain disruption—mounting a persistent attack with a single popular site and just one visit per user per epoch. PPA isn't currently vulnerable, lacking cross-advertiser support. But a similar attack works via single-advertiser reports:

Attack 2: Single-advertiser reports. Now the Sybils serve as conversion domains. When u visits X, X: (1) registers s impressions, each tied to a different Sybil; (2) auto-redirects s times, switching domains and requesting a single-advertiser report each time. This depletes global budget similarly. While aggressive redirection can be heuristically flagged, redirection is too common for browsers to ban outright.

Attack 3: Single-advertiser reports, subtler version. X (1) registers s impressions with Sybil conversion sites and (2) redirects once to load a new Sybil domain that requests a report. Reports use maximum attribution windows to draw budget across past epochs via impressions previously registered by X. If many users visit X's site roughly s times over each epoch's *data lifetime* (typically months), X can drive sustained global budget depletion—again with a single site but requiring more than one visit per user.

Limitations of existing defenses. Certain behaviors in these attacks clearly exceed reasonable use. For example: (1) PPA is intended for cross-site measurement, so identical values for `impressionSite` and `conversionSite` (Attack 1) should be disallowed. (2) A single user action should not drive both registration of an impression and trigger a conversion measurement of it—even under different domains (Attacks 1–2). (3) Excessive redirection (Attack 2) should be detectable. (4) Allowing an epoch's entire global budget to be exhausted in seconds is fundamentally flawed. These heuristics imply

minimal browser-enforced defenses, but more principled protections are needed to guard against subtler abuse like Attack 3.

In W3C discussions, several mitigations have been proposed—restricting which sites receive per-site filters (e.g., via mandatory registration), rate-limiting API calls, or capping the number of sites granted filters per epoch. While useful as part of browsers' defense-in-depth strategies, such measures risk over-constraining a nascent, evolving workload. Mandatory registration could limit API access, undermining the open-web ethos. Hard limits on per-site impression counts are tricky: some sites may show many ads, others few. The same with conversions, which can represent many things, from rare purchases to frequent XXX TODO(Ben): give an example of XXX. Capping the number of intermediaries per conversion might constrain advertisers' ability to work with diverse partners. And if the API is repurposed beyond advertising—for instance, to measure engagement or reach—workload characteristics may shift further. Fixed constraints that seem reasonable today could stifle innovation or penalize legitimate new use.

Our approach: Enforce stock-and-flow. We seek approaches grounded in *minimal workload assumptions*, offering a flexible springboard for implementing effective protection in browsers without over-constraining the API. As discussed in §2.3, intended API use follows a *stock-and-flow pattern*, where privacy loss is driven by explicit user actions—such as navigations or clicks—across distinct content and conversion domains. Attacks above break this pattern by relying on automatic flows and collapsing domain roles.

We restore this pattern through *quotas* that govern privacy consumption: (1) impression-site quotas cap stock creation per epoch, (2) conversion-site quotas cap triggered flow per epoch, and (3) a count-based limit bounds the number of new sites that can create the preceding quotas in response to a single user action. Unlike indirect metrics (e.g., such as API call counts per unit of time, number of intermediaries, or domains with per-site filters), our first two quotas operate directly on the core protected resource: the global filter. Each is grounded in a notion of *share*, calibrated to expected workloads. Finally, the third quota enforces the stock-and-flow pattern's behavioral anchor—explicit user action. Together, these quotas force adversaries to operate within the contours of expected workloads, significantly curbing their ability to drain the global budget from a single site with limited user interaction.

Gordon quota system. Fig. 2 highlights (yellow background) the internal state maintained by Gordon to manage PPA's global privacy filters. We use two types of quotas: (1) *quota filters* (`imp-quota`, `conv-quota`) implemented as *DP filters* not for privacy accounting but to regulate global filter consumption; and (2) a standard *count-based quota* that limits the number of new quota filters that can be instantiated per user action (e.g., a click or navigation). Algorithms XXX-ZZZ in appendix formalize the system's behavior.

The impression-site quota filter, `imp-quota`, is scoped per impression site and per epoch. It bounds the portion of global privacy loss attributable to flows leveraging stock created by impressions from that site. When a site `i` first invokes `saveImpression()` in epoch `e`, Gordon creates `imp-quota[e][i]` with a preconfigured capacity representing its share of the global filter. While any site `i` can receive a `imp-quota`, it is only consumed if a subsequent report matches an impression from `i` in that epoch—that is, if the site's stock is used.

The conversion-site quota filter, `conv-quota`, is defined symmetrically: scoped per conversion site and per epoch, it bounds the global privacy loss attributable to flows initiated by conversions on that site. `conv-quota[e][c]` is created when conversion site `c`, on or after epoch `e`, first calls `measureConversion()` in a way that could incur non-zero individual privacy loss in epoch `e`. It is consumed upon a `getResult()` – that is, when a privacy flow occurs.

Fig. 2 sketches Gordon's privacy loss accounting algorithm (box "Measuring Conversions;" full algorithm in appendix, Algorithms XXX-ZZZ). First, individual privacy losses are computed per epoch using the Cookie Monster algorithm. Then, on each `getReport()` call, we attempt to deduct these losses across all relevant filters and epochs in an atomic process that succeeds and alters any filter's state only if all checks pass. For each epoch `e`, the relevant filters are: `per-site` of the beneficiary, `global`, `conv-quota` of the conversion site, and `imp-quota` for each impression site with non-zero loss (as computed by Cookie Monster). To efficiently handle impression-site quotas, we scope loss computation to the (epoch, impression site) level and charge the resulting loss to the corresponding `imp-quota` filter. We also apply the cross-report optimizations from §4.2 to remove redundant charges. While the combined capacity and budget usage of all instantiated impression-site (or conversion-site) quotas may at any time exceed that of the global filter, our algorithm's checks against both the quotas and the global filter ensure that the global privacy guarantee is never breached.

Quota filters limit how much each first-party site can contribute to global privacy consumption. In a world without automatic redirects—where each domain change stems directly from a user action—this would suffice to reestablish PPA's intended user-driven stock-and-flow behavior. But since automatic redirects are pervasive on the web, we relax that assumption: following a single explicit user action, we permit a bounded number of unique first-party domains that trigger creation of new quota filters in an epoch. This bound, `quota-count`, is configurable, and while not necessarily one, we expect it to be small—typically in the low single digits.

Configuration to "normal" workload. A key question is how to configure privacy and quota filters to avoid impeding "normal" workloads in no-attack scenarios. Our approach has

"Normal" workload parameters: N, M, n, r (see text).	
Filter	**Capacity configuration**
Per-site filter	$\epsilon_{\text{per-site}}$: config. parameter
Global filter	$\epsilon_{\text{global}} = \max(N, n \cdot M)(1+r)\epsilon_{\text{per-site}}$
Impression-site quota	$\epsilon_{\text{imp-quota}} = n(1+r)\epsilon_{\text{per-site}}$
Conversion-site quota	$\epsilon_{\text{conv-quota}} = (1+r)\epsilon_{\text{per-site}}$

Tab. 1. Gordon filter configurations.

three steps. First, we define four browser-adjustable parameters that outline the scale of the intended workload: **N**, the maximum number of conversion sites that may request non-zero privacy loss from any epoch; **M**, the maximum number of impression sites in an epoch that may contribute toward non-zero privacy loss in that epoch; **n**, the maximum number of conversion sites that may request non-zero loss from a single (epoch, impression site) pair {Mathias: I don't get this one, and especially how it differs from *N*: do you mean that one conv requesting from imps counts for one in *N* and 2 in *n*? might be worth a quick example/tying to a previous example?}; and **r**, the maximum budget consumed by an intermediary's cross-advertiser queries on a single conversion site, as a fraction of the intermediary's $\epsilon_{\text{per-site}}$. Second, given N, M, n, r, and $\epsilon_{\text{per-site}}$, we specify constraints that configurations of ϵ_{global}, $\epsilon_{\text{imp-quota}}$, and $\epsilon_{\text{conv-quota}}$ must satisfy to support this workload: $\epsilon_{\text{conv-quota}} \geq (1+r)\epsilon_{\text{per-site}}$ (to support $\epsilon_{\text{per-site}}$ for single-advertiser queries plus $r\epsilon_{\text{per-site}}$ for cross-advertiser queries); $\epsilon_{\text{imp-quota}} \geq n\epsilon_{\text{conv-quota}}$ (to allow an impression site to be queried by n conversion sites using their full $\epsilon_{\text{conv-quota}}$); $\epsilon_{\text{global}} \geq N\epsilon_{\text{conv-quota}}$ and $\epsilon_{\text{global}} \geq M\epsilon_{\text{imp-quota}}$ (to support N conversion and M impression sites at full quota). Third, we solve these constraints to derive configuration formulas. Table 1 shows intuitive formulas from manual resolution. We use these formulas to establish Gordon 's analytical resilience below and evaluate it in §6.

Resilience to DoS depletion. Gordon's quotas are configured to enable normal workloads without interference, while bounding the power of an attacker. The `quota-count` bound on quota-filter creation (coupled with defense in-depth heuristics we expect browsers to deploy to detect sybil attacks) Roxana: What heuristics are needed to ensure what you specify next? Don't our quotas already ensure that? What the heuristics need to do is that these conditions are hard for an adversary to overcome, no? ensures that the number of `imp-quota` (privacy stock) and `conv-quota` (privacy flow) available to an adversary on any device d at epoch e is low, proportional to real user interactions with an adversary controlled site on that device, in that epoch.

Denote as M^{adv} and N^{adv} the number of `imp-quota` and `conv-quota`, respectively, that an adversary managed to create on a user's device d at epoch e (by controlling sites the user visited, and creating up-to `quota-count` quota features on each user action). This ensures the following constraint on how much our adversary above can consume from the global filter:

Theorem 1 (Resilience to DoS depletion)**.** *Consider a adversary controlling M^{adv} and N^{adv} `imp-quota` and `conv-quota`, respectively. The maximum budget ϵ^{adv}_{global} that this adversary can consume from the global filter is such that:*

$$\epsilon^{adv}_{global} \leq \min(M_{adv}\epsilon_{imp\text{-}quota},\ N_{adv}\epsilon_{conv\text{-}quota}).$$

Proof. Appendix **??**. □

As we can see, the impact of an adversary on the Global budget scales as the minimum of M^{adv} and N^{adv}, the number of `imp-quota` and `conv-quota` that the adversary was able to create on the user's device. The adversary hence needs a user to interact repeatedly with a site under their control, and to split the interactions between the creation of impressions and conversions, to be able to mount an attack.

A direct implication of Theorem 1 and DP composition is that benign users will retain at least $\epsilon_{\text{global}} - \epsilon^{\text{adv}}_{\text{global}}$ of the Global filter budget to use in epoch e on device d. Under the instantiations of Table 1, mounting a full DoS attack requires an adversary to create $M^{\text{adv}} \geq M$ `imp-quota` **and** $N^{\text{adv}} \geq N$ `conv-quota`, fully duplicating the entire legitimate workload. TODO(Mathias): Maybe add an implication that supports the "graceful degradation" claim for benign sites while under attack.

TODO(Mathias). {Mathias: In progress.} - What browsers need to do to implement complete defense on top of the resilience building blocks that our budget management provides. - For example, in terms of use, clearly prohibit sharding one's site for the purpose of getting more budget on a user's device. - In a concrete example, for Attacks 1 and 3, how does the attacker's effort (say measured in terms of number of per-user interactions he needs to get) increase for a full-on attack? - Give a sense for the profile of attacker that would still succeed in the attack: lots of per-user interaction, from many users. The attacker posts every link under a new domain. This is clearly illegal and browsers should develop methods to detect that. - End on: One approach to build further resilience within budget management is to release the global filter more gradually during the lifetime of an epoch, to ensure that an attack occurring during a unit of time does not affect benign sites' ability to do measurements in no-attack units of time. This leads to increased isolation but introduces utilization problems, as we next describe...

4.4 Batched global filter management (Gap 2)

As is often the case in scheduling, static partitioning of resources with our quota system can lead to underutilization of the global filter in benign cases. Consider a device that visits *only* two impression sites: *news.ex*, which contains ads for many conversion sites, and *blog.ex*, which contains only one. Due to the per-site filter, *blog.ex*'s single conversion site can only trigger consumption of a small fraction the global budget ($\epsilon_{\text{per-site}}$). On the other hand, due to the impression site quota, conversions drawing on *news.ex*'s impressions can

only consume up to half Roxana: Where do we get the half? What params M... are we assuming here? of the global filter, even if they register many legitimate conversions across different conversion sites. Therefore, almost half of the global filter's capacity is left unconsumed, even though from a DP perspective there would be no harm in allowing *news.ex* to consume more budget. This underutilization would especially hurt major impression sites that may host significant user traffic, and is therefore important to address.

Scheduling intervals. In order to address this problem, we can adopt a similar approach as max-min fairness scheduling [LPT+21], where we prioritize sites that request their "fair share", and only after scheduling all the "fair share" requests, we fully utilize the remaining unused resource. To this end, we split the data lifetime of each epoch into T *scheduling intervals*, where the length of each scheduling interval by default is a day. The scheduling interval has three phases: an initialization phase, an online phase and a batch phase, which we describe in more detail below. At the beginning of each initialization phase, the quotas get reset, and a fraction of the global filter's budget gets released ($\frac{\epsilon_{\text{global}}}{T}$).

