CHAPTER 1

Infinite Series

1.1 INTRODUCTION

Infinite sequences occur in many situations in scientific and engineering problems. In an infinite series, the number of terms is infinite and if the nth term is known, then we can determine all the terms of the series. Since the number of terms of an infinite series is not finite, the sum of all terms may or may not be finite. If the sum is finite, then the series is called *convergent series* otherwise it is called *divergent series*. The terms of the series form a set called a *sequence*. The terms of a sequence are ordered, i.e. if the nth term is known, then all the terms can be generated.

In this chapter, we first introduce the concept of sequence and then we study the convergence and divergence of an infinite series.

1.2 SEQUENCE

A sequence is a mapping from the set $\mathbb{N}$ (set of natural numbers) to the set $\mathbb{R}$ (the set of real numbers), i.e.

$$x : \mathbb{N} \to \mathbb{R}.$$

The terms of the sequence are denoted by $x_1, x_2, \cdots, x_n, \cdots$. The image of the nth element, x_n, is said to be the nth element of the sequence. A sequence is generally denoted by the symbol $\{x_n\}$ or $\{x_1, x_2, \cdots, x_n, \cdots\}$. The number of terms of a sequence is infinite.

Sometimes, the symbols like $\{u_n\}$, $\{a_n\}$, $\{s_n\}$, etc. may also be used to denote a sequence.

The terms of this sequence are $-1, 1, -1, 1, -1, \cdots$. That is, the sequence contains only two terms -1 and 1.

Again, $\{x_n\}$, where $x_n = \frac{1}{n}$, $n \in \mathbb{N}$ be the another sequence. The terms are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$. Here all the terms are distinct. This sequence is known as harmonic sequence.

The sequence $\{2, 2, \cdots\}$ is a constant sequence.

The *range* or *trace* is the set consisting of all distinct elements of a sequence without repetition. Thus, the range may be a finite set or an infinite set.

1.2.1 Bounds of a Sequence

A sequence $\{a_n\}$ is said to be *bounded above* if there exists a number M such that

$$a_n \le M \quad \text{for all} \quad n \in \mathbb{N}$$

M is called the upper bound of the sequence $\{a_n\}$.

A sequence $\{a_n\}$ is said to be *bounded below* if there exists a number m such that

$$a_n \ge m \quad \text{for all} \quad n \in \mathbb{N}$$

m is called the lower bound of $\{a_n\}$.

A sequence is said to be *bounded* if it is bounded both above and below. Thus a sequence $\{a_n\}$ is said to be bounded, if there exist two numbers m and M such that

$$m \le a_n \le M \quad \text{for all} \quad n \in \mathbb{N}.$$

Examples

1. The sequence $\{n^2\}$, i.e. the sequence $\{1, 4, 9, \cdots\}$ is bounded below, but unbounded above, since $n^2 \ge 1$ for all $n \in \mathbb{N}$.
2. The sequence $\{-n\}$ i.e. $\{-1, -2, -3, \cdots\}$ is bounded above and unbounded below, as $-n \le -1$, for all $n \in \mathbb{N}$.
3. The sequence $\left\{\frac{1}{n}\right\}$ is a bounded sequence since, $0 \le a_n \le 1$, where $a_n = \frac{1}{n}$, for all $n \in \mathbb{N}$.
4. Let $a_n = (-1)^n n$, $n \in \mathbb{N}$. The sequence $\{a_n\}$, i.e. $\{-1, 2, -3, 4, \cdots\}$ is unbounded above and unbounded below.

1.2.2 Monotone Sequence

A sequence $\{x_n\}$ is said to be *monotone increasing* (or non-decreasing) iff $x_{n+1} \ge x_n$ for all n.

If $x_{n+1} > x_n$ for all n, then the sequence $\{x_n\}$ is called *strictly monotone increasing*.

A sequence $\{x_n\}$ is said to be *monotone decreasing* (non-increasing) iff $x_{n+1} \le x_n$ for all n.

If $x_{n-1} < x_n$ for all n, then the sequence $\{x_n\}$ is called *strictly monotone decreasing*.

A sequence $\{x_n\}$ is said to be *simply monotone* if it is either monotone increasing or monotone decreasing.

Examples

1. The sequence $\{3, 9, 27, 81, \cdots, 3^n, \cdots\}$ is a strictly monotone increasing sequence.
2. The sequence $\{1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, \cdots\}$ is monotone increasing sequence, but not strictly monotone increasing.
3. The sequence $\left\{\frac{1}{n}\right\}$ i.e. $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ is a strictly monotone decreasing sequence.
4. The sequence $\{-1, 1, -1, 1, -1, 1, \cdots\}$ is neither monotone increasing nor monotone decreasing sequence.

EXAMPLE 1.1 Show that the sequence $\{x_n\}$, where $x_n = \dfrac{n+2}{2n+1}$ is monotone decreasing.

Solution

Since
$$x_n = \frac{n+2}{2n+1},\ x_{n+1} = \frac{n+3}{2n+3}.$$

Now,
$$x_{n+1} - x_n = \frac{n+3}{2n+3} - \frac{n+2}{2n+1} = \frac{-3}{(2n+3)(2n+1)} < 0,$$

i.e.
$$x_{n+1} - x_n < 0$$
or
$$x_{n+1} < x_n.$$

Hence the sequence $\{x_n\}$ is monotone decreasing.

EXAMPLE 1.2 Show that the sequence $\left\{\dfrac{n+1}{n}\right\}$ is monotone increasing.

Solution

Let
$$x_n = \frac{n+1}{n}.$$

Therefore
$$x_{n+1} = \frac{n+2}{n+1}.$$

Now,
$$\frac{x_n+1}{x_n} = \frac{n+2}{n+1} \times \frac{n+1}{n} = \frac{n+2}{n} = 1 + \frac{2}{n} > 1.$$

That is, $x_{n+1} > x_n$. Hence $\{x_n\}$ is monotone increasing sequence.

1.2.3 Convergence and Non-convergence of Sequences

A sequence $\{x_n\}$ is said to *converge* to a number l, if for $\varepsilon > 0$, there exists a positive integer m (depends on ε) such that

$$|x_n - l| < \varepsilon \quad \text{for all} \quad n \geq m.$$

Symbolically,

$$x_n \to l \quad \text{as} \quad n \to \infty$$

or

$$\lim_{n\to\infty} x_n = l.$$

A sequence which is not convergent is called *divergent.*

In the following we define the limit points of a sequence. By finding the limit points, if there be any, we can classify the convergent and divergent sequences.

A number x_i is said to be a *limit point* of a sequence $\{x_n\}$ if every nbd. of x_i contains an infinite number of members of the sequence.

Thus, x_i is a limit point of a sequence if given any positive number ε, however small, $x_n \in (x_i - \varepsilon, x_i + \varepsilon)$ for an infinite number of values of n, that is $|x_n - x_i| < \varepsilon$ for infinitely many values of n.

1. The constant sequence $\{x_n\}$, where $x_n = 5$ has the only one limit point 5.
2. The limit points of the sequence $\{(-1)^n\}$ are -1 and 1.
3. 0 is the limit point of the sequence $\left\{\frac{1}{n}\right\}$.

A bounded sequence which is not convergent and has at least two limit points, is said to *oscillate finitely.*

A bounded sequence is either convergent or oscillates finitely, but an unbounded sequence is either diverges to ∞ or $-\infty$ or oscillates infinitely.

Examples

1. The sequence $\{1 + (-1)^n\}$, i.e. $\{0, 2, 0, 2, \cdots\}$ oscillates finitely.
2. The sequence $\{n^2\}$ i.e. $\{1, 4, 9, \cdots\}$ diverges to $+\infty$, as

$$\lim_{n\to\infty} n^2 = \infty.$$

3. The sequence $\{-3^n\}$ i.e. $\{-3, -9, -27, \cdots\}$ diverges to $-\infty$, as

$$\lim_{n\to\infty} (-3^n) = -\infty.$$

4. The sequence $\{n(-1)^n\}$ oscillates infinitely.
5. The sequence $\left\{\frac{(-1)^{n+1}}{n!}\right\}$ converges to 0, as

$$\lim_{n\to\infty} \frac{(-1)^{n+1}}{n!} = 0.$$

6. The sequence $\left\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \cdots\right\}$ is bounded below but unbounded above.

Here, $$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

and $$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} (n+1) = \infty.$$

Thus, the given sequence oscillates infinitely.

7. The sequence $\{(-1)^n\}$ is oscillatory. Since

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} (-1)^{2n-1} = -1$$

and $$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} (-1)^{2n} = 1$$

that is, the sequence has two limit points, and hence it is oscillatory.

EXAMPLE 1.3 Show that the sequence $\{u_n\}$, where $u_n = 2(-1)^n$ does not converge. (WBUT 2005)

Solution

$$\lim_{n\to\infty} u_{2n-1} = \lim_{n\to\infty} 2(-1)^{2n-1} = \lim_{n\to\infty} (-2) = -2$$

and $$\lim_{n\to\infty} u_{2n} = \lim_{n\to\infty} 2(-1)^{2n} = 2.$$

Thus, the sequence has two distinct limit points and hence the sequence does not converge. It oscillates finitely.

1.3 INFINITE SERIES

Let $\{a_n\}$ be a real sequence. The series is the sum of the terms of the sequence $\{a_n\}$. Thus, the sum $a_1 + a_2 + a_3 + \cdots$ of all terms is called an *infinite series* and is denoted by $\sum_{n=1}^{\infty} a_n$ or simply Σa_n, a_n is called the nth term of the series.

But, it is difficult to add all the terms of an infinite series in the ordinary way. Thus, we start by associating with the given series, a sequence $\{s_n\}$, where s_n denotes the sum of the first n terms of the series. Therefore,

$$s_n = a_1 + a_2 + \cdots + a_n \text{ for all } n.$$

The sequence $\{s_n\}$ is called the *sequence of partial sums* of the series.

The series $\sum_{n=1}^{\infty} a_n$ is said to converge, diverge to $+\infty$ or $-\infty$ or be oscillatory, accordingly as the sequence $\{s_n\}$ converges, diverges to a limit s, then s is said to be the sum of the series Σa_n and we write $s = \Sigma a_n$.

A necessary condition for convergence of an infinite series is stated as follows.

Theorem 1.1 If Σa_n is convergent, then $\lim_{n\to\infty} a_n = 0$.

But, the converse of this theorem is not true. That is, if $\lim_{n\to\infty} a_n = 0$ does not imply the series Σa_n is convergent. But, if $\lim_{n\to\infty} a_n \neq 0$, then we surely conclude that the series Σa_n does not converge. This fact is illustrated in the following example.

EXAMPLE 1.4 Show that the series $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$ does not converge.

Solution

Let $$a_n = \frac{n}{n+1}$$

Therefore, $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+1/n} = 1 \neq 0.$$

Thus, the series is not convergent.

In the following, we state a necessary and sufficient condition for the convergence of an infinite series.

Theorem 1.2 (Cauchy's general principle for convergence).

A necessary and sufficient condition for convergence of Σa_n is that, given any $\varepsilon > 0$, we can find a positive integer m, depending on ε such that

$$|s_{n+p} - s_n| < \varepsilon$$

or

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \varepsilon$$

for every $n \geq m$ and every positive integer p.

EXAMPLE 1.5 Show that the series $\Sigma\frac{1}{n}$ does not converge.

Solution

If possible, let the series be convergent. Therefore, for any given ε (say, 1/4), there exists a positive integer m, such that

$$\left|\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+p}\right| < \varepsilon \text{ for all } n \geq m \text{ and } p \geq 1.$$

Let, in particular, $n = m$ and $p = m$, then

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+p} = \frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{2m} \geq m\cdot\frac{1}{2m} = \frac{1}{2} > \varepsilon.$$

Thus, there is a contradiction. Hence, the given series does not converge.

Note: It may be noted that $\lim_{n\to\infty}\frac{1}{n} = 0$ even though the series $\Sigma\frac{1}{n}$ does not converge.

Geometric series

The positive term series $1 + a + a^2 + \cdots = \Sigma a^n$ is convergent if $a < 1$ and is divergent if $a \geq 1$.

Proof: The nth partial sum

$$s_n = \frac{1-a^{n+1}}{1-a}, \ a \neq 1 \ \text{ and } \ s_n = n+1 \ \text{ if } \ a = 1.$$

If $a \geq 1$, then $s_n \to \infty$. Therefore Σa^n diverges.

If $a < 1$, then $s_n \to \dfrac{1}{1-a}$ and hence Σa^n converges.

p-series

The series $1 + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \cdots = \Sigma \dfrac{1}{n^p}$ is called *p-series*. The condition for convergence of this series is stated below.

Theorem 1.3 The positive term series $\Sigma \dfrac{1}{n^p}$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

From this theorem, we can say that the series

(i) $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots$ diverges

(ii) $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots$ converges

(iii) $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \cdots$ diverges

Some preliminary theorems

Theorem 1.4 If $\Sigma u_n = u$ then $\Sigma cu_n = cu$, where c is any real number independent of n.

Theorem 1.5 If $\Sigma u_n = u$ and $\Sigma v_n = v$, then $\Sigma w_n = u \pm v$ where $w_n = u_n \pm v_n$ for all n.

Theorem 1.6 A positive series converges iff the sequence of its partial sums is bounded above.

1.4 COMPARISON TESTS FOR POSITIVE TERM SERIES

1.4.1 (First Type)

If Σu_n and Σv_n are two positive term series and $k (\neq 0)$, a fixed positive real number (independent of n) and there exists a positive integer m such that $u_n \leq k\, v_n$ for all $n \geq m$, then

(i) Σu_n converges if Σv_n converges, and

(ii) Σv_n diverges if Σu_n diverges.

1.4.2 Limit Form

If Σu_n and Σv_n are two positive term series such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$, where l is a non-zero finite number, then the two series converge or diverge together.

EXAMPLE 1.6 Test the convergence of the series

$$\frac{1}{1.2^2}+\frac{1}{2.3^2}+\frac{1}{3.4^2}+\cdots$$

Solution

Let the series be Σu_n, where

$$u_n = \frac{1}{n(n+1)^2}.$$

Let
$$v_n = \frac{1}{n^3}.$$

Then
$$\lim_{x\to\infty}\frac{u_n}{v_n} = \lim_{x\to\infty}\frac{1}{n(n+1)^2}\times n^3 = \lim_{x\to\infty}\frac{1}{(1+1/n)^2} = 1 \neq 0.$$

The series Σv_n is convergent, therefore Σu_n is also convergent by comparison test.

EXAMPLE 1.7 Test the convergence of the series

$$\frac{1+2}{2^3}+\frac{1+2+3}{3^3}+\frac{1+2+3+4}{4^3}+\cdots$$

Solution

Let the series be Σu_n where

$$u_n = \frac{1+2+3+\cdots+(n+1)}{(n+1)^3}.$$

Since
$$1+2+3+\cdots+(n+1) = \frac{(n+1)(n+2)}{2}$$

so
$$u_n = \frac{(n+1)(n+2)}{2(n+1)^3} = \frac{n+2}{2(n+1)^2}.$$

Let
$$v_n = \frac{1}{n}.$$

Then
$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n+2}{2(n+1)^2}\cdot n = \lim_{n\to\infty}\frac{1+2/n^2}{2(1+1/n)^2} = \frac{1}{2} \neq 0.$$

But Σv_n is divergent, hence Σu_n is convergent by comparison test.

EXAMPLE 1.8 Test the convergence of the series $\Sigma(\sqrt[3]{n^3+1}-n)$.

(WBUT 2003, 2007)

Solution

Let $$u_n = \sqrt[3]{n^3+1}-n.$$

Then $$u_n = (n^3+1)^{1/3} - n = \{n^3\,(1+1/n^3)\}^{1/3} - n$$

$$= n\left[1+\frac{1}{3}\cdot\frac{1}{n^3}+\frac{1/3\,(1/3-1)}{2!}\left(\frac{1}{n^3}\right)^2+\cdots\right]-n$$

$$= \frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}+\cdots\right]$$

Let $$\Sigma v_n = \Sigma\frac{1}{n^2}.$$

Then $$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{1}{n^2}\left[\frac{1}{3}-\frac{1}{9}\cdot\frac{1}{n^3}+\cdots\right]}{\frac{1}{n^2}} = \frac{1}{3}\neq 0.$$

The series $\Sigma v_n = \Sigma\frac{1}{n^2}$ is convergent, hence by comparison test, the series Σu_n is convergent.

