Multi-transversals for Triangles and the Tuza's Conjecture*

Parinya Chalermsook† Samir Khuller‡ Pattara Sukprasert§ Sumedha Uniyal¶

Abstract

In this paper, we study a primal and dual relationship about triangles: For any graph G, let $\nu(G)$ be the maximum number of edge-disjoint triangles in G, and $\tau(G)$ be the minimum subset F of edges such that $G \setminus F$ is triangle-free. It is easy to see that $\nu(G) \leq \tau(G) \leq 3\nu(G)$, and in fact, this rather obvious inequality holds for a much more general primal-dual relation between k-hyper matching and covering in hypergraphs. Tuza conjectured in 1981 that $\tau(G) \leq 2\nu(G)$, and this question has received attention from various groups of researchers in discrete mathematics, settling various special cases such as planar graphs and generalized to bounded maximum average degree graphs, some cases of minor-free graphs, and very dense graphs. Despite these efforts, the conjecture in general graphs has remained wide open for almost four decades.

In this paper, we provide a proof of a non-trivial consequence of the conjecture; that is, for every $k \geq 2$, there exist a (multi)-set $F \subseteq E(G) : |F| \leq 2k\nu(G)$ such that each triangle in G overlaps at least k elements in F. Our result can be seen as a strengthened statement of Krivelevich's result on the fractional version of Tuza's conjecture (and we give some examples illustrating this.) The main technical ingredient of our result is a charging argument, that locally identifies edges in F based on a local view of the packing solution. This idea might be useful in further studying the primal-dual relations in general and the Tuza's conjecture in particular.

1 Introduction

The study of the relationship between primal and dual graph problems has been a cornerstone in combinatorial optimization, discrete mathematics, and design of approximation algorithms. Among the oldest such relations is perhaps the ratio between *maximum matching* (denoted by $\nu(G)$) and *minimum vertex cover* (denoted by $\tau(G)$), for which $\nu(G) \leq \tau(G) \leq 2\nu(G)$ holds for any graph G, and this inequality is tight. In special graph classes, the factor of two can be improved or even removed completely, for instance, in the famous König's theorem, we learn that $\nu(G) = \tau(G)$ for any bipartite graph G [6]. This notion of matching and covering can be generalized to any r-uniform hypergraphs, where $\nu(G)$ now denotes the cardinality of maximum hyper-matching (a collection of hyperedges that are pairwise disjoint) and $\tau(G)$ is called *transversal* of hypergraphs (a collection of vertices that hit every hyperedge). It is easy to see that the relation $\nu(G) \leq \tau(G) \leq r \cdot \nu(G)$ still holds, and in general, a factor r is tight: For instance, when $r = 3$, a natural set system of Fano plane provides a tight example where $\nu(G) = 1$ but $\tau(G) = 3$.

Knowing König's theorem, one wonders in what graph classes, the trivial relation $\tau(G) \leq r \cdot \nu(G)$ could be improved. An old conjecture, called Ryser's conjecture[1], states that $\tau(G) \leq (r-1)\nu(G)$ holds for any r-partite hypergraphs (this conjecture can easily be seen as generalizing König's theorem to any r-uniform hypergraphs.) The conjecture remains wide open for $r \geq 4$ (the $r = 3$ case was resolved [1]). Another direction that has received a lot of attention focuses on hypergraphs that are defined from specific (simple) graph structures: For any simple graph $G = (V, E)$, an r-uniform cycle hypergraph is created by having one vertex for each edge in E and a hyperedge for any r-cycle $\{e_1, e_2, \ldots, e_r\}$ in G. Now the matching and covering questions turn into studying the ratio between the maximum edge-disjoint r-cycle packing and the minimum edge set that hits all cycles of length r. In particular, when $r = 3$, we are interested in the relation between the number of disjoint triangles v.s. the set of edges that hit all triangles in G. Denote these numbers by $\nu^c(G)$ and $\tau^c(G)$ respectively. Tuza conjectured in 1981 that $\tau^c(G) \leq 2\nu^c(G)$ and verified the

*Part of the work was done while PS was visiting Aalto University and KTH, Royal Institute of Technology, and while PS was studying at the University of Maryland. Full version will be posted on ArXiV and also available at `https://users.aalto.fi/~uniyals1/resources/MultiTUZA-SODA20-full.pdf`

† Aalto University, Espoo, Finland
‡ Northwestern University, Evanston, USA
§ Northwestern University, Evanston, USA
¶ Aalto University, Espoo, Finland

[1] The conjecture was first stated in a thesis of his student, J. Henderson in 1971

conjecture for planar graphs [10, 11]. Tuza's conjecture has since received attention from various groups of researchers, leading to confirmations of the conjecture for many special cases such as $K_{3,3}$-free graphs [8], graphs with maximum average degree less than 7 [9], graphs of treewidth at most 6 [2], and graphs that contain quadratic-sized edge-disjoint triangles [12, 5].

In general graphs, Haxell showed that $\tau^c(G) \leq (3-\epsilon)\nu^c(G)$ [4], confirming at least the principles that the trivial ratio of 3 can really be improved. Haxell's result has remained the best known ratio for general graphs. Whether Tuza's conjecture holds or not has remained an intriguing open problem.

1.1 Our contributions In this paper, we study a natural question that is a consequence of the Tuza's conjecture and discuss potential applications of our work in making progress towards resolving the conjecture. From now on, we denote $\tau^c(G)$ and $\nu^c(G)$ simply by $\tau(G)$ and $\nu(G)$ respectively.

Our work is inspired from the mathematical programming perspectives of the Tuza's conjecture. For each edge $e \in E$, we have a real variable x_e. Define $\tau^*(G)$ to be an optimal value to the following linear program:

$$\text{(LP)} \quad \min \quad \sum_e x_e$$
$$\text{s.t.} \quad \sum_{e:e\in E(t)} x_e \geq 1 \text{ for all triangles } t$$

Clearly, since the above LP is a relaxation of the covering number, we have that $\tau^*(G) \leq \tau(G)$. Observe that the Tuza's conjecture would imply the weaker inequality $\tau^*(G) \leq 2\nu(G)$, and this can be seen as a fractional variant of the Tuza's conjecture. Krivelevich [8] showed that $\tau^*(G) \leq 2\nu(G)$ (later strengthened by [3]).

Another way to phrase Krivelevich's result (without bothering about linear programs) is through the notion of multi-transversals: For some $k \in \mathbb{N}$, there exist a multi-set $F \subseteq E(G) : |F| \leq 2k\nu(G)$ such that each triangle in G contains at least k elements of F. We call such set a *multi-transversal* set of order k (or k-multi-transversal), and define $\tau_k^*(G)$ as the minimum value of $\frac{|F|}{k}$ for any k-multi-transversal set F. Observe that the existence of k-multi-transversal set F is equivalent to saying that there exists a fractional cover $z : E(G) \to \{0, 1/k, 2/k, \ldots, 1\}$ that is feasible for (LP) such that $\sum_{e\in E(G)} z(e) = \frac{|F|}{k}$. We call such a feasible solution a $(1/k)$-*integral solution*. Notice that, the case when $k = 1$ (the 1-multi-transversal set) is equivalent to the transversal set in the sense of Tuza's conjecture, and that $(\forall k :)\tau^*(G) \leq \tau_k^*(G) \leq \tau_1^*(G) = \tau(G)$.

It is then natural to consider the following question:

CONJECTURE 1.1. (STRONG FRACTIONAL TUZA'S CONJECTURE) *For any $k \in \mathbb{N}$ and $k \geq 2$, we have $\tau_k^*(G) \leq 2\nu(G)$ for any graph G.*

Clearly, Conjecture 1.1 is a consequence of the Tuza's conjecture and it implies the standard fractional version of Tuza's conjecture that was resolved by Krivelevich. Krivelevich's proof is based on induction and only implies that $\tau_k^*(G) \leq 2\nu(G)$ for some (very large) $k \in \mathbb{N}$. [2]

In this paper, we resolve the above question in the affirmative.

THEOREM 1.1. *For all integer $k \geq 2$ and any graph G, $\tau_k^*(G) \leq 2\nu(G)$. Moreover, we can efficiently find the triangle packing solution and the k-multi-transversal that together achieve this bound.*

The proof of this theorem is inspired by the local search technique. Based on an optimal packing solution, we define a collection of edges in the covering solution via the local view of the packing solution. Our proofs are constructive in nature. Starting with any packing solution, we can either find a multi-transversal or improve the current packing solution. Because the packing solution can be improved at most $O(n^3)$ times, hence we get an algorithm to find the triangle packing solution and the k-multi-transversal that together achieve this bound.

We believe that the structural insights from this work would be useful in attacking the Tuza's conjecture in the future.

1.2 Comparisons with Fractional Transversals We discuss two showcases which illustrate that our concept of multi-transversal of lower order is stronger than the standard fractional LP solution.

First we show that proving Conjecture 1.1 can be reduced to the cases when $k = 2$ and $k = 3$.

PROPOSITION 1.1. *If $\tau_2^*(G) \leq 2\nu(G)$ and $\tau_3^*(G) \leq 2\nu(G)$, then $\tau_k^*(G) \leq 2\nu(G)$ for all $k \geq 2$.*

Proof. Consider any graph G. Let F_2 and F_3 be multi-transversal sets of orders 2 and 3 respectively, so any

[2] We note that the values of τ_k^* are not monotone in k: There is an infinite family of graphs G for which $\tau_2^*(G) \leq (\frac{3}{4} + o(1))\tau_3^*(G)$, and another family of graphs G for which $\tau_3^*(G) \leq (\frac{5}{6} + o(1))\tau_2^*(G)$. See Section A for more detail.

triangle t in G is hit by two elements of F_2 and 3 elements of F_3. We will use (multiple copies of) F_2 and F_3 to hit every triangle in G at least k times. For even $k = 2q$, we can simply use q copies of F_2, and this would hit each triangle $2q$ times. For odd $k > 3$, we can write $k = 2q + 3$, and so we can use q copies of F_2 together with 1 copy of F_3. □

In Section A, we show that $\tau_2^*(G)$ and $\tau_3^*(G)$ are not subsumed by each other, so we would need to consider both cases.

Next, we will present evidence that $\tau_2^*(G)$ and $\tau_3^*(G)$ are stronger lower bound of $\tau(G)$ than the standard fractional bound $\tau^*(G)$.

The following proposition shows that $\tau_3^*(G)$ is strictly stronger than $\tau^*(G)$ in general.

PROPOSITION 1.2. *For any graph G, $\tau(G) \leq 1.5\tau_3^*(G)$, while there exists a family of graphs G for which $\tau(G) = (2 - o(1))\tau^*(G)$.*

Proof. We prove this by turning any LP solution $\{z_e\}_{e \in E}$ that is $\frac{1}{3}$-integral into an integral solution of cost at most $1.5(\sum_{e \in E} z_e)$. For any subset of edges $S \subseteq E$, let $z(S) := \sum_{e \in S} z_e$. First, we define $E' = \{e : z_e \geq 2/3\}$. Notice that $|E'| \leq \frac{3}{2} z(E')$. We consider the remaining edges $\tilde{E} \subseteq E - E'$ in the support of z, i.e. all edges with non-zero z value. Note that for every $e \in \tilde{E}$, we have $z_e = 1/3$, which implies $z(\tilde{E}) = |\tilde{E}|/3$. This implies that every triangle t in $G - E'$ must satisfy $|E(t) \cap \tilde{E}| = 3$.Moreover, there exists a collection of edges $E'' \subseteq \tilde{E}$ whose size is $|E''| \leq |\tilde{E}|/2$ and whose removal creates a bipartite graph; in particular, E'' hits every triangle in $G - E'$. Notice that $|E''| \leq |\tilde{E}|/2$ while $z(\tilde{E}) = |\tilde{E}|/3$, so we have $|E''| \leq \frac{3}{2} z(\tilde{E})$.

The set $E' \cup E''$ is our integral solution.

For the family of graphs G, for which $\tau(G) = (2 - o(1))\tau^*(G)$, we refer to [7]. □

Another example illustrates that $\tau_2^*(G)$ can be stronger than $\tau^*(G)$ in complete graphs.

PROPOSITION 1.3. *For any even integer n, we have that $\tau(K_n) = \tau_2^*(K_n) = n(n-2)/4 = \frac{n^2}{4}(1 - o(1))$, while $\tau^*(K_n) = \frac{1}{3}\binom{n}{2} = \frac{n^2}{6}(1 - o(1))$.*

Proof. Notice that $\tau(K_n) \leq n(n-2)/4$ because we can partition the vertex set into two equal sets of size $n/2$ and remove edges in the same set. Therefore, we have removed $\binom{n}{2} - \frac{n^2}{4} = \frac{n(n-2)}{4}$ edges.