Response time. In addition, we extend PPA's API to allow requests to specify their *response time*. The response time determines when the response to the requests will be returned. If a request sets a short request time, it will receive a more timely response, but it is less likely to actually consume the privacy budget and is more likely to get a null response. By default, we expect requests will use a response time that spans multiple scheduling intervals (*e.g.,* a week), so requests have multiple opportunities to get allocated budget. We now elaborate the three phases of the scheduling intervals, starting with the online phase.

Online phase. In the online phase of each interval, requests consume budget by the order of arrival. In this phase, only requests that have *sufficient budget left in their quotas* get scheduled. Since they are limited by their quotas, they have to request less or equal to their *fair share*, otherwise they are requesting more budget than their quotas allow. If requests arrive and are not scheduled (either because they have no more quota or filter budget left, or because they requested more than their fair share), they are added to a *queue* and may get another chance at being scheduled in the batch phase.

Batch phase. At the end of the scheduling interval, Gordon tries to utilize all the remaining global budget by scheduling from the remaining requests in the queue that have not already generated responses. To this end, Gordon tries to schedule the requests in the queue one by one, making sure they have sufficient per-site and global filter budgets, but ignoring the quotas. There are different possible algorithms for sorting the requests in the queue, with different possible objective functions. For example, we could try to maximize per-site or global budget consumption, or conversely optimize for fairness across the different impression sites. By default, we choose the following algorithm, which optimizes for fairness. Gordon schedules outstanding requests one by one from the queue. Each time after it schedules a request from the queue Gordon sorts the requests in the queue based on the impression site that received the least budget so far, and among the requests that belong to the same impression site it sorts the requests based on their requested privacy budget (from small to large). It continues greedily scheduling requests until it reaches a point where either there are no more requests to schedule, or all the remaining requests cannot be scheduled, since they do not have sufficient remaining budget in the filters.

Initialization phase. All the requests that have not been allocated budget at the end of the batch phase and still do not need to be returned to the querier site, are forwarded to the queue of the next scheduling interval. As mentioned before, at start of the new scheduling interval, we reset the quotas and $\frac{\epsilon_{\text{global}}}{T}$ of the global filter's budget is released. Before starting the online phase of the new scheduling interval, Gordon first tries to schedule any outstanding requests from the queue that requested less or equal to their fair share, using the newly-reset quotas and newly-released global budget. After this initial step, the online phase of the next scheduling interval begins.

{Asaf: this paragraph below can be moved to appendix}

Response time tradeoff. We can prove by induction that the longer the requested response time, the higher the probability requests will get allocated privacy budget. We provide a proof sketch below. The base case is a response time that results in a response within the same scheduling interval as the request. In this case the request is only granted the opportunity to be scheduled during the online phase. If the response time is longer than a single scheduling interval, the request has the opportunity to be scheduled either: in the online phase of the scheduling interval it was requested in, in the batch phase of that scheduling interval, and in the initialization phase of the next scheduling interval, thereby increasing its probability of getting scheduled. The same property holds true for any response time of size $n = k$ scheduling intervals, where if we increase the response time by one interval ($n = k + 1$), the request would have two additional opportunities (at the batch phase of scheduling interval k and the initialization phase of scheduling interval $k+1$) to get scheduled, thereby increasing its probability of getting allocated budget.

4.5 Recommendations for browsers

Our research provide a springboard on which browsers to build effective defenses against privacy and depletion attacks. We articulate XXX core directions/principles of focus in developing heuristics and mechanisms for this. TODO(Write this at the end. What leverage do we give the browsers? What should they focus on? How does our work direct what they need to do? Etc. Short but strong and leveling.).

5 Prototype

Roxana: WIP. Do not read.

We implement Gordon in two components: (1) `pdslib`, a generic on-device individual DP library and (2) its integration into Mozilla Firefox's Private Attribution, a minimal PPA implementation. **`pdslib`:** TODO(Mark). Roxana: The description needs to be much more focused on the implementation. Do not repeat what's already written in the paper; the reader knows that by now. Also, you implement Gordon, not just Cookie Monster. Spell out the name first – `pdslib`, standing for Private Data Service. The mention about being generic is good. You can list which features you implement of the ones we discuss in the paper. You can say that we plan to release it as a reference for W3C (see intro statement, you can repeat that one, it appeared long ago :)). Keep it to one single short paragraph, we're crunching on space terribly. `pdslib` is our implementation of a generic Rust API that extends Cookie Monster's individual DP accounting with structured privacy budget management. `pdslib` aims to address key challenges in DoS attacks, especially large-scale Sybil attacks, by implementing our concerted filter architecture consisting of both non-collusion (NC) filters, collusion (C) filters, and specific quota filters of these c-filters, namely the *conversionsites* and *impressionsites* filters (§A for privacy and isolation guarantees).

The core innovation of `pdslib` is its enabling of cross-advertiser optimization queries while maintaining strong privacy guarantees against colluding advertisers. Through multiple coordinated filters, `pdslib` prevents adversaries from depleting global privacy resources through Sybil attacks, protecting legitimate queriers' utility. Furthermore, the library's modular traits-based architecture written in the generic Rust style separates event storage, privacy accounting, and query execution into composable components, allowing it better generality beyond the web advertising motivation Cookie Monster started with to many more applications that require on-device DP guarantees against adaptive adversaries.

Firefox integration: TODO(Giorgio): 1 very short paragraph. Go a bit through how you embedded `pdslib` into FF by changing ... whatever... Don't use too low-level code terms, like the specific name of the component you changed, but just what kind of components you plugged in. Mention Private Attribution, a very primitive version of PPA. Mention LoC of changes (added, changed).

6 Evaluation

Pierre: WIP. Do not read.

Evaluation questions:

- **Q1:** What parameters define "normal" operation in the Criteo workload?
- **Q2:** In the online setting, how do query error rates vary with different quota capacities?
- **Q3:** Can quotas preserve low error rates for benign queries under DoS attacks?
- **Q4:** Do quotas lead to under-utilization, and can batching mitigate this?

6.1 Methodology

Datasets. Pierre: Criteo PrivateAd, resampled. Multiple advertisers, multiple publishers, one intermediary (Criteo). Training on the first 10 days, evaluating on the last 20 days.

Evaluation scenario. Pierre: Each advertiser runs single-advertiser measurements. Histogram queries.

Benign workload process. Pierre: Generate conversions. Keep the 73 conversion sites that have more than 100 reports per day on average. Define the histogram buckets. Calibrate the batch size.

Attack workload process. Pierre: ...

Metric. Pierre: RMSRE, histogram generalization from ARA paper.

Roxana: Move a very short version of the below to the first section that studies error attribution. We further analyze how much each type of filter contributes to the overall error in a query, by introducing an "error cause" metric defined as follows. For each report, we compute how many and which filters were out-of-budget while computing the report. If any per-site filter filter is out-of-budget, we label the whole report as potentially biased and attribute the error to the per-site filter filter. We look at global filter filters next, then conversion-site quota filters and finally impression-site quota filters. Next, for each query we compute the number of reports in each category, and divide it by the number of reports in the query. This gives us the fraction of reports in the query affected by the per-site filter (resp. global filter, impression-site quota, conversion-site quota) filter. Finally, we average the fractions over all the queries in the workload.

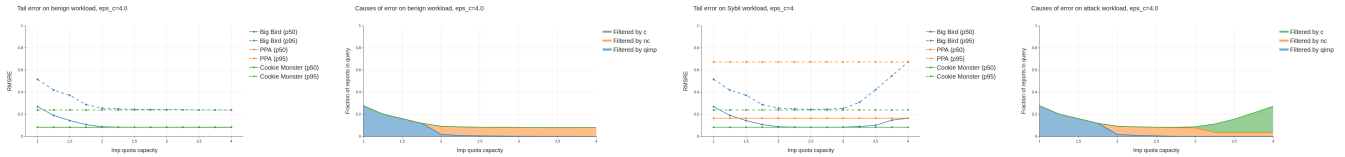
Pierre: Expand on other sources of error, DP error and RMSRE calibration. Bias variance.

Baselines. We compare Gordon against two baselines. The first baseline is **Cookie Monster**, which only maintains per-site filter filters. The second baseline, which we call **PPA**, is Cookie Monster augmented by a global filter filter, which corresponds to the current state of the PPA draft specification with the global filter filter as a safety limit.

Defaults. $\epsilon_{\text{per-site}} = 1$.

6.2 "Normal" workload parameters in Criteo (Q1)

In §4.3 we established sufficient conditions under which filters do not harm normal operation, thanks to parameters r, N, M and n. In our Criteo workload we have $r = 0$ since we only consider measurement queries, and can thus take $\epsilon_{\text{conv-quota}} = \epsilon_{\text{per-site}}$. However, determining the values of N, M and n is not immediate since it would require executing a benign workload to count how many non-zero reports each device-epoch sent to or from various parties. Pierre: TODO: do that, it's tighter and easier to explain in the text. Keeping



(a) Error, normal workload **(b)** Error causes, normal workload **(c)** Error under attack **(d)** Error causes under attack

Fig. 3. Online quota system evaluation. (a), (b): Query error and its root causes in a benign case. (c), (d): Benign-query error and root causes under attack. Roxana: Changes to graphs: (1) Move the legends inside the graphs (there's space on top given the [0,1] y axis (which you should keep!). You can order the legend entries 2x2 to fit atop each graph. (2) Enlarge all fonts and line widths and point sizes significantly so they become visible when areas are squeezed. (3) Remove graph titles. (4) Turn everything into black-and-white – no color differentiations pls! You can differentiate based on point types (consistent across the colors hence system types). For area graphs, you can use different textures in place of colors. (5) Make the PPA points much larger than the Cookie Monster points so you can see them from behind CM. (6) Rename Big Bird to Gordon. (7) Label Cookie Monster as "Cookie Monster $\epsilon_{\text{global}} = \infty$", PPA as "PPA $\epsilon_{\text{global}} = 4$, Gordon as "Gordon $\epsilon_{\text{global}} = 4$". (8) x axis label: "Impression site quota capacity ($\epsilon_{\text{imp-quota}}$). x axis label: RMSRE becomes "RMSRE (error, lower is better)"; "Fraction of reports in query" becomes "% of reports in query impacted by a filter." For (b) and (d), legend should say: "per-site filter, "global filter," "quota filter". (9) Please transform y axis for (b) and (d) to 0-100%, to not confuse it with the other 0-1 RMSRE axes. Pierre: Also probably combine all the 4 graphs as a single pdf on the Python side, to share captions etc.

rough statistics as a placeholder, but they are overestimations (*e.g.,* count of "authorized" conversion sites for each impression site, instead of actual conversions that happen). Instead, we can derive upper bounds $\tilde{N} \geq N, \tilde{M} \geq M, \tilde{n} \geq n$ by computing simple statistics from a dataset of impressions and conversions, as detailed next. For $\tilde{N}$, we compute a percentile of the distribution of number of unique conversion sites in each device-epoch, and multiply by the maximum attribution window length (2 in Criteo). This gives a crude upper bound on the maximum number of times most devices are queried, which is itself an upper bound on the maximum number of times most devices are queried *with non-zero loss*. For $\tilde{M}$, we take a percentile of the number of unique impressions sites across device-epochs. For $\tilde{n}$, we take a percentile of the number of unique conversion sites across impression sites and device-epochs, and multiply by the maximum attribution window length.

Percentile	$\tilde{N}$	$\tilde{M}$	$\tilde{n}$	ϵ_{global}	$\epsilon_{\text{imp-quota}}$
50	2	1	2	2	2
90	4	2	4	8	4
95	4	2	4	8	4
99	6	3	6	18	6
100	12	7	14	98	14

Tab. 2. Percentile values for $\tilde{N}, \tilde{M}$, and $\tilde{n}$.

Finally, we take $\epsilon_{\text{conv-quota}} = \epsilon_{\text{per-site}}$, $\epsilon_{\text{imp-quota}} = \tilde{n}\epsilon_{\text{per-site}}$ and $\epsilon_{\text{global}} = \max(\tilde{N}, \tilde{n} \cdot \tilde{M})\epsilon_{\text{per-site}}$. For instance, taking values at the 95th percentile gives $\epsilon_{\text{global}} = 8$, $\epsilon_{\text{imp-quota}} = 4$, $\epsilon_{\text{conv-quota}} = 1$.

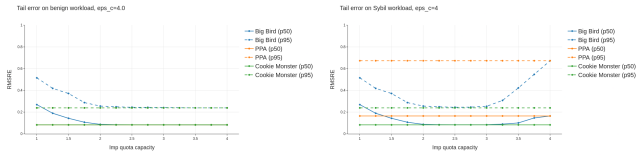
6.3 Query errors under normal workload (Q2)

We fix $\epsilon_{\text{global}} = 4$, a value that is large enough to accommodate all the queries without triggering the global filter filter. Pierre: TODO: rerun experiments with $\epsilon_{\text{global}} = 8$ or whatever we get from Q1. We vary $\epsilon_{\text{imp-quota}}$ and measure its impact on query error. Fig. 3a reports both median and 95th percentile RMSRE. Since Cookie Monster and PPA lack a impression-site quota filter, their errors remain constant across $\epsilon_{\text{imp-quota}}$ values. Moreover, Cookie Monster and PPA exhibit identical error, as we set ϵ_{global} high enough to deactivate PPA's global filter filter—effectively reducing it to Cookie Monster. In contrast, Gordon 's error increases at low $\epsilon_{\text{imp-quota}}$, as the impression-site quota filter blocks some reports. For $\epsilon_{\text{imp-quota}} \geq 2$, the filter no longer affects benign-query error, suggesting that reasonably-configured quotas can preserve utility.