EXAMPLE 1.9 Test the convergence of the series Σu_n where

$$u_n = \sqrt{n^4+1}-\sqrt{n^4-1}.$$ (WBUT 2008)

Solution

Let $$v_n = \frac{1}{n^2}.$$

$$u_n = \sqrt{n^4+1}-\sqrt{n^4-1} = \frac{(\sqrt{n^4+1})^2-(\sqrt{n^4-1})^2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

$$= \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}.$$

Then

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \lim_{n\to\infty}\frac{2}{\sqrt{1+1/n^4}+\sqrt{1-1/n^4}} = 1\neq 0.$$

But $\Sigma\, v_n = \Sigma \dfrac{1}{n^2}$ is convergent. Therefore, $\Sigma\, u_n$ is convergent by comparison test.

EXAMPLE 1.10 Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin \frac{1}{n}. \qquad \text{(WBUT 2001)}$$

Solution

Let $$u_n = \frac{1}{\sqrt{n}} \sin \frac{1}{n} \text{ and } v_n = \frac{1}{n^{3/2}}.$$

Then $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}} \sin \frac{1}{n}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1 \neq 0.$$

But $\Sigma\, v_n = \Sigma \dfrac{1}{n^{3/2}}$, here $p = 3/2 > 1$, is convergent. Hence by comparison test, the series $\Sigma\, u_n$ is convergent.

EXAMPLE 1.11 Test the convergence of the following series.

$$\sin\left(\frac{1}{1^{3/2}}\right) + \sin\left(\frac{1}{2^{3/2}}\right) + \sin\left(\frac{1}{3^{3/2}}\right) + \sin\left(\frac{1}{4^{3/2}}\right) + \cdots$$

(WBUT 2005)

Solution

Let $$u_n = \sin\left(\frac{1}{n^{3/2}}\right) \text{ and } v_n = \frac{1}{n^{3/2}}.$$

Then $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^{3/2}}\right)}{\frac{1}{n^{3/2}}} = \lim_{x\to 0} \frac{\sin x}{x}, \left(\text{where } x = \frac{1}{n^{3/2}}\right) = 1 \neq 0.$$

But $\Sigma\, v_n = \Sigma \dfrac{1}{n^{3/2}}$ is convergent. Therefore, the series $\Sigma\, u_n$ is convergent by comparison test.

EXAMPLE 1.12 Test the convergence of the series

$$\frac{\sqrt{1}}{a\cdot 1^{3/2} + b} + \frac{\sqrt{2}}{a\cdot 2^{3/2} + b} + \frac{\sqrt{3}}{a\cdot 3^{3/2} + b} + \cdots,\ a > 0.$$

(WBUT 2004)

Solution

Let $$u_n = \frac{\sqrt{n}}{a \cdot n^{3/2} + b} \text{ and } v_n = \frac{1}{n}.$$

But $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{a \cdot n^{3/2} + b} \cdot n = \lim_{n\to\infty} \frac{n^{3/2}}{a \cdot n^{3/2} + b}$$

$$= \lim_{n\to\infty} \frac{1}{a + b\,n^{-3/2}} = \frac{1}{a} \neq 0.$$

Here $\Sigma v_n = \Sigma \frac{1}{n}$ is a divergent series. Hence by comparison test, the series Σu_n, i.e. given series is divergent.

EXAMPLE 1.13 Test the convergence of the series

$$\Sigma \frac{1}{n \log n}, n > 2.$$

Solution

Let $$u_n = \frac{1}{n \log n}.$$

We know for $n > 2$, $n \log n > n$

or $$\frac{1}{n \log n} < \frac{1}{n}$$

Thus, $$\Sigma \frac{1}{n \log n} < \Sigma \frac{1}{n}.$$

Also, we know $\Sigma \frac{1}{n}$ is a divergent series. Hence by comparison test (first form) the series $\Sigma \frac{1}{n \log n}$ is divergent.

EXAMPLE 1.14 Test the convergence of the series $\Sigma \frac{1}{n^{1+1/n}}$.

Solution

Let $$u_n = \frac{1}{n^{1+1/n}} \quad \text{and} \quad v_n = \frac{1}{n}.$$

Now, $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^{1+1/n}} \cdot n = \lim_{n\to\infty} \frac{1}{n^{1/n}}.$$

Now, let us consider $y = n^{1/n}$. Taking logarithm of both sides, we get

$$\log y = \frac{1}{n}\log n.$$

Therefore,
$$\lim_{n\to\infty} \log y = \lim_{n\to\infty} \frac{\log n}{n} = 0.$$

or
$$\log\left(\lim_{n\to\infty} y\right) = 0$$

or
$$\lim_{n\to\infty} y = e^0 = 1$$

or
$$\lim_{n\to\infty} n^{1/n} = 1$$

Thus,
$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1 \neq 0.$$

But, $\Sigma\, v_n = \Sigma \frac{1}{n}$ is a divergent series. Hence by comparison test $\Sigma\, u_n$, i.e. the given series is divergent.

1.4.3 Comparison Test (Second Type)

If $\Sigma\, u_n$ and $\Sigma\, v_n$ are two positive term series and there exists a positive integer m such that $\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}$ for all $n \geq m$, then

(i) $\Sigma\, u_n$ is convergent if $\Sigma\, v_n$ is convergent,
(ii) $\Sigma\, v_n$ is divergent if $\Sigma\, u_n$ is divergent.

EXAMPLE 1.15 Test the convergence of the series

$$1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \cdots \qquad \text{(WBUT 2005)}$$

Solution

Let the series be $1 + \sum_{n=1}^{\infty} u_n$, where $u_n = \frac{n^n}{(n+1)^{n+1}}$.

Let $v_n = \frac{1}{n}$ since the difference between the degrees of denominator and numerator is 1.

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} \cdot n = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{1+n}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{1+n}} \cdot \frac{1}{1+\frac{1}{n}}$$

$$= \frac{1}{e} \cdot \frac{1}{1+0} = \frac{1}{e} \neq 0 \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

Since the series $\Sigma\, v_n = \Sigma \frac{1}{n}$ is divergent, by comparison test, the given series $1 + \Sigma\, u_n$ is divergent.

1.5 D'ALEMBERT'S RATIO TEST

Statement. If $\Sigma\, u_n$ is a positive term series, such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then the series

(i) converges, if $l < 1$,
(ii) diverges, if $l > 1$, and
(iii) the test fails, if $l = 1$.

EXAMPLE 1.16 Test the convergence of the series

$$1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots$$

Solution

Let $\sum_{n=1}^{\infty} u_n$ be the given series.

Then $$u_n = \frac{2n-1}{n!}.$$

$\therefore$ $$u_{n+1} = \frac{2n+1}{(n+1)!}$$

and $$\frac{u_{n+1}}{u_n} = \frac{2n+1}{(n+1)!} \times \frac{n!}{2n-1} = \frac{2n+1}{(n+1)\,(2n-1)}$$

Now, $$\lim_{n\to\infty} \frac{u_{n+1}}{v_n} = \lim_{n\to\infty} \frac{2n+1}{(n+1)\,(2n-1)} = \lim_{n\to\infty} \frac{\frac{1}{n}\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(2-\frac{1}{n}\right)} = 0 < 1.$$

By D'Alembert's ratio test, $\Sigma\, u_n$ is convergent.

EXAMPLE 1.17 Test the convergence of the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots, x > 0.$

Solution

Let Σu_n be the given series. Then $u_n = \frac{x^n}{n}$. Since $x > 0$, the series is positive term.

Now, $$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{n+1} \times \frac{n}{x^n} = \lim_{n\to\infty} \frac{x^n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = x.$$

Hence, by D'Alembert's ratio test, Σu_n is convergent if $x < 1$, Σu_n is divergent if $x > 1$.

When $x = 1$, the series becomes $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, which is divergent. Hence, the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

EXAMPLE 1.18 Test the convergence of the series

$$\Sigma \frac{n^2 - 1}{n^2 + 1} x^n, x > 0. \qquad \text{(WBUT 2006)}$$

Solution

Let $$u_n = \frac{n^2 - 1}{n^2 + 1} x^n.$$

Then $$u_{n+1} = \frac{(n+1)^2 - 1}{(n+1)^2 + 1} x^{n+1}$$

$\therefore$ $$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^2+1}{n^2-1} \times \frac{(n+1)^2 - 1}{(n+1)^2 + 1} \frac{x^{n+1}}{x^n}$$

$$= \lim_{n\to\infty} \frac{\left(1 + \frac{1}{n^2}\right)}{\left(1 - \frac{1}{n^2}\right)} \cdot \frac{\left(1 + \frac{1}{n}\right)^2 - \frac{1}{n^2}}{\left(1 + \frac{1}{n}\right)^2 + \frac{1}{n^2}} x = x.$$

Hence by D'Alembert's ratio test, the series converges if $x < 1$ and diverges if $x > 1$.

The test fails when $x = 1$. But, when $x = 1$, then

$$u_n = \frac{n^2 - 1}{n^2 + 1}$$

and $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1 - \frac{1}{n^2}}{1 + \frac{1}{n^2}} = 1 \neq 0.$$

Hence by necessary conditions, the series is divergent. Thus, the series converges if $x < 1$ and diverges if $x \geq 1$.

EXAMPLE 1.19 Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}. \qquad \text{(WBUT 2003)}$$

Solution

Let $\Sigma\, u_n$ be the given series, where

$$u_n = \frac{n!\,2^n}{n^n}.$$

Then
$$u_{n+1} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}}$$

and
$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!\,2^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{n!\,2^n} = \frac{2n^n}{(n+1)^n}$$

$\therefore$
$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1$$

$$\left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

Hence by D'Alembert's ratio tests, the given series is convergent.

EXAMPLE 1.20 Test the convergence of the series

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 + \cdots \qquad \text{(WBUT 2002, 2007)}$$

Solution

Let $\Sigma\, u_n$ be the given series, where

$$u_n = \left(\frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdot 7\cdots (n+1)}\right)^2.$$

Then
$$u_{n+1} = \left(\frac{1\cdot 2\cdot 3\cdots n\,(n+1)}{3\cdot 5\cdot 7\cdots (2n+1)\,(2n+3)}\right)^2.$$

and
$$\frac{u_{n+1}}{u_n} = \left(\frac{1\cdot 2\cdot 3\cdots n\,(n+1)}{3\cdot 5\cdot 7\cdots (2n+1)\,(2n+3)}\right)^2 \times \left(\frac{3\cdot 5\cdot 7\cdots (2n+1)}{1\cdot 2\cdot 3\cdots n}\right)^2$$

$$= \left(\frac{n+1}{2n+3}\right)^2$$

$$\therefore \qquad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{n+1}{2n+3}\right)^2 = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right)^2 = \left(\frac{1}{2}\right)^2 < 1.$$

Hence by D'Alembert's ratio tests, the series is convergent.

EXAMPLE 1.21 Examine the convergence of the series

$$1 + \frac{x^2}{2^p} + \frac{x^4}{4^p} + \frac{x^6}{6^p} + \cdots$$

Solution

Let the given series be $1 + \sum_{n=1}^{\infty} u_n$,

where $$u_n = \frac{x^{2n}}{(2n)^p}.$$

Then $$u_{n+1} = \frac{x^{2n+2}}{(2n+2)^p}$$

and $$\frac{u_{n+1}}{u_n} = \frac{x^{2n+2}}{(2n+2)^p} \times \frac{(2n)^p}{x^{2n}} = x^2 \frac{n^p}{(n+1)^p}$$

$$\therefore \qquad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} x^2 \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} x^2 \left(\frac{1}{1+1/n}\right)^p = x^2.$$

Thus, by D'Alembert's ratio test, the given series is convergent if $x^2 < 1$ or $|x| < 1$ and divergent if $|x| > 1$.

When $x = 1$,

then $$u_n = \frac{1}{(2n)^p} < \frac{1}{n^p}$$

$$\therefore \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n)^p} = 0,\ p > 0$$

since $$\lim_{n\to\infty} u_n = 0.$$

The series may or may not be converge when $x = 1$. But, $\Sigma u_n < \Sigma \frac{1}{n^p}$ and $\Sigma \frac{1}{n^p}$ is convergent when $p > 1$. Thus, the given series is convergent when $x = 1$ and $p > 1$.

EXAMPLE 1.22 Test the convergence of the series

$$[illegible]$$

Solution

Let Σu_n be the given series.

Then $$u_n = \frac{n^2 (n+1)^2}{n!}$$

$\therefore$ $$u_{n+1} = \frac{(n+1)^2 (n+2)^2}{(n+1)!}$$

and $$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2 (n+2)^2}{(n+1)!} \times \frac{n!}{n^2 (n+1)^2} = \frac{(n+2)^2}{(n+1)n^2}$$

Now, $$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+2)^2}{n^2 (n+1)} = \lim_{n\to\infty} \frac{1}{n+1}\left(1+\frac{2}{n}\right)^2 = 0 < 1.$$

Hence the given is convergent by D'Alembert's ratio test.

EXAMPLE 1.23 Examine the convergence and divergence of the series

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots \infty. \qquad \text{(WBUT 2003)}$$

Solution

Let $\sum_{n=0}^{\infty} u_n$ be the given series, where $u_n = \frac{x^n}{n^2+1}$. Then $u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$.

Now, $$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)^2+1} \times \frac{n^2+1}{x^n} = x \cdot \frac{n^2+1}{(n+1)^2+1}$$

$\therefore$ $$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} x \cdot \frac{1+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2 + \frac{1}{n^2}} = x.$$

Hence, by D'Alembert's ratio test, the series is convergent if $x < 1$ and divergent if $x > 1$.

When $x = 1$, then

$$u_n = \frac{1}{n^2+1} < \frac{1}{n^2}$$

$\therefore$ $$\Sigma u_n < \Sigma \frac{1}{n^2}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{n+1}{2n+3}\right)^2 = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right)^2 = \left(\frac{1}{2}\right)^2 < 1.$$

Hence by D'Alembert's ratio tests, the series is convergent.

EXAMPLE 1.21 Examine the convergence of the series

$$1 + \frac{x^2}{2^p} + \frac{x^4}{4^p} + \frac{x^6}{6^p} + \ldots$$

Solution

Let the given series be $1 + \sum_{n=1}^{\infty} u_n$,

where $\quad u_n = \dfrac{x^{2n}}{(2n)^p}$,

Then $\quad u_{n+1} = \dfrac{x^{2n+2}}{(2n+2)^p}$

and $\quad \dfrac{u_{n+1}}{u_n} = \dfrac{x^{2n+2}}{(2n+2)^p} \times \dfrac{(2n)^p}{x^{2n}} = x^2 \dfrac{n^p}{(n+1)^p}$

$$\therefore \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} x^2 \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} x^2 \left(\frac{1}{1+1/n}\right)^p = x^2.$$

Thus, by D'Alembert's ratio test, the given series is convergent if $x^2 < 1$ or $|x| < 1$ and divergent if $|x| > 1$.

When $x = 1$,

then $\quad u_n = \dfrac{1}{(2n)^p} < \dfrac{1}{n^p}$

$$\therefore \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n)^p} = 0,\ p > 0$$

since $\quad \lim_{n\to\infty} u_n = 0.$

The series may or may not be converge when $x = 1$. But, $\Sigma u_n < \Sigma \dfrac{1}{n^p}$ and

1 ... when $p > 1$. Thus, the given series is convergent when $x = 1$

EXAMPLE 1.22 Test the convergence of the series

$$\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \frac{4^2 \cdot 5^2}{4!} + \cdots$$

Solution

Let $\Sigma\, u_n$ be the given series.