We will now argue that $\tau_2^*(K_n) \geq n(n-2)/4$. Consider an LP solution $\{z_e\}_{e \in E(K_n)}$ that is half-integral. For any subset of edges $S \subseteq E(K_n)$, let $z(S) := \sum_{e \in S} z_e$. We will prove that the value of z is at least $n(n-2)/4$ by induction. It is easy to see that this is true for the base case of K_2. Now consider $n \geq 4$ and graph $G = K_n$. If there is no edge in G with zero LP value, we are immediately done, since the total LP value must be at least $\frac{1}{2}\binom{n}{2}$. Otherwise, let (u, v) be such an edge with zero LP value. Consider $V' = V(G) \setminus \{u, v\}$. The induced subgraph $G[V']$ is K_{n-2}, hence by IH, we get that the total LP-value $z(E(G[V']))$ is at least $(n-2)(n-4)/4$. Now we analyze the LP-values on the edges between V' and $\{u, v\}$. For each $w \in V'$, notice that we have $z_{wu} + z_{wv} \geq 1$ since $z_{uv} = 0$ (and the triangle $\{u, v, w\}$ must be hit). Therefore, the total LP-values on the edges incident to u or v must be at least

$$\sum_{w \in V'} (z_{wu} + z_{wv}) \geq |V'| = (n-2)$$

which leads to the total of $\frac{(n-2)(n-4)}{4} + (n-2) = \frac{n(n-2)}{4}$. □

1.3 Open problems We propose the notion of multi-transversal of any order and show that the Tuza's conjecture holds in this relaxed setting. It is interesting to see whether our results and techniques would be useful in making further progress. Open problems that are most directly related to our work are to characterize the graph classes where $\tau_2^*(G) = \tau(G)$ or $\tau_3^*(G) = \tau(G)$. This condition would be sufficient for graph G to satisfy the Tuza's conjecture. The fractional notion may appear weak, but it has in fact been used to obtain non-trivial results of Tuza's conjecture in very dense graphs [12] (through a non-trivial use of Szemeredi regularity lemma). It is intriguing to see whether our stronger notion of fractional covering would lead to similar kind of results.

Another open problem is whether our techniques would help in improving the best known duality gap ratio, as shown by Haxell [4]. For instance, can we achieve the bound of $(3 - \epsilon)$ using our technique? Even simpler, can we prove that for every graph G, $\tau(G) \leq (1.5 - \epsilon)\max\{\tau_2^*(G), \tau_3^*(G)\}$? Combining with our result, this would imply the $(3 - \epsilon)$ bound.

1.4 Further related work In this paper, we focus on the relation between (a stronger variant of) $\tau^*(G)$ and $\nu(G)$. Another fractional Tuza problem is the relation between $\nu^*(G)$ (the fractional packing number) and $\tau(G)$. In that case, Krivelevich also showed that $\tau(G) \leq 2\nu^*(G)$ [8], and this is known to be tight (see, [7] for a tight example.)

2 Overview of the proof

2.1 The setup In this section, we introduce the technical terms that will be used in our proof. Let $\mathcal{T}$ be the set of all triangles in G and $\mathcal{V} \subseteq \mathcal{T}$ be an optimal packing solution. Triangles and edges in $\mathcal{V}$ are referred to as solution triangles and solution edges respectively. Notice that each (non-solution) triangle $t \in \mathcal{T} \setminus \mathcal{V}$ overlaps (or conflicts) with at most 3 triangles in $\mathcal{V}$ and that if t conflicts with $\psi \in \mathcal{V}$, then $|E(t) \cap E(\psi)| = 1$.

We categorize non-solution triangles in $\mathcal{T} \setminus \mathcal{V}$.

- If t overlaps with exactly one triangle $\psi \in \mathcal{V}$, we say that t is *singly-attached*[3] to ψ, and the unique edge in $E(t) \cap E(\psi)$ is called a *base-edge*. The vertex $anchor(t) := V(t) \setminus V(\psi)$[4] is called the *anchoring vertex* of t.
- If t overlaps with exactly two triangles $\psi_1, \psi_2 \in \mathcal{V}$, we say that t is *doubly-attached* to these triangles.
- Otherwise, if t overlaps with three triangles in $\mathcal{V}$, we say that t is a *hollow triangle*.

Types of solution triangles: For each $\psi \in \mathcal{V}$, denote the conflict list of ψ by $\mathcal{CL}(\psi)$ which contains triangles in $\mathcal{T} - \mathcal{V}$ that overlap with ψ. We naturally partition $\mathcal{CL}(\psi)$ into $\mathcal{CL}_{sin}(\psi) \cup \mathcal{CL}_{dou}(\psi) \cup \mathcal{CL}_{hol}(\psi)$ which denote the sets of singly attached, doubly-attached, and hollow triangles overlapping with ψ.

We denote by $base(\psi)$ the set of all the base-edges contained in ψ. The triangle ψ is said to be type-i (or $type(\psi) := i$) if $|base(\psi)| = i$. This naturally partitions the set of solution triangles $\mathcal{V}$ into sets of type-i triangles $\mathcal{V}_i$ for each $i \in \{0, 1, 3\}$ (next proposition implies $|\mathcal{V}_2| = 0$).

Type-1 triangles: For any type-1 triangle $\psi^1 \in \mathcal{V}_1$, we use $base(\psi^1)$ to refer to the only *base-edge* in ψ^1 and *non-base* edges to refer to edges in $E(\psi^1) \setminus base(\psi^1)$. If $CL_{sin}(\psi^1) = \{t\}$ (which can only happen for type-1 triangles), then we define $anchor(\psi^1) := anchor(t)$.

Figures' drawing convention: In the figures, we label vertices using roman characters. We will use Δ_{abc} to refer to a triangle consisting of vertices $\{a, b, c\}$. Moreover, if we want to specify the type of the triangle, we will put it in the superscript, so Δ^1_{abc} will be a type-1 triangle. If the type is not specified, then it can be any type, including singly-attached, doubly-attached or hollow triangle. All the solution triangles will be filled with gray color. All the solution edges are drawn in solid style and the non-solution edges are drawn using dashed style.

The proofs for the propositions stated in this section can be found in the Section B.

PROPOSITION 2.1. (STRUCTURES OF SOLUTION TRIANGLES) *The following properties hold:*

1. $|base(\psi)| \neq 2$ *(there is no type-2 solution triangle.)*
2. *If ψ is type-3, then $|\mathcal{CL}_{sin}(\psi)| = 3$ and all triangles in $\mathcal{CL}_{sin}(\psi)$ share a common anchoring vertex. In this case, $V(\psi)$ together with such an anchoring vertex induce a K_4.*

Common anchoring vertex for type-3: If $\psi \in \mathcal{V}$ is a type-3 triangle, then we denote the common anchoring vertex for the $CL_{sin}(\psi)$ triangles by $anchor(\psi)$ (see Proposition 2.1.2).

Types of doubly-attached triangles: After having defined the types of triangles in $\mathcal{V}$, each doubly-attached triangle $t \in \mathcal{T} - \mathcal{V}$ can also be given a type in a natural way, i.e., $type(t)$ is a list of types of triangles in $\mathcal{V}$ conflicting with t; for instance, if t is a doubly-attached triangle adjacent to solution triangles ψ_1 and ψ_2, then $type(t) = [type(\psi_1), type(\psi_2)]$ where $type(\psi_1) \leq type(\psi_2)$ (see Figure 1). If there is no restriction on some dimension, then we put a star ($[*]$) there.

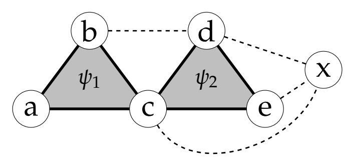


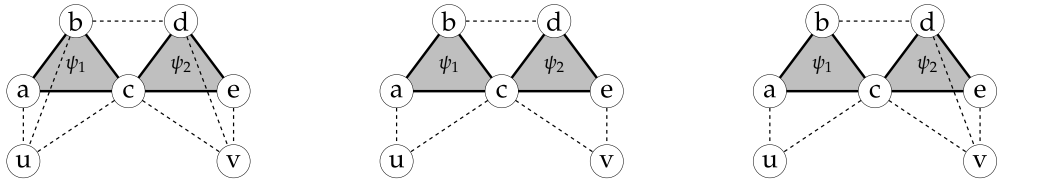
Figure 1: A doubly-attached triangle Δ_{bcd} of type-$[0, 3]$. In this case, $\psi_1 = \Delta^0_{abc}$, $\psi_2 = \Delta^3_{cde}$.

PROPOSITION 2.2. (STRUCTURES OF DOUBLY ATTACHED TRIANGLES) *Consider any doubly attached triangle t adjacent to solution triangles $\psi_1, \psi_2 \in \mathcal{V} \setminus \mathcal{V}_0$.*

1. $type(t) \neq [3, 3]$. *So the type is either $[1, 3]$ or $[1, 1]$.*
2. *If $type(t) = [1, 3]$ where $type(\psi_1) = 1$ and $type(\psi_2) = 3$, then (i) $base(\psi_1) \in E(t)$ OR (ii) $|\mathcal{CL}_{sin}(\psi_1)| = 1$ where the only anchoring vertex is in $V(t)$; in this case, $base(\psi_1)$ does not contain the common vertex in $V(\psi_1) \cap V(\psi_2)$.*

[3]The notion of "attachment" we use here is motivated by the view that triangles in $\mathcal{V}$ are the "skeleton" of the graph, and the rest of the graphs are "attachments" hanging around the skeleton.

[4] For singleton sets, we abuse notations a bit by interchangeably using e and $\{e\}$.

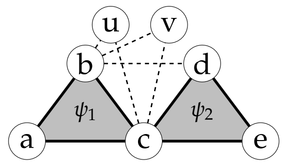


(a) type-$[3,3]$ triangle

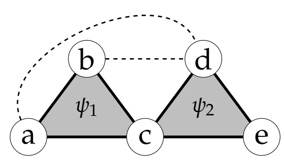
(b) type-$[1,1]$ triangle not containing any base-edge

(c) type-$[1,3]$ triangle without base-edge of the type-1 triangle

Figure 2: Three possible non-existing scenarios for doubly-attached triangles Δ_{bcd} adjacent to type-1 or type-3 triangles.



(a) type-$[1,*]$ triangle containing the base-edge of the type-1 triangle ψ_1



(b) type-$[1,*]$ triangle Δ_{bcd} with the ψ_1's unique singly-attached triangle Δ_{abd} having anchoring vertex $d = anchor(\psi_1)$ in $V(\Delta_{bcd}) \cap V(\Delta_{cde})$. Notice the two type-$[1,*]$ doubly-attached triangles Δ_{bcd} and Δ_{acd}.

Figure 3: Two of the possible scenarios for doubly-attached triangles Δ_{bcd} of type-$[1,*]$ adjacent to a type-1 triangle $\psi_1 = \Delta^1_{abc}$ and a type-1 or type-3 triangle $\psi_2 = \Delta_{cde}$.

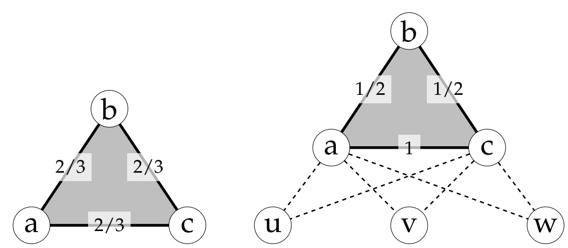
2.2 A Warm-up: Multi-transversal of order six We illustrate the essence of our techniques by showing a weaker statement that there is a multi-transversal set of order 6. Notice that this is equivalent to finding the assignment $f : E \to \mathbb{R}^+$ such that $f(e) \in \{\frac{j}{6} : j \in \{0,\ldots,6\}\}$ where $\sum_e f(e) \leq 2|\mathcal{V}|$ and for all $t \in \mathcal{T}$, $\sum_{e \in E(t)} f(e) \geq 1$.

Our analysis is done locally in the sense that each triangle $t \in \mathcal{V}$ will have two credits and distribute these credits to the nearby edges.

Charge Distribution: For any solution triangle ψ, the distribution of credit is done as follows.

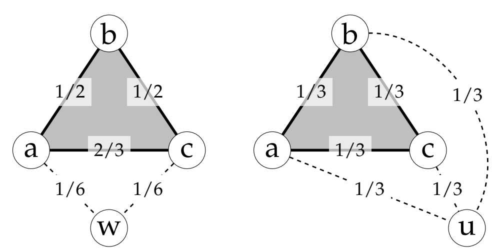
1. If $type(\psi) = 0$, let $f(e) = \frac{2}{3}$ for all $e \in E(\psi)$. Total distribution is 2 credits.
2. If $type(\psi) = 1$, we first define $f(e) = \frac{1}{2}$ for the two non-base edges. Then there are two sub-cases.
   (a) If $|\mathcal{C}L_{sin}(\psi)| > 1$, we give 1 credit to the base-edge. Clearly, total credit distribution for ψ is two.
   (b) Otherwise, let $\mathcal{C}L_{sin}(\psi) = \{t\}$. We give $\frac{2}{3}$ credit to the base-edge and $\frac{1}{6}$ each to two other edges of t. Total credit distribution is two in this case.
3. If $type(\psi) = 3$, let $f(e) = \frac{1}{3}$ for every edge e in the K_4-subgraph induced by $V(\psi) \cup anchor(\psi)$ (refer to Proposition 2.1.2). Since six edges get $\frac{1}{3}$ each, hence total credits distribution is two.

Refer to Figure 4 for illustration of this scheme.



(a) Δ^0_{abc}

(b) $|\mathcal{C}L_{sin}(\Delta^1_{abc})| > 1$



(c) $|\mathcal{C}L_{sin}(\Delta^1_{abc})| = 1$

(d) Δ^3_{abc}

Figure 4: Fractional Charging Scheme of order six

Analysis: It is easy to see that all solution triangles are covered. We analyze three cases of non-solution triangles.