Fig. 3b attributes the error to each filter type, for Gordon. Roxana: I had initially misunderstood the y axis. Explain this graph and the process of attributing error here, not in methodology, as people lose track. Mention just extremely briefly that errors in queries can come from multiple sources in Gordon: DP itself, but also filters and quotas. Remind them how quotas/filters OOB add bias into the query results (the may not know this as we never reminded them of bias etc.! Don't say too much, just find a very simple, intuitive way of explaining it in one short sentence). Say that the graph is showing attribution of errors that come from the filters or quotas. Pierre: Add Cookie Monster and PPA for reference as numbers in the text? They only have per-site filter reports, the same as Gordon. For $\epsilon_{\text{imp-quota}} = 1$, almost a third of reports in each query are filtered out by the impression-site quota filter. This explains the high error seen at low values of $\epsilon_{\text{imp-quota}}$ in Fig. 3a. For $\epsilon_{\text{imp-quota}} = 2$ and higher, almost all the biased reports come from per-site filter filters. These filters also exist in Cookie Monster and PPA, which explains why Gordon's error converges to Cookie Monster's error in Fig. 3a.

6.4 Query errors under DoS attack (Q3)

We augment the benign scenario with a DoS depletion attack akin to Attacks 2 and 3 (since our dataset does not permit cross-advertiser queries for more than one intermediary site). First, we create a malicious impression site X, by duplicating the impression site X^* that had the most impressions in Criteo. This ensures that a non-negligible fraction of the devices visit X at some point, thus enabling the attack to have visible effect on query accuracy. Next, we create new malicious conversion sites $Y_1^1, \ldots, Y_1^s, \ldots, Y_k^1, \ldots, Y_k^s$ by duplicating s



(a) Error, normal workload **(b)** Error under attack

Fig. 4. Batched system evaluation. Roxana: BOGUS FIGURES, DO NOT LOOK.

times each of the $k = 10$ conversion sites $Y_1^*, \ldots, Y_k^*$ with most conversions attributed to X^* in Criteo. Finally, we create impressions for X by copying all the impressions X^* has for $Y_1^*, \ldots, Y_s^*$, and we create conversions for Y_i^j by copying all the conversions that Y_i^* has for X. $s = 1$ corresponds to Attack 3, since conversions happen over time, following the same pattern as the benign conversions $Y_1^*, \ldots, Y_k^*$ have for X^*. A larger value for s corresponds to Attack 2, since we have a chain of s conversion sites $Y_i^1, \ldots, Y_i^s$ (using automatic redirection or tricking the user into clicking) instead of a single conversion site Y_i^*.

Fig. 3c shows that impression-site quota limits the impact X has on the c-filter, for an attack with $s = 10$. We do not include X's queries in the error. For a well-configured impression-site quota filter, honest queriers are perfectly isolated from X, and are still able to run their own queries without being perturbed by the impression-site quota.

Fig. 3d further drills down into the causes for high error at both ends of the graph in Fig. 3c. For low $\epsilon_{\text{imp-quota}}$, the error comes from the impression-site quota filter, as in Fig. 3b. The attack has no direct impact on error, but impression-site quota is hurting even honest queries. For high $\epsilon_{\text{imp-quota}}$, the error comes from the global filter filter. This is because the impression-site quota filter is too loose and lets X deplete the global filter filter, thereby denying reports for honest queriers.

6.5 Batched system evaluation (Q4)

7 Related Work

8 Conclusions

References

[LPT+21] Tao Luo, Mingen Pan, Pierre Tholoniat, Asaf Cidon, Roxana Geambasu, and Mathias Lécuyer. "Privacy Budget Scheduling". In: *15th USENIX Symposium on Operating Systems Design and Implementation (OSDI 21)*. USENIX Association, July 2021, pp. 55–74. ISBN: 978-1-939133-22-9. URL: https://www.usenix.org/conference/osdi21/presentation/luo.

[TKM+24] Pierre Tholoniat, Kelly Kostopoulou, Peter McNeely, Prabhpreet Singh Sodhi, Anirudh Varanasi, Benjamin Case, Asaf Cidon, Roxana Geambasu, and Mathias Lécuyer. "Cookie Monster: Efficient On-Device Budgeting for Differentially-Private Ad-Measurement Systems". In: *Proceedings of the ACM SIGOPS 30th Symposium on Operating Systems Principles*. SOSP '24. New York, NY, USA: Association for Computing Machinery, Nov. 15, 2024, pp. 693–708. ISBN: 9798400712517. DOI: 10.1145/3694715.3695965.

A Formal analysis

A.1 Model with Single-Beneficiary Reports

We start from the same model as Cookie Monster, and then refine it by adding constraints on the data and the queries. §A.1 makes the following assumptions, that we relax in §A.2.

- The set of public events for a beneficiary site b is the set of all conversions where b appears as beneficiary site. In other words, we hardcode $P = C_b$. This prevents publisher reports or certain adtech optimization queries, that were technically supported by Cookie Monster. [1]
- Each report has a single beneficiary site: either the conversion site (measurement report) or another site (optimization report). In this formalism, a conversion site can register the same conversion data multiple times with different beneficiary sites if it wants to execute both measurement and optimizations.

A.1.1 Sites and roles.

We use a different terminology compared to Cookie Monster, to better align with PPA. Moreover, we make sites appear explicitly in the data and query model. We take a set of sites $\mathcal{S}$ (*e.g.,* domain name). The same site can appear under different roles:

- impression site: site where an impression occurs (publisher in Cookie Monster)
- conversion site: site where a conversion occurs (advertiser in Cookie Monster)
- beneficiary site: site that receives the results of a DP query (querier in Cookie Monster)

A.1.2 Data model.

A database D is a set of device-epoch records where each record $x = (d, e, F) \in \mathcal{X} = \mathcal{D} \times \mathcal{E} \times \mathcal{P}(\mathcal{S} \times \mathcal{I} \cup \mathcal{S} \times \mathcal{S} \times C)$ contains a device d, an epoch e and a set of impression and conversion events F. Each event $f \in F$ contains the site (impression site i or conversion site c) where the event occurred: $f = (i, \text{imp}) \in \mathcal{S} \times \mathcal{I}$ or $f = (c, b, \text{conv}) \in \mathcal{S} \times \mathcal{S} \times C$. Additionally, conversions contain the beneficiary site b that will receive the conversion report. [2]

Definition 1 (Filter ($\mathcal{F}_x$))**.** *Pierre: Thanks for adding this! This definition can become particularly useful if you actually define "canConsume" and "tryConsume" for a Pure DP filter, with the explicit sum of parameters from the Rogers 2016 paper. Then the isolation proofs will be able to point to the equation when they need to justify that the c-filter passes. For each device-epoch record $x = (d, e, F)$, we maintain several types of privacy filters:*

- $\mathcal{F}_x^{nc[b]}$*: Non-collusion filter for beneficiary site b, with capacity* $\epsilon_{nc[b]}$
- $\mathcal{F}_x^{c}$*: Collusion filter with capacity* ϵ_c.
- $\mathcal{F}_x^{q\text{-}conv[c]}$*: Conversion site quota filter for site c, with capacity* $\epsilon_{q\text{-}conv}$
- $\mathcal{F}_x^{q\text{-}imp[i]}$*: Impression site quota filter for site i, with capacity* $\epsilon_{q\text{-}imp}$

Each filter maintains a state of consumed budget and provides operations:

- canConsume(ϵ)*: Returns* TRUE *if the filter can accommodate additional privacy loss* ϵ.
- tryConsume(ϵ)*: Deducts privacy loss* ϵ *from the filter's remaining capacity*

A.1.3 Query model.

Definition 2 (Attribution function, adapted from Cookie Monster)**.** *Fix a set of relevant impression sites* $\mathbf{i}_A \subset \mathcal{S}$ *and a set of* impressions [3] *relevant to the query* $F_A \subset \mathbf{i}_A \times \mathcal{I}$. *Fix* $k, m \in \mathbb{N}^*$ *where* k *is a number of epochs. An* attribution function *is a function* $A : \mathcal{P}(\mathcal{I} \cup C)^k \to \mathbb{R}^m$ *that takes* k *event sets* $F_1, \ldots, F_k$ *from* k *epochs and outputs an* m*-dimensional vector* $A(F_1, \ldots, F_k)$*, such that only* relevant events *contribute to* A*. That is, for all* $(F_1, \ldots, F_k) \in \mathcal{P}(\mathcal{I} \cup C)^k$*, we have:*

$$A(F_1, \ldots, F_k) = A(F_1 \cap F_A, \ldots, F_k \cap F_A). \tag{1}$$

Definition 3 (Report identifier and attribution report, same as Cookie Monster)**.** *Fix a domain of report identifiers* $\mathbb{Z}$*. Consider a mapping* $d(\cdot)$ *from report identifiers* R *to devices* $\mathcal{D}$ *that gives the device* d_r *that generated a report* r. [4]

Given an attribution function A*, a set of epochs* E *and a report identifier* $r \in \mathbb{Z}$*, the* attribution report $\rho_{r,A,E}$*, or* ρ_r *for short, is a function over the whole database* D *defined by:*

$$\rho_r : D \in \mathbb{D} \mapsto A(D_{d_r}^E). \tag{2}$$

Definition 4 (Query, same as Cookie Monster)**.** *Consider a set of report identifiers* $R \subset \mathbb{Z}$*, and a set of attribution reports* $(\rho_r)_{r \in R}$ *each with output in* $\mathbb{R}^m$*.* [5] *The* query *for* $(\rho_r)_{r \in R}$ *is the function* $Q : \mathbb{D} \to \mathbb{R}^m$ *is defined as* $Q(D) := \sum_{r \in R} \rho_r(D)$ *for* $D \in \mathbb{D}$.

Algorithm 1 Gordon Notations and Setup

Input
  Database D // *Fixed for simplicity here, could be adaptive.*
  Stream of adaptively chosen queries
function $\mathcal{M}(D)$
  $(S_b)_{b\in\mathcal{S}} = (\emptyset)_{b\in\mathcal{S}}$
  for $(d, e, F) \in D$ **do**
    for $f \in F : f = (c, b, \text{conv})$ **do**
      Generate report identifier $r \xleftarrow{\$} U(\mathbb{Z})$
      // *Save mapping from r to the device that generated it*
      $d_r \leftarrow d$
      $S_b \leftarrow S_b \cup \{(r, f)\}$
  // *Each beneficiary receives its public events and corresponding report identifiers*
  for $b \in \mathcal{S}$ **do**
    output S_b to b
  // *Beneficiaries ask queries interactively*
  for $t \in [t_{\max}]$ **do**
    receive $Q_t^{b_t}$ from beneficiary site b_t.
    output AnswerQuery$(Q_t^{b_t})$ to b_t
// *Collect, aggregate and noise reports to answer Q*
function AnswerQuery(Q)
  Get report identifiers R and noise parameter σ from Q
  for $r \in R$ **do**
    Read Q to get conversion site c, beneficiary site b, impression sites $\mathbf{i}$, target epochs E, attribution function A for report r.
    $\rho_r \leftarrow$ GenerateReport$(d, c, b, \mathbf{i}, E, A, \sigma)$
  Sample $X \sim \mathcal{L}(\sigma)$
  return $\sum_{r\in R} \rho_r + X$

A.1.4 Privacy guarantees.

Mechanisms. Alg. 1 defines two types of interactive mechanisms. First, for each beneficiary site b we can denote by $\mathcal{M}^b$ the interactive mechanism that only interacts with b. Second, $\mathcal{M}$ is the interactive mechanism that interacts with all the beneficiary sites concurrently.

Levels of accounting. EpochImpSiteBudget computes privacy loss at a different granularity than EpochBudget, using Def. 5.