Then $$u_n = \frac{n^2\,(n+1)^2}{n!}$$

$\therefore$ $$u_{n+1} = \frac{(n+1)^2\,(n+2)^2}{(n+1)!}$$

and $$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2\,(n+2)^2}{(n+1)!} \times \frac{n!}{n^2\,(n+1)^2} = \frac{(n+2)^2}{(n+1)n^2}$$

Now, $$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+2)^2}{n^2\,(n+1)} = \lim_{n\to\infty} \frac{1}{n+1}\left(1+\frac{2}{n}\right)^2 = 0 < 1.$$

Hence the given is convergent by D'Alembert's ratio test.

EXAMPLE 1.23 Examine the convergence and divergence of the series

$$1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \cdots \infty. \qquad \text{(WBUT 2003)}$$

Solution

Let $\sum\limits_{n=0}^{\infty} u_n$ be the given series, where $u_n = \dfrac{x^n}{n^2+1}$. Then $u_{n+1} = \dfrac{x^{n+1}}{(n+1)^2+1}$.

u . x^{n+1} n^2+1 n^2+1

But, $\Sigma \frac{1}{n^2}$ is a convergent series, hence by comparison test (first form) the series $\Sigma \frac{1}{n^2+1}$ is convergent. Hence the given series is convergent if $x \le 1$ and divergent if $x > 1$.

EXAMPLE 1.24 Examine the convergence of the series

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$$

Solution

Let the given series be $\sum_{n=1}^{\infty} u_n$, where $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$.

Then
$$u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

and
$$\frac{u_{n+1}}{u_n} = \frac{x^{2n}}{(n+2)\sqrt{n+1}} \times \frac{(n+1)\sqrt{n}}{x^{2n-2}} = x^2 \frac{n+1}{n+2}\frac{\sqrt{n}}{\sqrt{n+1}}.$$

$\therefore$
$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} x^2 \frac{1+\frac{1}{n}}{1+\frac{2}{n}} \times \frac{1}{\sqrt{1+\frac{1}{n}}} = x^2.$$

Hence, by D'Alembert's ratio test the given series is convergent when $x^2 < 1$, i.e., $|x| < 1$ and divergent when $|x| > 1$.

When $x^2 = 1$, then

$$u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n\sqrt{n}\left(1+\frac{1}{n}\right)}.$$

Let
$$v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

$\therefore$
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n\sqrt{n}\left(1+\frac{1}{n}\right)} \times n^{3/2} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \ne 0.$$

Thus, Σu_n and Σv_n behave alike.

Again, $\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$ is a convergent series, hence by comparison test the given series is convergent.

Thus, the given series is convergent when $|x| \le 1$ and divergent when $|x| > 1$.

EXAMPLE 1.25 Test the series

$$1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\cdots$$

Solution

Let Σu_n be the given series, where $u_n=\dfrac{n^{n-1}x^{n-1}}{n!}$.

Then
$$u_{n+1}=\frac{(n+1)^n x^n}{(n+1)!}$$

and
$$\frac{u_{n+1}}{u_n}=\frac{(n+1)^n\,x^n}{(n+1)!}\times\frac{n!}{n^{n-1}\,n^{n-1}}=\frac{(n+1)^{n-1}}{n^{n-1}}\,x.$$

$\therefore$
$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}x\left(1+\frac{1}{n}\right)^{n-1}=xe.$$

$$\left[\because\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n-1}=e.\right]$$

Hence, by D'Alembert's test, the series is convergent if $xe<1$, i.e. $x<\dfrac{1}{e}$ and divergent if $x>\dfrac{1}{e}\cdot$

When $xe=1$, i.e. $x=\dfrac{1}{e}$ then

$$u_n=\frac{n^{n-1}}{n!\,e^{n-1}}\cdot$$

Again
$$\frac{n^{n-1}}{n!}=\frac{n}{n}\,\frac{n}{n-1}\,\frac{n}{n-2}\cdots\frac{n}{2}<1 \text{ (as each term is <1).}$$

$\therefore$
$$u_n<\frac{1}{e^{n-1}},\text{ i.e., }\Sigma u_n<\Sigma\frac{1}{e^{n-1}}\cdot$$

The series $\Sigma\dfrac{1}{e^{n-1}}=\Sigma\left(\dfrac{1}{e}\right)^{n-1}$ is a geometric series with common ratio $\dfrac{1}{e}\,(<1)$, therefore it is convergent. Hence by comparison test Σu_n is convergent when $x=\dfrac{1}{e}\cdot$ Thus the given series is convergent when $x\le\dfrac{1}{e}$ and divergent when $x>\dfrac{1}{e}\cdot$

EXAMPLE 1.26 Discuss the convergence of the series

$$x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{6}+\cdots$$

Solution

Let the series be $x+\sum_{n=1}^{\infty}u_n$, where

$$u_n=\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\frac{x^{2n+1}}{2n+1}.$$

Therefore,
$$u_{n+1}=\frac{1\cdot 3\cdots(2n-1)(2n+1)}{2\cdot 4\cdots(2n)(2n+2)}\frac{x^{2n+3}}{2n+3}.$$

Now,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{1\cdot 3\cdots(2n-1)(2n+1)}{2\cdot 4\cdots(2n)(2n+2)}\frac{x^{2n+3}}{2n+3}\times\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n-1)}\frac{2n+1}{x^{2n+1}}$$

$$=\lim_{n\to\infty}\frac{2n+1}{2n+2}\frac{2n+1}{2n+3}x^2$$

$$=\lim_{n\to\infty}\frac{1+\frac{1}{2n}}{1+\frac{1}{n}}\frac{1+\frac{1}{2n}}{1+\frac{3}{2n}}x^2=x^2.$$

Hence, by D'Alembert's ratio test, the series is convergent if $x^2<1$, i.e. $|x|<1$ and divergent if $|x|>1$.

EXAMPLE 1.27 Discuss the convergence of the series

$$\frac{1}{3}+\left(\frac{1}{3}\right)^{1+\frac{1}{2}}+\left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{3}}+\cdots$$

Solution

We know,
$$\left(\frac{1}{3}\right)^{1+\frac{1}{2}}<\left(\frac{1}{3}\right)^{2}$$

$$\left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{3}}<\left(\frac{1}{3}\right)^{3}$$

$$\left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}<\left(\frac{1}{3}\right)^{4}\text{ and so on.}$$

Thus, $$\frac{1}{3}+\left(\frac{1}{3}\right)^{1+\frac{1}{2}}+\left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{3}}+\left(\frac{1}{3}\right)^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}+\cdots$$

$$<\frac{1}{3}+\left(\frac{1}{3}\right)^{2}+\left(\frac{1}{3}\right)^{3}+\left(\frac{1}{3}\right)^{4}+\cdots$$

The series $\frac{1}{3}+\left(\frac{1}{3}\right)^{2}+\left(\frac{1}{3}\right)^{3}+\left(\frac{1}{3}\right)^{4}+\cdots$ is a geometric series with common ratio $\frac{1}{3}<1$, hence it is convergent. Therefore, by comparison test the given series is convergent.

EXAMPLE 1.28 Test the convergence of the series

$$\frac{1+x}{1!}+\frac{(1+2x)^2}{2!}+\frac{(1+3x)^3}{3!}+\cdots,\quad x>0.$$

Solution

Let the given series be Σu_n, where $u_n=\dfrac{(1+nx)^n}{n!}$.

Now, $$u_{n+1}=\frac{\{1+(n+1)\,x\}^{n+1}}{(n+1)!}$$

$$\therefore\quad \frac{u_{n+1}}{u_n}=\frac{\{1+(n+1)\,x\}^{n+1}}{(n+1)!}\times\frac{n!}{(1+nx)^n}=\frac{\{(n+1)x\}^{n+1}\left\{1+\dfrac{1}{(n+1)\,x}\right\}^{n+1}}{(n+1)\,(nx)^n\left\{1+\dfrac{1}{nx}\right\}^n}$$

$$=x\left(\frac{n+1}{n}\right)^n\frac{\left\{1+\dfrac{1}{(n+1)\,x}\right\}^{n+1}}{\left\{1+\dfrac{1}{nx}\right\}^n}$$

Now, $$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}x\left(1+\frac{1}{n}\right)^n\frac{\left\{1+\dfrac{1}{(n+1)\,x}\right\}^{n+1}}{\left\{1+\dfrac{1}{nx}\right\}^n}=xe\cdot\frac{e^{1/x}}{e^{1/x}}=xe.$$

Hence by D'Alembert's ratio test the given series is convergent if $xe<1$, i.e. $x<\frac{1}{e}$ and divergent if $x>\frac{1}{e}$.

EXAMPLE 1.29 Discuss the convergence of the series

$$\sum_{n=1}^{\infty} n^4 e^{-n^2}. \qquad \text{(WBUT 2005)}$$

Solution

Let $u_n = n^4 e^{-n^2}$. Then the given series is Σu_n.

$\therefore$
$$u_{n+1} = (n+1)^4 e^{-(n+1)^2}.$$

Now,
$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^4 e^{-(n+1)^2}}{n^4 \cdot e^{-n^2}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^4 e^{-(n^2+2n+1)+n^2}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^4 e^{-2n} \cdot e^{-1} = 1.0 \cdot e^{-1} = 0 < 1.$$

$$[\because e^{-\infty} = 0]$$

Hence by D'Alembert's ratio test the given series is convergent.

1.6 CAUCHY'S ROOT TEST

Statement: If Σu_n is a positive term series, such that $\lim\limits_{n\to\infty} (u_n)^{1/n} = l$, then the series

(i) converges, if $l < 1$
(ii) diverges, if $l > 1$, and
(iii) test fails, if $l = 1$

EXAMPLE 1.30 Test for convergence of the series

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}. \qquad \text{(WBUT 2004)}$$

Solution

Let Σu_n be the given series, where

$$u_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}$$

$\therefore$
$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$$

$$\left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

Hence by Cauchy's root test, the series is convergent.

EXAMPLE 1.31 Examine the convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots \quad \text{(WBUT 2001, 2005)}$$

Solution

Let the series be Σu_n, where $u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$.

$$\therefore \qquad \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\left\{\left(\frac{n+1}{n}\right)^n - 1\right\}\right]^{-1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\left\{\left(1+\frac{1}{n}\right)^n - 1\right\}^{-1}$$

$$= (1+0)^{-1}(e-1)^{-1} = \frac{1}{e-1} < 1$$

$$\left[\text{as } 2<e<3 \text{ or } 1<e-1<2 \text{ or } \frac{1}{e-1}<1\right]$$

Hence, the given series is convergent by Cauchy's root test.

EXAMPLE 1.32 Test for convergence of the series

$$r+\frac{r^2}{2^2}+\frac{r^3}{3^3}+\frac{r^4}{4^4}+\cdots, \quad r>0.$$

Solution

Let the given series be Σu_n, where $u_n = \frac{r^n}{n^n}$.

$$\therefore \qquad \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\frac{r}{n} = 0 < 1$$

Hence, by Cauchy's root test, the series is convergent.

EXAMPLE 1.33 Test for convergence of the series

$$\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\cdots+\left(\frac{n}{2n+1}\right)^n+\cdots$$

Solution

Let the series be Σu_n, where $u_n = \left(\dfrac{n}{2n+1}\right)^n$.

Now, $$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+1/n}\frac{1}{2} < 1.$$

Hence, by Cauchy's root test, the series is convergent.

EXAMPLE 1.34 Examine the convergence of the series Σu_n, where $u_n = 2^{-n-(-1)^n}$.

Solution

Here $$u_n = 2^{-n-(-1)^n}.$$

$$\therefore \quad \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left\{2^{-n-(1)^n}\right\}^{1/n} = \lim_{n\to\infty} 2^{-1-\frac{(-1)^n}{n}} = 2^{-1} < 1.$$

Hence, by Cauchy's root test, the given series is convergent.

1.7 ALTERNATING SERIES

A series whose terms are alternatively positive and negative is called *alternating series*.

For example, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is an alternating series.

There are many formulae available to test the convergence of alternating series among them Leibnitz's test is simple and widely used, which is stated below.

Leibnitz's test: Let the alternating series be $u_1 - u_2 + u_3 - u_4 + \cdots$ ($u_n > 0$ for all n) and

(i) $u_{n+1} \le u_n$ for all n ($\{u_n\}$ is monotonic decreasing)

(ii) $\lim_{n\to\infty} u_n = 0$,

then the series is convergent.

It may be noted that Leibnitz's test can test the convergence of an alternating series.

EXAMPLE 1.35 Show that the series $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \cdots$ converges for $p > 0$.

Solution

Let $u_n = \dfrac{1}{n^p}\cdot$ Then $u_{n+1} = \dfrac{1}{(n+1)^p}\cdot$

Now, $$\frac{u_{n+1}}{u_n} = \frac{n^p}{(n+1)^p} = \left(\frac{n}{n+1}\right)^p < 1 \text{ if } p > 0.$$

$\therefore$ $$u_{n+1} < u_n \text{ for } p > 0.$$

Also, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^p} = 0 \text{ if } p > 0.$$

Hence by Leibnitz's test the given series is convergent when $p > 0$.

Note: From the above example we note the following:

(i) the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent

(ii) the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is convergent

(iii) the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ is convergent.

EXAMPLE 1.36 Show that the following series is convergent

$$\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \cdots \quad (x \neq \text{any negative integer}).$$

Solution

Let $u_n = \dfrac{1}{x+n}$. Then $u_{n+1} = \dfrac{1}{x+(n+1)}$.

Hence, $$\frac{u_{n+1}}{u_n} = \frac{x+n}{x+(n+1)} < 1$$

or $$u_{n+1} < u_n$$

i.e. $\{u_n\}$ is monotonic decreasing.

Also, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{x+n} = 0$$

Hence by Leibnitz's test, the given series is convergent.

EXAMPLE 1.37 Show that the following series is convergent

$$1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$$

Solution

Let the series be $u_1 - u_2 + u_3 - u_4 + \cdots$, where $u_n = \dfrac{1}{n\sqrt{n}}$.

Then $$u_{n+1} = \frac{1}{(n+1)\sqrt{n+1}}.$$

Now, $$\frac{u_n}{u_{n+1}} = \frac{(n+1)\sqrt{n+1}}{n\sqrt{n}} = \left(1+\frac{1}{n}\right)\sqrt{\left(1+\frac{1}{n}\right)} > 1$$

or $u_n > u_{n+1}$. Therefore, $\{u_n\}$ is monotone decreasing.

Also, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0.$$

Hence, by Leibnitz's test, the given series is convergent.

EXAMPLE 1.38 Show that the following series is convergent

$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \cdots$$

Solution

Let the given series be $u_1 - u_2 + u_3 - \cdots$, where $u_n = \dfrac{\log(n+1)}{(n+1)^2}$.

Then $$u_{n+1} = \frac{\log(n+2)}{(n+2)^2}.$$

Now,

$$u_{n+1} - u_n = \frac{\log(n+2)}{(n+2)^2} - \frac{\log(n+1)}{(n+1)^2} = \frac{(n+1)^2\log(n+2) - (n+2)^2\log(n+1)}{\{(n+2)(n+1)\}^2} < 0$$

i.e. $u_{n+1} < u_n$. Therefore, $\{u_n\}$ is monotone decreasing.

Also, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\log(n+1)}{(n+1)^2} = \lim_{n\to\infty} \frac{\log(n+1)}{(n+1)} \times \frac{1}{n+1} = 0 \times 0 = 0.$$

Hence, by Leibnitz's test the series is convergent.

EXAMPLE 1.39 Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}. \qquad \text{(WBUT 2002)}$$

Solution

The given series is $\sum\limits_{n=1}^{\infty} \dfrac{\cos n\pi}{n^2+1} = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^2+1}$, i.e. the series is alternating.

$[\because \cos n\pi = (-)^n]$

Let $u_n = \dfrac{1}{n^2+1}$. Then obviously $\lim\limits_{n\to\infty} u_n = 0$.

Now, $$\frac{u_{n+1}}{u_n} = \frac{n^2+1}{(n+1)^2+1} < 1, \text{ i.e. } u_{n+1} < u_n$$

Therefore, the sequence $\{u_n\}$ is divergent. Hence, by Leibnitz's test, the given series is convergent.