- If t is a hollow triangle, since each solution edge has at least $\frac{1}{3}$ credit, it is obvious that t is covered.
- Now consider the case when t is a singly-attached triangle of ψ. If $type(\psi) = 3$, we are immediately done since $\frac{1}{3}$ credit would be placed on every edge in t. Otherwise, if $type(\psi) = 1$, we are also done: Either we have one credit on the base of ψ or we have $\frac{2}{3} + \frac{1}{6} + \frac{1}{6}$.
- The final case is when t is doubly-attached to $\{\psi_1, \psi_2\}$. There is only one case which is not obvious, when $type(\psi_1) = 1$ and $type(\psi_2) = 3$, in which it could be possible that two overlapping edges of t only get $\frac{1}{2} + \frac{1}{3}$ credit. From Proposition 2.2, one of the two scenarios must happen:
  1. $base(\psi_1) \in E(t)$, in which case we would be done since $base(\psi_1)$ has at least $\frac{2}{3}$ credits and any solution edge of type-3 has $\frac{1}{3}$ credit.
  2. The anchor of ψ_1 is a vertex in $V(t)$, in which case, we have that the edge $E(t) \setminus (E(\psi_1) \cup E(\psi_2))$ must be an edge of the unique singly-attached triangle to ψ_1. This edge receives the credit of $\frac{1}{6}$.

3 Multi-transversals of order three

Recall that in this case we want to find the assignment $f : E \to \mathbb{R}^+$ such that $f(e) \in \{\frac{j}{3} : j \in \{0, 1, 2, 3\}\}$ where $\sum_e f(e) \leq 2|\mathcal{V}|$ and for all $t \in \mathcal{T}$, $\sum_{e \in E(t)} f(e) \geq 1$. This case requires one more idea in comparison to the previous case. In particular, compared to the proof in the previous section, we need to understand the structures of type-$[1,1]$ doubly-attached triangles.

PROPOSITION 3.1. (STRUCTURES OF DOUBLY-ATTACHED TYPE-$[1,1]$ TRIANGLES) *For any doubly-attached type-$[1,1]$ triangle t adjacent to solution triangles $\psi_1, \psi_2 \in \mathcal{V}_1$, one of the following holds (up to renaming ψ_1 and ψ_2):*

1. *$base(\psi_1) \in E(t)$ (see Figure 3a).*
2. *$|\mathcal{C}L_{sin}(\psi_1)| = 1$ where the only anchoring vertex is in $V(t)$ (see Figure 3b).*
3. *Both $\psi_1 = \{v, u_1, w_1\}$ and $\psi_2 = \{v, u_2, w_2\}$ have exactly one singly-attached triangle each (say, ω_1 and ω_2 respectively) such that $E(\omega_1) \cap E(\omega_2) = \{(v, a)\}$ where a is a common anchoring vertex of ω_1 and ω_2. If this case happens, we say that ψ_1 and ψ_2 form a* pair *(see Figure 5).*

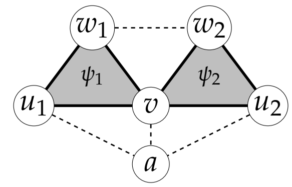


Figure 5: type-$[1,1]$ doubly-attached triangle $\Delta_{vw_1w_2}$ belonging to Proposition 3.1.3

Charge Distribution: Starting with $f(e) = 0$ for each edge in G, for any solution triangle $\psi \in \mathcal{V}_0 \cup \mathcal{V}_3$, distribute the credits exactly the same as in the previous section (see Figure 4). The charging for type-1 triangles is more subtle in this case (because we used $\frac{1}{2}$ credits in the previous case which we cannot use here.) We will distribute credits to type-1 triangles iteratively to make sure that all the type-$[1,1]$ triangles are covered.

Charging Scheme for type-1 triangles: Let $\mathcal{V}_1^1 \subseteq \mathcal{V}_1$ be the subset of type-1 triangles having exactly one singly-attached triangle.

For $\psi \in \mathcal{V}_1 \setminus \mathcal{V}_1^1$, set $f(e) = 1$ if e is the base-edge of ψ and $f(e) = \frac{1}{3}$ for each e which is a non-base edge of ψ. We use exactly $\frac{5}{3}$ credit for each $\psi \in \mathcal{V}_1 \setminus \mathcal{V}_1^1$. See Figure 6 for the charging illustration of triangles in $\mathcal{V}_1 \setminus \mathcal{V}_1^1$.

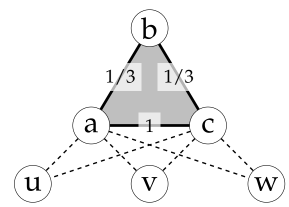


Figure 6: Order-3 Fractional Charging Scheme for type-1 triangle Δ^1_{abc} such that $|CL_{sin}(\Delta^1_{abc})| > 1$

Now let $\pi : \mathcal{V}_1^1 \to [|\mathcal{V}_1^1|]$ be a bijection that denotes an arbitrary permutation of $\mathcal{V}_1^1$. We will distribute charge corresponding to triangles in $\mathcal{V}_1^1$ in this order.

For $1 \leq i \leq |\mathcal{V}_1^1|$, let $\psi_i = \pi^{-1}(i)$ be the i^{th} triangle in the permutation. Let u_i and v_i be two end points of the edge $base(\psi_i)$. Let e_{u_i} and e_{v_i} be the non-base solution edges adjacent to u_i and v_i, respectively. Similarly, let e'_{u_i} and e'_{v_i} be the non-solution edges of the unique singly-attached triangle of ψ_i adjacent to u_i and v_i, respectively.

Starting from $i = 1$, we first assign $f(base(\psi_i)) = \frac{2}{3}$. We then inspect the charge on e'_{u_i}. If $f(e'_{u_i}) = 0$, then we set $f(e'_{u_i}) = f(e_{u_i}) = \frac{1}{3}$. Otherwise, we leave $f(e'_{u_i})$ as-is and set $f(e_{u_i}) = \frac{2}{3}$. Similarly, we distribute the charge to e_{v_i} and e'_{v_i}.

We continue assigning charges for ψ_i in order until we have processed all the triangles in $\mathcal{V}_1^1$ for every $1 \le i \le |\mathcal{V}_1^1|$. See Figure 7 for illustration.

The following observation lists the properties of the charging scheme $\{f(e)\}_{e\in E}$ above.

OBSERVATION 3.1. *The following properties hold for the charging scheme* $\{f(e)\}_{e\in E}$.

1. *For every solution edge* $e \in E(\mathcal{V})$, $f(e) \ge \frac{1}{3}$.
2. *For every* $\psi \in \mathcal{V}_0$ *and for each* $e \in E(\psi)$, $f(e) = \frac{2}{3}$.
3. *For every* $\psi \in \mathcal{V}_1$, $f(base(\psi)) \ge \frac{2}{3}$. *If* $|CL_{sin}(\psi)| > 1$, *then* $f(base(\psi)) = 1$.
4. *For every* $\psi \in \mathcal{V}_1^1$, *if* e_1 *and* e_2 *are the non-solution edges of the unique singly-attached triangle of* ψ, *then* $f(e_1) = f(e_2) = \frac{1}{3}$.
5. *For every* $\psi \in \mathcal{V}_3$, *for each edge* e *in the* K_4 *structure associated with it*, $f(e) = \frac{1}{3}$.

As we assign additional 2 credits per ψ_i, it is easy to see that $\sum_e f(e) \le 2|\mathcal{V}|$.

Analysis: It is easy to see from Observation 3.1 that all solution and singly-attached triangles are covered. We analyze non-solution triangles as follows.

- If t is a hollow triangle, since each solution edge has at least $\frac{1}{3}$ credit, hence it is covered.
- If t is a doubly-attached triangle of type-$[0,*]$, by Observation 3.1, since the edge of t adjacent to the type-0 triangle has $\frac{2}{3}$ credit and the other solution edge has at least $\frac{1}{3}$ credit, hence it is covered.
- If t is a doubly-attached triangle of type-$[1,3]$, let e_t be the non-solution edge of t. By Proposition 2.2, it must be that (i) $E(t)$ contains a base-edge of the type-1 triangle or (ii) e_t is an edge of the unique singly-attached triangle of the type-1 triangle in $\mathcal{V}_1^1$ adjacent to t. In (i), t is covered by the base-edge which has at least $\frac{2}{3}$ credit and the other solution edge which has at least $\frac{1}{3}$ credit. Otherwise, in (ii) t is also covered since any solution edge has at least $\frac{1}{3}$ credit and $f(e_t) = \frac{1}{3}$.
- If t is a doubly-attached triangle of type-$[1,1]$, then we use Proposition 3.1 to argue that t is covered. If t contains any base-edge, then since $f(e) \ge \frac{2}{3}$ for any base-edge of type-1 triangles and the other solution edge in t has at least $\frac{1}{3}$ credit, hence t is covered. Otherwise, t does not have any base-edge. Let ψ_1 and ψ_2 be two solution triangles which t is adjacent to. By renaming we assume that $\pi(\psi_1) < \pi(\psi_2)$. Let e_t be the non-solution edge of t. If t falls into Proposition 3.1.2 case, then $f(e_t) = \frac{1}{3}$ which implies every edge of t has at least $\frac{1}{3}$ credit, hence t is covered. If t falls into the pair case in Proposition 3.1.3, since $anchor(\psi_1) = anchor(\psi_2)$, it must be that $f(E(t) \cap E(\psi_2)) = \frac{2}{3}$, hence t is covered.

4 Multi-transversals of order two

Again, this is equivalent to finding the assignment $f : E \to \mathbb{R}^+$ such that $f(e) \in \{\frac{j}{2} : j \in \{0,1,2\}\}$ where $\sum_e f(e) \le 2|\mathcal{V}|$ and for all $t \in \mathcal{T}$, $\sum_{e\in E(t)} f(e) \ge 1$.

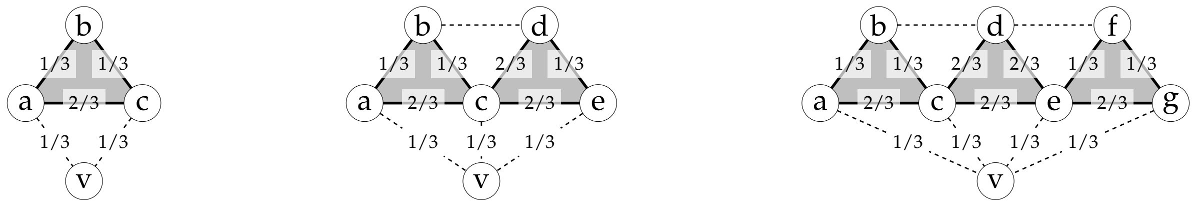
We will show this result in the following three steps.

1. First, we start with an initial charge distribution shown in Section 4.1. Notice that each type-0 triangle has extra half credit remaining, which will be used later.
2. Then in Section 4.2, we will discharge credits from type-1 solution triangles, using what we call the *alternating chains structure*. After this step, all triangles in G that are not adjacent to any type-0 triangle will be covered.
3. Finally in Section 4.3, we show an iterative process to use the extra half credits of type-0 triangles along with *rotating* the configuration of credit distribution for some type-1 and type-3 triangles to cover all the triangles in G.

4.1 Initial Charge Distribution For any solution triangle ψ, the distribution of credits is done as follows.

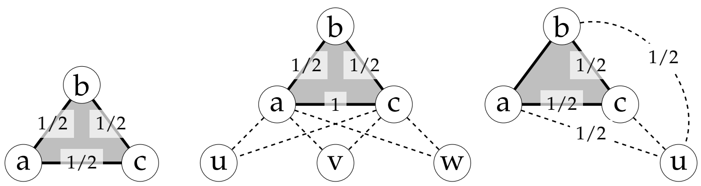
1. If $type(\psi) = 0$, let $f(e) = \frac{1}{2}$ for all $e \in E(\psi)$. Total distribution is 1.5 credits.
2. If $type(\psi) = 1$, let $f(e) = \frac{1}{2}$ for two non-base edges e and we give 1 credit to the base edge. Total distribution is two credits in this case as well.
3. If $type(\psi) = 3$, let H be any C_4 subgraph of the induced subgraph on $V(\psi)$ and the anchoring vertex of $\mathcal{C}L_{sin}(\psi)$. Define $f(e) = \frac{1}{2}$ for every edge in H. (refer to Proposition 2.1 and Figure 8). We refer to the solution edge without any credit as *poor type*-3 edge.

We use the following fact several times in the proof of our result.



(a) Δ_{abc} gets assigned credit first (b) Δ_{cde} gets assigned credit after Δ_{abc} (c) Δ_{cde} gets assigned credit after Δ_{abc} and Δ_{efg}

Figure 7: Credit distributions for triangles in $\mathcal{V}_1^1$ in different scenarios



(a) Δ^0_{abc} (b) Δ^1_{abc} with full-base edge $\overline{ac}$ (c) Δ^3_{abc}

Figure 8: Initial Order-2 Fractional Charging Scheme

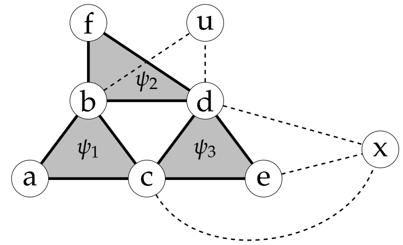


Figure 9: A hollow triangle Δ_{bcd} of type-$[0, 1, 3]$. In this case, $\psi_1 = \Delta^0_{abc}, \psi_2 = \Delta^1_{bdf}, \psi_3 = \Delta^3_{cde}$.