Definition 5 (Device-epoch-site neighborhood relation)**.** *Consider a device $d \in \mathcal{D}$, an epoch $e \in \mathcal{E}$, an impression site $i \in \mathcal{S}$ and a set of impression events happening on i: $F_i \in \{i\} \times \mathcal{I}$.* [10]

We say $D \sim_{d,e,i,F_i} D'$ if there exists D_0 and $F \in \mathcal{S} \setminus \{i\} \times \mathcal{I}$ such that:

$$\{D, D'\} = \{D_0 + (d, e, F), D_0 + (d, e, F \cup F_i)\} \tag{4}$$

Theorem 2. *Consider $x \in \mathcal{X}$ on device d.*

Algorithm 2 Gordon On-Device Algorithm

Input
  Per-site budget $\epsilon_{\text{per-site}}$
  Normal number of conversion sites N
  Normal number of impression sites M
  Fraction of budget for optimization queries r
  EpochBudget (Alg. 4) and EpochImpSiteBudget (Alg. 5) subroutines
// *Query at time t*
function Capacities$(\epsilon_{\text{per-site}}, N, M, r)$[6]
  $\epsilon_{\text{global}} \leftarrow N(1 + r)\epsilon_{\text{per-site}}$
  $\epsilon_{\text{conv-quota}} \leftarrow \epsilon_{\text{global}}/N$
  $\epsilon_{\text{imp-quota}} \leftarrow \epsilon_{\text{global}}/M$
  return $\epsilon_{\text{global}}, \epsilon_{\text{imp-quota}}, \epsilon_{\text{conv-quota}}$
// *Generate report and update on-device budget*
function GenerateReport$(d, c, b, \mathbf{i}, E, A, \sigma)$
  for $e \in E$ **do**
    $x \leftarrow (d, e, D_d^e)$
    if $\mathcal{F}_x$ is not defined **then**
      $\epsilon_{\text{global}}, \epsilon_{\text{imp-quota}}, \epsilon_{\text{conv-quota}} \leftarrow$ Capacities$(\epsilon_{\text{per-site}}, N, M, r)$
      $\mathcal{F}_x \leftarrow$ InitializeFilters$(\epsilon_{\text{global}}, \epsilon_{\text{per-site}}, \epsilon_{\text{imp-quota}}, \epsilon_{\text{conv-quota}})$
    $F_e \leftarrow D_d^e$
    // *Compute individual device-epoch-level privacy losses, Alg. 4*
    $\epsilon_x^t \leftarrow$ EpochBudget$(x, d, E, A, \mathcal{L}, \sigma)$
    $\epsilon_{\mathbf{x}}^{\mathbf{i},t} \leftarrow \{\}$ // *Initialize empty map*
    for $i \in \mathbf{i}$ **do**
      // *Compute individual device-epoch-site-level privacy losses, Alg. 5*
      $\epsilon_x^i \leftarrow$ EpochImpSiteBudget$(x, i, d, E, A, \mathcal{L}, \sigma)$
      $\epsilon_{\mathbf{x}}^{\mathbf{i},t}[i] \leftarrow \epsilon_x^i$
    // *Atomic filter check and update, Alg. 3*
    if AtomicFilterCheckAndConsume$(\mathcal{F}_x, b, c, \mathbf{i}, \epsilon_x^t, \epsilon_{\mathbf{x}}^{\mathbf{i},t})$ = FALSE **then**
      $F_e \leftarrow \emptyset$ // *Empty the report if any filter check fails*
  $\rho \leftarrow A((F_e)_{e\in E})$ // *Clipped attribution report*
  return ρ

- *For each beneficiary site $b \in \mathcal{S}$, $\mathcal{M}^b$ satisfies individual device-epoch $\epsilon_{per\text{-}site}(x)$-DP for x under public information C_b.*
- *$\mathcal{M}$ satisfies individual device-epoch $\epsilon_{global}(x)$-DP for x under public information C.* [11]

Proof. Pierre: TODO, might need more formal privacy games.
Pierre: TODO: prove that the atomic filter doesn't break the privacy proofs, because we add a "canConsume" API to the filter to do the 2PC.

□

A.1.5 Isolation guarantees.

Algorithm 3 2-Phase Commit Subroutine

Input:

ϵ_x^t: epoch-level privacy loss for a particular query

$\epsilon_{\mathbf{x}}^{\mathbf{i},t}$: epoch-site-level privacy loss for a particular query

canConsume: function that returns FALSE only if

$$\epsilon_x^t > \epsilon_{\text{nc/c/q-conv}}(b \text{ or } c) =: \epsilon_{\text{nc/c/q-conv}} - \sum_{k=1}^{t-1} \epsilon_x^k(b \text{ or } c) \cdot \mathbb{I}_{\text{pass}}[k] \tag{3}$$

(resp. $\epsilon_{\mathbf{x}}^{\mathbf{i},t}[i] > \epsilon_{q-imp}[i] = \epsilon_{\text{q-imp}} - \sum_{k=1}^{t-1} \epsilon_{\mathbf{x}}^{\mathbf{i},k}[i] \cdot \mathbb{I}_{\text{pass}}[k]$ for epoch-site filter for site i), where $\left\{(\epsilon_x^k, \mathbb{I}_{\text{pass}}(k))\right\}_{k\in[t-1]}$ are stored for canConsume.

In particular, we let $\epsilon_{\text{nc}}(b)$, ϵ_{c}, $\epsilon_{\text{q-conv}}(c)$, and $\epsilon_{\text{q-imp}}(i)$ respectively represent nc budget left on b, c budget left overall, q-conv budget left on c, and q-imp budget left on i, whereas $\epsilon_{\text{nc/c/q-conv/q-imp}}$ represent initial privacy budgets. Budget consumption is done by basic pure DP composition

tryConsume: function that consumes ϵ_x^t (resp. $\epsilon_{\mathbf{x}}^{\mathbf{i},t}[i]$ for site i) from the corresponding filter

Output:

Boolean function if all filters have enough budget for the privacy loss ϵ_x^t or not.

function AtomicFilterCheckAndConsume($\mathcal{F}_x, b, c, \mathbf{i}, \epsilon_x^t, \epsilon_{\mathbf{x}}^{\mathbf{i},t}$)[7]

// Phase 1: Prepare - check if all filters can consume

if $\mathcal{F}_x^{\text{per-site filter}[b]}$.canConsume($\epsilon_x^t$) = FALSE **then**

return FALSE

if $\mathcal{F}_x^{\text{global filter}}$.canConsume($\epsilon_x^t$) = FALSE **then**

return FALSE

if $\mathcal{F}_x^{\text{conversion-site quota}[c]}$.canConsume($\epsilon_x^t$) = FALSE **then**

return FALSE

for $i \in \mathbf{i}$ **do**

if $\mathcal{F}_x^{\text{impression-site quota}[i]}$.canConsume($\epsilon_{\mathbf{x}}^{\mathbf{i},t}[i]$) = FALSE **then**

return FALSE

// Phase 2: Commit - consume from all filters

// Privacy filters under no collusion

$\mathcal{F}_x^{\text{per-site filter}[b]}$.tryConsume($\epsilon_x^t$)

// Privacy filter under collusion

$\mathcal{F}_x^{\text{global filter}}$.tryConsume($\epsilon_x^t$)

// Quota filter for conversion site [8]

$\mathcal{F}_x^{\text{conversion-site quota}[c]}$.tryConsume($\epsilon_x^t$)

// Privacy filters for impression sites

for $i \in \mathbf{i}$ **do**

// Invididual device-epoch-impression site loss.[9]

$\mathcal{F}_x^{\text{impression-site quota}[i]}$.tryConsume $\epsilon_{\mathbf{x}}^{\mathbf{i},t}[i]$)

return TRUE

Definition 6. *We define the following notations for privacy loss accounting:*

Algorithm 4 Compute epoch-level privacy budget

Input:

x: device-epoch record (d, e, F)

d: device identifier

E: set of epochs

A: attribution function

$\mathcal{L}$: parametrized noise distribution

σ: noise parameter

Output:

Returns the epoch-level individual privacy loss for device-epoch x

function EPOCHBUDGET($x, d, E, A, \mathcal{L}, \sigma$)

// Extract components from device-epoch record

$(d', e, F) \leftarrow x$

$relevantEvents \leftarrow \{f \in F | f \text{ is relevant to } A\}$

if $relevantEvents = \emptyset$ **then**

// Case 1: No relevant events in this epoch

return 0

if $|E| = 1$ **then**

// Case 2: Single epoch query

$attributionOutput \leftarrow A(relevantEvents)$

// L1-norm of attribution output

$individualSensitivity \leftarrow$ compute_attribution($attributionOutput, \mathcal{L}$)

else

// Case 3: Multiple epochs query

$individualSensitivity \leftarrow report\_global\_sensitivity$

$requestedEpsilon \leftarrow 1/\sigma \cdot \sqrt{2}$

$epochBudget \leftarrow requestedEpsilon \cdot individualSensitivity / query\_global\_sensitivity$

return $epochBudget$

- $\epsilon_c^{\leq t}$*: the cumulative privacy loss charged to the c-filter up to step t before 2PC is triggered at step t.*
- $D^{\leq e}$*: The subset of database D containing records with epochs up to and including e.*

Lemma 1 (2-phase commit filter guarantees)**.** *For query k, let:*

$$\mathbb{I}_{pass}(k) \triangleq \begin{cases} 1 & \text{if AtomicFilterCheckAndConsume returns} \\ & \quad \textit{TRUE for query } k \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The AtomicFilterCheckAndConsume *function in Algorithm 3 guarantees the following properties:*

For any query k processed by AtomicFilterCheckAndConsume*, if* $\mathbb{I}_{pass}(k) = 1$*, then*

1. ***Epoch-level Consistency Property:*** *the nc-filter, c-filter, and q-conv-filter all consume exactly the same amount of budget* ϵ_x^t *for that query.*

Algorithm 5 Compute epoch-site level privacy budget

Input:

x : device-epoch record (d, e, F)
i : impression site
d : device
E : set of epochs, from which we can extract set of impression sites in these epochs relevant to A
A : attribution function
$\mathcal{L}$: parameterized noise distribution
σ : noise scale

Output:

Returns the epoch-site-level individual privacy loss for impression site i at device-epoch x

function EPOCHIMPSITEBUDGET($x, i, d, E, A, \mathcal{L}, \sigma$)
  epochSiteBudget $\leftarrow 0$
  Extract epoch e from $x = (d, e, F)$
  // Relevant epoch-site level events for current epoch
  relevantSiteLevelEvents
    $\leftarrow \{f \in F | f$ is an impression site i and is relevant$\}$
  // Compute epoch-site-level privacy losses
  if *relevantSiteLevelEvents* $= \emptyset$ **then**
    // Case 2 of equations (57) or (58): No relevant epoch-site events in epoch
    individualSensitivity $\leftarrow 0$
  // Check if we have single or multiple epoch-sites in E
  if $|E| = 1$ AND $|\{$site i relevant to A in epoch $e \in E\}| = 1$ **then**
    // Case 1 of equation (57): Single epoch, and only one impression site with relevant events to A in this epoch
    individualSensitivity
      $\leftarrow$ single_epoch_individual_sensitivity(
        A(*relevantSiteLevelEvents*), $\mathcal{L}, \sigma$)
  else
    // Case 1 of equation (58): Request either touches multiple epochs or multiples epoch-sites with relevant events
    individualSensitivity
      $\leftarrow$ report_global_sensitivity()

  return *individualSensitivity* / σ

2. ***Epoch-site-level Consistency Property:*** *the q-imp filter consumes exactly $\epsilon_x^i[i]$, which represents the device-epoch-impressionsite-level individual privacy loss.*

Proof. We can prove both properties at the same time. Fix an arbitrary query k, for which $\mathbb{I}_{\text{pass}}(k) = 1$. From Algorithm 3, we observe that AtomicFilterCheckAndConsume returns TRUE if and only if: (1) all canConsume checks in Phase 1 pass, and (2) All tryConsume operations in Phase 2 are executed. For a query from conversion site c with beneficiary site b and impression sites $\mathbf{i}$, the function calls:

- $\mathcal{F}_x^{\text{per-site filter}[b]}$.tryConsume($\epsilon_x^t$)
- $\mathcal{F}_x^{\text{global filter}}$.tryConsume($\epsilon_x^t$)
- $\mathcal{F}_x^{\text{conversion-site quota}[c]}$.tryConsume($\epsilon_x^t$)
- For each $i \in \mathbf{i}$: $\mathcal{F}_x^{\text{impression-site quota}[i]}$.tryConsume($\epsilon_{\mathbf{x}}^{\mathbf{i},t}[i]$)

Note that ϵ_x^t is computed once at EpochBudget in Line 6 of Algorithm 2 and represents the device-epoch-level individual privacy loss. Similarly, each $\epsilon_{\mathbf{x}}^{\mathbf{i},t}[i]$ is computed once via EpochImpSiteBudget and represents the device-epoch-impressionSite-level individual privacy loss. Therefore, when $\mathbb{I}_{\text{pass}}(k) = 1$, the nc-filter, c-filter, and q-conv filter all consume exactly the same amount ϵ_x^t, while each q-imp filter consumes its specific amount $\epsilon_{\mathbf{x}}^{\mathbf{i},t}[i]$, which is proportional to its sensitivity at the impression site level. □

Theorem 1 (global filter isolation properties). *Consider a adversary controlling M^{adv} and N^{adv} `imp-quota` and `conv-quota`, respectively. The maximum budget ϵ_{global}^{adv} that this adversary can consume from the Global filter is such that:*

$$\epsilon_{global}^{adv} \leq \min(M_{adv}\epsilon_{imp\text{-}quota},\ N_{adv}\epsilon_{conv\text{-}quota}).$$

Theorem 3 (Isolation properties for global filter-filter)**.** *{Mathias: This theorem statement is way too long and verbose. You can put notations/context in text before the statement.} Consider an execution of Alg. 1. A report at time step k concerns with* one *conversion site c_k, and some subset of impression sites $\mathbf{i}_k \subseteq S$. The union of attackers, in particular, can control an arbitrary subset of conversion sites $\mathbf{bad_c} \subseteq S$, which may or may not contain c_k at a given k. Similarly, it can control an arbitrary subset of impression sites $\mathbf{bad_i} \subseteq S$, which may or may not intersect with $\mathbf{i}_k$. We let $\mathbf{bad} = \mathbf{bad_c} \cup \mathbf{bad_i}$ and $\mathbf{good} = S \backslash \mathbf{bad}$.*

At step t in Line 16, suppose that beneficiary site b requests a report $\rho_{r,E,A}$ with noise σ through conversion site c for impression sites $\mathbf{i}$. Consider a device-epoch x, with individual budget ϵ_x^t computed at Line 6 in Alg. 2. Denote by $N_x^{\leq t}$ the number of conversion sites in $\mathbf{bad_c}$ with respect to x that were queried with non-zero budget by step t. Denote by $M_x^{\leq t}$ the number of impression sites in $\mathbf{bad_i}$ with respect to x that were queried with non-zero budget by step t.