1.7.1 Absolute and Conditional Convergence

A series $u_1 - u_2 + u_3 - u_4 + \cdots$ ($u_n > 0$ for all n) is said to be *absolutely convergent* if the series $u_1 + u_2 + u_3 + u_4 + \cdots$ of positive term is convergent.

A series $u_1 - u_2 + u_3 - u_4 + \cdots$ ($u_n > 0$ for all n) is called *conditionally convergent* if $u_1 + u_2 + u_3 + u_4 + \cdots$ of positive term is not convergent.

A conditionally convergent series is also called a *semi-convergent* or a *non-absolutely convergent* series.

Theorem 1.7 Every absolutely convergent series is convergent.

EXAMPLE 1.40 Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent.

Solution

Let the given series be $u_1 - u_2 + u_3 - u_4 + \cdots$. Then $u_n = \frac{1}{n}$.

Now,
$$\frac{u_{n+1}}{u_n} = \frac{n}{n+1} < 1.$$

i.e.
$$u_{n+1} < u_n.$$

Therefore, $\{u_n\}$ is monotone decreasing.

Also,
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Hence, by Leibnitz's test, the given series is convergent. But, the series $u_1 + u_2 + u_3 + \cdots$ i.e., $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (the p-series with $p = 1$) is divergent. Hence, the given series is conditionally convergent.

EXAMPLE 1.41 Show that the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is absolutely convergent.

Solution

The positive term series corresponding to the given series is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

The above series is a p-series with $p = 2 > 1$. Therefore, it is convergent. Hence, the given series is absolutely convergent.

EXAMPLE 1.42 Show that for any fixed value of x, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is convergent.

Solution

Let $$u_n = \frac{\sin nx}{n^2}$$

Therefore, $$|u_n| = \frac{|\sin nx|}{n^2} \leq \frac{1}{n^2} \text{ for all } n.$$

That is, $\Sigma |u_n| = \Sigma \frac{1}{n^2}$. But, the series $\Sigma \frac{1}{n^2}$ is convergent.

Hence, by comparison test the series $\Sigma |u_n|$ is convergent. Therefore, the given series is absolutely convergent.

EXAMPLE 1.43 Show that the series $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ converges absolutely for all values of x.

Solution

Let $$u_n = \frac{x^n}{n!}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \times \frac{n!}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \text{ for all } x.$$

Hence, by D'Alembert's ratio test the series $\Sigma |u_n|$ is convergent. Therefore, the given series is absolutely convergent for all x.

Note: We know that if the series Σu_n is convergent, then $\lim_{n\to\infty} u_n = 0$. Thus from the above example, we have $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

EXAMPLE 1.44 Show that the series $\Sigma \frac{(-1)^{n+1}}{n^p}$ is absolutely convergent for $p > 1$, but, conditionally convergent for $0 < p \leq 1$.

Solution

Let the given series be Σu_n, where $u_n = \frac{(-1)^{n+1}}{n^p}$.

Now, $|u_n| = \frac{1}{n^p}$. Then $\Sigma |u_n| = \Sigma \frac{1}{n^p}$, the p-series, which is convergent for $p > 1$ and divergent if $p \leq 1$.

Therefore, the given series is absolutely convergent if $p > 1$ and $\Sigma |u_n|$ is divergent if $p \leq 1$.

Let $$p \le 1.$$

Then $$\frac{|u_n|}{|u_{n+1}|} = \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p > 1, \text{i.e.} |u_n| > |u_{n+1}|.$$

Also, $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{1}{n^p} = 0, \quad \text{if } p > 0.$$

Therefore, $$\Sigma u_n = \Sigma \frac{(-1)^{n+1}}{n^p} \text{ is convergent if } p > 0.$$

Hence the given series is absolutely convergent if $p > 1$ and conditionally convergent if $0 < p \le 1$.

EXAMPLE 1.45 Test the convergence of the series

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \cdots$$

Solution

Let the series be Σu_n, where $u_n = \dfrac{n+1}{n}(-1)^{n+1}$.

Now, $$\frac{|u_{n+1}|}{|u_n|} = \frac{n+2}{n+1} \times \frac{n}{n+1} = \frac{n^2+2n}{n^2+2n+1} < 1, \text{ i.e. } |u_{n+1}| < |u_n|.$$

Therefore, $\{|u_n|\}$ is monotone decreasing.

But, $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) 1 \ne 0.$$

Therefore, the conditions of Leibnitz's test are not satisfied, and hence the given series does not converge.

EXAMPLE 1.46 Prove that the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

is absolutely convergent when $|x| < 1$ and conditionally convergent when $x = 1$.

(WBUT 2001)

Solution

Let the series be Σu_n, when $u_n = (-1)^{n+1}\dfrac{x^n}{n}$.

Now, $$\Sigma |u_n| = \frac{|x|^n}{n}$$

$$\therefore \quad \lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \times \frac{n}{|x|^n}$$

$$= \lim_{n\to\infty} \frac{n|x|}{n+1} = \lim_{n\to\infty} \frac{|x|}{1+\frac{1}{n}} = |x|.$$

If $|x|<1$ then by D'Alembert's ratio test $\Sigma|u_n|$ is convergent, i.e. Σu_n is absolutely convergent. But, when $x=1$ then the series $1-\frac{1}{2}+\frac{1}{3}-\cdots$ is convergent by Leibnitz's test while the series $1+\frac{1}{2}+\frac{1}{3}+\cdots$, the p-series with $p = 1$, is divergent. Hence the given series is conditionally convergent.

EXERCISES

Short Answer Questions

(Section A)

1. The series $2 + 2^2 + 2^3 + 2^4 + \cdots$ is .... (convergent/divergent)
2. The series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$ is .... (convergent/divergent)
3. The series $1 + r + r^2 + r^3 + \cdots$ is convergent if $r <$ ....
4. If the series $\Sigma\, u_n$ is convergent, then $\lim\limits_{n\to\infty} u_n$ ....
5. The limit point of the sequence $\left\{\frac{1}{n}\right\}$ is ....
6. The sequence $\left\{\frac{n+1}{n}\right\}$ is monotone ....
7. If $\Sigma\, u_n$ is a convergent series, then $\lim\limits_{n\to\infty} nu_n =$ ....
8. The series $\sum\limits_{n=1}^{\infty} \frac{\log n}{\sqrt{n+1}}$ is .... (convergent/divergent)
9. The series $\sum\limits_{n=2}^{\infty} \frac{1}{\sqrt{n\,(n-1)}}$ is .... (convergent/divergent)
10. The series $\sum\limits_{n=1}^{\infty} \frac{n}{3n^3-2}$ is .... (convergent/divergent)

CHAPTER 4

Mean Value Theorem

4.1 INTRODUCTION

The Lagrange's and Cauchy's Mean Value Theorems (MVTs) are very important results in calculus and they are used in many branches in science and engineering. The general form of MVT due to Taylor and Maclaurin are presented here. These two theorems are used to expand a function into finite and infinite series.

4.2 ROLLE'S THEOREM

Let a function f be defined on a closed interval $[a, b]$. Further suppose that

(i) f is continuous on $[a, b]$,

(ii) f is derivable in the open interval (a, b), and

(iii) $f(a) = f(b)$.

Then there exists at least one value of x say c, where $a < c < b$, such that $f'(c) = 0$.

Note: The three conditions of Rolle's theorem are a set of sufficient conditions. Sometimes all these conditions are not necessary to get the result.

Geometrical interpretation of Rolle's theorem: If the graph, $y = f(x)$ be continuous throughout the interval from a to b; and if the curve has a tangent at every point on it from a to b except possibly at the two extreme points a and b; and has the ordinates at two points a, b equal, then there must exist at least one point on the curve between a and b where the tangent is parallel to the x-axis (Figures 4.1(a) and (b)).

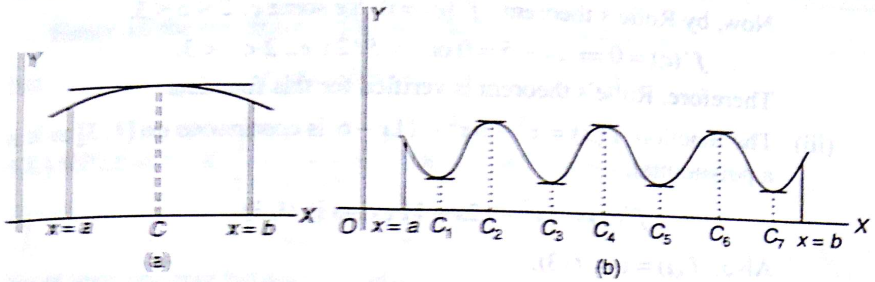

Figure 4.1 Geometrical interpretation of Rolle's theorem.

EXAMPLE 4.1 Verify Rolle's theorem for each of the following cases:

(i) $f(x)=|x|, -1\le x\le 1$; (WBUT 2003)

(ii) $f(x)=x^2-5x+6$ in $2\le x\le 3$ (WBUT 2001)

(iii) $f(x)=x^3-6x^2+11x-6$ in $1\le x\le 3$ (WBUT 2002)

(iv) $f(x)=\sin x$, in $[c,\pi]$

(v) $f(x)=3+(x-1)^{1/3}$ in $0\le x\le 2$.

Solution

(i) The function $f(x)=|x|$ is continuous on $[-1, 1]$. But, it is not differentiable at $x=0$, shown below.

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|-0}{h}$$

$$=\begin{cases}\lim\limits_{h\to 0^-}\dfrac{-h}{h}=\lim\limits_{h\to 0^-}(-1)=-1\\ \lim\limits_{h\to 0^+}\dfrac{h}{h}=\lim\limits_{h\to 0^+}(1)=1.\end{cases}$$

This shows that the left and right derivatives are not equal and hence $f'(x)$ does not exist at $x=0$, i.e. in $(-1, 1)$. Therefore, all the conditions of Rolle's theorem do not be satisfied. Hence Rolle's theorem is not applicable for $f(x)=|x|$.

(ii) The function $f(x)=x^2-5x+6$ is continuous on $[2, 3]$ as it is a polynomial (every polynomial is continuous everywhere).

$$f'(x)=2x-5 \text{ exists for all } x \text{ in } (2, 3).$$

$$f(2)=0=f(3).$$

Hence all the conditions of Rolle's theorem are satisfied for $f(x)=x^2-5x+6$.

Now, by Rolle's theorem, $f'(c)=0$ for some c, $2<c<3$.

$$f'(c)=0 \Rightarrow 2x-5=0 \text{ or } c=5/2 \text{ i.e., } 2<c<3.$$

Therefore, Rolle's theorem is verified for this function.

(iii) The function $f(x)=x^3-6x^2+11x-6$ is continuous on [1, 3] as it is a polynomial.

$$f'(x)=3x^2-12x+11 \text{ exists in } (1, 3).$$

Also, $f(1)=c=f(3)$.

Hence, $f(x)$ satisfied all the conditions of Rolle's theorem.

Now, $f'(c)=0, 1<c<3$ implies $3c^2-12c+11=0$

or $$c=2\pm 1/\sqrt{3}.$$

Hence, $c=2+1/\sqrt{3}, 2-1/\sqrt{3}$ and $1<2+1/\sqrt{3}<3$

and $1<2-1/\sqrt{3}<3$, i.e. $1<c<3$ for both c.

Hence, Rolle's theorem is verified for this function.

(iv) $f(x)=\sin x$ is continuous on $[0, \pi]$.

$f'(x)=\cos x$ exists for all values of x in $(0, \pi)$.

Also, $f(0)=0=f(\pi)$.

Hence $f(x)$ satisfied all the conditions of Rolle's theorem.

Hence $f'(c)=0$ for some c, $0<c<\pi$,

implies $\cos c=0$ or $c=\pi/2$, i.e. $0<\pi/2<\pi$.

Therefore, Rolle's theorem is verified for this function.

(v) Here the function $f(x)=3+(x-1)^{1/3}$ is continuous on [0, 2].

$f'(x)=\frac{1}{3}(x-1)^{-2/3}$ does not exist at $x=1$.

That is, $f(x)$ is not differentiable at $x=1$, i.e., in (0, 2).

Thus, the function $f(x)$ does not satisfy all the conditions of Rolle's theorem. Hence, Rolle's theorem is not applicable for this function.

EXAMPLE 4.2 Show that for the function $f(x)=\frac{1}{x}+\frac{1}{1-x}; 0\le x\le 1$ all the conditions of Rolle's theorem do not satisfied, but $f'(c)=0$ for $0<c<1$.

Solution

The function is continuous in $0<x<1$, not in $0\le x\le 1$, and $f'(x)=\frac{1}{(1-x)^2}-\frac{1}{x^2}$ exists in $0<x<1$ and $f(0)\ne f(1)$ (both being undefined).

Hence all the conditions of Rolle's theorem do not hold.

But, $$f'(c) = 0 \text{ when } c = 1/2, 0 < c < 1.$$

EXAMPLE 4.3 If $\frac{a}{5}+\frac{b}{4}+\frac{c}{3}+\frac{d}{2}+e=0$ show that the equation

$$ax^4+bx^3+cx^2+dx+e=0$$

has at least one root between 0 and 1.

Solution

Let $f(x)=\frac{ax^5}{5}+\frac{bx^4}{4}+\frac{cx^3}{3}+\frac{dx^2}{2}+ex, 0 \le x \le 1.$

Therefore, $$f(0)=0 \text{ and } f(1)=\frac{a}{5}+\frac{b}{4}+\frac{c}{3}+\frac{d}{2}+e=0$$

(by the given condition).

Thus $$f(0)=f(1).$$

As $f(x)$ is a polynomial, it is continuous on [0, 1] and its derivative $f'(x) = ax^4+bx^3+cx^2+dx+e$ exists in (0, 1).

$\therefore$ by Rolle's theorem, $f'(c_1)=0, 0<c_1<1,$

or $$ac_1^4+bc_1^3+c\,c_1^2+dc_1+e=0.$$

Hence $c_1, 0<c_1<1,$ is a root of the equation

$$ax^4+bx^3+cx^2+dx+e=0.$$

EXAMPLE 4.4 Show that the root of the equation $e^x \cos x + 1 = 0$ lies between any pair of roots of $e^x \sin x - 1 = 0$.

Solution

Let $F(x)=e^{-x}(e^x \sin x - 1) = \sin x - e^{-x}$ and a and b be any two roots of $e^x \sin x - 1 = 0$, i.e. $F(a)=F(b)=0$.

Now,

(i) $F(x)$ is continuous on $[a, b]$, as $\sin x$ and e^{-x} are both continuous there

(ii) $F'(x)=\cos x + e^{-x}$, exists in (a, b) and

(iii) $F(a)=F(b)$.

Therefore, by Rolle's theorem $F'(c)=0$ in $a<c<b$.

That is, $\cos c + e^{-c} = 0$ in (a, b)

or $$e^c \cos c + 1 = 0, a<c<b.$$

Therefore, c, in (a, b), is a root of the equation $e^x \cos x + 1 = 0$.

EXAMPLE 4.5 Show that Rolle's theorem is not applicable to $f(x) = \tan x$ on $[0, \pi]$, although $f(0) = f(\pi)$. (WBUT 2004)

Solution

The function $f(x) = \tan x$ is continuous on $[0, \pi]$ except at $x = \pi/2$ and $f'(x) = \sec^2 x$ does not exist at $x = \pi/2$. But, $f(0) = f(\pi)$. Hence $f(x)$ does not satisfy all the conditions of Rolle's theorem and consequently Rolle's theorem is not applicable for $f(x) = \tan x$ on $[0, \pi]$.

4.3 LAGRANGE'S MEAN VALUE THEOREM

If a function f is

(i) continuous in the closed interval $[a, b]$

(ii) derivable in the open interval (a, b)

Then there exists at least one value of x, say c, such that

$$\frac{f(b) - f(a)}{b - a} = f'(c), \text{ for } a < c < b \tag{4.1}$$

Note: The Lagrange's mean value theorem is known as mean value theorem or in short MVT.

Corollary 4.1 $h - \theta$ form of MVT.