FACT 4.1. (HALF-INTEGRAL K_4 CHARGING) *In K_4, assigning $\frac{1}{2}$ credits to any C_4 subgraph covers every triangle.*

We refer to this charge distribution as *half-integral* K_4 charging.

DEFINITION 4.1. *An edge e is a* null-edge *if $f(e) = 0$.*

DEFINITION 4.2. *An edge e is a* half-edge *if $f(e) = \frac{1}{2}$.*

DEFINITION 4.3. *An edge e is a* full-edge *if $f(e) = 1$.*

For convenience, when we say that an edge e is at least a half-edge, it means that $f(e) \geq \frac{1}{2}$, i.e., e is either a half-edge or a full-edge. Also, we might drop the dash for brevity, e.g., a null type-3 edge e would mean that e is a null-edge and e is a solution edge of a type-3 triangle.

Types of hollow triangles: Following a similar naming convention as for doubly-attached triangles, if t is a hollow triangle adjacent to solution triangles ψ_1, ψ_2 and ψ_3, then $type(t) = [type(\psi_1), type(\psi_2), type(\psi_3)]$ where $type(\psi_1) \leq type(\psi_2) \leq type(\psi_3)$ (see Figure 9).

The following proposition captures some structural properties for hollow triangles.

PROPOSITION 4.1. (STRUCTURES OF HOLLOW TRIANGLES) *Consider any hollow triangle t adjacent to solution triangles ψ_1, ψ_2 and ψ_3.*

1. *$type(t) \neq [3, 3, 3]$.*
2. *If $type(t) = [1, *, *]$[5] where $type(\psi_1) = 1$, then t must be adjacent to the base edge of one of the ψ_1, ψ_2, ψ_3 which is a type-1 triangle or two type-1 triangles from $\{\psi_1, \psi_2, \psi_3\}$ share a sole anchoring vertex.*

4.2 Type-1 and type-3 instances In this section we assume that $\mathcal{V}_0 = \varnothing$ and show a valid charging scheme to cover all triangles G. First we do the initial charge distribution as shown in Section 4.1.

Covered triangles: We first argue about all triangles that are covered by the initial charge distribution. Type-1 solution triangles and their corresponding singly-attached triangles are covered since the base edge is a full-edge. Type-3 solution triangle and all their corresponding singly-attached triangles are covered by Fact 4.1. Type-$[1, 1]$ doubly-attached triangles are covered since each solution edge of type-1 is at least a half-edge. Type-$[1, 3]$ doubly-attached triangles containing base edge of type-1 triangle are covered since any base edge of type-1 triangle is a full-edge. Type-$[1, 1, \star]$ hollow triangles are covered since each solution edge of type-1 is at least a half-edge. Type-$[1, 3, 3]$ hollow triangles are covered since it contains the base edge of type-1 triangle which is a full-edge. Proposition 4.1 implies no other hollow triangles exist.

[5]If there is no restriction on some dimension, then we put a star ($[*]$) there.

Using Proposition 2.2, we can infer that only some type-$[1,3]$ doubly-attached triangles may not be covered. Those triangles must be adjacent to null type-3 edges.

Twin-doubly-attached triangles By Proposition 2.2.2, if any doubly-attached type-$[1,3]$ triangle contains the base-edge of type-1 triangle, it is immediately covered. In the other case, it may not be covered and will look like Figure 11a , containing a null type-3 edge. Notice that there are two doubly-attached triangles $\Delta_{v_0v_1c_1}, \Delta_{v_0c_1c_2}$ in this structure that share the null type-3 edge $\overline{v_0c_1}$. Whenever a type-1 triangle ψ has an anchoring vertex a belonging to some adjacent solution triangle, then the K_4 structure induced by $V(\psi) \cup \{a\}$ contains two adjacent doubly-attached triangles which we call *twin-doubly-attached* triangles.

From now on we develop a discharging strategy to cover twin-doubly-attached type-$[1,3]$ triangles while maintaining the coverage of the rest of the triangles. In fact, we will even guarantee much stronger extra properties which would help in the subsequent section where we deal with the general case.

Lending Structure We define a relation that will be useful in the discharging credits and cover the not-yet-covered twin-doubly-attached triangles.

DEFINITION 4.4. (LEND RELATION AND GAIN FUNCTION) *Let lend* $\subseteq \mathcal{V}_1 \times (\mathcal{V}_1 \cup \mathcal{V}_3)$ *be a relation (possibly* $\varnothing$*) from type-*1 *triangles* $\mathcal{V}_1$ *to a type-*1 *or type-*3 *solution triangle in* $\mathcal{V}_1 \cup \mathcal{V}_3$*. If there exists twin-doubly-attached triangles* t, t' *adjacent to triangles* $\psi^1 \in \mathcal{V}_1$ *and* $\psi \in \mathcal{V}_1 \cup \mathcal{V}_3$ *such that* $|\mathcal{C}L_{sin}(\psi^1)| = 1$ *where the only anchoring vertex is in* $V(\psi)$*; in this case,* $(\psi^1, \psi) \in lend$ *(see Figures 10 and 11a). In addition, we also define a function* $gain\colon lend \to E$*, where* $gain((\psi^1, \psi)) := E(t) \cap E(\psi) = E(t) \cap E(t')$.

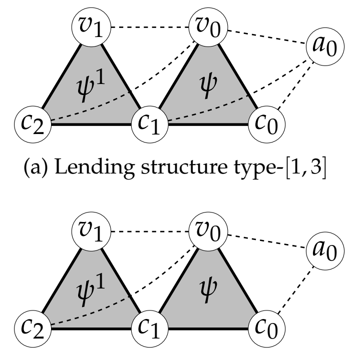
For brevity, we will drop a pair of parenthesis and use the notation $gain(\psi^1, \psi)$ instead of $gain((\psi^1, \psi))$.

The following structural lemma is crucially used to define a valid charging.

LEMMA 4.1. *In the lend relation, any* $\psi^1 \in \mathcal{V}_1$ *is related to at most one* $\psi \in \mathcal{V}$.

Proof. Let v be the vertex not adjacent to the base-edge of ψ^1 and let $a = anchor(\psi^1)$ be the anchoring vertex. For any ψ such that $(\psi^1, \psi) \in lend$, it must be that $\overline{va} \in E(\psi)$. Since any solution edge must be a part of exactly one triangle, this lemma is true. ∎

FACT 4.2. *If* $(\psi_1, \psi_2) \in lend$ *and* t^1, t^2 *are the corresponding twin-doubly-attached triangles and* ω_1 *is the singly-attached triangle of* ψ_1*, then each of* t^1, t^2 *shares a different non-solution edge with* ω_1 *and a different non-base edge of* ψ_1*. Also, both* t^1, t^2 *contain the* $gain(\psi_1, \psi_2)$ *edge. Moreover,* $V(\psi_1) \cup anchor(\psi_1)$ *induce a* K_4 *which contains precisely* $\psi_1, \omega_1, t^1, t^2$ *as its four triangles.*



(a) Lending structure type-$[1,3]$

(b) Lending structure type-$[1,1]$

Figure 10: Lending structure. The twin doubly-attached triangles are $\Delta_{v_0v_1c_1}, \Delta_{v_0c_0c_1}$. The gain-edge of this lend relation is $\overline{v_0c_1}$.

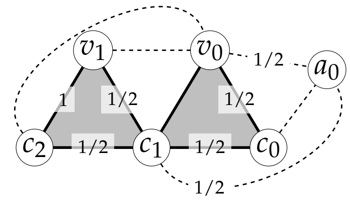
4.2.1 Alternating Chains of Triangles First we start with the high-level overview of our charging strategy and its proof. Our structures may be viewed as a natural extension of the alternating paths structure for matching and the property that for any optimal matching there cannot exist any *augmenting path* in the graph. We will contrast and compare them at different points in the paper. We refer to our alternating structure of triangles as *alternating chain*. We will use the alternating chain structure and the optimality of the solution $\mathcal{V}$ in two ways. The alternating chains will also have some sense of direction and based on that we will refer the first triangle of the chain as *head* and the last triangle as *tail*. First we use the alternating chain of solution and non-solution triangles to *discharge* credits from type-1 triangles participating in this chain from tail to head which will be a type-3 triangle. Secondly, we argue that the tail of any alternating chain cannot be adjacent to some special kind of doubly-attached or hollow triangles, otherwise it would lead to an *augmenting* structure for the solution $\mathcal{V}$ violating its optimality. These special doubly-attached or hollow triangles can be thought of as an unmatched edge between the two odd leveled vertices from two different alternating trees in the alternating forest in the blossom's algorithm. The exclusion of such structures allows us to discharge credits from tail triangle iteratively to the type-3 triangle to cover the type-$[1,3]$ triangles that were not covered while maintaining the coverage for all the other triangles in the graph.

Discharging-via-Chains construction: We start with any pair of twin-doubly-attached type-$[1,3]$ triangles such that the type-3 triangle has one null-solution edge. Let the type-3 triangle be ψ_0 and the type-1 triangle be ψ_1. Note that $(\psi_1, \psi_0) \in$ *lend* (see Figure 11). We cover them by *discharging* $\frac{1}{2}$ credit from the type-1 triangle to the type-3 triangle by making any solution edge and any two non-solution edges in the graph induced by $V(\psi_0) \cup \{anchor(\psi_0)\}$ half-edges. This ensures that $f(gain(\psi_1, \psi_0)) = \frac{1}{2}$. Then we complete the assignment of credits in the graph induced by $V(\psi_1) \cup \{anchor(\psi_1)\}$ such that all edges except a matching of $G[V(\psi_1) \cup \{anchor(\psi_1)\}]$ are half-edges (as shown in Figure 11). Note that in total we still use exactly 4 credits for ψ_0 and ψ_1. We call the base-edge of discharged type-1 triangle as half-base edge and the solution edge without any credit as null non-base edge. After this discharging, it is clear that the solution and the singly-attached triangles remain covered, triangles in $G[V(\psi_1) \cup \{anchor(\psi_1)\}]$ are covered by Fact 4.1. But there could be some doubly-attached or hollow triangles adjacent to ψ_1's null non-base edge which may not be covered. We will later show that all such triangles, except some type-$[1,1]$ triangles, are either already covered or cannot exist by using optimality of our packing solution $\mathcal{V}$. If there exists any type-$[1,1]$ triangles corresponding to twin-doubly-attached triangles having a lend relationship to ψ_1 (See Figure 12), then we let any such type-1 triangle to be ψ_2. Hence, $(\psi_2, \psi_1) \in$ *lend* and $gain(\psi_2, \psi_1)$ is a non-base edge of ψ_1 (which may not be the one which is the null-edge). We again discharge half credit from ψ_2 to the null-non-base edge of ψ_1 by making it a half-edge and then completing the assignment of credits in the graph induced by $V(\psi_2) \cup \{anchor(\psi_2)\}$ such that all edges except a matching of $G[V(\psi_2) \cup \{anchor(\psi_2)\}]$ are half-edges (as shown in Figure 12). For any $i \geq 1$, we continue discharging credits by finding any arbitrary type-1 triangle ψ_{i+1} corresponding to $(\psi_{i+1}, \psi_i) \in$ *lend*, until we reach some type-1 triangle (say ψ_k) which does not have any twin doubly-attached triangles adjacent to it.

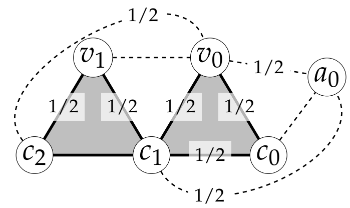
Let $\mathcal{C}_1 = \{\psi_k, \psi_{k-1}, \ldots, \psi_0\}$ be our first finished chain. For any such chain, we define $head(\mathcal{C}_1) := \psi_0$ and $tail(\mathcal{C}_1) := \psi_k$. We will also refer to ψ_0 as the *head* of $\mathcal{C}_1$ and ψ_k as the *tail* of $\mathcal{C}_1$. We define the *size* of such a chain to be k.

OBSERVATION 4.1. *Any size-k chain contains one type-3 triangle as its head and $k \geq 1$ type-1 triangles.*

We repeat this procedure starting with another pair of twin-doubly-attached type-$[1,3]$ triangles which are not covered to form $\mathcal{C}_2$. We keep on repeating this procedure until we cannot find any uncovered type-$[1,3]$ triangle. Let $\chi := \{\mathcal{C}_1, \mathcal{C}_2, \ldots \mathcal{C}_\ell\}$ be the set of all finished chains which we get. Since at any iteration $i \geq 1$, we continue discharging credits every time when there exists some $\psi \in \mathcal{V}_1$ such that $(\psi, \psi_i) \in$ *lend*, we get the following property.



(a) Before discharging



(b) After discharging

Figure 11: type-$[1,3]$ discharging. After discharging, the base-edge $\overline{v_1c_2}$ in this case has $\frac{1}{2}$ credit and one of the non-base edges has zero credit. After this discharging, there is no null-edge in this type-3 triangle.