The two following sufficient conditions each imply that attackers cannot block out benign users from executing queries that consume the global filter-filter:

- *If $N_x^{\leq t} < N$, then the benign users have at least $\frac{N - N_x^{\leq t}}{N} \cdot \epsilon_c$ much budget throughout the first t steps.*
- *If $M_x^{\leq t} < M$, then the benign users have at least $\frac{M - M_x^{\leq t}}{M} \cdot \epsilon_c$ much budget throughout the first t steps.*

Proof (Part 1). We will show budget left for benign users given $N_x^{\leq t} < N$. The total privacy loss in the global filter-filter incurred by the attackers for the c-filter by step t, not inclusive, Pierre: It doesn't seem right to sum over all k. We should only sum over k where $c_k \in bad$ no? is:

$$\epsilon_{\text{used}}^{\mathbf{bad}} = \epsilon_{c,\mathbf{bad}}^{\leq t-1} = \sum_{k=1}^{t-1} \epsilon_c^k \cdot \mathbb{I}_{pass}(k). \tag{6}$$

By composition under a pure DP filter and the definition of individual privacy loss, the total privacy loss equals:

$$= \sum_{c \in \mathbf{bad_c}} \sum_{k<t:c_k=c} \Delta_x \rho^k_{b_k} \cdot \mathbb{I}_{\text{pass}}(k) \tag{7}$$

$$= \sum_{c \in \mathbf{bad_c}} \sum_{k<t:c_k=c} \Delta_x \rho^k_{b_k} \cdot \mathbb{I}_{\text{pass}}(k), \tag{8}$$

where $\Delta_x \rho^k_{b_k}$ represents how much the attribution report generated for query k from beneficiary b_k can change with respect to x.

By the consistency property of lemma 1, Alg. 3 ensures that privacy loss is only incurred on k where $\mathbb{I}_{\text{pass}}(k) = 1$. So, for each conversion site c, the filter $\mathsf{q\text{-}conv}[c]$ precisely tracks the privacy losses incurred, so

$$\epsilon_{\mathsf{q\text{-}conv}}[c]^{\leq t-1} = \sum_{k<t:c_k=c} \Delta_x \rho^k_{b_k} \cdot \mathbb{I}_{\text{pass}}(k). \tag{9}$$

Substitute this equality into equation (8), we get:

$$\epsilon^{\leq t-1}_{c,\mathbf{bad}} = \sum_{c \in \mathbf{bad_c}} \epsilon_{\mathsf{q\text{-}conv}}[c]^{\leq t-1}. \tag{10}$$

This sum can be restricted to conversion sites with non-zero privacy loss, i.e.:

$$= \sum_{c \in \mathbf{bad_c}: \epsilon_{\mathsf{q\text{-}conv}}[c]^{\leq t-1} > 0} \epsilon_{\mathsf{q\text{-}conv}}[c]^{\leq t-1} \tag{11}$$

$$\leq \sum_{c \in \mathbf{bad_c}: \epsilon_{\mathsf{q\text{-}conv}}[c]^{\leq t-1} > 0} \epsilon_{\mathsf{q\text{-}conv}}, \tag{12}$$

where $\epsilon_{\mathsf{q\text{-}conv}}$ is the capacity of each $\mathsf{q\text{-}conv}$ filter. It follows that the number of conversion sites with non-zero privacy loss is precisely $N_x^{\leq t}$, so:

$$\leq \|\{c \in \mathbf{bad_c} : \epsilon_{\mathsf{q\text{-}conv}}[c]^{\leq t-1} > 0\}\| \cdot \epsilon_{\mathsf{q\text{-}conv}} \tag{13}$$

$$= N_x^{\leq t-1} \cdot \epsilon_{\mathsf{q\text{-}conv}}. \tag{14}$$

So, given the parameter set in algorithm 2, $\epsilon_{\mathsf{q\text{-}conv}} = \epsilon_c / N$ and the assumption that $N_x^{\leq t-1} \leq N_x^{\leq t} < N$, we have:

$$\epsilon^{\mathbf{bad}}_{\text{used}} = \epsilon_c^{\leq t-1} \leq N_x^{\leq t-1} \cdot \frac{\epsilon_c}{N}. \tag{15}$$

Now, during the 2PC for time t, we have the following cases:

- Suppose ϵ_x^t is a reasonable value, in the sense that it's bounded by the capacity $\epsilon_{\mathsf{q\text{-}conv}}$. Then,

$$\epsilon^{\leq t}_{c,\mathbf{bad}} = \epsilon^{\mathbf{bad}}_{\text{used}} + \epsilon_x^t \leq N^{\leq t} \cdot \frac{\epsilon_c}{N}. \tag{16}$$

- Otherwise, ϵ_x^t is unreasonable, in which case ϵ_x^t exceeds the capacity $\epsilon_{\mathsf{q\text{-}conv}}$. In this case,

$$\epsilon_{\mathsf{q\text{-}conv}}[c_t]^{\leq t-1} + \epsilon_x^t \geq \epsilon_{\mathsf{q\text{-}conv}}, \tag{17}$$

causing $\mathcal{F}_x^{\mathsf{q\text{-}conv}[c]}.\text{canConsume}(\epsilon_x^t)$ to return FALSE by definition, so no budget is spent at all. In such a case,

$$\epsilon^{\leq t}_{c,\mathbf{bad}} = \epsilon^{\mathbf{bad}}_{\text{used}} + 0 = \epsilon^{\mathbf{bad}}_{\text{used}} \leq N_x^{\leq t-1} \cdot \frac{\epsilon_c}{N}, \tag{18}$$

by equation (15).

In either case, the attackers can consume at most $\frac{N_x^{\leq t}}{N} \cdot \epsilon_c$ of the global filter-filter budget by the end of time t, which means the c-filter has at least $\frac{N - N_x^{\leq t}}{N} \cdot \epsilon_c$ capacity left for benign users to accommodate queries up to the end of time t. □

Proof (Part 2). By basic composition of pure DP, we know ϵ_c is by definition the sum of c-filter consumption at each time up to $t-1$: Pierre: Same here, the sum is the total budget consumed, not just the budget consumed by queries that requested a bad impression site.

$$\epsilon^{\leq t-1}_{c,\mathbf{bad}} = \sum_{k \in [t-1]} \epsilon_x^k \cdot \mathbb{I}_{\text{pass}}(k) \tag{19}$$

$$= \sum_{k \in [t-1]} \Delta_x \rho^k_{b_k} \cdot \mathbb{I}_{\text{pass}}(k), \tag{20}$$

where $\Delta_x \rho^k_{b_k}$ is how much the attribution report generated for query k from beneficiary b_k can change with respect to x, which we substitute next. Let $\vec{x} = (x_{i_1}, \ldots, x_{i_m})$ be the vector of $|\mathbf{i}_k| = m$ device-epoch-sites, where $x_i = (d, e, F_i)$ has all its events on site i. Then, we let the corresponding neighboring dataset of $D \in \mathbb{D}$ be $D + \vec{x}$, so that:

$$\rho(D + \vec{x}) - \rho(D) = \sum_{j=1}^{m} \rho(D + x_{i_j}) - \rho(D + x_{i_{j-1}}), \tag{21}$$

where $x_{i_0} = 0$. Thus, we substitute by the following decomposition of a query's sensitivity across the impression sites relevant to that query, by how it is assigned in Alg. 2 Line 6:

$$\Delta_x \rho^k_{b_k} = \max_{D, D' \in \mathcal{D}' = D + x} \|\rho^k_{b_k}(D') - \rho^k_{b_k}(D)\|_1 \tag{22}$$

$$= \max_{D \in \mathcal{D}} \|\rho^k_{b_k}(D + x) - \rho^k_{b_k}(D)\|_1 \tag{23}$$

$$\leq \max_{D \in \mathcal{D}} \sum_{j \in [m]: i_j \in \mathbf{bad_i}} \|\rho^k_{b_k}(D + x_1 + \ldots + x_i) - \tag{24}$$

$$\rho^k_{b_k}(D + x_1 + \ldots + x_{i-1})\|_1 \tag{25}$$

$$\leq \sum_{j \in [m]: i_j \in \mathbf{bad_i}} \max_{\hat{D} \in \mathcal{D}} \|\rho^k_{b_k}(\hat{D} + x_{i_j}) - \rho^k_{b_k}(\hat{D})\|_1 \tag{26}$$

$$= \sum_{j \in [m]: i_j \in \mathbf{bad_i}} \Delta_x^{i_j} \rho^k_{b_k} = \sum_{i \in \mathbf{bad_i}} \Delta_x^i \rho^k_{b_k} \tag{27}$$

$$= \sum_{i \in \mathbf{bad_i}} \epsilon_{\mathbf{x}}^{\mathbf{i},k}[i], \tag{28}$$

Plugging in this, we get

$$\epsilon^{\leq t-1}_{c,\mathbf{bad}} \leq \sum_{k \in [t-1]} \sum_{i \in \mathbf{bad_i}} \epsilon_{\mathbf{x}}^{\mathbf{i},k}[i] \cdot \mathbb{I}_{\text{pass}}(k) \tag{29}$$

$$= \sum_{i \in S} \sum_{k \in [t-1]: i \in \mathbf{bad_i}} \epsilon_{\mathbf{x}}^{\mathbf{i},k}[i] \cdot \mathbb{I}_{\text{pass}}(k), \tag{30}$$

where the second equality is by changing order of summation, and letting the first summation to be over all S while restricting what i it can be in each round in the second summation. But note that

$$\epsilon_{\text{q-imp}}^{\leq t-1}[i] = \sum_{k\in[t-1]:i\in S} \epsilon_{\mathbf{x}}^{\mathbf{i},k}[i] \cdot \mathbb{I}_{\text{pass}}(k), \tag{31}$$

because, by epoch-site-level consistency property in lemma 1, we know that only relevant site i at time k, where every filter has enough budget to pass the 2-PC check, will have epoch-site level privacy losses incurred. Substituting this equality into equation (30), we get:

$$\begin{aligned}
\epsilon_{c,\mathbf{bad}}^{\leq t-1} &\leq \sum_{i\in\mathbf{bad_i}} \epsilon_{\text{q-imp}}^{\leq t-1}[i] && (32)\\
&= \sum_{i\in\mathbf{bad_i}:\epsilon_{\text{q-imp}}^{\leq t-1}[i]>0} \epsilon_{\text{q-imp}}^{\leq t-1}[i] && (33)\\
&\leq \sum_{i\in\mathbf{bad_i}:\epsilon_{\text{q-imp}}^{\leq t-1}[i]>0} \epsilon_{\text{q-imp}} && (34)\\
&= \left|\left\{i \in \mathbf{bad_i} : \epsilon_{\text{q-imp}}^{\leq t-1}[i] > 0\right\}\right| \cdot \epsilon_{\text{q-imp}}, && (35)
\end{aligned}$$

because only non-zero privacy losses that were incurred contribute meaningfully to the composition. Finally, we note that $\left|\left\{i \in \mathbf{bad_i} : \epsilon_{\text{q-imp}}^{\leq t-1}[i] > 0\right\}\right| \leq M^{\leq t-1}$ by definition and:

$$\epsilon_{c,\mathbf{bad}}^{\leq t-1} \leq M^{\leq t-1} \cdot \epsilon_{\text{q-imp}}. \tag{36}$$

Following this result, similar to the proof for part 1:

- Suppose $\epsilon_x^t \leq \frac{\epsilon_c}{M}$,

$$\epsilon_{c,\mathbf{bad}}^{\leq t} \leq M_x^{\leq t} \cdot \frac{\epsilon_c}{M}. \tag{37}$$

- Else, $\epsilon_x^t > \frac{\epsilon_c}{M}$, then $\epsilon_{\text{q-conv}}$ will be exceeded, causing canConsume to return FALSE, so,

$$\epsilon_{c,\mathbf{bad}}^{\leq t} = \epsilon_{c,\mathbf{bad}}^{\leq t-1} + 0 = \epsilon_{c,\mathbf{bad}}^{\leq t-1}, \tag{38}$$

by equation (36).

In either case, the total privacy loss incurred by attackers by the end of time t is at most $\frac{M_x^{\leq t}}{M} \cdot \epsilon_c$, leaving at least $\frac{M-M_x^{\leq t}}{M} \cdot \epsilon_c$ privacy budget for benign parties throughout the first t time steps.

Corollary 4. *Suppose in any t time range before and after which all filters receive refreshed privacy budgets, the union of attackers is expected to control at most $N_x^{\leq t}$ conversion sites and $M_x^{\leq t}$ impression sites with high probability, and the workload of benign users is expected to be $N_{x,\mathbf{good}}^{\leq t}$ conversion sites and $M_{x,\mathbf{good}}^{\leq t}$ impression sites.*

Then, setting $N = N_x^{\leq t} + N_{x,\mathbf{good}}^{\leq t}$ and $M = M_x^{\leq t} + M_{x,\mathbf{good}}^{\leq t}$, and setting $\epsilon_{conv\text{-}quota}$ to ϵ_c/N and setting $\epsilon_{imp\text{-}quota}$ to ϵ_c/M guarantee benign users to have sufficient privacy global filter-filter budgets for their workloads with high probability.