When $b = a + h$ then $c = a + \theta h$, $0 < \theta < 1$ where as $a < c < b$.
Therefore, the Lagrange's MVT in $[a, a + h]$ is

$$f(a + h) - f(a) = hf'(a + \theta h), \; 0 < \theta < 1.$$

Corollary 4.2 Substituting $h = x$ and $a = 0$ to the above result.

$$f(x) - f(0) = xf'(\theta x), \; 0 < \theta < 1 \text{ in the interval } [0, x]$$

or $$f(x) = f(0) + xf'(\theta x), \; 0 < \theta < 1 \text{ in } [0, x].$$

Geometrical interpretation of MVT

From Figure 4.2, $\dfrac{f(b) - f(a)}{b - a} = \dfrac{BN}{AN} = \tan \angle BAN$. Since $f'(c) = \tan \angle CTX$. From MVT $\tan \angle BAN = \tan \angle CTX$ or $\angle BAN = \angle CTX$, i.e., AB is parallel to CT.

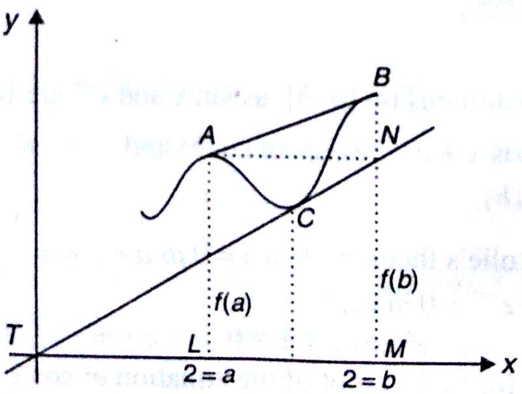

Figure 4.2 Geometrical interpretation of MVT.

Thus, if the graph ACB of $f(x)$ is a continuous curve having everywhere a tangent, then there must be at least one point C between A and B at which the tangent is parallel to the chord AB.

4.3.1 Deduction of Rolle's Theorem from Lagrange's MVT

If $f(x)$ is continuous on $[a, b]$ and derivable in (a, b) then by Lagrange's MVT, we have for at least one c,

$$\frac{f(b)-f(a)}{b-a}=f'(c), \quad a<c<b.$$

In addition, if $f(a)=f(b)$ (which is the third condition of Rolle's Theorem) then $f'(c)=0$, $a<c<b$. Hence, the Rolle's theorem follows from Lagrange's theorem.

EXAMPLE 4.6 Verify Lagrange's mean value theorem for the following functions:

(i) $f(x)=x^2+3x+2$ in $1 \le x \le 2$

(ii) $f(x)=\dfrac{1}{x}$ in $-1 \le x \le 1$

(iii) $f(x)=1+x^{2/3}$ in $[-8, 1]$

(iv) $f(x)=\begin{cases} x\sin\dfrac{1}{x}, & x \ne 0 \\ 0, & x=0 \end{cases}$ in $[-1, 1]$

Solution

(i) Since $f(x) = x^2 + 3x + 2$ is a polynomial and every polynomial is continuous and differentiable everywhere, $f(x) = x^2 + 3x + 2$ is continuous and differentiable in $1 \le x \le 2$. Also, $f'(x) = 2x + 3$, $1 \le x \le 2$.

Therefore, $f(x)$ satisfies all the conditions of Lagrange's MVT.

Let c, $1 < c < 2$, be any value of x. Then by Lagrange's MVT

$$\frac{f(2)-f(1)}{2-1}=f'(c)$$

or $$\frac{12-6}{1}=2c+3$$

or $$2c+3=6 \text{ or, } c=3/2,$$

i.e. $$1<3/2<2.$$

Hence Lagrange's MVT is verified for this function.

(ii) Here $f(x)=\dfrac{1}{x}$ and $f'(x)=-\dfrac{1}{x^2}$ in $-1 \le x \le 1$. It is easy to observe that both the functions $f(x)$ and $f'(x)$ do not exist at $x=0$. Thus, $f(x)$ is neither continuous nor derivable on $-1 \le x \le 1$.

Therefore, the Lagrange's MVT is not applicable for this function.

(iii) The function $f(x) = 1 + x^{2/3}$ is continuous on $[-8, 1]$ as the function is finite for all values of x on $[-8, 1]$.

But, $f'(x) = \frac{2}{3}x^{-2/3}$ does not exist at $x = 0$, i.e. $f(x)$ is not derivable in $(-8, 1)$

Hence Lagrange's MVT is not applicable for this function.

(iv) Here $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

To test continuity

Let ε be any pre-assigned positive number and $\delta > 0$ be another number that depends on ε.

Then $$|f(x) - f(0)| = \left|x\sin\frac{1}{x} - 0\right| = |x|\left|\sin\frac{1}{x}\right| \leq |x|$$

as $$\left|\sin\frac{1}{x}\right| \leq 1$$

$\therefore$ $$|f(x) - f(0)| < \varepsilon \text{ if } |x| < \varepsilon$$

or $$|f(x) - f(0)| < \varepsilon \text{ if } |x - 0| < \delta \quad \text{where } \delta = \varepsilon.$$

Hence $f(x)$ is continuous at $x = 0$ and obviously it is continuous for all values of x on $[-1, 1]$.

To test differentiability

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x\sin\frac{1}{x} - 0}{x} = \lim_{x\to 0}\sin\frac{1}{x}.$$

But, it is well known that $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist. Therefore, $f(x)$ is not differentiable at $x = 0$ and consequently $f(x)$ is not differentiable in $(-1, 1)$.

Thus Lagrange's MVT is not applicable for this function.

EXAMPLE 4.7 Use Lagrange's MVT to prove the following inequalities:

(i) $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$

(ii) $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$ for all $x > 0$

(iii) $0 < \frac{1}{x}\log\left(\frac{e^x - 1}{x}\right) < 1$, for $x > 0$

(iv) $\dfrac{x}{\sqrt{1-x^2}} \geq \sin^{-1} x \geq x$, if $0 \leq x < 1$

Solution

(i) Let $f(x) = \log(1+x)$.

Therefore, $f(0) = 0$ and $f'(x) = \dfrac{1}{1+x}$ or $f'(\theta x) = \dfrac{1}{1+\theta x}, 0 < \theta < 1.$

Then by MVT, for the interval $[0, x]$, we have

$$f(x) = f(0) + x f'(\theta x), \quad 0 < \theta < 1$$

or $$\log(1+x) = \frac{x}{1+\theta x}. \tag{1}$$

Since $0 < \theta < 1, 0 < \theta x < x$ $(\because x > 0)$

or $1 < 1 + \theta x < 1 + x$ or $1 > \dfrac{x}{1+\theta x} > \dfrac{1}{1+x}$

or $\dfrac{x}{1+x} < \dfrac{1}{1+\theta x} < x$ or $\dfrac{x}{1+x} < \log(1+x) < x$ [Using (1)]

(ii) From previous examples, we have

$$\frac{x}{1+x} < \log(1+x) < x$$

or $\dfrac{1+x}{x} > \dfrac{1}{\log(1+x)} > \dfrac{1}{x}$ or $0 < \dfrac{1}{\log(1+x)} - \dfrac{1}{x} < 1.$

(iii) Let $f(x) = e^x$ $\therefore f'(x) = e^x, f(0) = 1$

From MVT, on $[0, x]$, we have

$$f(x) = f(0) + x f'(\theta x), \quad 0 < \theta < 1$$

or $$e^x = 1 + x\theta x \text{ or } \frac{e^x - 1}{x} = e^{\theta x}$$

or $$\theta = \frac{1}{x}\log\left(\frac{e^x-1}{x}\right).$$

Since, $$0 < \theta < 1, \quad 0 < \frac{1}{x}\log\left(\frac{e^x-1}{x}\right) < 1.$$

(iv) Let $f(x) = \sin^{-1} x$.

$\therefore$ $$f(0) = 0, \; f'(x) = \frac{1}{\sqrt{1-x^2}}.$$

From MVT, on $[0, x]$, we have

$$f(x) = f(0) + x f'(\theta x), \quad 0 < \theta < 1$$

or
$$\sin^{-1} x = \frac{x}{\sqrt{1-(\theta x)^2}}, \quad 0 < \theta < 1.$$

Since, $\quad 0 < \theta < 1, 0 \le \theta x \le x \quad$ as $\quad 0 \le x < 1$

or $\quad -x^2 \le -(\theta x)^2 \le 0 \quad$ or $\quad 1 - x^2 \le 1 - (\theta x)^2 \le 1$

$$1 \le \frac{1}{\sqrt{1-(\theta x)^2}} \le \frac{1}{\sqrt{1-x^2}} \quad \text{or} \quad x \le \sin^{-1} x \le \frac{x}{\sqrt{1-x^2}}.$$

EXAMPLE 4.8 Prove that if $0 < a < b$ then

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}.$$

Solution

Let $f(x) = \tan^{-1}x$. $f(x)$ satisfies all the conditions of Lagrange's theorem on $[a, b]$.

Therefore, by Lagrange's MVT we have

$$\frac{f(b) - f(a)}{b-a} = f'(c), a < c < b$$

or
$$\frac{\tan^{-1} b - \tan^{-1} a}{b-a} = \frac{1}{1+c^2}, a < c < b.$$

Now, $\quad a < c < b \quad$ or $\quad a^2 + 1 < c^2 + 1 < b^2 + 1$

or
$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

or
$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

or
$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

EXAMPLE 4.9 In the MVT applied to $f(x)$ in $[0, h]$, i.e. in

$$f(h) = f(0) + hf'(\theta h), 0 < \theta < 1$$

prove that $\lim\limits_{h \to 0^+} \theta = \frac{1}{2}$ when $f(x) = \cos x$.

Solution

Here

$$f(x) = \cos x.$$

$$f(0) = 1 \text{ and } f'(x) = -\sin x$$

Now, from $f(h) = f(0) + hf'(\theta h)$, we have

$$\cos h = 1 + h(-\sin \theta h) \text{ or } \frac{\cos h - 1}{h} = -\sin \theta h$$

or
$$\frac{2\sin^2(h/2)}{h} = \sin \theta h \text{ or } \left(\frac{\sin(h/2)}{h/2}\right)^2 = 2\frac{\sin \theta h}{h}$$

or
$$2 \lim_{h \to 0^+} \left(\frac{\sin \theta h}{\theta h} \cdot \theta\right) = \lim_{h \to 0^+} \left(\frac{\sin(h/2)}{h/2}\right)^2$$

or
$$2 \lim_{h \to 0^+} \theta = 1 \left[\because \lim_{h \to 0^+} \left(\frac{\sin \theta h}{\theta h}\right) = 1\right].$$

Hence $\lim_{h \to 0^+} \theta = \frac{1}{2}$.

EXAMPLE 4.10 Apply Lagrange's mean value theorem, find the derivative of a function assuming that the derivatives which occur are continuous.

Solution

The Lagrange's MVT on $[a, a + h]$ is

$$f(a + h) - f(a) = hf'(a + \theta h), 0 < \theta < 1$$

or
$$f'(a + \theta h) = \frac{f(a + h) - f(a)}{h}.$$

Taking limit $h \to 0$, we get

$$\lim_{h \to 0} f'(a + \theta h) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

or
$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

$$\left[\because f'(x) \text{ is continuous (given), so } \lim_{h \to 0} f'(a + h) = f'(a)\right]$$

This is the well known formula for derivative.

EXAMPLE 4.11 Use Lagrange's MVT to prove that, if $f(x)$ is continuous and $f'(x) > 0$, then $f(x)$ is an increasing function.

Solution

Let $f(x)$ be continuous on $[a, b]$ and differentiable in (a, b) where $a < b$. Then by Lagrange's MVT,

$$f(b) - f(a) = (b - a) f'(c), \qquad a < c < b.$$

Since $f'(x) > 0$ on $[a, b]$ and $b > a$, the right hand side of the above equation is positive, and hence

$$f(b) - f(a) > 0 \text{ or } f(b) > f(a)$$

Thus f is an increasing function on $[a, b]$.

EXAMPLE 4.12 Use MVT to prove that

$$\sin 46° \simeq \frac{1}{2}\sqrt{2}\left(1 + \frac{\pi}{180}\right)$$

Is this estimate high or how? (WBUT 2003)

Solution

Let $f(x) = \sin x$. The Lagrange's MVT on $[a, a+h]$ is

$$f(a+h) = f(a) + hf'(a + \theta h), 0 < \theta < 1.$$

Let $a = 45°$, $h = 1°$.

Then
$$f(a) = f(45°) = \sin 45° = \frac{1}{\sqrt{2}}.$$

$$f'(a) = \cos x, f'(a + \theta h) = f'(45° + \theta.1°) = \cos(45° + \theta°).$$

Therefore,
$$\sin(45° + 1°) = \frac{1}{\sqrt{2}} + 1° \times \cos(45° + \theta°)$$

or
$$\sin 46° = \frac{1}{\sqrt{2}} + \frac{\pi}{180}\cos(45° + \theta°)$$

$$\simeq \frac{1}{\sqrt{2}} + \frac{\pi}{180}\cos 45° \left[\because \theta° \text{ is too small compared to } 45°\right]$$

$$\simeq \frac{1}{\sqrt{2}} + \frac{\pi}{180} \cdot \frac{1}{\sqrt{2}}$$

$$\simeq \frac{1}{\sqrt{2}}\left(1 + \frac{\pi}{180}\right)$$

Hence $\sin 46° \simeq \frac{1}{2}\sqrt{2}\left(1 + \frac{\pi}{180}\right)$.

For $\cos(45° + \theta°) < \cos 45°$.

$\therefore$ Exact value of $\sin 46°$ < approximate value of $\sin 46°$.

Thus the estimate is high.

4.4 CAUCHY'S MEAN VALUE THEOREM

If two functions f and g

(i) be both continuous on $a \le x \le b$

(ii) are both differentiable in $a < x < b$

(iii) $g'(x)$ does not vanish at any value of x in $a < x < b$, then there exists at least one value of x, say c, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \quad \text{for } a < c < b. \tag{4.2}$$

Corollary 4.3 $\theta - h$ form of Cauchy's MVT

Let $b = a + h$. Then $c = a + \theta h$, $0 < \theta < 1$.

Then from Equation (4.2)

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, \quad 0<\theta<1 \tag{4.3}$$

on $[a, a+h]$.

Corollary 4.4 Cauchy's MVT on $[0, x]$.

Substituting $h = x$ and $a = 0$ to the above expression.

Then $$\frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(\theta x)}{g'(\theta x)}, \quad 0<\theta<1.$$

4.4.1 Reduction of Lagrange's MVT from Cauchy's MVT

The Cauchy's MVT for the functions f and g on $[a, b]$ is

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \quad a<c<b.$$

We consider $g(x) = x$ which is continuous and differentiable everywhere, and $g'(x) = 1$.

Then the above expression reduces to

$$\frac{f(b)-f(a)}{b-a} = \frac{f'(c)}{1}, \quad a<c<b$$

or $$f(b)-f(a) = (b-a)\,f'(c), \quad a<c<b$$

This is the well known form of Lagrange's MVT.

EXAMPLE 4.13 Verify Cauchy's mean value theorem for the following functions:

(i) $f(x) = 2x^2$, $g(x) = 4x+1$, for $1 \le x \le 2$

(ii) $f(x) = \sin x$, $g(x) = x^3$, for $-1 \le x \le 1$.

Solution

(i) The functions $f(x) = 2x^2$ and $g(x) = 4x+1$ both are continuous on $[1, 2]$.

$$f'(x) = 4x \text{ and } g'(x) = 4 \text{ both exist in } (1, 2)$$

Also, $\qquad g'(x) \neq 0$ for all x.

Thus Cauchy's MVT is applicable for these functions.

Now, by Cauchy's MVT,

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}, 1 < c < 2$$

or
$$\frac{8-2}{9-5} = \frac{4c}{4} \text{ or } \frac{6}{4} = c \text{ or } c = \frac{3}{2}.$$

That is c lies between 1 and 2. Hence Cauchy's MVT is verified for these functions.