FACT 4.3. *For any chain $\mathcal{C} = \{\psi_k, \psi_{k-1}, \ldots, \psi_0\} \in \chi$, there is no $\psi \in \mathcal{V}_1$ such that $(\psi, \psi_k) \in$ lend. This implies that any arbitrary K_4 half-integral charging for $V(\psi_k) \cup \{anchor(\psi_k)\}$ such that $f(gain(\psi_k, \psi_{k-1})) = \frac{1}{2}$ for the tail triangle covers all the twin doubly-attached triangles adjacent to it.*

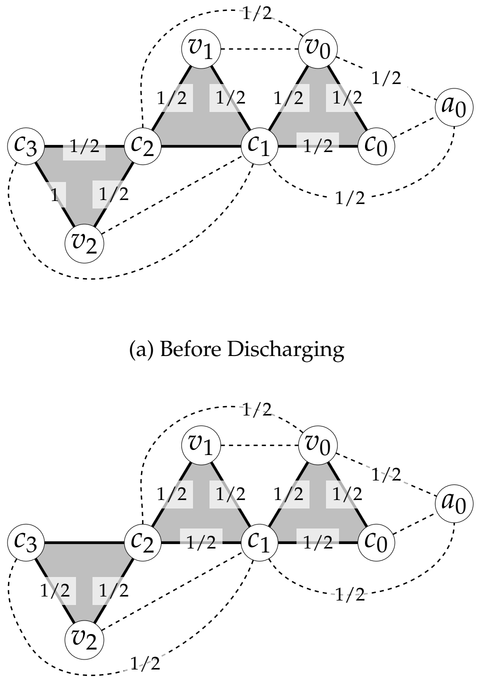
The following claim shows that the Discharging-via-Chains construction is well-defined. Moreover, for any two chains $\mathcal{C}, \mathcal{C}' \in \chi$, $\mathcal{C} \cap \mathcal{C}' = \emptyset$.

CLAIM 4.1. *Any type-1 triangle will be required to discharge credits at most once.*

Proof. The proof directly follows by Lemma 4.1 and the way we construct the chains. ◻

The following observation lists the properties of the charging $\{f(e)\}_{e \in E}$ at end of the Discharging-via-Chains procedure.

OBSERVATION 4.2. *The charging $\{f(e)\}_{e \in E}$ at end of the Discharging-via-Chains procedure has the following properties:*



(a) Before Discharging

(b) After Discharging

Figure 12: type-[1, 1] discharging. After discharging, the base-edge $\overline{v_2c_3}$ in this case has $\frac{1}{2}$ credit and one of the non-base edges has zero credits.

1. *For any type-3 triangle $\psi \in \mathcal{V}_3$ such that $head(\mathcal{C}) = \psi$ for some $\mathcal{C} \in \chi$ and , $\forall e \in E(\psi)$, $f(e) = \frac{1}{2}$.*

2. *For any type-1 triangle $\psi \in \mathcal{V}_1$ such that $\psi \in \mathcal{C}$ and $tail(\mathcal{C}) \neq \psi$ for some $\mathcal{C} \in \chi$, $\forall e \in E(\psi)$, $f(e) = \frac{1}{2}$.*

3. *For any type-1 triangle $\psi \in \mathcal{V}_1$ such that $tail(\mathcal{C}) = \psi$ for some $\mathcal{C} \in \chi$, the base-edge and one non-base edge have $\frac{1}{2}$ credits and there is one null-non-base edge with no credit.*

4. *Except the null type-3 edges and null-non-base type-1 edge of any type-1 triangle which is a tail, for any solution edge e, e is at least a half-edge, i.e., $f(e) \geq \frac{1}{2}$.*

The following observation pin points the edges for which there is some *credit loss*.

OBSERVATION 4.3. (CREDIT LOSS) *There are two types of edges for which the credits reduce after the Discharging-via-Chains construction. Non-base edge of every tail type-1 triangle is reduced from a half-edge to a null-edge. Base-edge of every type-1 triangle participating in any chain is reduced from a full-edge to a half-edge.*

Covered Triangles: After the Discharging-via-Chains construction, the covered triangles are as follows. Any type-1 triangle and its singly-attached triangles are covered either by the half-integral K_4 charging (Fact 4.1), used during the Discharging-via-Chains construction or the initial charge distribution. Type-3 triangles and their singly-attached triangles are covered as they did not lose any credit (Observation 4.3). Type-[1, 1] twin-doubly-attached triangles are covered. The coverage of intermediate twin-doubly-attached triangles of any chain follows from the fact that both the solution edges are half-edges. Twin-doubly-attached triangles not involved in any chain remain covered. By Fact 4.3, twin-doubly-attached triangles adjacent to any tail triangle are covered. Type-[1, 3] twin-doubly-attached triangles are covered since the procedure terminated. Type-[1, 1, 1] hollow triangles with at most one null-edge or two null-edges and one full-base edge. Type-[1, 1, 3] hollow triangles with no null-non-base edges for type-1 or one null-edge and one full-base edge of type-1. Type-[1, 3, 3] hollow triangle containing the full-base edge of type-1.

Potentially uncovered triangles: Now we enumerate the triangles which may not remain covered because of the reduction in credits for these edges. To do so we use Observation 3.1 and Propositions 2.2, 3.1 and 4.1. Type-[1, 1] doubly-attached triangles containing at least one null-non-base edge of some type-1 triangle which is a tail of some chain. The second solution edge can either be a half-base edge or some non-base edge. Type-[1, 3] doubly-attached triangles containing half-base edge of some type-1. Type-[1, 1, 1] hollow triangles with at least two null-non-base edges of two type-1 triangles which are tails of two chains. Type-[1, 1, 3] hollow triangle containing at least one null-non-base edge of some type-1 which is a tail of some chain. Another type-1 solution edge can either be a half-base edge or a non-base edge. Type-[1, 3, 3] hollow triangle containing half-base edge of type-1 triangle and null-edges of the other two type-3 triangles.

We will prove the following lemma which implies that at the end of Discharging-via-Chains procedure we get a valid assignment.

LEMMA 4.2. *For any optimal solution $\mathcal{V}$ of our graph G, such that $\mathcal{V}_0 = \varnothing$, starting with the initial charging distribution and then applying the Discharging-via-Chains construction, any potentially uncovered triangle cannot exist in G. In other words, the construction covers every triangle in G.*

Proof-of-Concept: We look at the solution edges adjacent to some potentially uncovered triangle t. The main idea is to show the existence of an improving swap involving t to derive a better packing solution, thus contradicting the fact that $\mathcal{V}$ is optimal. Notice that one of these solution edges is either a null-non-base edge g_{k+1} of the type-1 tail triangle of some k-sized chain $\mathcal{C}$ ($k > 1$) or a half-base edge of any type-1 triangle participating in some chain. We use the structure of chains that allows for several alternate solutions which *frees-up* the g_{k+1} edge and/or the half-base edge without making the triangle packing solution $\mathcal{V}$ worse. We observe that there exists simple alternate solutions which frees-up the other solution edges of t, while maintaining an optimal solution. If we flip the solution triangles with these alternate solutions, then we can free-up all the edges of t and then include it to the new solution thereby getting a solution of higher cardinality. This is not hard to see if all the non-solution edges which we deal with are different. Things become complex when these structures start sharing vertices leading to shared non-solution edges. To deal with such situations, we crucially exploit the rich structure of our alternating chains which allows for various choices of alternate solutions. In some sub-cases, using the charging scheme for chains, we can deduce that t is covered, leading to a different contradiction.

Moreover we get the following observations for this charging which will be crucially used in the next section to prove the general case. They follow from the fact that we arbitrary fix any K_4 half-integral charging for every tail triangle and for every type-3 triangle which is not part of any chain.

OBSERVATION 4.4. *Lemma 4.2 remains true for any arbitrary but fixed K_4 half-integral charging for any tail triangle ψ_k of any k-sized chain $\mathcal{C} \in \chi$ such that $f(base(\psi_k)) = f(gain(\psi_k, \psi_{k-1})) = \frac{1}{2}$.*

OBSERVATION 4.5. *Lemma 4.2 remains true for any arbitrary but fixed K_4 half-integral charging for any type-3 triangle which is not part of any chain.*

4.3 Covering everything using the remaining half credits and rotations For any fixed optimal solution $\mathcal{V}$ for our graph G, we first tentatively do the initial charge distribution for all triangles as shown in Section 4.1 and then we run the Discharging-via-Chains construction described in Section 4.2.

Covered triangles: Lemma 4.2 implies that all triangles not adjacent to any type-0 triangles are covered. Using the fact that all type-0 solution edges are half-edges implies that type-0 solution triangles, all the doubly-attached triangles of type-$[0, 0]$ and hollow triangles of type-$[0, 0, *]$ are also covered. By Propositions 2.2 and 4.1, the rest of the possibly uncovered triangles in G are type-$[0, 3]$, type-$[0, 1]$, type-$[0, 1, 1]$, type-$[0, 1, 3]$ and type-$[0, 3, 3]$. Out of these triangles, the triangles adjacent to any solution edge of non-tail type-1 triangles, any solution edge head type-3 triangles or base edge of tail type-1 triangles are surely covered because they get at least half credit from these type-1 or type-3 solution edges and half credits from the type-0 edge.

In this section, we show a strategy to cover the *rest* of these triangles while maintaining the coverage of the previously argued triangles. Let $\mathcal{A}$ be the set of *free* solution triangles consisting of (1) all type-0 triangles, (2) tail type-1 triangles, (3) type-3 triangles which are not part of any chain. Let $\mathcal{A}_i = \mathcal{V}_i \cap \mathcal{A}$ be the set of type-i triangles in $\mathcal{A}$. Also, let $\mathcal{A}_{13} := \mathcal{A}_1 \cup \mathcal{A}_3$.

Demanding triangles: Let $\mathcal{D}$ be the set of all the non-solution triangles such that each triangle contains exactly one type-0 solution edge, does not contain any solution edge outside $E(\mathcal{A})$, and may only contain non-base edges of type-1 triangles in $\mathcal{A}_1$. Note that, these are precisely the triangles of type-$[0, 3]$, type-$[0, 1]$, type-$[0, 1, 1]$, type-$[0, 1, 3]$ and type-$[0, 3, 3]$ minus the ones which we argue to be covered. We refer to any triangle in $\mathcal{D}$ as a *demanding triangle*.

We will show that apart from a few very special cases, any type-0 triangle can share at most one edge e with any triangle in $\mathcal{D}$.

For any solution triangle ψ, let $\mathcal{D}(\psi) \subseteq \mathcal{D}$ be the set of all the triangles in $\mathcal{D}$, adjacent to ψ.

Due to space constrains, we only sketch the proof idea of the following technical Lemma 4.3; the proof will be included in the full version. The proof technique used is very similar to the proof of Lemma 4.2. We prove it by contradiction by showing that for any type-0 triangle $\psi \in \mathcal{V}_0$, if any two triangles $t_1, t_2 \in \mathcal{D}(\psi)$ are edge disjoint, then we can find an improving swap by finding alternate solutions using the triangle(s) adjacent to t_1 (but not to ψ) and the triangle(s) adjacent to t_2 (but not to ψ) and finally swapping in both t_1, t_2 instead of ψ.

LEMMA 4.3. *For any type-0 triangle $\psi \in \mathcal{V}_0$, for any $t, t' \in \mathcal{D}(\psi)$, $|E(t') \cap E(t)| = 1$.*

Now we prove a generic claim which will be handy to prove the next lemma.

CLAIM 4.2. *Given a set $\mathcal{S}$ of triangles with $|\mathcal{S}| > 1$, if it holds that for any $t, t' \in \mathcal{S}$, $|E(t') \cap E(t)| = 1$ then one of the two cases is true. Either $| \cap_{t \in \mathcal{S}} E(t)| = 1$ or $|\mathcal{S}| \leq 4$, such that $\cup_{t \in \mathcal{S}} V(t)$ induce a K_4.*

Proof. Let us assume that $| \cap_{t \in \mathcal{S}} E(t)| \neq 1$, i.e., $| \cap_{t \in \mathcal{S}}$

$E(t)| = 0$. It implies that there exists some triangles $t_1, t_2, t_3 \in \mathcal{S}$, such that $E(t_1) \cap E(t_2) \neq E(t_1) \cap E(t_3)$. Let $E(t_1) \cap E(t_2) = e_{12}$, $E(t_1) \cap E(t_3) = e_{13}$ and $E(t_2) \cap E(t_3) = e_{23}$ Since e_{12}, e_{23} belongs to t_2 and e_{13}, e_{23} belongs to t_3, it implies that e_{23} is incident to the common vertex between e_{12}, e_{13}, say u. Also, let the other end points of e_{23}, e_{13}, e_{12} be v_1, v_2, v_3 respectively. Clearly $V(t_1) = \{u, v_2, v_3\}$, hence $V(t_1) \cup \{v_1\}$ induces a K_4. This implies that there is another triangle $t_4 = \Delta_{v_1 v_2 v_3}$ which shares edge $\overline{v_2 v_3}, \overline{v_1 v_3}, \overline{v_1 v_2}$ with t_1, t_2, t_3 respectively. Hence, any three triangles pair-wise sharing different edges, share a common vertex and induces a K_4. Moreover, t_4 is a valid triangle to be in $\mathcal{S}$. If there exists triangle $t \in \mathcal{S} \setminus \{t_1, t_2, t_3, t_4\}$, then by the pairwise-intersection property, it has to share one edge each with all the three triangles t_1, t_2, t_3. Since there is no common edge between all three triangles, it should contain at least two edges (say f_1, f_2) of the K_4 graph induced by $V(t_1) \cup \{v_1\}$. Now f_1, f_2 cannot be adjacent otherwise t will be one of t_1, t_2, t_3, t_4 otherwise they cannot be part of same triangle t. Hence, no such t exists in this case, which implies our claim. ◻

Using Lemma 4.3 and the claim above we prove the following structural lemma which is the key for defining our final discharging process.