Proof. Same analysis as theorem 1 by plugging in the new total values of conversion sites and impression sites that receive queries with non-zero sensitivities, respectively. □

□

Analysis in the proof for part 2 can be further optimized when we restrict the class of attributions. We first define a class of histogram attribution based on definition 8:

Definition 7 (Single-touch attribution function)**.** *An attribution function A is considered single-touch if we attribute to the entire conversion to exactly one impression event $f \in (F_1 \cap F_A) \cup \cdots \cup (F_k \cap F_A)$, where the selection criterion of that one event may be, for example, last touch. As such, we have*

$$a_{\mathbf{F}}(f) = \begin{cases} a_{\mathbf{F}}(f^*) & \text{, if } f = f^* \\ 0 & \text{, otherwise} \end{cases} \tag{39}$$

$$\implies A(\mathbf{F}) = a_{\mathbf{F}}(f^*) \cdot H(f^*), \tag{40}$$

where $H(f^*)$ is the one hot-encoding defined same way as in definition 8.

Mark: Point out which equation would be changed and TODO on what that can accomplish for us. Instead of consuming on a epoch-site level, we can consume on an epoch level but only pay when the impression site is requested by the query. This should help us consume ϵ_x instead of a sum of ϵ_x^i, which should be better by getting rid of the triangular inequality, but it can be bad because sometimes ϵ_x can be greater than ϵ_x^i with certain contrived attribution functions at least. Pierre: Alternative for later: instead of doing device-epoch-site accounting in each qimp[i] filter, what if we do device-epoch accounting but only for queries that requested each site i? i.e. qimp[i] pays eps_x^k if it is requested at step k, otherwise it pays 0. There is a tradeoff here: the alternative is nice because we don't have the triangle inequality ($eps_x <= \sum_i eps_x^i$ which is loose). But it can be bad because sometimes $eps_x^i << eps_x$ (but not always, sometimes we can even make crazy queries where $eps_x^i > eps_x$?). Example of $eps_x^i = 0 << eps_x$: the report requests impression site i, but x has zero impressions for i

Corollary 5. *If the attribution function A at every time k up to including t is single-touch, then, the second sufficient condition of theorem 1 instead implies at least $\frac{M-1}{M}\epsilon_c$ budget for benign users through the first t times.*

Proof. In the single-touch case, Line 23 would be a max over the multiple impression sites touched on, not a sum, so,

instead of using triangular inequality, we have:

$$\Delta_x \rho^k_{b_k} = \max_{D\in\mathcal{D}} \|\rho^k_{b_k}(D+x) - \rho^k_{b_k}(D)\|_1 \quad (41)$$

$$= \max_{i\in\mathbf{bad}_\mathbf{i}} \max_{D\in\mathcal{D}} \|\rho^k_{b_k}(D+x) - \rho^k_{b_k}(D)\|_1 \quad (42)$$

$$= \max_{i\in\mathbf{bad}_\mathbf{i}} \epsilon^{\mathbf{i},k}_{\mathbf{x}}[i] \quad (43)$$

$$\implies \epsilon^{\leq t-1}_{c,\mathbf{bad}} \leq \max_{i\in\mathbf{bad}_\mathbf{i}} \epsilon^{\leq t-1}_{\text{q-imp}}[i] \leq \epsilon_{\text{q-imp}}. \quad (44)$$

This is without regard to t, so by the end of t, there is at least $\epsilon_c - \epsilon_{\text{q-imp}} = \frac{M-1}{M}\epsilon_c$ budget left in the global filter-filter. □

Theorem 6. *Consider the same set-up as theorem 1. Further denote by $\epsilon_{nc}(q)$ the remaining budget for beneficiary b in the nc-filter. Denote by $\epsilon_{q\text{-}conv}(c)$ the remaining budget for conversion site c in the q-conv filter. Denote by $\epsilon_{q\text{-}imp}(c)$ the remaining budget for impression site i in the q-imp filter.*

If the following conditions all hold, then $\mathbb{I}_{pass}(t) = \mathsf{TRUE}$, i.e. the AtomicFilterCheckAndConsume *returns* TRUE*:*

1. *Denote the number of new conversion sites that benign queriers need to query with non-zero sensitivity by $N^{=t}_{x,\mathbf{good}}$; that of new impression sites by $M^{=t}_{x,\mathbf{good}}$. Either $N^{\leq t}_x < N$ and the benign queries have consumed less budget than $\frac{N-N^{\leq t}_x - N^{=t}_{x,\mathbf{good}}}{N}\cdot\epsilon_c$ from the global filter-filter, or $M^{\leq t}_x \leq M$ and the benign queries have consumed less budget than $\frac{M-M^{\leq t}_x - M^{=t}_{x,\mathbf{good}}}{M}\cdot\epsilon_c$ from the global filter-filter.*
2. $\epsilon^t_x \leq \epsilon_{nc}(b)$
3. $\epsilon^t_x \leq \epsilon_{q\text{-}conv}(c)$
4. $\epsilon^{\mathbf{i},t}_{\mathbf{x}}[i] \leq \epsilon_{q\text{-}imp}(i)$ *for* $i \in \mathbf{i}$

Proof. In theorem 1, we have proved the budget left for benign queriers throughout the first t times in the respective case. So, if the benign queriers have enough of the budget left by the beginning of time t for their query at time t, we have:

$$\mathcal{F}^c_x.\,\text{canConsume}(\epsilon^t_x) = \mathsf{TRUE}. \quad (45)$$

In Alg. 3 Line 3, we define canConsume to return TRUE iff enough budget is left to consume the specified budget in the corresponding filter (by basic pure DP composition). So:

- $\epsilon^t_x \leq \epsilon_{\text{nc}}(b) = \epsilon_{\text{nc}} - \sum_{k=1}^{t-1}\epsilon^k_x(b)\cdot\mathbb{I}_{\text{pass}}[k]$ means that, by definition:

$$\mathcal{F}^{\text{nc[b]}}_x.\,\text{canConsume}(\epsilon^t_x) = \mathsf{TRUE}. \quad (46)$$

- $\epsilon^t_x \leq \epsilon_{\text{q-conv}}(c) = \epsilon_{\text{q-conv}} - \sum_{k=1}^{t-1}\epsilon^k_x(c)\cdot\mathbb{I}_{\text{pass}}[k]$ means that, by definition:

$$\mathcal{F}^{\text{q-conv[c]}}_x.\,\text{canConsume}(\epsilon^t_x) = \mathsf{TRUE}. \quad (47)$$

- For any $i \in \mathbf{i}$, $\epsilon^{\mathbf{i},t}_{\mathbf{x}}[i] \leq \epsilon_{\text{q-imp}}(i) = \epsilon_{\text{q-imp}} - \sum_{k=1}^{t-1}\epsilon^{\mathbf{i},k}_{\mathbf{x}}(i)\cdot\mathbb{I}_{\text{pass}}[k]$ means that, by definition:

$$\mathcal{F}^{\text{q-imp[i]}}_x.\,\text{canConsume}(\epsilon^t_x) = \mathsf{TRUE}. \quad (48)$$

Thus, when all conditions are satisfied, all of equations (**??**)-(48) hold, which means all filter checks should pass and AtomicFilterCheckAndConsume should return TRUE. □

A.1.6 Sensitivity analysis and privacy budget.

Theorem 7 (Global sensitivity of reports per epoch-site)**.** *Fix a report identifier r, a device d, a set of epochs $E_r = \{e^{(r)}_1, \ldots, e^{(r)}_k\}$, and a set of sites $I^{(e)}_r = \{i^{(e)}_1, \ldots, i^{(e)}_{m_e}\}$ for each epoch $e \in E_r$. Let $A(\cdot)$ be the attribution function of interest, so that the corresponding report is $\rho : D \mapsto A(D^{E_r,\{I^{(e)}_r\}_{e\in E_r}}_d)$. We have $\Delta(\rho)$*

$$= \max_{\substack{i\in[k],e:=e^{(r)}_i,j\in[m_e],e'=e^{(r)}_k \\ \mathcal{F}:=(F_{1,1},\ldots,F_{k,m_{e'}})\subseteq(\mathcal{S}\times\mathcal{S}\times\mathcal{C}\cup\mathcal{S}\times\mathcal{I})^k}} \|A(\mathcal{F}: F_{i,j} := \emptyset) - A(\mathcal{F})\|_1. \quad (49)$$

Proof. Fix a report ρ. Let k be $|E_r|$. Then, for $i \in [k]$, we enumerate through the epochs in E_r, and call the i-th epoch e_i. By definition of global sensitivity:

$$\Delta(\rho) = \max_{D,D'\in\mathbb{D}:\exists x\in\mathcal{X},D'=D+x} \|\rho(D) - \rho(D)\|_1, \quad (50)$$

from which we expand what ρ function does:

$$\Delta(\rho) = \max_{\substack{D,D'\in\mathbb{D}: \\ \exists x\in\mathcal{X},D'=D+x}} \|A(D^{E_r,\{I^{(e)}_r\}_{e\in E_r}}_d) - A((D')^{E_r,\{I^{(e)}_r\}_{e\in E_r}}_d)\|_1 \quad (51)$$

$$= \max_{\substack{D,D'\in\mathbb{D}: \\ \exists x=(d,e,F)\in\mathcal{X}: \\ e\in E_r,D'=D+x, \\ \forall(i,imp)\in F: i\in I^{(e)}_r}} \|A(D^{E_r,\{I^{(e)}_r\}_{e\in E_r}}_d) - A((D')^{E_r,\{I^{(e)}_r\}_{e\in E_r}}_d)\|_1, \quad (52)$$

because for $(d', e', F) \in \mathcal{X}$ with $d' \neq d$ or $e' \notin E_r$, or any $(i, imp) \in F$ where $i \notin I^{(e)}_r$, they would not be considered as in the computation of $A(\cdot)$ and $A(D^{E_r,\{I^{(e)}_r\}_{e\in E_r}}_d) = A((D')^{E_r,\{I^{(e)}_r\}_{e\in E_r}}_d)$.

Next, we show that the following sets are equal:

- $\{(D^{E_r,\{I^{(e)}_r\}_{e\in E_r}}_d, (D')^{E_r,\{I^{(e)}_r\}_{e\in E_r}}_d) : D, D' \in \mathbb{D} : \exists x = (d, e, F) \in \mathcal{X} : e \in E_r, D' = D + x, \forall(i, imp) \in F : i \in I^{(e)}_r\}$.
- $\{(\mathcal{F} : F_{i,j} = \emptyset, \mathcal{F}) : i \in [k], e := e^{(r)}_i, j \in [m_e], e' = e^{(r)}_k, \mathcal{F} := (F_{1,1}, \ldots, F_{k,m_{e'}}) \subseteq (\mathcal{S}\times\mathcal{S}\times\mathcal{C}\cup\mathcal{S}\times\mathcal{I})^k\}$

We show that for all instances on the one side, there exists a corresponding instance on the other. In one direction, take any tuple from the first set, where $(D')^{E_r,\{I^{(e)}_r\}_{e\in E_r}}_d = D^{E_r,\{I^{(e)}_r\}_{e\in E_r}}_d + x$, for some $x = (d, e, F)$. Firstly, the x satisfies $d = d_r$, $e \in E_r$, and $\forall f = (i, imp) \in F, i \in I^{(e)}_r$. Therefore, we

construct $\mathcal{F}$ as follows:

$F_{i,j} = \{f \in F : f = (j, imp) \text{ or } (c, q, conv) \text{ is relevant to site } j\}$.

Note that every $F_{i,j}$ is set independently from x, except for the one that contains the events for site j_0 such that F contains either (j_0, imp) or $(c, q, conv)$ that is associated with j_0, in epoch i_0 such that $e_{i_0}^{(r)} = e \in E_r$. Since $x \notin D$, we know that F_{i_0,j_0} is set to $\emptyset$ on the side corresponding to D, and it is set to contain all events related to j_0 in epoch i_0 on the side corresponding to D'. That is, the corresponding instance in the second set would be $(\mathcal{F} : F_{i_0,j_0} = \emptyset, \mathcal{F})$.

Conversely, let $(\mathcal{F} : F_{i,j} = \emptyset, \mathcal{F})$ be from the second set. Then, we know the set of events corresponding to site j in epoch i is empty for the first element. So, we let $x := (d, e, F) \in \mathcal{X}$, where d is the same devices as was fixed, $e = e_i^{(r)}$, and F contains events related to site j in epoch i. Then, we let $(D, D') \in \mathbb{D} \times \mathbb{D}$, where $D' = D + x$ and everything in D and D' other than x corresponds to the rest of the $F_{i,j}$ that are the same in both $\mathcal{F}$ and $\mathcal{F} : F_{i,j} = \emptyset$. As such, $(D_d^{E_r, \{I_r^{(e)}\}_{e \in E_r}}, (D')_d^{E_r, \{I_r^{(e)}\}_{e \in E_r}})$ is the corresponding element in the first set.