(ii) Here both $f(x) = \sin x$ and $g(x) = x^3$ are continuous on $[-1, 1]$ and both are differentiable in $(-1, 1)$, where $f'(x) = \cos x$ and $g'(x) = 3x^2$. But, $g'(x) = 0$ at $x = 0$. Thus, all the conditions of Cauchy's MVT do not satisfy. Hence Cauchy's MVT is not applicable for these functions.

EXAMPLE 4.14 Use Cauchy's MVT to prove that $0 < \frac{1}{x}\log\left(\frac{e^x-1}{x}\right) < 1$.

Solution

Let $f(x) = e^x$ and $g(x) = x$. $f'(x) = e^x$ and $g'(x) = 1$.

The Cauchy's MVT in $[0, x]$ is

$$\frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(\theta x)}{g'(\theta x)}, 0 < \theta < 1$$

or
$$\frac{e^x-1}{x-0} = \frac{e^{\theta x}}{1} \text{ or } e^{\theta x} = \frac{e^x-1}{x}$$

or
$$\theta x = \log\left(\frac{e^x-1}{x}\right), 0 < \theta < 1 \text{ or } \theta = \frac{1}{x}\log\left(\frac{e^x-1}{x}\right).$$

Since,
$$0 < \theta < 1, 0 < \frac{1}{x}\log\left(\frac{e^x-1}{x}\right) < 1.$$

4.5 GENERALISED MEAN VALUE THEOREM: TAYLOR'S THEOREM

If a function f be such that

(i) the $(n-1)$th derivative f^{n-1} is continuous on $[a, a+h]$
(ii) the nth derivative f^n exists in the open interval $(a, a+h)$, then there exists at least one number θ, $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n,$$

(i) If $R_n = \dfrac{h^n}{n!} f^n (a + \theta h), 0 < \theta < 1,$ then the theorem is called Taylor's theorem with Lagrange's form of remainder.

(ii) If $R_n = \dfrac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n (a + \theta h), 0 < \theta < 1,$ then the theorem is known as Taylor's theorem with Cauchy's form of remainder.

4.5.1 Alternative form of Taylor's Theorem

If a function f be such that

(i) the $(n-1)$th derivative f^{n-1} is continuous on $[a, x]$.

(ii) the nth derivative f^n exists in the open interval (a, x), then there exists at least one member θ, $0 < \theta < 1$, such that

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1} (a) + R_n,$$

(i) $R_n = \dfrac{(x-a)^n}{n!} f^n (a + \theta (x-a)), 0 < \theta < 1,$ due to Lagrange.

(ii) $R_n = \dfrac{(x-a)^n (1-\theta)^{n-1}}{(n-1)!} f^n (a + \theta (x-a)), 0 < \theta < 1,$ due to Cauchy.

This form of Taylor's theorem is used to expand the functions $f(x)$ in powers of $(x-a)$, i.e. about the point a. The Taylor's theorem is also called the Mean Value Theorem of nth order. The above form of the Taylor's series is the finite form.

EXAMPLE 4.15 Apply Cauchy's MVT to the functions $f(x) = e^x$ and $g(x) = e^{-x}$ in $[x, x+h]$ and obtain the value of θ. Interpret your result.

(WBUT 2003)

Solution

Here $f(x) = e^x$ and $g(x) = e^{-x}$

Therefore, $f'(x) = e^x$ and $g'(x) = -e^{-x}$

The Cauchy's MVT on $[x, x+h]$ is

$$\frac{f(x+h) - f(x)}{g(x+h) - g(x)} = \frac{f'(x+\theta h)}{g'(x+\theta h)}, 0 < \theta < 1,$$

or
$$\frac{e^{x+h} - e^x}{e^{-(x+h)} - e^{-x}} = \frac{e^{x+\theta h}}{-e^{-(x+\theta h)}}$$

or
$$\frac{e^x (e^h - 1)}{e^{-x} (e^{-h} - 1)} = \frac{e^x \cdot e^{\theta h}}{-e^{-x} \cdot e^{-\theta h}}$$

or $$\frac{e^h - 1}{e^{-h}(1-e^h)} = -e^{2\theta h}$$

or $$e^{2\theta h} = e^h \quad \text{or} \quad 2\theta h = h \quad \text{or} \quad 2\theta = 1, \quad \text{i.e.} \quad \theta = \frac{1}{2}.$$

It may be noted that the condition $0 < \theta < 1$ is satisfied by this θ and it is independent of both x and h.

Note: The condition (i) of Taylor's theorem implies that $f, f', f'', \ldots, f^{n-1}$ all are continuous on the interval $[a, a+h]$.

A particular case of Taylor's theorem known as Maclaurin's theorem is stated below.

4.5.2 Maclaurin's Theorem

If a function f be such that

(i) the $(n-1)$th derivative f^{n-1} is continuous on $[0, x]$,

(ii) the nth derivative f^n exists in $(0, x)$, then there exists at least one number θ, where $0 < \theta < 1$, such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$$

where $R_n = \frac{x^n}{n!} f^n(\theta x), 0 < \theta < 1$ the theorem is known as Maclaurin's theorem with Lagrange's form of remainder, and when $R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^n(\theta x), 0 < \theta < 1$, the theorem is known as Maclaurin's theorem with Cauchy's form of remainder.

Note: The Maclaurin's theorem is used to expand a function in powers of x i.e. to expand in the neighbourhood of the origin.

EXAMPLE 4.16 Expand $7x^3 + 4x + 8$ in powers of $x - 1$.

Solution

Let $f(x) = 7x^3 + 4x + 8$ and $a = 1$. $f(1) = 19$.

$f'(x) = 21x^2 + 4$ $\qquad f'(1) = 25$

$f''(x) = 42x$ $\qquad f''(1) = 42$

$f'''(x) = 42$ $\qquad f'''(1) = 42$

$f^{iv}(x) = 0$ $\qquad f^{iv}(1) = 0.$

Then by Taylor's theorem

$$f(x) = f(1) + (x-1)\, f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \frac{(x-1)^4}{4!} f^{\text{iv}}(1)$$

$$= 19 + 25\,(x-1) + 42 \times \frac{(x-1)^2}{2!} + 42 \times \frac{(x-1)^3}{3!}$$

$$= 19 + 25\,(x-1) + 21\,(x-1)^2 + 7\,(x-1)^3.$$

EXAMPLE 4.17 Apply Maclaurin's theorem to the function $f(x) = (1+x)^4$ to deduce $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$.

Solution

Here $f(x) = (1+x)^4$. Then $f(0) = 1$.

$f'(x) = 4(1+x)^3$	$f'(0) = 4$
$f''(x) = 12(1+x)^2$	$f''(0) = 12$
$f'''(x) = 24(1+x)$	$f'''(0) = 24$
$f^{\text{iv}}(x) = 24$	$f^{\text{iv}}(0) = 24$
	$f^{\text{v}}(x) = 0.$

Then by Maclaurin's theorem

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{\text{iv}}(0)$$

or $$(1+x)^4 = 1 + 4x + \frac{x^2}{2!} \times 12 + \frac{x^3}{3!} \times 24 + \frac{x^4}{4!} \times 24$$

or $$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

EXAMPLE 4.18 Obtain the expansions of the following functions with the remainder in Lagrange's form

(i) $e^x = 1 + x + \dfrac{1}{2!}x^2 + \dfrac{1}{3!}x^3 + \dfrac{1}{4!}x^4 e^{\theta x},$

(ii) $\cos x + \sin x = 1 + x - \dfrac{1}{2!}x^2 - \dfrac{1}{3!}x^3 + \dfrac{1}{4!}x^4 (\sin \theta x + \cos \theta x),$

(iii) $(x+h)^{3/2} = x^{3/2} + \dfrac{3}{2}x^{1/2}\, h + \dfrac{3}{2} \cdot \dfrac{1}{2} \cdot \dfrac{h^2}{2!} \dfrac{1}{\sqrt{(x+\theta h)}}$

(iv) $a^x = 1 + x \log a + \dfrac{x^2}{2!}(\log a)^2 + \cdots + \dfrac{x^{n-1}}{(n-1)!}(\log a)^{n-1} + \dfrac{x^n}{n!} a^{\theta x} (\log a)^n.$

Solution

(i) Let $f(x) = e^x$, $f(0) = 1$.

$$f'(x) = e^x,\ f'(0) = 1,\ f''(x) = e^x,\ f''(0) = 1.$$

Similarly, $\quad f'''(0) = 1,\ f^{\text{iv}}(x) = e^x$ and $f^{\text{iv}}(\theta x) = e^{\theta x}, 0 < \theta < 1.$

Then by Maclaurin's theorem up to fifth term, with Lagrange's form of remainder

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{\text{iv}}(\theta x), 0 < \theta < 1,$$

or $$e^x = 1 + x\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} e^{\theta x}, 0 < \theta < 1.$$

(ii) Let $f(x) = \cos x + \sin x$, $f(0) = 1$

$f'(x) = -\sin x + \cos x, \qquad f'(0) = +1$

$f''(x) = -\cos x - \sin x, \qquad f''(0) = -1$

$f'''(x) = \sin x - \cos x, \qquad f'''(0) = -1$

$f^{\text{iv}}(x) = \cos x + \sin x, \qquad f^{\text{iv}}(\theta x) = \cos \theta x + \sin \theta x$

Therefore, by Maclaurin's theorem with Lagrange's form of remainder,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{\text{iv}}(\theta x), 0 < \theta < 1$$

or $$\cos x + \sin x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} (\cos \theta x + \sin \theta x), 0 < \theta < 1.$$

(iii) Let $$f(h) = (x+h)^{3/2},\ f(0) = x^{3/2}$$

$$f'(h) = \frac{3}{2}(x+h)^{1/2},\ f'(0) = \frac{3}{2} x^{1/2}$$

$$f''(h) = \frac{3}{2} \cdot \frac{1}{2} (x+h)^{1/2},\ f''(\theta h) = \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{(x+\theta h)}}$$

The Maclaurin's theorem in powers of h with Lagrange's form of remainder

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h)$$

$$= x^{3/2} + \frac{3}{2} h\, x^{1/2} + \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{h^2}{2!} \cdot \frac{1}{\sqrt{(x+\theta h)}}, 0 < \theta < 1.$$

(iv) Let
$$f(x) = a^x,\ f(0) = 1$$
$$f'(x) = a^x \cdot \log a,\ f'(0) = \log a$$
$$f''(x) = a^x (\log a)^2,\ f''(0) = (\log a)^2$$

Similarly,
$$f^{n-1}(x) = a^x (\log a)^{n-1},\ f^{n-1}(0) = (\log a)^{n-1}$$
$$f^n(x) = a^x (\log a)^n,\ f^n(\theta x) = a^{\theta x} (\log a)^n.$$

Therefore, by Maclaurin's theorem with Lagrange's form of remainder,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x), 0<\theta<1$$

or

$$a^x = 1 + x \log a + \frac{x^2}{2!} (\log a)^2 + \cdots + \frac{x^{n-1}}{(n-1)!} (\log a)^{n-1} + \frac{x^n}{n!} e^{\theta x} (\log a)^n, 0<\theta<1.$$

EXAMPLE 4.19 Use MVT, to prove that, if $0 \le x \le 1$,

$$\left| \log(1+x) - x + \frac{1}{2} x^2 \right| \le \frac{1}{3} x^3.$$

Solution

Let $f(x) = \log(1+x),\ f(0) = 0$

$$f'(x) = \frac{1}{1+x},\ f'(0) = 1$$

$$f''(x) = \frac{1}{(1+x)^2},\ f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3},\ f'''(\theta x) = \frac{2}{(1+\theta x)^3}.$$

Now, by Maclaurin's finite series up to fourth term,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x), 0<\theta<1$$

or
$$\log(1+x) = 0 + x - \frac{x^2}{2!} + \frac{x^3}{3!} \cdot \frac{2}{(1+\theta x)^3}, 0<\theta<1$$

or
$$\log(1+x) - x + \frac{x^2}{2} = \frac{x^3}{3} \cdot \frac{1}{(1+\theta x)^3}, 0<\theta<1$$

Since $0<\theta<1$ and $0 \le x \le 1, 0 \le \theta x \le 1$.

Therefore, $$(1+\theta x)^3 \geq 1 \quad \text{or} \quad \frac{1}{(1+\theta x)^3} \leq 1.$$

Hence $$\left| \log(1+x) - x + \frac{x^2}{2} \right| \leq \frac{1}{3} x^3.$$

EXAMPLE 4.20 If f'' is continuous on some neighbourhood of c, prove that

$$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

Solution

From Taylor's series in $[c, c+h]$ we have

$$f(c+h) = f(c) + hf'(c) + \frac{h^2}{2!} f''(c + \theta_1 h), 0 < \theta_1 < 1 \tag{1}$$

This result is true for all h, negative or positive or 0. Replacing h by $-h$ in equation (1), we get

$$f(c-h) = f(c) - hf'(c) + \frac{h^2}{2!} f''(c - \theta_2 h), 0 < \theta_2 < 1 \tag{2}$$

Adding (1) and (2) we get

$$f(c+h) + f(c-h) = 2f(c) + \frac{h^2}{2!} \{f''(c+\theta_1 h) + f''(c - \theta_2 h)\}$$

or $$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = \frac{1}{2!} \lim_{h \to 0} \{f''(c+\theta_1 h) + f''(c-\theta_2 h)\}$$

$$= \frac{1}{2} \times \{2 f''(c)\}$$

Because f'' is continuous in the neighbourhood of c,

$$\lim_{h \to 0} f''(c + \theta_1 h) = f''(c)$$

and also $$\lim_{h \to 0} f''(c - \theta_2 h) = f''(c)]$$

Hence $$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

4.6 MACLAURIN'S INFINITE SERIES

If (i) f be defined on $[-h, h]$

(ii) for each positive integer n, $f^n(x)$ exists for $-h \leq x \leq h$

(iii) $\lim_{n \to \infty} R_n = 0$ for each x in $[-h, h]$, where R_n is the remainder after n terms, then for each x in $[-h, h]$

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n\,(0) + \cdots$$

This series is called Maclaurin's infinite series expansion of $f(x)$ about $x = 0$.

EXAMPLE 4.21 Obtain the Maclaurin's infinite series for $\sin x$ and show that the series is valid for all real x.

Solution

Let $f(x) = \sin x,$ $\quad f(0) = 0$

$f'(x) = \cos x = \sin(\pi/2 + x),$ $\quad f'(0) = 1$

$f''(x) = \cos(\pi/2 + x) = \sin(2.\pi/2 + x),$ $\quad f''(0) = 0$

$f'''(x) = \cos(2.\pi/2 + x) = \sin(3.\pi/2 + x),$ $\quad f'''(0) = -1$

and so on.

Therefore, $\quad f^n\,(x) = \sin(n\pi/2 + x), f^n(0) = \sin(n\pi/2)$

and $\quad f^n\,(\theta x) = \sin(n\pi/2 + \theta x).$

Also, R_n (the Lagrange's form of remainder)

$$= \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} \sin(n\pi/2 + \theta x), 0 < \theta < 1.$$

Maclaurin's infinite series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

or
$$\sin x = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

This series converges to $\sin x$ for all x, iff $R_n \to 0$ as $n \to \infty$ for all x.

Now,
$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \left| \frac{x^n}{n!} \sin(n\pi/2 + \theta x) \right|$$

$$\le \lim_{n\to\infty} \frac{|x|^n}{n!} = 0 \text{ for all } x$$

$$[\because |\sin(n\pi/2 + \theta x)| \le 1]$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \text{ for all } x.$$

EXAMPLE 4.22 Find the expansion of $f(x) = (1+x)^m$, where m is a negative fraction.