LEMMA 4.4. *For any type-*0 *triangle* $\psi \in \mathcal{V}_0$ *and the corresponding set* $\mathcal{D}(\psi)$ *only one of the following three cases are possible:*

1. *Either there exists a common e, such that* $\{e\} = \bigcap_{t \in \mathcal{D}(\psi)} E(t) \subseteq E(\psi)$ *(see Figure 13a),*
2. *or there are precisely two doubly-attached triangles* $t_1, t_2 \in \mathcal{D}(\psi)$ *and one hollow triangle* $t_3 \in \mathcal{D}(\psi)$ *adjacent to each solution edge of* ψ *respectively (see Figure 13b),*
3. *or there are precisely three hollow triangles* $t_1, t_2, t_3 \in \mathcal{D}(\psi)$ *adjacent to each solution edge of* ψ *respectively (see Figure 13c).*

Proof. Lemma 4.3 implies that for any $t, t' \in \mathcal{D}(\psi) \cup \{\psi\}$, $|E(t) \cap E(t')| = 1$. Let us assume that Item 1 is not true. This implies that there exists at least two triangles $\{t_1, t_2\} \in \mathcal{D}(\psi)$ such that $E(t_1) \cap E(\psi) \neq E(t_2) \cap E(\psi)$. Using Claim 4.2, we deduce that the graph induced by $V(\psi) \cup V(\mathcal{D}(\psi))$ is a K_4. We need to prove that the fourth triangle in the K_4 structure also belongs to $\mathcal{D}(\psi)$ and that the type of triangles which we get here falls in Item 2 or 3. Let t_3 be the fourth triangle in the induced K_4 graph. Let the vertices of ψ be $\{v_1, v_2, v_3\}$ such that it shares $\overline{v_2 v_3}, \overline{v_1 v_3}$ with triangles t_1 and t_2 respectively. Let the fourth vertex in $V(\psi) \cup V(\mathcal{D}(\psi))$ be u. This implies $V(t_1) = \{u, v_2, v_3\}$, $V(t_2) = \{u, v_1, v_3\}$ and $V(t_3) = \{u, v_1, v_2\}$. Clearly, ψ share edge $\overline{v_1 v_2}$ with t_3. Now since $t_1, t_2 \in \mathcal{D}(\psi)$, this implies that the edges $\{\overline{uv_1}, \overline{uv_2}, \overline{uv_3}\}$ are either non-solution edges or solution edges adjacent to some triangles in $\mathcal{A}_{13}$. Hence $t_3 \in \mathcal{D}(\psi)$.

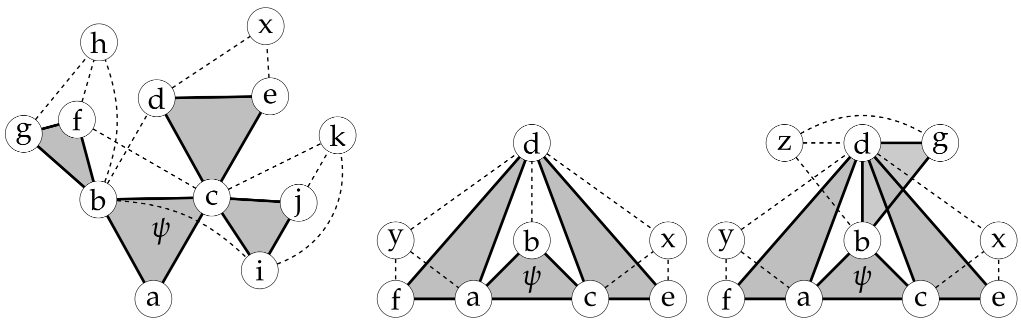
Note that at least two edges out of $\{\overline{uv_1}, \overline{uv_2}, \overline{uv_3}\}$ should be solution edges as ψ is a type-0 triangle. This implies that at least one of t_1, t_2, t_3 is a hollow triangle. If the third edge is a non-solution edge, then the other two triangles will be doubly-attached triangles else all three will be hollow triangles which proves the two cases. ◻

Discharging overview: Using Lemma 4.4, we describe our discharging scheme by giving some high-level intuition before the formal description.

This discharging scheme aims at covering the set of demanding triangles $\mathcal{D}$, while maintaining the coverage of the triangles outside $\mathcal{D}$. Note that even if some of these triangles may be initially covered, we still put them in $\mathcal{D}$ and *declare* them to be *covered* only during our process to make sure that we do not uncover any of them along the way. We also make sure that the process does not uncover any of the triangles outside of $\mathcal{D}$.

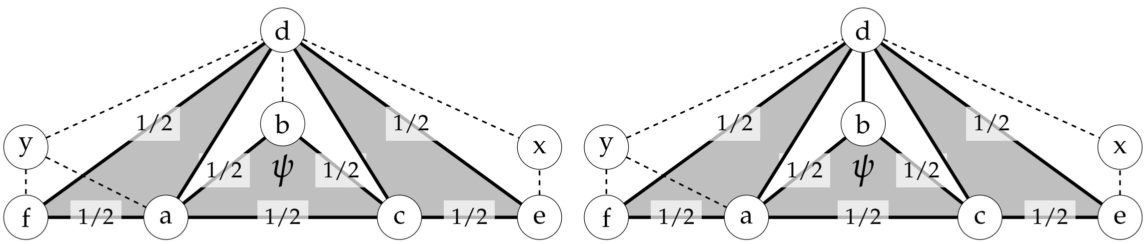
Remember that after the initial charge distribution, each type-0 triangle has an unused half credit. The first natural attempt is to allocate these half credits in a clever way to cover all triangles in $\mathcal{D}$. Clearly it does not affect the coverage of any triangle outside of $\mathcal{D}$, but unfortunately only using this allocation does not work. If a type-0 triangle is in the second or the third sub-case of Lemma 4.4, then it is actually impossible to put this half credit anywhere to cover all t_1, t_2, t_3 at the same time (see Figure 14). Then arises the need to exploit credits from type-3 and type-1 triangles in $\mathcal{A}_{13}$ to cover these demanding triangles. For any type-1 triangle $\psi^1 \in \mathcal{A}_1$ we use the property in Observation 4.4 that allows us to freely *rotate* the allocated credits by picking any one of the non-base edge to be the null-edge. Hence, if for ψ^1 there is a non-base solution edge e^1 that is not adjacent to any demanding triangle in $\mathcal{D}$, then we can choose the charging that makes e^1 a null-edge without affecting the coverage of triangles outside of $\mathcal{D}$.

Similarly, for any type-3 triangle $\psi^3 \in \mathcal{A}_3$ (by Observation 4.5), we are free to *rotate* the allocated credits by specifying the one null-solution edge. Hence, if there is a solution edge e^3 of ψ^3 that is not adjacent to any demanding triangle in $\mathcal{D}$, then we can make e^3 a null-edge without affecting the coverage of triangles outside of $\mathcal{D}$.



(a) $\overline{bc}$ is the common edge (b) Two doubly-attached case (c) Three hollow case

Figure 13: Demanding structure.



(a) Two doubly-attached case (b) Three hollow case

Figure 14: Discharging $\frac{1}{2}$ credit from ψ to cover all the three triangles $\Delta_{abd}, \Delta_{cbd}, \Delta_{acd}$ is not possible.

But we cannot use these operation for the $\psi \in \mathcal{A}_1$ or $\psi' \in \mathcal{A}_3$ triangles which have demanding triangles sharing both the non-base or all three solution edges respectively. Again, in such a situation we can try to use the extra half credit from a type-0 to cover some of these triangles.

Note that the two rotation operations have a very similar structure. Also in the $\mathcal{D}$ these type-1 and type-3 triangles in $\mathcal{A}_1$ and $\mathcal{A}_3$ are structurally very similar. The only difference is that, for type-3 we have three possibilities for the null-edge whereas for the tail type-1 there are only two (non-base edges) possibility for the null-edge. But also the triangles in $\mathcal{D}$ are not adjacent to the base-edges of triangles in $\mathcal{A}_1$, hence we can perform the rotation operations analogously everywhere.

At this point, the main difficulty lies in taking advantage of the unused half credit from type-0 triangles in $\mathcal{A}_0$ and the rotation operations for triangles in $\mathcal{A}_{13}$ in a coherent and synchronized way. Fortunately, we can show that a very simple and natural greedy procedure can cover all the demanding triangles without affecting any triangle outside of $\mathcal{D}$. We call it *Discharge-and-Pin* algorithm.

Our procedure works in iterations as follows. We maintain a dynamic set (initially $D = \mathcal{D}$) which contains our potentially uncovered demanding triangles and remove from D the triangles which we *declare* as *covered*. We also keep track of triangles in $\mathcal{A}$ for which we have not discharged or rotated the credits yet, so that we perform these operations at most once for any triangle in $\mathcal{A}$. We repeat until $D = \varnothing$ and declare that all triangles in G are covered.

First, for any type-0 triangle $\psi^0 \in \mathcal{A}_0$, if any triangle in D is adjacent to at most one solution edge $e \in \psi^0$ (similar to Lemma 4.4 Item 1), then we assign the extra half credit of ψ^0 to e making it a full-edge. Then we remove from D all triangles adjacent to e and go to the next iteration.

Otherwise, it would be true that the demanding triangles adjacent to the type-0 triangle ψ^0 must lie in Lemma 4.4 Item 2 or 3.

Then we start an iteration by fixing a rotation of any arbitrary triangle $\psi \in \mathcal{A}_{13}$, which covers all the demanding triangles except the ones adjacent to the only null-edge $e \in E(\psi)$. We first try to find an edge of ψ (excluding the base-edge in case when $\psi \in \mathcal{A}_1$) that is not adjacent to any triangle in D. If we can find such an edge, then we fix e to be this edge, remove covered triangles from D and move on to the next iteration.

Else, we fix an edge e arbitrarily and rotate the credits of ψ so that e becomes a null-edge. We look at any demanding triangle t adjacent to the null-edge e of ψ and *discharge* half credit from the type-0 triangle (say ψ^0), adjacent to t, to e by making it a half-edge. Remove all the triangles adjacent to ψ from D, as they are covered. Now for the type-0 triangle ψ^0, once it spends the extra half credit, two out of three demanding triangles adjacent to it are covered. For the third demanding triangle t', we pick any triangle $\psi' \in \mathcal{A}_{13}$ which is adjacent to t' via edge e'. To make sure that t' gets covered, we fix any one of the rotations for ψ' in which e' will be a half-edge. Then we continue with the iteration as we did for ψ, via the null-edge e'' of ψ', covering and removing triangles from D using discharge and rotate operations, until we arrive at a case where we reach a null-edge which is not adjacent to any triangle in D. Then we start with a new iteration. We show that we can repeat this natural depth-first-search like process until every triangle is covered without ever getting stuck. We already know that any of these operations cannot affect the coverage of any triangle outside of $\mathcal{D}$. We can show that any step of the procedure does not affect the coverage of any triangle in $\mathcal{D} \setminus D$ and we discharge or rotate any triangle in $\mathcal{A}$ at most once. Also, note that we only move around initially assigned credits and unused credits of type-0 triangles, hence we do not overcharge. All these properties together imply that the procedure is well-defined and gives a valid charging.

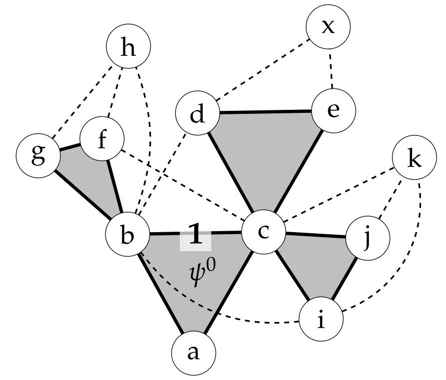
Now we give the formal description of our algorithm.

Discharge-and-Pin Operations: We define three operations, one for type-0 triangles, one for type-1 triangles in $\mathcal{A}_1$ and one for type-3 triangles in $\mathcal{A}_3$.

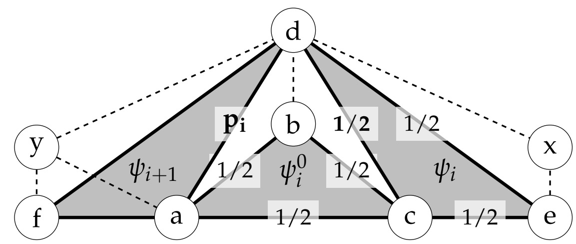
1. For any type-0 triangle ψ^0 with extra half credit, we want to utilize this half credit to cover some demanding triangles closed to ψ^0 in $\mathcal{D}$, the set of demanding triangle adjacent to ψ^0. Let $discharge(\psi^0, e)$ be the function that, when called, discharge the extra half credit of ψ^0 to the edge e (not necessarily in $E(\psi^0)$, see Figure 15).