Finally, we substitute the first set in equation line (52) by the second set accordingly, and the resulting max should be equal as the two sets are equivalent, and the resulting form will be the one claimed. □

Lemma 2. *Suppose the attribution function $A(\cdot)$ has an m-dimensional outputs, and each dimension of any $\mathcal{F} \subseteq (\mathcal{S} \times \mathcal{S} \times \mathcal{I} \cup \mathcal{S} \times \mathcal{C})^k$ satisfy $\forall i \in [m]$, $A(\mathcal{F})_i \in [0, A^{\max}]$, then we have $\Delta(\rho) \le mA^{\max}$.*

Proof. Given the form in theorem 7, we know $\Delta(\rho)$ equals the max over $\|A(\mathcal{F} : F_{i,j} = \emptyset) - A(\mathcal{F})\|_1 = \sum_{i=1}^m |A(\mathcal{F} : F_{i,j} = \emptyset)_i - A(\mathcal{F})_i|$. But, we know for each dimension, $A(\mathcal{F}_i) \in [0, A^{\max}]$, so the absolute value of the difference in each dimension is $\in [0, A^{\max}]$. Thus, the sum over m dimensions is $\|A(\mathcal{F} : F_{i,j} = \emptyset) - A(\mathcal{F})\|_1 \le mA^{\max}$. □

Theorem 8 (Global sensitivity of queries per epoch-site)**.** *Let R be the set of report identifiers relevant to a query Q. Then, we write $(\rho_r, d_r, E_r)_{r \in R}$ as the reports, devices, and epoch windows corresponding to each $r \in R$. Then, we have*

$$\Delta(Q) \le \max_{(d,e,F) \in \mathcal{X}} \sum_{\substack{r \in R: d = d_r, e \in E_r, \\ \exists i \in I_r^{(e)} : F \text{ contains events related to } i}} \Delta(\rho_r).$$

Proof. By definition of global sensitivity,

$$\Delta(Q) = \max_{D, D' \in \mathbb{D} : \exists x \in \mathcal{X}, D' = D + x} \|Q(D) - Q(D')\|_1 \tag{53}$$

$$= \max_{x \in \mathcal{X}} \max_{D, D' \in \mathbb{D} : D' = D + x} \|Q(D) - Q(D')\|_1. \tag{54}$$

Let $x = (d, e, F) \in \mathcal{X}$, where F contains events associated with site j. For $r \in R$ such that $d \neq d_r$ or $e \notin E_r$, or F contains events related to site $i \notin I_r^{(e)}$, we have $\rho(D) = \rho(D')$. So, by applying triangle-inequality in the mean time:

$$\|Q(D) - Q(D')\|_1 \le \sum_{r \in R: d = d_r, e \in E_r, i \in I_r^{(e)}} \|\rho_r(D) - \rho_r(D')\|_1 \tag{55}$$

$$\le \sum_{r \in R: d = d_r, e \in E_r, i \in I_r^{(e)}} \Delta(\rho_r), \tag{56}$$

where the second inequality is by definition of $\Delta(\rho_r)$. Note that this bound is independent from the choice of D and D', so we take the max over (d, e, i) and get

$$\Delta(Q) \le \max_{d,e,i} \sum_{r \in R: d = d_r, e \in E_r, i \in I_r^{(e)}} \Delta(\rho_r).$$

But, the choice of i can be used to set F accordingly, so, equivalently, this suffices to prove the theorem. □

Lemma 3. *If each device-epoch-site participates in at most one report, then $\Delta(Q) = \max_{r \in R} \Delta(\rho_r)$.*

Proof. Since each device-epoch-ste participates in at most one report, $\sum_{r \in R: d = d_r, e \in E_r, i \in I_r^{(e)}} \Delta(\rho_r) = \Delta(\rho_r)$. So,

$$\Delta(Q) \le \max_{(d,e,F) \in \mathcal{X}} \Delta(\rho_r).$$

But then, $\Delta(Q)$ must match the sensitivity of one of the r's in this case, so the inequality is tight: $\Delta(Q) = \max_{r \in R} \Delta(\rho_r)$. Particularly, $\forall r$, $\exists D \sim D'$, such that $\|\rho_r(D') - \rho_r(D)\|_1 = \Delta(\rho_r)$. □

Theorem 9 (Individual sensitivity of reports per epoch-site)**.** *Fix a report identifier r, a device d_r, a set of epochs $E_r = \{e_1^{(r)}, \ldots, e_k^{(r)}\}$, a set of sites $I_r^{(e)} = \{i_1^{(e)}, \ldots, i_{m_e}^{(e)}\}$ for each epoch $e \in E_r$, an attribution function A with relevant events F_A, and the corresponding report $\rho : D \mapsto A(D_{d_r}^{E_r, \{I_r^{(e)}\}_{e \in E_r}})$. Fix a device-epoch record $x = (d, e, F) \in \mathcal{X}$, where $F \subseteq \mathcal{S} \times \mathcal{S} \times \mathcal{C} \cup \mathcal{S} \times \mathcal{I}$, so that $x_i = (d, e, F_i)$ is the projection where F_i contains only events related to site i.*

If the report requests a single epoch $E_r = \{e_r\}$ as well as a single site in the one epoch, $I_r^{(e_r)} = \{i_r\}$, then we have:

$$\Delta_{x_i}(\rho) = \begin{cases} \|A(F_i) - A(\emptyset)\|_1 & \text{, if } d = d_r, e = e_r \text{ and } i = i_r \\ 0 & \text{, otherwise.} \end{cases} \tag{57}$$

In particular, it should be intuitive to see why $\mathcal{F}$ is a single F_i in the first case, because we only have one epoch which contains one site, so there should be only one set of events corresponding to the one site in the one epoch.

Otherwise, either $|E_r| \ge 2$ or $|I_r^{(e)}| \ge 2$ for some $e \in E_r$, or both, and so we have:

$$\Delta_{x_i}(\rho) \le \begin{cases} \Delta(\rho) & \text{, if } d = d_r, e \in E_r, i \in I_r^{(e)} \text{ and } F_i \cap F_A \neq \emptyset \\ 0 & \text{, otherwise.} \end{cases} \tag{58}$$

Proof. Fix a report ρ and $x_i = (d, e, F_i) \in \mathcal{X}$. Consider any $D, D' \in \mathbb{D}$ such that $D' = D + x_i$. We have $\rho(D) = A(D_{d_r}^{E_r, \{I_r^{(e)}\}_{e \in E_r}})$ and $\rho(D') = A((D')_{d_r}^{E_r, \{I_r^{(e)}\}_{e \in E_r}})$.

- First, if $d \neq d_r$, $e \notin E_r$, or $i \notin I_r^{(e)}$, then $(D')_{d_r}^{E_r, \{I_r^{(e)}\}_{e \in E_r}} = D_{d_r}^{E_r, \{I_r^{(e)}\}_{e \in E_r}}$. Hence, $\|\rho(D) - \rho(D')\|_1 = 0$ for all such D, D', which implies $\Delta_{x_i}(\rho) = 0$.
- Next, suppose that the report requests a single epoch $E_r = \{e_r\}$ with a single site $I_r^{(e_r)} = \{i_r\}$:
  - If $d = d_r$, $e = e_r$, and $i = i_r$, then since $D + x_i = D'$, we must have $(d_r, e_r, F_i) \notin D$, and thus $D_{d_r}^{e_r, i_r} = \emptyset$. On the other hand, $(D')_{d_r}^{e_r, i_r} = F_i$ (restricted to events relevant to site i_r). Thus, $\|\rho(D) - \rho(D')\|_1 = \|A(F_i) - A(\emptyset)\|_1$.
  - If $d \neq d_r$, $e \neq e_r$, or $i \neq i_r$, then (d, e, F_i) doesn't change the outcome and $(D')_{e_r}^{i_r} = D_{e_r}^{i_r}$. Hence, $\|\rho(D) - \rho(D')\|_1 = 0$.
- Now, suppose that the report requests either an arbitrary range of epochs E_r each of whom has at least one site, or a single epoch that has multiple sites $I_r^{(e_r)}$:
  - If $d \neq d_r$, $e \notin E_r$, or $i \notin I_r^{(e)}$, then $A((D')_{d_r}^{E_r, \{I_r^{(e)}\}_{e \in E_r}}) = A(D_{d_r}^{E_r, \{I_r^{(e)}\}_{e \in E_r}})$, i.e., $\|\rho(D') - \rho(D)\|_1 = 0$.
  - If we have $d = d_r$, $e = e_j^{(r)} \in E_r$, and $i \in I_r^{(e)}$, but F_i is simply not related to the attribution request, i.e. $F_i \cap F_A = \emptyset$. Then, by definition of F_A, we have $A((D')_{d_r}^{E_r, \{I_r^{(e)}\}_{e \in E_r}}) = A(D_{d_r}^{E_r, \{I_r^{(e)}\}_{e \in E_r}})$, i.e., $\|\rho(D) - \rho(D')\|_1 = 0$.
  - Otherwise, it must be the case that $d = d_r$, $e = e_j^{(r)} \in E_r$, $i \in I_r^{(e)}$ and $F_i \cap F_A \neq \emptyset$ and there are events in the intersection that is related to some site i in epoch e, so we have:

$$\|\rho(D) - \rho(D')\|_1 = \|A(\mathcal{F} : F_{j,i} = \emptyset) - A(\mathcal{F})\|_1, \tag{59}$$

  where j is the index of epoch e in E_r, and $F_{j,i}$ represents the relevant events for site i in epoch $e_j^{(r)}$.

The first two cases are independent over choices of $D \sim D'$, so taking the max over such choices still gives $\Delta_{x_i}(\rho) = 0$. Unfortunately, the third identity does depend on the choice of $D \sim D'$, and taking the max only gives the general definition of global sensitivity, in the worst case. Particularly,

$$\Delta_{x_i}(\rho) = \max_{\mathcal{F} = \{F_{j,i}\} : F_{j,i} \subseteq \mathcal{S} \times \mathcal{S} \times \mathcal{C} \cup \mathcal{S} \times \mathcal{I}} \|\rho(D) - \rho(D')\|_1 \tag{60}$$

$$= \max_{\mathcal{F} = \{F_{j,i}\} : F_{j,i} \subseteq \mathcal{S} \times \mathcal{S} \times \mathcal{C} \cup \mathcal{S} \times \mathcal{I}} \|A(\mathcal{F} : F_{j,i} = \emptyset) - A(\mathcal{F})\|_1 \tag{61}$$

$$\leq \Delta(\rho), \tag{62}$$

where the last inequality is tight due to the definition of global sensitivity and theorem 7, for the worst case choice of x_i in general. □

Theorem 10 (Individual sensitivity of queries per device-epoch-site)**.** *Fix a query Q with corresponding report identifiers R and reports $(\rho_r)_{r \in R}$. Fix a device-epoch-site record $x_i = (d, e, F_i) \in \mathcal{X}$, where F_i contains only events related to site i. We have:*

$$\Delta_{x_i}(Q) \leq \sum_{r \in R} \Delta_{x_i}(\rho_r) \tag{63}$$

In particular, if x_i participates in at most one report ρ_r, then $\Delta_{x_i}(Q) = \Delta_{x_i}(\rho_r)$.

Proof. Take $D, D' \in \mathcal{D}$ such that $D' = D + x_i$. By the triangle inequality:

$$\Delta_{x_i}(Q) = \max_{D' = D + x_i \in \mathcal{D}} \|Q(D) - Q(D')\|_1 \tag{64}$$

$$= \max_{D' = D + x_i \in \mathcal{D}} \left\| \sum_{r \in R} \rho_r(D) - \rho_r(D') \right\|_1 \tag{65}$$

$$\leq \sum_{r \in R} \max_{D' = D + x_i \in \mathcal{D}} \|\rho_r(D) - \rho_r(D')\|_1 \tag{66}$$

$$\leq \sum_{r \in R} \Delta_{x_i}(\rho_r), \tag{67}$$

where the last inequality is by the definition of individual sensitivity.

When x_i participates in at most one report ρ_{r_0}, we have $\Delta_{x_i}(\rho_r) = 0$ for all $r \neq r_0$. Therefore, $\Delta_{x_i}(Q) \leq \Delta_{x_i}(\rho_{r_0})$. This inequality is tight because there exists a pair (D^*, D'^*) with $D'^* = D^* + x_i$ such that $\|\rho_{r_0}(D'^*) - \rho_{r_0}(D^*)\|_1 = \Delta_{x_i}(\rho_{r_0})$, and for this pair, $\|Q(D^*) - Q(D'^*)\|_1 = \Delta_{x_i}(\rho_{r_0})$. □

A.2 Cross-report privacy loss optimization

In practice, a single conversion event can trigger multiple reports being sent to different beneficiary sites. When reports have specific structures (*e.g.,* a large histogram across all adtechs, where each adtech receives a piece of the histogram), it is more efficient in terms of c-filter accounting to analyze all the reports at once.