Solution

Here $f(x) = (1+x)^m$, where $x \neq -1$

$$f'(x) = m(1+x)^{m-1}$$

$$f''(x) = m(m-1)(1+x)^{m-2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f^k(x) = m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k}$$

$$f^k(0) = m(m-1)(m-2)\cdots(m-k+1)$$

$$f^n(x) = m(m-1)\cdots(m-n+1)\,x^{m-n},\ x \neq -1.$$

We consider Cauchy's remainder after n terms

$$R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f(1+\theta x), \quad 0<\theta<1$$

$$= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} m(m-1)(m-1)\cdots(m-n+1)(1+\theta x)^{m-n}$$

$$= \frac{m(m-1)(m-1)\cdots(m-n+1)}{(n-1)!} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} (1+\theta x)^{m-1}.$$

We see that

$$\lim_{n\to\infty} \frac{m(m-1):\cdots(m-n+1)}{(n-1)!} x^n = 0 \quad \text{for} \quad |x|^n < 1$$

and $$\frac{1-\theta}{1+\theta x} < 1, \text{ so that } \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0 \text{ as } n \to \infty.$$

Also, $$(1+\theta x)^{m-1} < (1+|x|)^{m-1}, \qquad m>1, 0<\theta<1$$

and $$(1+\theta x)^{m-1} = \frac{1}{(1+\theta x)^{1-m}} < \frac{1}{(1-|x|)^{1-m}}, \qquad \text{when } m<1.$$

Thus $R_n \to 0$ when $n \to \infty$ for $|x|<1$.

Hence $$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots, \quad \text{for } |x|<1.$$

EXAMPLE 4.23 Find the expansion of $\log_e(1+x)$ in a power series of x and indicate the region of validity of the expansion.

Solution

Let
$$f(x) = \log_e(1+x) \qquad f(0) = 0$$
$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$$
$$f''(x) = -\frac{1}{(1+x)^2} \qquad f''(0) = -1$$
$$f'''(x) = \frac{2\cdot 1}{(1+x)^3} \qquad f'''(0) = 2!$$
$$f^{\text{iv}}(x) = \frac{-3\cdot 2\cdot 1}{(1+x)^4} \qquad f^{\text{iv}}(0) = -3!$$

and in this way
$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

Taking Lagrange's form of remainder,
$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{(-1)^{n-1} x^n}{n(1+\theta x)^n} = (-1)^{n-1}\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n$$

Therefore, $f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0) + \cdots + \dfrac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n$

or,
$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + R_n$$

Case I: When $0 \le x \le 1$, then $0 < \theta x < x \le 1$ and
$$|R_n| = \left|\frac{x^n}{n}\right|\left|\frac{1}{(1+\theta x)^n}\right| \le \frac{x^n}{n} \le \frac{1}{n} \text{ since } 0 \le x \le 1.$$

Also, $\dfrac{1}{n} \to 0$ as $n \to \infty$.

Therefore, $R_n \to 0$ as $n \to \infty$ for $0 \le x \le 1$.

Thus the conditions of Maclaurin's infinite series expansion are satisfied for $0 \le x \le 1$.

Case II: When $-1 < x < 0$.

In this case x may or may not be numerically less than $1+\theta x$, so that nothing can be said about the limit of $\left(\dfrac{x}{1+\theta x}\right)^n$ when $n \to \infty$. Thus from Lagrange's form of remainder, we cannot draw any definite conclusion. Now, we consider Cauchy's form of remainder,

$$R_n = \frac{x^n (1-\theta)^{n-1} f^n(\theta x)}{(n-1)!} = \frac{(-1)^{n-1} x^n (1-\theta)^{n-1}}{(1+\theta x)^n}$$

$$= (-1)^{n-1} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \frac{1}{1+\theta x}.$$

Now, $1-\theta < 1+\theta x$ so that $\left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$.

Also, $x^n \to 0$ as $n \to \infty$ and $\dfrac{1}{1+\theta x} < \dfrac{1}{1-|x|}$ and moreover it is independent of n.

Thus $R_n \to 0$ as $n \to \infty$.

Hence, the conditions of Maclaurin's series expansion are satisfied also when $-1 < x < 0$.

Thus $$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \text{ for } -1 < x \le 1.$$

EXAMPLE 4.24 Expand e^x in powers of x in infinite series. (WBUT 2004)

Solution

Let

$$f(x) = e^x,\ f(0) = 1.$$

$$f'(x) = e^x,\ f'(0) = 1,\ f''(x) = e^x,\ f''(0) = 1,$$

and in this way

$$f^n(x) = e^x,\ f^n(\theta x) = e^{\theta x}.$$

The Maclaurin's series for $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$$

where $$R_n = \frac{x^n}{n!} f^n(\theta x), \quad 0 < \theta < 1 \qquad \text{(Due to Lagrange's)}$$

or,
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and
$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} e^{\theta x}.$$

Now,
$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \frac{|x|^n}{n!} e^{\theta x}$$

$$= e^{\theta x} \lim_{n\to\infty} \frac{|x|^n}{n!} = 0. \qquad \left[\because \lim_{n\to\infty} \frac{|x|^n}{n!} = 0\right]$$

4.7 INDETERMINATE FORM: L'HOSPITAL RULE

Suppose we have to calculate the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$. If $\lim_{x\to a} g(x)$ is 0 but $\lim_{x\to a} f(x)$ is non-zero then this limit does not exist. But, when both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are 0 then this limit is of the form $\frac{0}{0}$ and it may have a define value. The form $\frac{0}{0}$ is called an indeterminate form. Other types of indeterminate forms are

$$\frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, 1^{\pm\infty}, \infty^0, \text{etc}.$$

The limit of the form $\frac{0}{0}$ can be obtained by L'Hospital rule.

(a) $\frac{0}{0}$ *form*

L'Hospital rule: If $f(x)$ and $g(x)$ are

(i) continuous in the closed interval $[a, a+h]$,
(ii) derivable in the open interval $(a, a+h)$, and
(iii) $f(a) = g(a) = 0$,

then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \text{ provided the limit exists.}$$

Note: If
$$f(a) = f'(a) = \cdots = f^{n-1}(a) = 0$$

and
$$g(a) = g'(a) = \cdots = g^{n-1}(a) = 0,$$

but,
$$\lim_{x\to a} g^n(a) \neq 0$$

then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f^n(x)}{g^n(x)}.$$

It may be noted that the evaluation of the limit of the forms $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, $1^{\pm\infty}$, ∞^0, etc., depends on the evaluation of the limit of the form $\frac{0}{0}$.

EXAMPLE 4.25 Evaluate the following limits:

(i) $\lim_{x \to 0} \frac{\sin x}{x}$ (ii) $\lim_{x \to 0} \frac{e^x + \sin x - 1}{\log(1+x)}$

(iii) $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ (iv) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

Solution

(i) $\lim_{x \to 0} \frac{\sin x}{x} \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

(ii) $\lim_{x \to 0} \frac{e^x + \sin x - 1}{\log(1+x)} \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{e^x + \cos x}{1/(1+x)} = \frac{1+1}{1} = 2.$$

(iii) $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x} \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{e^x - \cos x\, e^{\sin x}}{1 - \cos x} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x + \sin x\, e^{\sin x} - \cos^2 x\, e^{\sin x}}{\sin x} \left(\frac{0}{0} \text{ form}\right)$$

(Again applying L'Hospital rule)

$$= \lim_{x \to 0} \frac{e^x + \cos x\, e^{\sin x} + \sin x \cos x\, e^{\sin x} - \cos^3 x\, e^{\sin x} + 2 \sin x \cos x\, e^{\sin x}}{\cos x}$$

$$= \frac{1+1+0-1+0}{1} = 1.$$

(iv) $$\lim_{x\to 0}\frac{\tan x - x}{x-\sin x}\left(\frac{0}{0}\text{ form}\right)=\lim_{x\to 0}\frac{\sin x - x\cos x}{x\cos x-\sin x\cos x}\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{\cos x-\cos x+x\sin x}{\cos x - x\sin x-\cos 2x}\left(\frac{0}{0}\text{ form}\right)\quad\left[\because \sin x\cos x=\frac{1}{2}\sin 2x\right]$$

$$=\lim_{x\to 0}\frac{\sin x + x\cos x}{-\sin x-\sin x - x\cos x+2\sin 2x}\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{\cos x+\cos x - x\sin x}{-2\cos x-\cos x+x\sin x+4\cos 2x}$$

$$=\frac{1+1}{-2+4}=2.$$

(b) $\frac{\infty}{\infty}$ *form*: This form is similar to the form $\frac{0}{0}\cdot$ The same technique is applicable for this form, i.e. if

$$\lim_{x\to a} f(x)=\infty,\text{ and }\lim_{x\to a} g(x)=\infty$$

then $$\lim_{x\to a}\frac{f(x)}{g(a)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},\text{ provided }\lim_{x\to a} g'(x)\text{ exists.}$$

EXAMPLE 4.26 Find the value of $\lim_{x\to\infty}\frac{\log(1+x)}{x}$.

Solution

$$\lim_{x\to\infty}\frac{\log(1+x)}{x}\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{d}{dx}[\log(1+x)]}{\frac{d}{dx}[x]}=\lim_{x\to\infty}\frac{\frac{1}{1+x}}{1}=\lim_{x\to\infty}\frac{1}{1+x}=0.$$

(c) $0\cdot\infty$ *form*: This form can be converted to either in $\frac{0}{0}$ form or in $\frac{\infty}{\infty}$ form.

That is, if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$ then $\lim_{x\to a} f(x)\,g(x)$ can be written in the following two forms:

(i) $$\lim_{x\to a} f(x)\,g(x)=\lim_{x\to a}\frac{f(x)}{1/g(x)}\quad\left(\frac{0}{0}\text{ form}\right)$$

(ii) $$\lim_{x\to a} f(x)\,g(x)=\lim_{x\to a}\frac{1/f(x)}{g(x)}\quad\left(\frac{\infty}{\infty}\text{ form}\right).$$

EXAMPLE 4.27 Find $\lim_{x\to 0} \cot x \cdot \log \dfrac{1+x}{1-x}$.

Solution

$$\lim_{x\to 0} \cot x \cdot \log \frac{1+x}{1-x} \quad (\infty \cdot 0 \text{ form})$$

$$= \lim_{x\to 0} \frac{\log(1+x) - \log(1-x)}{\tan x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\dfrac{1}{1+x} + \dfrac{1}{1-x}}{\sec^2 x} = 2.$$

(d) $\infty - \infty$ *form*: If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a} [f(x) - g(x)]$ is of the form $\infty - \infty$. This form can be written as

$$\lim_{x\to a} \frac{1/g(x) - 1/f(x)}{1/\{g(x)\, f(x)\}} \quad \left(\frac{0}{0} \text{ form}\right).$$

EXAMPLE 4.28 Find the following limits:

(i) $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$ (ii) $\lim_{x\to 0} \left\{\dfrac{1}{x} - \dfrac{\log(1+x)}{x^2}\right\}$

(iii) $\lim_{x\to 1} \left\{\dfrac{x}{x-1} - \dfrac{1}{\log x}\right\}$.

Solution

(i) $$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) \quad (\infty - \infty \text{ form})$$

$$= \lim_{x\to 0} \frac{\sin x - x}{x \sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-\sin x}{\cos x + \cos x - x \sin x} = \frac{0}{1+1} = 0.$$

(ii) $$\lim_{x\to 0}\left\{\frac{1}{x}-\frac{\log(1+x)}{x^2}\right\} \qquad (\infty-\infty \text{ form})$$

$$= \lim_{x\to 0}\frac{x-\log(1+x)}{x^2} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{1-\dfrac{1}{1+x}}{2x} = \lim_{x\to 0}\frac{1}{2(1+x)} = \frac{1}{2}.$$

(iii) $$\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\log x}\right) \qquad (\infty-\infty \text{ form})$$

$$= \lim_{x\to 1}\frac{x\log x-(x-1)}{(x-1)\log x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 1}\frac{\log x + x\cdot\dfrac{1}{x}-1}{\log x+(x-1)\cdot\dfrac{1}{x}} = \lim_{x\to 1}\frac{x\log x}{x\log x+x-1} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 1}\frac{\log x + x\cdot\dfrac{1}{x}}{\log x + x\cdot\dfrac{1}{x}+1} = \frac{1}{2}.$$

(e) *Other forms* : $0^0, 1^{\pm\infty}, \infty^0$: These forms occur when the limit is of the form $\lim_{x\to a}[f(x)]^{g(x)}$. This limit can be evaluated by taking logarithm of the function $[f(x)]^{g(x)}$. That is, if we take $y=[f(x)]^{g(x)}$ then $\log y = g(x)\log f(x)$ and $\lim_{x\to a}\log y = \lim_{x\to a} g(x)\log f(x)$ or $\log(\lim_{x\to a} y) = \lim_{x\to a} g(x)\log f(x)$. The right hand side becomes one of the forms discussed earlier.

EXAMPLE 4.29 Evaluate the following limits:

(i) $\displaystyle\lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)^{1/x}$, (ii) $\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}$,

(iii) $\displaystyle\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$, (iv) $\displaystyle\lim_{x\to\infty} x^{1/x}$,

(v) $\displaystyle\lim_{x\to \pi/2}(\sin x)^{\tan x}$.

Solution

(i) The given limit is of the form 1^∞.

Let $y=\left(\dfrac{\sin x}{x}\right)^{1/x}$. Both sides taking logarithm we get,

$$\log y=\frac{1}{x}\log\left(\frac{\sin x}{x}\right).$$

Now, $$\lim_{x\to 0^+}\log y=\lim_{x\to 0^+}\frac{\log\left(\dfrac{\sin x}{x}\right)}{x}\left(\frac{0}{0}\text{ form}\right)\quad\left[\because \lim_{x\to 0^+}\frac{\sin x}{x}=1\right]$$

$$=\lim_{x\to 0^+}\frac{\dfrac{x}{\sin x}\,\dfrac{x\cos x-\sin x}{x^2}}{1}=\lim_{x\to 0^+}\frac{x\cos x-\sin x}{x\sin x}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0^+}\frac{\cos x-x\sin x-\cos x}{\sin x+x\cos x}=\lim_{x\to 0^+}\frac{-x\sin x}{\sin x+x\cos x}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0^+}\frac{-\sin x-x\cos x}{\cos x+\cos x-x\sin x}=\frac{0}{2}=0$$

or $$\log\left(\lim_{x\to 0^+}y\right)=0\ \text{ or }\ \lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)^{1/x}=e^\circ=1.$$

(ii) Let $y=\left(1+\dfrac{2}{x}\right)^x$. Taking logarithm both sides.

$$\log y=x\log\left(1+\frac{2}{x}\right).$$

$$\therefore\quad \lim_{x\to\infty}\log y=\lim_{x\to\infty}\frac{\log\left(1+\dfrac{2}{x}\right)}{1/x}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to\infty}\frac{\dfrac{1}{1+2/x}\left(-\dfrac{2}{x^2}\right)}{-\dfrac{1}{x^2}}=\lim_{x\to\infty}\frac{2}{1+\dfrac{2}{x}}=2$$

i.e. $$\lim_{x\to\infty}\log y=2\ \text{ or }\ \log\left(\lim_{x\to\infty}y\right)=2$$

or $$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2.$$

(iii) Let $y=\left(\dfrac{\tan x}{x}\right)^{1/x^2}$. Then $\log y = \dfrac{1}{x^2}\log\left(\dfrac{\tan x}{x}\right)$

$$\therefore \lim_{x\to 0}\log y = \lim_{x\to 0}\frac{\log\left(\dfrac{\tan x}{x}\right)}{x^2} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$\left[\because \lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x\cos x} \quad \left(\frac{0}{0}\text{ form}\right) = \lim_{x\to 0}\frac{\cos x}{\cos x - x\sin x} = 1\right]$$

$$= \lim_{x\to 0}\frac{\dfrac{x}{\tan x}\cdot\dfrac{x\sec^2 x - \tan x}{x^2}}{2x} = \lim_{x\to 0}\frac{x - \dfrac{1}{2}\sin 2x}{x^2\sin 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{1-\cos 2x}{2x\sin 2x + 2x^2\cos 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{2\sin 2x}{2\sin 2x + 4x\cos 2x + 4x\cos 2x - 4x^2\sin 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{4\cos 2x}{4\cos 2x + 8\cos 2x - 12\,x\sin 2x - 8x\sin 2x - 8x^2\cos 2x}$$

$$= \frac{4}{12} = \frac{1}{3}.$$

$$\therefore \qquad \lim_{x\to 0}\log y = \frac{1}{3} \text{ or } \log\left\{\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}\right\} = \frac{1}{3}$$

or $$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}.$$

(iv) $\lim\limits_{x\to\infty} x^{1/x}$ $\quad$ (∞^0 form).