2. For type-1 triangle $\psi^1 \in \mathcal{A}_1$, we want to rotate the credit of ψ^1 to cover some triangles in $\mathcal{D}(\psi^1)$. For a non-base edge $e \in E(\psi^1) \setminus base(\psi^1)$, let $pin(\psi^1, e)$ be the function that, when called, rotate the credit of ψ^1 in a way that e becomes a null-edge, (see Figure 16). To be more specific, let $v = V(\psi^1) \setminus V(e)$ and let $a = anchor(\psi^1)$, we put half credit on four edges in of K_4 induced by $V(\psi^1) \cup \{a\}$ except e and $\overline{va}$.

3. For type-3 triangle $\psi^3 \in \mathcal{A}_3$, we want to rotate the credit of ψ^3 to cover some triangles in $\mathcal{D}(\psi^3)$. For an edge $e \in E(\psi^3)$, let $pin(\psi^3, e)$ be the function that, when called, rotate the credit of ψ^3 in a way that e becomes a null-edge. To be more specific, let $v = V(\psi^3) \setminus V(e)$ and let $a = anchor(\psi^3)$, we put half credit on four edges in of K_4 induced by $V(\psi^3) \cup \{a\}$ except e and $\overline{va}$.

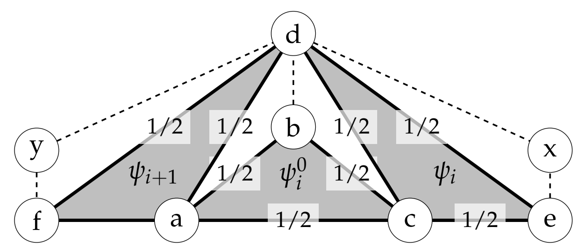


(a) $discharge(\psi^0, \overline{bc})$ to the demanding edge $\overline{bc}$

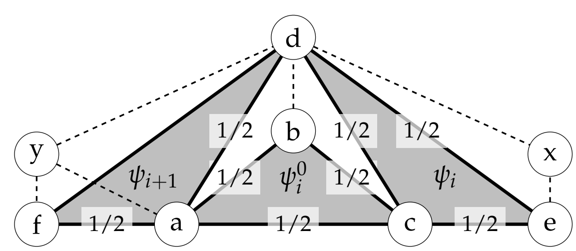


(b) $discharge(\psi_i^0, \overline{cd})$ and fixing $p_i = \overline{ad}$ for ψ_{i+1}

Figure 15: Discharge operations for a type-0 triangle.



(a) $pin(\psi_{i+1}, \overline{af})$



(b) $pin(\psi_{i+1}, \overline{df})$

Figure 16: Pin operation for type-1 and type-3 triangles.

By Observations 4.4 and 4.5 the pin operation does not uncover anything. Combining this with the fact that the discharge operation for type-0 does not reduce credits on any edge, we get the following lemma.

LEMMA 4.5. *Let σ be any sequence of Discharge-and-Pin operations called on each triangle in $\mathcal{A}$ at most once. Let t be a triangle not in $\mathcal{D}$, then t is covered after the initial charge distribution and is still covered after applying σ.*

Now we prove the following lemma by showing an iterative algorithm to find the sequence σ and cover the triangles in $\mathcal{D}$ along the way, such that σ performs discharge or pin operation on any triangle in $\mathcal{A}$ at most once.

LEMMA 4.6. *There exists a sequence of Discharge-and-Pin operations σ which performs these operations on each triangle in $\mathcal{A}$ at most once, such that applying σ covers all triangles in $\mathcal{D}$.*

Discharge-and-Pin Algorithm: This algorithm will find the sequence σ which satisfies Lemma 4.6.

Let $D = \mathcal{D}$ be the current demanding triangles that we need to cover. For brevity, we define $D(e) = \{t \in D : e \in E(t)\}$ to be the set of triangles in D containing the edge e. Also, for any triangle t, let $D(t) = \bigcup_{e \in E(t)} D(e)$.

Recall the set of triangles $\mathcal{A}$ consisting of (1) all type-0 triangles, (2) tail type-1 triangles, (3) type-3 triangles which are not part of any chain. Let $A = \mathcal{A}$ be the set of triangles for which we have not performed the pin or discharge operation yet. At every step, we will find a triangle $\psi \in A$ to perform discharge or pin operation (based on the type of triangles) to cover some triangles in D. Then we remove ψ from A and the covered triangles from D. We can show that until D is empty, we can always find some $\psi \in A$ to perform discharge or pin operation to cover some triangles in D. This together with the fact that we just pin or discharge any triangle in $\mathcal{A}$ at most once, implies that we cover all the triangles in D. In the description of the algorithm we assume that we can always find some triangle in A at every step, which we prove later. Each while loop corresponds to an iteration which performs a sequence of discharge and pin operations in a greedy depth-first way starting from some triangle in A. At any point in the algorithm, we use A_i to refer to $A \cap \mathcal{A}_i$ and A_{13} to refer to $A \cap \mathcal{A}_{13}$.

while $D \neq \varnothing$, we will do the following.

1. If for any type-0 triangle $\psi^0 \in A_0$, there exists exactly one edge $e \in E(\psi^0)$ such that $D(e) \neq \varnothing$, then we can apply $discharge(\psi^0, e)$. Once the operation is applied on ψ^0, we remove ψ^0 from A and remove $D(e)$ from D and repeat.
2. Else we pick an arbitrary triangle $\psi_1 \in A_{13}$ to be the starting point of the current iteration. Then we greedily alternate between triangles in A_0 and A_{13} while fixing their distribution. ψ_i would be the i^{th} inspected triangle in A_{13}. Similarly, ψ_i^0 would be the i^{th} inspected triangle in A_0. We inspect ψ_i^0 right after ψ_i. To make sure that we cover all the triangles, we keep track of some *critical* edge $p_i \in \psi_{i+1}$ which needs to be a half-edge, based on the charging of previously fixed triangles ψ_i^0 and ψ_i.

   To begin with the current iteration, let $p_0 = \varnothing$. For any $i > 0$, based on the information in p_{i-1} we will decide to pin one edge of ψ_i.

   (a) If $\psi_i \in A_3$ and there exists an edge $e_i \neq p_{i-1}$, such that $D(e_i) = \varnothing$, then we can pin this edge and end this iteration. Similarly, If $\psi_i \in A_1$ and there exists a non-base edge $e_i \neq p_{i-1}$, such that $D(e_i) = \varnothing$, then we can pin this edge and end this iteration. That is, we apply $pin(\psi_i, e_i)$, set $D \leftarrow D \setminus D(\psi_i)$ and remove ψ_i from A and set $A \leftarrow A - \psi_i$ and start a new iteration.

   Observe that any demanding triangle in $D(\psi_i)$ is now covered since they are adjacent to a half-solution edge of ψ_i ($D(e_i) = \varnothing$) and a half-solution edge of a type-0 triangle.

   (b) Otherwise in case $\psi_i \in A_3$, for each $e \in E(\psi_i)$, $D(e) \neq \varnothing$. We pick e_i^3 to be any one of at least two edges in $E(\psi_i) \setminus p_{i-1}$. Similarly, in case $\psi_i \in A_1$, for each $e \in E(\psi_i) \setminus base(\psi_i)$, $D(e) \neq \varnothing$. We pick e_i to be the non-base edge in $E(\psi_i) \setminus \{p_{i-1}, base(\psi_i)\}$. We apply $pin(\psi_i, e_i)$, set $D \leftarrow D \setminus (D(\psi_i) \setminus D(e_i))$ and remove ψ_i from A and set $A \leftarrow A - \psi_i$.

   In this case, since $D(e_i) \neq \varnothing$ and e_i is a null-edge, hence we continue with the current iteration, as there will be some uncovered demanding triangles adjacent to e_i. Pick t_i arbitrarily from $D(e_i)$. Let e_i^0 be the solution edge of a type-0 triangle in $E(t_i)$. Let ψ_i^0 be the type-0 triangle containing e_i^0. Here we assume that $\psi_i^0 \in A$. Let e'^0_i be the edge of ψ_i^0 not adjacent to $V(\psi_i) \cap V(\psi_i^0)$. Once we have $\psi_i^0, e_i, e_i^0, e'^0_i$, we will discharge half credit of ψ_i^0. We call $discharge(\psi_i^0, e_i)$. Now we cover all triangles in $D(e_i)$ since they will be adjacent to one half-solution edge of a type-0 triangle and e_i which is now a half-edge. In particular, we cover the two demanding triangles adjacent to $E(\psi_i^0) \setminus e'^0_i$.

Hence, we can remove $D(e_i)$ from D and ψ_i^0 from A.

i. If $D(e'^0_i) = \varnothing$, then we are done with this iteration. This could happen when the triangle $D(e'^0_i)$ got covered when one of the type-3 triangle(s) adjacent to this non-solution triangle got pinned previously.

ii. Else, we let $t'_i = D(e'^0_i)$. Note that there is exactly one such triangle, since for any type-0 belonging to Lemma 4.4.1, the algorithm would finish discharging them in the initial iterations. We let $p_i \in E(t'_i)$ be a solution edge of some $\psi_{i+1} \in A_{13}$ (there can be one or two such triangles, breaking ties arbitrarily). Also here we assume that there is such $\psi_{i+1} \in A$. Continue with the current iteration using p_i and ψ_{i+1}. Note that, fixing this edge as p_i ensures that we cover the last triangle $t'_i \in D$ adjacent to ψ_i^0.

4.3.1 Proof of Lemma 4.6

By the way we define the algorithm, it is clear that we discharge or pin any triangle in $\mathcal{A}$ at most once. Now we show a series of claims to prove this lemma.

First we show that at every step when we remove any triangle in D, they are covered. For triangles $D(e)$ falling in Lemma 4.4.1 for some $\psi \in \mathcal{A}_0$ and a common demanding edge $e \in \psi$, we call *discharge*(ψ, e) which makes e a full edge, hence all triangles in $D(e)$ are covered.

For the other steps we prove the following claims.

CLAIM 4.3. *In any iteration, for any $i > 0$, $D(\psi_i) = \varnothing$ after calling discharge(ψ_i^0, e_i) and removing $D(e_i)$ from D.*

Proof. We did call *pin*(ψ_i, e_i). Hence, any triangle in $D(\psi_i) \setminus D(e_i))$ is covered and removed from D. When we call *discharge*(ψ_i^0, e_i), we cover $D(e_i)$ and remove them. ∎

CLAIM 4.4. *In any iteration, for any $i > 1$, $D(\psi_{i-1}^0) = \varnothing$ after calling pin(ψ_i, e_i) and setting $D \leftarrow D \setminus (D(\psi_i) \setminus D(e_i))$.*

Proof. Since we have already covered and removed triangles from D falling in Lemma 4.4.1, it implies that, $|D(e)| = 1$ for all $e \in E(\psi_{i-1}^0)$. When we call *discharge*(ψ_{i-1}^0, e_{i-1}), two triangles in $D(\psi_{i-1}^0) \setminus D(e'^0_i)$ are covered and removed from D. Calling *pin*(ψ_i, e_i) cover the triangle in $D(e'^0_i)$. ∎

Now we prove the two assumptions we made in the algorithm.

CLAIM 4.5. *In any iteration, for any $i > 0$, we can find some $\psi_{i+1} \in A_{13}$, such that $p_i \in \psi_{i+1}$.*

Proof. This is true as $D(\psi_{i+1}) \neq \varnothing$ when we choose ψ_{i+1} but $D(\psi) = \varnothing$ for any $\psi \in \mathcal{A}_{13} \setminus A_{13}$, because we remove them from D when we remove ψ from A. ∎

CLAIM 4.6. *In any iteration, for any $i > 0$, we can find $\psi_i^0 \in A_0$ which contains e_i^0.*

Proof. This is true as $D(\psi_i^0) \neq \varnothing$ when we choose ψ_i^0 but $D(\psi) = \varnothing$ for any $\psi \in \mathcal{A}_0 \setminus A_0$, because we remove them from D when we remove ψ from A. ∎

Now we show that triangles in $\mathcal{D} \setminus D$ remain covered which implies that when $D = \varnothing$, everything is covered.

CLAIM 4.7. *After any discharge or pin operation, every triangle in $\mathcal{D} \setminus D$ is covered.*

Proof. By the above reasoning, it is clear that whenever we remove any triangle from D, it is covered at that point in time. Also, discharge operation cannot uncover any triangle as it does not reduce credits on any edge. What remains to argue is that every triangle $\mathcal{D} \setminus D$ *stay covered* after any pin operation. Recall that we start with $D = \mathcal{D}$ to be all the triangles which may not be covered, which implies that they are oblivious to the initial tentative charging of any triangle in $\mathcal{A}_{13}$. Now by the way we proceed, we only rely on the half credits of type-0 triangles edges, the half credits of the edges of the triangles which are pinned or the credits of the edges to which a type-0 triangle discharge credits to cover and remove triangles from D. Since we only discharge or pin at most once, the algorithm never removes credits from these edges hence triangles in $\mathcal{D} \setminus D$ stay covered during the execution of the algorithm. ∎

The following claim finishes the proof by implying that the Discharge-and-Pin algorithm does not terminate until $D = \varnothing$.

CLAIM 4.8. *As long as $D \neq \varnothing$, we can always find a triangle $\psi \in A$ as the starting point of the next iteration.*

Proof. First note that for any triangle in $t \in D$ which is adjacent to any type-0 triangle ψ, such that it lies in Lemma 4.4.1, the algorithm will always find $\psi \in A$, discharge it, cover t and remove it from D.