{Mathias: we need intuition in english for this def (for me and the generic reader both :)): what are the F_k in the sense of what they represent/why F needs to be decomposed this way? can they overlap (right now yes)? what do the conditions mean?}

Definition 8 (Histogram attribution function)**.** *Let A be an attribution function $A : \mathcal{P}(\mathcal{I} \cup \mathcal{C})^k \to \mathbb{R}^m$, for a fixed k and m. Fix a set of relevant impressions F_A and a set of event sets $\mathbf{F} = (F_1, \ldots \cup F_k$. Let $a_{\mathbf{F}}(f)$ denote the value attributed to event $f \in (F_1 \cap F_A) \cup \cdots \cup (F_k \cap F_A)$ by $A(\mathbf{F})$. Pierre: $\mathbf{F}$ should be a vector of event sets, and we don't need $F_1, \cup F_k$ here yet. A is a histogram attribution function, if the following conditions all hold:*

1. *For each event $f \in (F_1 \cap F_A) \cup \cdots \cup (F_k \cap F_A)$, we have that $a_F(f) \geq 0$*

2. *There exists a one-hot encoding function* $H : \mathbb{R}^m \to \{0,1\}^m$ *such that* $||H(f)||_1 = 1$, *for each event* $f \in (F_1 \cap F_A) \cup \cdots \cup (F_k \cap F_A)$.
3. $A(F_1, ..., F_k) = \sum_{i=1}^{k} \sum_{f \in F_i \cap F_A} a_F(f) \cdot H(f)$

Definition 9 (Query partitions for beneficiary sites)**.** *Let* A *be a histogram attribution function* $A : \mathcal{P}(\mathcal{I} \cup \mathcal{C})^k \to \mathbb{R}^m$, *for a fixed* k *and* m *and with an associated one-hot encoding function* $H : \mathbb{R}^m \to \{0,1\}^m$. *Fix* $n \in \mathbb{N}_+$. *Let* F_A *be a fixed set of relevant events. Let* $a_F : F \to \mathbb{R}$ *be a function that gets the value attributed to a given event. We have that* $a_F(f)$ *denotes the value attributed to event* $f \in (F_1 \cap F_A) \cup \cdots \cup (F_k \cap F_A)$ *by* $A(\mathbf{F})$. *Alison: I don't partition the inputs anymore here. I still need to fix the rest of the definitions to reflect this. We define the following:*

- H_j *is the* encoding partition *for* $j \in [n]$ *such that* $H_j : \mathbb{R}^m \to \{0,1\}^{m_j}$ *and* $H_j(f) = H(f)$ *if* $f \in F_A^j$ *and* $H_j(f) = 0$ *otherwise, where* $\sum_{i=0}^{n} m_i = m$. *{Mathias: we we also partition the output dimensions!? Is it not enough to just partition the inputs and require that the output on the whole input set is the sum of outputs on each subset?}*
- A_j *is the* attribution partition *for* $j \in [n]$ *where*

$$A_j(F_1, \ldots F_k) := \sum_{i=1}^{k} \sum_{f \in F_i \cap F_A^j} a_{\mathbf{F}}(f) \cdot H_j(f) \tag{68}$$

- ρ^j *is the* report partition *for* $j \in [n]$ *for a fixed report identifier* $r \in \mathbb{Z}$ *where*

$$\rho_r^j(D) = A_j(D_d^E) \tag{69}$$

Let $B = \{b_1, \ldots, b_n\}$ *be a set of beneficiary sites, where each* b_j *has a corresponding set of relevant events* F_A^j. *Consider a set of report identifiers* R. *We define* Q_j *as the* query partition *that results by considering only* F_A^j *and the corresponding report partitions* ρ^j. *We have:*

$$Q_j(D) := \sum_{r \in R} \rho_r^j(D) \tag{70}$$

Proposition 1 (Individual sensitivity of a report partition)**.** *Fix a report identifier* r, *a device* d_r, *a set of epochs* E_r, *a beneficiary* $j \in B$, *an attribution partition function* A_j *with relevant events* F_A^j *and the corresponding report partition* $\rho^j : D \to A_j(D_{d_r}^{E_r})$. *We define*

$$A^{\max} := \max_{F \in \mathcal{P}(\mathcal{I} \cup \mathcal{C})^k} \sum_{i=1}^{k} \sum_{f \in F_i \cap F_A} a_F(f) \tag{71}$$

For a fixed device-epoch record $x = (d, e, F) \in \mathcal{X}$, *we have that the individual sensitivity of* ρ_j *is*

$$\Delta_x(\rho^j) \leq \begin{cases} 0 & \text{if } d \neq d_r, e \notin E_r \text{ or } F \cap F_A^j = \emptyset \\ \|A_j(F)\|_1 & \text{if } d = d_r \text{ and } E_r = \{e\} \\ 2A^{\max} & \text{if } d = d_r, e \in E_r \text{ and } F \cap F_A^j \neq \emptyset \end{cases}$$

Proof. Follows directly from Theorem 18 of [TKM+24]. □

Proposition 2 (Individual sensitivity of a concatenated report)**.** *Fix a report identifier* r, *a device* d, *a set of epochs* E_r, *a histogram attribution function* A *and a set of beneficiaries* $B = \{b_1, b_2, ..., b_n\}$. *Suppose each beneficiary has a corresponding report partition (i.e.* ρ_j *is the report partition for beneficiary* b_j*) that results from histogram attribution function* A. *For a fixed device-epoch record* $x = (d, e, F) \in \mathcal{X}$, *the individual sensitivity of the concatenated report* ρ *is*

$$\Delta_x(\rho) \leq \begin{cases} 0 & \text{if } d \neq d_r, e \notin E_r \text{ or } F \cap F_A = \emptyset \\ \|A(F)\|_1 & \text{if } d = d_r \text{ and } E_r = \{e\} \\ 2A^{\max} & \text{if } d = d_r, e \in E_r \text{ and } F \cap F_A \neq \emptyset \end{cases}$$

Alison: I will add these details, but it should be the same as proposition 1

Proposition 3 (Sensitivity of query partitions Q_j)**.** *Fix a relevant event set* F_A *and a corresponding partition* F_A^j. *Let* Q_j *be the query partition of beneficiary* j *that considers only events in* F_A^j. *Fix a device-epoch record* $x = (d, e, F) \in \mathcal{X}$, *where* F. *Then if* x *participates in a single report* r', *we have that*

$$\Delta_x(Q_j) = \Delta_x(\rho_r^j) \tag{72}$$

Proof. Take $D, D' \in \mathcal{D}$ such that $D' = D + x$. We have that

$$\Delta_x(Q_j) = \max_{D' = D + x_i \in \mathcal{D}} \|Q_j(D) - Q_j(D')\|_1 \tag{73}$$

$$= \max_{D' = D + x \in \mathcal{D}} \left\| \sum_{r \in R} \rho_r^j(D) - \rho_r^j(D') \right\|_1 \tag{74}$$

$$= \max_{D' = D + x \in \mathcal{D}} \left\| \rho_{r'}^j(D) - \rho_{r'}^j(D') \right\|_1 \tag{75}$$

$$= \Delta_x(\rho_{r'}^j) \tag{76}$$

by the definition of individual sensitivity. □

{Mathias: I can't quite get what Proposition 2 and Theorem 9 say. I was expecting something like $\Delta_x(Q) \leq \max_{i \in [n]} \Delta_x(\rho_r^i)$ or something, to basically say "if we can decompose to queries over the event set, we only pay once, and we pay the max sensitivity if it's not equal" or something like this.}

Theorem 11 (Multi-beneficiary optimization for query Q)**.** *Let* F_A *be a fixed relevant event set, such that* F_A *is partitioned into* n *disjoint subsets and* $F_A = F_A^1 \sqcup F_A^2 \sqcup ... \sqcup F_A^n$. *Let* $B = \{b_1, \ldots, b_n\}$ *be a set of beneficiary sites, where each* b_j *has a set of relevant events* F_A^j. *Consider a set of report identifiers* R. *Let each beneficiary site* b_j *also have a corresponding query partition* Q_j *that results by considering only* F_A^j *and the corresponding report partitions* ρ_r^j *for* $r \in R$ *and attribution partition* A_j. *Let* $Q = Q_1, Q_2, ..., Q_n$ *be the query that results from processing all of the beneficiaries queries* Q_j *at once. Fix a device-epoch record* $x = (d, e, F) \in \mathcal{X}$. *We have that the sensitivity of* Q *is such that*

$$\Delta_x(Q) \leq \sum_{r \in R} \Delta_x(\rho_r) \tag{77}$$

Proof. We first notice that for a fixed r we have that $\sum_{j=1}^{n} \rho_r^j(D) = \rho_r(D)$. This follows directly from our definition of report partition.

$$\sum_{j=1}^{n} \rho_r^j(D) = \sum_{j=1}^{n} A_j(D_d^E) \tag{78}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{k} \sum_{f \in F_i \cap F_A^j} a_{\mathbf{F}}(f) \cdot H_j(f) \tag{79}$$

$$= \sum_{i=1}^{k} \sum_{f \in (F_i \cap F_A^1) \cup \ldots \cup (F_i \cap F_A^n)} a_{\mathbf{F}}(f) \cdot H(f) \tag{80}$$

$$= \sum_{i=1}^{k} \sum_{f \in (F_i \cap F_A)} a_{\mathbf{F}}(f) \cdot H(f) \tag{81}$$

$$= \rho_r(D) \tag{82}$$

Now, take $D, D' \in \mathcal{D}$, such that $D' = D + x$. We have that

$$\Delta_x(Q) = \max_{D'=D+x \in \mathcal{D}} \|Q(D) - Q(D')\|_1 \tag{83}$$

$$= \max_{D'=D+x \in \mathcal{D}} \| \sum_{j=1}^{n} Q_j(D) - Q_j(D')\|_1 \tag{84}$$

$$= \max_{D'=D+x \in \mathcal{D}} \| \sum_{j=1}^{n} \sum_{r \in R} \rho_r^j(D) - \rho_r^j(D')\|_1 \tag{85}$$

$$= \max_{D'=D+x \in \mathcal{D}} \| \sum_{r \in R} \sum_{j=1}^{n} \rho_r^j(D) - \sum_{j=1}^{n} \rho_r^j(D')\|_1 \tag{86}$$

$$= \max_{D'=D+x \in \mathcal{D}} \| \sum_{r \in R} \rho_r(D) - \rho_r(D')\|_1 \tag{87}$$

$$\leq \sum_{r \in R} \max_{D'=D+x \in \mathcal{D}} \|\rho_r(D) - \rho_r(D')\|_1 \tag{88}$$

$$\leq \sum_{r \in R} \Delta_x(\rho_r) \tag{89}$$

□

Example We provide an example to illustrate the c-filter budget optimizations of processing Q as a whole rather than processing each individual beneficiary site's Q_j.

- Without Multi-Beneficiary Optimization
  - *individual sensitivity used for nc-filter deduction for beneficiary site j*: $\sum_{r \in R} \Delta_x(\rho_r^j)$
  - *individual sensitivity used for c-filter deduction*: $\sum_{j=1}^{n} \sum_{r \in R} \Delta_x(\rho_r)$
- With Multi-Beneficiary Optimization
  - *individual sensitivity used for nc-filter deduction for beneficiary site j*: $\sum_{r \in R} \Delta_x(\rho_r)$
  - *individual sensitivity used for c-filter deduction*: $\sum_{r \in R} \Delta_x(\rho_r)$

Since we have that $\sum_{r \in R} \Delta(\rho_r) \leq \sum_{j=1}^{n} \sum_{r \in R} \Delta(\rho_r^j)$, processing Q in its entirety spends less privacy budget.

Notes

1. Pierre: We could even completely remove public information from the data model and use auxiliary information, now that we can refer to Cookie Monster as an example of how to generate and distribute auxiliary information?

2. PPA further requires impressions to come with a list of authorized conversion sites and beneficiary sites. Pierre: I will add them to the data model only when the need becomes obvious.

3. Cookie Monster defines a set of *relevant events*, potentially including conversions, but we only consider *relevant impressions* for simplicity. In particular, this hardcodes "Case 1" from Cookie Monster Thm. 1.

4. This mapping is a sort of environment variable, not instantiated explicitly but maintained in a distributed way through network connections.

5. For simplicity, we do not explicitly require all the reports identifiers in a query to be associated with the same beneficiary site. In practice, browsers can implement an additional authorization mechanism. This is reflected in Alg. 2, where a beneficiary site only knows identifiers that were associated with conversions where it was the beneficiary site, and has zero probability of guessing the identifiers of reports associated with other beneficiary sites.

6. Pierre: Might become an ILP if we introduce extra constraints like R, n, m.

7. Pierre: Deducting from filters sequentially should work, but we might be overpaying: if one filter passes but another halts we send a null report, in which case it would be more efficient to deduct zero from all the filters. What is the right abstraction for a "conjunction filter"? For RDP we have a conjunction of filters across the tracked alphas, and that works by creating a new filter that takes the conjunction of all the continuation rules. That means we can't directly use each filter as a black box. Does atomicity add non-obvious interactions? *e.g.,* q-imp sending a null report in one world but not the other, thereby rolling back the c-filter in one world but not the other, etc. {Mathias: isn't this a 2 Phase Commit, where you prepare for all filter, and either commit/release depending on the outcome?} Pierre: Yes, to implement this we cqn definitely do a 2PC. My concerns are (1) whether this has implications in terms of privacy, because now we don't have a single vanilla filter but a collection of filters, that we might need to analyze jointly, and (2) whether we actually want a 2PC for all the filters, in particular q-imp: if one impSite is out of budget, it sounds better to drop it and keep running the transaction on the other filters. (1) is simple and worked when I sketched it, but I think it'd be good to write down explicitly why it works and why we can reuse existing filter proofs under a 2PC. Especially when we have IDP filters, I tend to worry about the state of the filters being different across worlds. For (2), we might need a more detailed order in which we run the filters, instead of a single 2PC. e.g. q-imp can be OOB without causing the transaction to fail (and dropping some OOB impsites can actually lower the privacy loss for other filters ?), but if subsequent filters are OOB then we can still revert the q-imp.

8. If quotas have no privacy meaning, it would be nice to treat them as a "query pre-processing step" that can only reduce the query sensitivity, after which we go through the real privacy filters. We could then swap quotas for any other rate limiting mechanism that has the same pre-processing properties, without having to re-do the privacy proofs.

9. Could be $F_{e,i} \leftarrow \emptyset$ instead. It seems natural to leave impression-site quota out of the atomicity requirement.

10. We might not need a variant with "public impression information" if we hardcode the fact that only conversions can be public.

11. If capacities are set system-wide we don't even need IDP for the end-to-end guarantees. IDP can be an internal accounting tool that only appears in the proofs, with no need for exotic definitions in the body of the paper.