Now any remaining triangle $t \in D$, is adjacent to some triangle $\psi \in A_{13}$, which implies $D(\psi) \neq \varnothing$. But this implies that $\psi \in A$ because for any triangle $\psi' \notin A$, $D(\psi') = \varnothing$ by Claims 4.3 and 4.4. $\square$

Lemmas 4.5 and 4.6 together imply that we do not overcharge any triangle and we cover every triangle in G which finishes the proof for the main result of this section.

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A Relations between multi-transversals of various orders

In this section, we explain why $\tau_2^*(G)$ or $\tau_3^*(G)$ are not subsumed by each other.

CLAIM A.1. *There is an infinite family of graphs G for which $\tau_2^*(G) \leq (\frac{3}{4} + o(1))\tau_3^*(G)$*

Proof. We will illustrate a family of n-vertex graphs G_n where $\tau_2^*(G_n) = \frac{n}{2}$ but $\tau_3^*(G_n) = \frac{2}{3}(1 - o(1))n$. By [7], it suffices to construct a triangle-free graph G_n where the gap exists between the fractional and integral optimal solutions of the vertex cover problem. We consider a triangle-free graph G_n where $\mathsf{VC}(G_n) = (1 - o(1))n$ (this is just a random graph $G(n,p)$ with appropriate parameter and using the alteration steps to remove all triangles). In any graph, there is a fractional feasible cover of value $\frac{n}{2}$ (by assigning $\frac{1}{2}$ everywhere), so we have that $\tau_2^*(G) \leq \frac{n}{2}$.

Now we analyze $\tau_3^*(G)$. Consider any $\frac{1}{3}$-integral assignment z on vertices $V(G_n)$. Partition $V(G_n)$ based on the assigned z-values into $V_0, V_{1/3}$, $V_{2/3}$, and V_1. Notice that $V_0 \cup V_{1/3}$ must be an independent set, for otherwise an edge $e \in G[V_0 \cup V_{1/3}]$ would violate the covering constraint. Therefore, $|V_0 \cup V_{1/3}| = o(n)$ (since the size of any independent set in G is at most $o(n)$). This implies that the total assignment must be at least $\frac{2}{3}(1 - o(1))n$. $\square$

CLAIM A.2. *There is an infinite family of graphs G for which $\tau_3^*(G) \leq \frac{5}{6}\tau_2^*(G)$.*

Proof. We show that this gap holds in the case of a complete graph on 6 vertices. In particular, $\tau_3^*(K_6) \leq 5$ but $\tau_2^*(K_6) \geq 6$. The first claim is easy, since we can simply assign $\frac{1}{3}$ on each edge, and there are 15 edges in the graph. The second claim relies on Proposition 1.3 to say that $\tau_2^*(K_6) \geq 6$. $\square$

B Omitted Claims from Section 2

B.1 Proof of Proposition 2.1

CLAIM B.1. *For any solution triangle ψ, if there are two triangles singly-attached to ψ, then these two attached triangles either have the same base, or the same anchoring vertex.*

Proof. Otherwise if ψ is the solution triangle and ω, ω' are the singly-attached triangles of ψ with different base edges and different anchoring vertices, then $\mathcal{V} \cup \{\omega, \omega'\} \setminus \{\psi\}$ is a better solution, contradicting the optimality of $\mathcal{V}$. □

COROLLARY B.1. *There is no type-*2 *solution triangle.*

Proof. Byclaim B.1, for any solution type-2 triangle ψ, the two singly-attached triangles attached to different edges of ψ must share their anchoring vertex and there cannot be any other singly-attached triangle attached to ψ. This implies that the vertices of ψ plus the anchoring vertex induce a K_4, hence in fact it is a type-3 triangle. □

COROLLARY B.2. *For any type-*3 *triangle in $\mathcal{T}$, there are exactly three attached triangles and one anchoring vertex.*

Proof. Follows directly by claim B.1. □

B.2 Proof of Proposition 2.2 Let the doubly-attached triangle $t \in \mathcal{T}$ be adjacent to the two solution triangles ψ_1 and ψ_2. Clearly, the three triangles t, ψ_1, ψ_2 share a vertex, say c.

First we argue that there is no $t : type(t) = [3, 3]$. Assume there is such a triangle. Note that:

- The anchoring vertices (possibly same) of ψ_1 and ψ_2 cannot be in $V(\psi_1) \cup V(\psi_2)$; otherwise, some of the attachments would have been doubly-attached, rather than singly-attached.
- Let e_1 and e_2 be edges in $E(\psi_1)$ and $E(\psi_2)$ not adjancent to c. Notice that these two edges are vertex-disjoint.

There exist two edge-disjoint singly-attached triangles $\omega_1 \ni e_1$ and $\omega_2 \ni e_2$ attached to ψ_1 and ψ_2 respectively. ω_1 and ω_2 are edge-disjoint because e_1 and e_2 are vertex-disjoint. Hence, $\mathcal{V} \cup \{\omega_1, \omega_2, t\} \setminus \{\psi_1, \psi_2\}$ is a larger set of edge disjoint triangles, which contradicts the optimality of $\mathcal{V}$.

Now consider the case when $type(t) = [1, 3]$. WLOG, let ψ_1 be type-1. Similar to the previous case, $anchor(\psi_2)$ cannot be in $V(\psi_1) \cup V(\psi_2)$. If $base(\psi_1) \in E(t)$, then we are done. For the rest of the proof, we assume that the base $e_1 \notin E(t)$.

First, if $|\mathcal{C}L_{sin}(\psi_1)| > 1$, then there exists a singly-attached triangle (say ω_1), such that $anchor(\omega_1) \notin V(t)$. Notice that $E(\omega_1)$ and $E(t)$ are disjoint. Since ψ_2 is type-3, there is a singly-attached triangle ω_2 whose base does not contain c. Notice that t, ω_1, ω_2 are edge-disjoint triangles (an easy way to see this is that each pair of them has at most one vertex in common). We could exchange the solution by removing $\{\psi_1, \psi_2\}$ and adding $\{t, \omega_1, \omega_2\}$. Hence, it must be that $|\mathcal{C}L_{sin}(\psi_1)| = 1$ and that the anchoring vertex is in $V(t)$.

C Omitted Claims from Section 3

C.1 Proof of Proposition 3.1 Recall that t is a type-$[1, 1]$ triangle. Let ψ_1 and ψ_2 be solution triangles which t is doubly-attached to. Let e be the non-solution edge of t and $c = V(t) \cap V(\psi_1) \cap V(\psi_2)$. If $E(t)$ contains $base(\psi_1)$ or $base(\psi_2)$, then t belongs to Proposition 3.1.1 and we are done. Now we assume that $\{base(\psi_1), base(\psi_2)\} \cap E(t) = \varnothing$. If $|CL_{sin}(\psi_1)| \geq |CL_{sin}(\psi_2)| \geq 2$, then in both cases when $base(\psi_1) \cap base(\psi_2) = \varnothing$ or $base(\psi_1) \cap base(\psi_2) = c$, there exists two singly-attached triangles ω_1, ω_2 attached to $base(\psi_1), base(\psi_2)$ respectively such that both $anchor(\omega_1) \neq anchor(\omega_2)$ and ω_1, ω_2 are edge-disjoint from t. Hence, we let $\mathcal{V}' = (\mathcal{V} \setminus \{\psi_1, \psi_2\}) \cup \{t, \omega_1, \omega_2\}$. Since $\mathcal{V}'$ has more triangle than $\mathcal{V}$, which contradicts the optimality of $\mathcal{V}$.

Now by renaming, let us assume $|CL_{sin}(\psi_1)| = 1$. Let ω_1 be the singly-attached triangle of ψ_1. If $anchor(\omega_1) \in V(t)$, then t belongs to Proposition 3.1.2 and we are done. If not, then if $|CL_{sin}(\psi_2)| > 1$ we have an improving swap using the an argument similar to the previous case. If $|CL_{sin(\psi_2)}| = 1$, let ω_2 be the singly-attached triangle. If $anchor(\omega_2) \in V(t)$ then we are done by renaming. Else if $base(\psi_1) \cap base(\psi_2) = \varnothing$ or $anchor(\omega_1) \neq anchor(\omega_1)$, then still $\mathcal{V}' = (\mathcal{V} \setminus \{\psi_1, \psi_2\}) \cup \{t, \omega_1, \omega_2\}$ is an improving swap because ω_1, ω_2, t are edge disjoint since they share at most one vertex with each other. Hence the only remaining possibility is that $anchor(\omega_1) = anchor(\omega_2)$ and $base(\psi_1) \cap base(\psi_2) \neq \varnothing$, which implies t belongs to Proposition 3.1. 3. Hence, we conclude the proof.

D Omitted proofs for Section 4

D.1 Proof of Proposition 4.1 First we argue that there is no $t : type(t) = [3, 3, 3]$. Assume there is a hollow triangle $t \in \mathcal{T}$ attached to three solution triangles of type-3, ψ_1, ψ_2 and ψ_3. Notice that $|\bigcup_{i=1}^{3} V(\psi_i)| = 6$ since these solution triangles cannot share edges. For any i, let $v_i = V(\psi) \setminus V(t)$ be an only vertex of ψ_i that is not in $V(t)$. Let $cv_{ij} \in V(\psi_i) \cap V(\psi_j)$ be the vertex shared between ψ_i and ψ_j. It is possible that all $a_i = anchor(\psi_i)$ are the same vertex for all i. Consider the set S of four disjoint triangles $\{\Delta_{v_1 cv_{12} a_1}, \Delta_{v_2 cv_{23} a_2}, \Delta_{v_3 cv_{13} a_3}, t\}$. Since $\mathcal{V}' = (\mathcal{V} \setminus \bigcup_{i=1}^{3} \psi_i) \cup S$ is a set of disjoint triangle of size $|\mathcal{V}| + 1$, then it contradicts the fact that $\mathcal{V}$ is optimal.

Now we consider the case where $type(\psi_1) = 1$. There are three sub cases.

The first sub case is when $type(t) = [1, 3, 3]$. Assume that $base(\psi_1)$ is not in t. In this case, $anchor(\psi_2) \neq anchor(\psi_3)$ or the base edge of ψ_1 will be in t. WLOG, assume that the $base(\psi_1)$ is $\overline{v_1 cv_{12}}$.

We then can select the set S of four disjoint triangles $\{\Delta_{v_2cv_{23}a_2}, \Delta_{v_3cv_{23}a_3}, \omega_1 \in \mathcal{CL}_{sin}(\psi_1), t\}$. Since $\mathcal{V}' = (\mathcal{V} \setminus \bigcup_{i=1}^{3} \psi_i) \cup S$ is a set of disjoint triangle of size $|\mathcal{V}| + 1$, then it again is a contradiction.

The second sub case is when $type(t) = [1,1,3]$. Assume that $\{base(\psi_1), base(\psi_2)\} \cap E(t) = \varnothing$. Let $\omega_1 \in \mathcal{CL}_{sin}(\psi_1)$ and $\omega_2 \in \mathcal{CL}_{sin}(\psi_2)$ be two disjoint singly-attached triangles. There exists such triangles or ψ_1 and ψ_2 will share a single anchoring vertex. Let ω_3 be a singly-attached triangle of ψ_3, which is disjoint from ω_1 and ω_2. There exists such ω_3 or ω_1 and ω_2 would share their anchoring vertex. Let $S = \{\omega_1, \omega_2, \omega_3, t\}$ be a set of four disjoint triangles. Since $\mathcal{V}' = (\mathcal{V} \setminus \bigcup_{i=1}^{3} \psi_i) \cup S$ is a set of disjoint triangle of size $|\mathcal{V}| + 1$, then it again is a contradiction.

Now come the third sub case when $type(t) = [1,1,1]$. Assume that $\{base(\psi_1), base(\psi_2), base(\psi_3)\} \cap E(t) = \varnothing$. If no two out of these three base edges share a vertex, then it is easy to see that there exists three disjoint singly-attached triangles ω_1, ω_2, and ω_3, attaching to ψ_1, ψ_2, and ψ_3 in respective order. Let $S = \{\omega_1, \omega_2, \omega_3, t\}$ be a set of four disjoint triangles. Since $\mathcal{V}' = (\mathcal{V} \setminus \bigcup_{i=1}^{3} \psi_i) \cup S$ is a set of disjoint triangle of size $|\mathcal{V}| + 1$, then it again is a contradiction.

Otherwise, WLOG, assume that $base(\psi_1)$ and $base(\psi_2)$ share the vertex $cv(\psi_1, \psi_2)$. We are still able to select two disjoint singly-attached triangle $\omega_1 \in \mathcal{CL}_{sin}(\psi_1)$ and $\omega_2 \in \mathcal{CL}_{sin}(\psi_2)$ as they are not sharing a single anchoring vertex. Let $S = \{\omega_1, \omega_2, \omega_3 \in \mathcal{CL}_{sin}(\psi_3), t\}$ be a set of four disjoint triangles. Since $\mathcal{V}' = (\mathcal{V} \setminus \bigcup_{i=1}^{3} \psi_i) \cup S$ is a set of disjoint triangles of size $|\mathcal{V}| + 1$, then it again is a contradiction.

This concludes the proof.