Let $y = x^{1/x}$. Taking logarithm both sides, we get

$$\log y = \frac{1}{x}\log x.$$

$$\therefore \lim_{x\to\infty} \log y = \lim_{x\to\infty} \frac{\log x}{x}\left(\frac{\infty}{\infty}\text{ form}\right) = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

$$\therefore \quad \log\left(\lim_{x\to\infty} y\right) = 0 \quad \text{or} \quad \lim_{x\to\infty} x^{1/x} = e^{0} = 1.$$

(v) $\lim\limits_{x\to\pi/2} (\sin)^{\tan x}$ (1^{∞}).

Let $y = (\sin x)^{\tan x}$. Taking logarithm both sides, we get

$$\log y = \tan x \log \sin x.$$

$$\therefore \quad \lim_{x\to\pi/2} \log y = \lim_{x\to\pi/2} \tan x \log \sin x \qquad (\infty\cdot 0\text{ form})$$

$$= \lim_{x\to\pi/2} \frac{\log \sin x}{\cot x}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to\pi/2} \frac{\dfrac{1}{\sin x}\cdot\cos x}{-\operatorname{cosec}^2 x} = \lim_{x\to\pi/2} (-\cos x \sin x) = 0.$$

$$\therefore \quad \lim_{x\to\pi/2} \log y = 0 \quad \text{or} \quad \log\left(\lim_{x\to\pi/2} y\right) = 0$$

$$\text{or} \quad \lim_{x\to\pi/2} (\sin x)^{\tan x} = e^{0} = 1.$$

EXAMPLE 4.30

(i) Find the value of $\lim\limits_{x\to\infty}\left[x - \sqrt{(x-a)(x-b)}\right]$

(ii) Determine a such that

$$\lim_{x\to 0} \frac{a\sin x - \sin 2x}{\tan^3 x} \text{ exists and it is equal to 1.}$$

(iii) Find the values of a, b, c such that

$$\lim_{x\to 0} \frac{ae^{x} - b\cos x + ce^{-x}}{x \sin x} = 2.$$

Solution

(i) Substituting $x = \dfrac{1}{y}$. When $x \to \infty$ then $y \to 0$.

Now, $\lim\limits_{x\to\infty}\left[x - \sqrt{(x-a)(x-b)}\right]$ $\quad(\infty - \infty\text{ form})$

$$= \lim_{y \to 0} \left[\frac{1}{y} - \sqrt{\left(\frac{1}{y} - a\right)\left(\frac{1}{y} - b\right)} \right]$$

$$= \lim_{y \to 0} \left[\frac{1 - \sqrt{(1 - ay)(1 - by)}}{y} \right] \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{y \to 0} \frac{-\frac{d}{dy}\left[\sqrt{(1 - ay)(1 - by)} \right]}{1}$$

$$= \lim_{y \to 0} \left\{ -\frac{1}{2} \frac{1}{\sqrt{(1 - ay)(1 - by)}} [-(a + b) + 2aby] \right\}$$

$$= \frac{a + b}{2}.$$

(ii) Let $\quad l = \lim_{x \to 0} \dfrac{a \sin x - \sin 2x}{\tan^3 x} \quad \left(\dfrac{0}{0} \text{ form} \right)$

$$= \lim_{x \to 0} \frac{a \cos x - 2 \cos 2x}{3 \tan^2 x \cdot \sec^2 x}.$$

Here the numerator is $a - 2$ (when $x \to 0$), but, denominator is 0. So, if $a - 2$ is not equal to 0 then the limit does not exist. But, the limit is finite.

Therefore, $a - 2 = 0 \quad$ or $\quad a = 2$.

(iii) Let $l = \lim_{x \to 0} \dfrac{ae^x - b \cos x + ce^{-x}}{x \sin x}$

Here the numerator is $a - b + c$ and the denominator is 0. For a finite limit

$$a - b + c = 0 \tag{1}$$

$\therefore \quad l = \lim_{x \to 0} \dfrac{ae^x - b \cos x + ce^{-x}}{x \sin x} \quad \left(\dfrac{0}{0} \text{ form} \right)$

$$= \lim_{x \to 0} \frac{ae^x + b \sin x - ce^{-x}}{\sin x + x \cos x}.$$

Again, denominator is zero, but, numerator is $a - c$ and it should be zero for finite limit.

Thus,

$$a - c = 0 \tag{2}$$

Now, $\quad l = \lim_{x \to 0} \dfrac{ae^x + b \sin x - ce^{-x}}{\sin x + x \cos x} \quad \left(\dfrac{0}{0} \text{ form} \right)$

$$= \lim_{x \to 0} \frac{ae^x + b\cos x - ce^{-x}}{\cos x + \cos x - x \sin x}$$

$$= \frac{a+b+c}{2}.$$

Given that $l = 2$. Therefore,

$$\frac{a+b+c}{2} = 2 \quad \text{or,} \quad a+b+c = 4. \qquad (3)$$

Solving equations (1), (2) and (3), we get

$$a = 1, b = 2, c = 1.$$

These are the required values of a, b and c.

EXERCISES

Short Answer Questions

(Section A)

1. In the mean value theorem $f(x+h) = f(x) + hf'(x + \theta h)$, if $f(x) = ax^2 + bx + c$, $a \neq 0$ then $\theta =$ ....
2. In the mean value theorem $f(a+h) = f(a) + hf'(a + \theta h)$, if $a = 1$, $h = 3$ and $f(x) = \sqrt{x}$, then $\theta =$ ....
3. If $f(x) = x^2$, $\phi(x) = x$, $a \leq x \leq b$ then the value of c in terms of a, b in Cauchy's mean value theorem is ....
4. Applying Cauchy's mean value theorem to the functions $f(x) = e^x$ and $g(x) = e^{-x}$ in the interval $[a, b]$ then c is ....
5. The value of c in the MVT

   $f(b) - f(a) = (b - a) f'(c)$, if $f(x) = x(x-1) \quad (x-2)$, $a = 0$, $b = \frac{1}{2}$ is ....
6. The value of $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x}$ is ....
7. $\lim_{x \to \infty} x^{1/x}$ is equal to ....
8. $\lim_{x \to 0} (\cos 2x)^{1/x^2}$ is equal to ....
9. The value of $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$ is equal to ....
10. The value of $\lim_{x \to \infty} \left[x - \sqrt{(x-a)(x-b)}\right] =$ ....

CHAPTER 7

Functions of Several Variables: Limit, Continuity and Partial Derivatives

7.1 INTRODUCTION

We have already discussed about the limit, continuity and differentiability of a function containing single variable. But, in many practical situations, the functions of several variables may occur. In these situations, computation of limit, testing of continuity and computation of partial derivatives are necessary. In this chapter, we shall mainly consider the problems where two variables will appear. The approach of three and more variables are similar.

Let x, y be two variables and they are connected by a functional relation, say, $z = f(x, y)$, then we say z is a function of x, y. The ordered pair of numbers (x, y) is called a point and the aggregate of the pairs of numbers (x, y) is said to be the *domain* (or region) *of definition* of the function.

The neighbourhood of a point

The set of values x, y other than a, b that satisfy the conditions

$$|x - a| < \delta, \quad |y - b| < \delta$$

where δ is an arbitrary small positive number said to form a neighbourhood of the point (a, b) or sometimes it is called δ-neighbourhood of the point (a, b). That is, δ-neighbourhood of the point (a, b) is the square $(a - \delta, a + \delta; b - \delta, b + \delta)$ where x takes any value from $a - \delta$ to $a + \delta$ except a and y from $b - \delta$ to $b + \delta$ except b.

Also, the δ-neighbourhood of a point (a, b) is the region (circular region)

$$(x - a)^2 + (y - b)^2 < \delta^2.$$

7.2 THE LIMIT OF A FUNCTION

Let $f(x, y)$ be a function defined over a certain domain R. Then the function f is said to tend to a limit l as a point (x, y) tends to (a, b) if for any arbitrary positive number ε, there corresponds to a positive number δ, such that

$$|f(x, y) - l| < \varepsilon$$

for every point (x, y) [other then (a, b)] which satisfies

$$|x - a| < \delta, \quad |y - b| < \delta \quad \text{or} \quad (x-a)^2 + (y-b)^2 < \delta^2.$$

Symbolically, we write

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l \quad \text{or} \quad \lim_{(x, y) \to (a, b)} f(x, y) = l \quad \text{or} \quad f(x, y) \to l \text{ as } (x, y) \to (a, b),$$

l is called the *limit* (the *double limit* or the *simultaneous limit*) of f when x, y tend to a, b simultaneously.

In this definition, we have allowed (x, y) to vary over the region R and approach (a, b). We may often restrict the path along which (x, y) should move. The point (x, y) may approach (a, b) along infinite many paths, but, the important point is that the limit l must be unique, along every possible path.

Thus, if we can find two functions $y = \phi_1(x)$ and $y = \phi_2(x)$ such that the limits of $f(x, \phi_1(x))$ and $f(x, \phi_2(x))$ are different, then the simultaneous limit does not exist.

EXAMPLE 7.1 Show that $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ does not exist for

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } x^4 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0. \end{cases}$$

Solution

If we approach the origin along any axis $f(x, y) = 0$.

If we approach the origin along any line $y = mx$ then

$$f(x, y) = f(x, mx) = \frac{mx^3}{x^4 + m^2x^2} = \frac{mx}{x^2 + m^2} \to 0 \text{ as } x \to 0$$

so any straight line approach gives $\lim\limits_{(x, y) \to (0, 0)} f(x, y) = 0.$

But putting $y = mx^2$

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} f(x, mx^2) = \frac{m}{1 + m^2}$$

which is different for different line (m).

Hence, $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

EXAMPLE 7.2 Show that $\lim_{(x, y) \to (0,0)} xy \dfrac{x^2 - y^2}{x^2 + y^2} = 0.$

Solution

Put $x = r\cos\theta, y = r\sin\theta$ to

$$\left| xy\frac{x^2 - y^2}{x^2 + y^2} - 0 \right| = \left| r^2 \sin\theta \cos\theta \cos 2\theta \right| = \left| \frac{r^2}{4} \sin 4\theta \right|$$

$$\leq \frac{r^2}{4} = \frac{x^2 + y^2}{4} < \varepsilon$$

if
$$\frac{x^2}{4} < \frac{\varepsilon}{2} \text{ and } \frac{y^2}{4} < \frac{\varepsilon}{2}$$

$$\text{if } |x| < \sqrt{2\varepsilon} = \delta$$

or

and
$$|y| < \sqrt{2\varepsilon} = \delta$$

where
$$\delta = \sqrt{2\varepsilon}.$$

$\Rightarrow$
$$\lim_{(x, y) \to (0,0)} xy \frac{x^2 - y^2}{x^2 + y^2} = 0.$$

Repeated limits: If a function f is defined in some nbd of (a, b) then the limit $\lim_{y \to b} f(x, y)$ if it exists, is a function of x, say $\phi(x)$. If the limit $\lim_{x \to a} \phi(x)$ exists and is equal to λ, we write

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = \lambda$$

and say that λ is a repeated limit of f as $y \to b, x \to a$.

If we change the order of taking the limits, we get the other repeated limits.

$$\lim_{y \to b} \lim_{x \to a} f(x, y) = \lambda \text{ (say)}$$

When first $x \to a$ and then $y \to b$.

These two limits may or may not be equal.

EXAMPLE 7.3 Find the repeated limits of $\dfrac{x^2 - y^2}{x^2 + y^2}$ at (0, 0).

Solution

We have for all $\lim_{x \to 0} \dfrac{x^2 - y^2}{x^2 + y^2} = -1.$

$$\therefore \qquad \lim_{y \to 0} \lim_{x \to 0} \frac{x^2 - y^2}{x^2 + y^2} = -1.$$

Again,
$$\lim_{y \to 0} \frac{x^2 - y^2}{x^2 + y^2} = 1 \ \forall \ x$$

$$\therefore \qquad \lim_{x \to 0} \lim_{y \to 0} \frac{x^2 - y^2}{x^2 + y^2} = 1.$$

Thus, while the two repeated limits exist in this case, they are not equal.

EXAMPLE 7.4 Let $f(x, y) = \dfrac{xy}{x^2 + y^2}$ when $(x, y) \neq (0, 0)$. Show that both the repeated limits exist.

Solution

We have
$$\lim_{x \to 0} f(x, y) = 0 \ \forall \ y$$

and then
$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = 0.$$

Again,
$$\lim_{y \to 0} f(x, y) = 0 \ \forall \ x$$

$$\therefore \qquad \lim_{x \to 0} \lim_{y \to 0} f(x, y) = 0.$$

Hence the repeated limits exist and they are equal.

EXAMPLE 7.5 Show that the limit exists at the origin but the repeated limits do not

$$f(x, y) = \begin{cases} x \sin \dfrac{1}{y} + y \sin \dfrac{1}{x}, & xy \neq 0 \\ 0, & xy = 0. \end{cases}$$

Solution

Since $\lim\limits_{x \to 0} \sin \dfrac{1}{x}$ does not exist

$$\therefore \qquad \lim_{y \to 0} f(x, y) \text{ and } \lim_{x \to 0} f(x, y)$$

do not exist and therefore

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y); \ \lim_{y \to 0} \lim_{x \to 0} f(x, y)$$

do not exist.

Again,

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| < |x| + |y| \le 2\sqrt{x^2 + y^2} < \varepsilon$$

if
$$x^2 < \left(\frac{\varepsilon}{4}\right)^2 \text{ and } y^2 < \left(\frac{\varepsilon}{4}\right)^2$$

or
$$|x| < \frac{\varepsilon}{2} = \delta, \ |y| < \frac{\varepsilon}{2} = \delta.$$

Thus for $\varepsilon > 0, \exists\, \delta > 0$ such that

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| < \varepsilon$$

when
$$|x| < \delta, \ |y| < \delta$$

Hence
$$\lim_{(x,y)\to(0,0)} \left(x \sin \frac{1}{y} + y \sin \frac{1}{x} \right) = 0.$$

7.3 CONTINUITY

A function f is said to be continuous at a point (a, b) of its domain of definition if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

Alternatively: A function f is said to be continuous at a point (a, b) of its domain of definition if for $\varepsilon > 0, \exists$ a *nbd* N of (a, b) such that $|f(x, y) - f(a, b)| < \varepsilon$ for all $(x, y) \in N$.

EXAMPLE 7.6 Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at the origin.

Solution

Let $x = r\cos\theta, y = r\sin\theta$

$$\therefore \quad \left| \frac{xy}{\sqrt{x^2 + y^2}} - 0 \right| = |r\cos\theta\sin\theta| \le r = \sqrt{x^2 + y^2} < \varepsilon$$

if
$$x^2 < \frac{\varepsilon^2}{2} \text{ and } y^2 < \frac{\varepsilon^2}{2}, \text{ or}$$

if $$|x| < \frac{\varepsilon}{\sqrt{2}},\ |y| < \frac{\varepsilon}{\sqrt{2}}$$

Thus $$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon$$

when $$|x| < \frac{\varepsilon}{\sqrt{2}} = \delta, |y| < \frac{\varepsilon}{\sqrt{2}} = \delta.$$

Thus $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 = f(0,0).$$

$\therefore$ $$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

Hence, f is continuous at (0, 0).

EXAMPLE 7.7 Show that the function f defined by

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & \text{when } (x,y) \to (0,0) \\ 0, & \text{when } (x,y) = (0,0) \end{cases}$$

is continuous at the origin.

Solution

Put $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $$\left|xy\,\frac{x^2-y^2}{x^2+y^2}\right| = \left|r^2\cos\theta\sin\theta\,\frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2(\cos^2\theta+\sin^2\theta)}\right|$$

$$= r^2\,|\cos\theta\sin\theta\cos 2\theta| \le r^2 = x^2+y^2 < \varepsilon$$

if $$x^2 < \frac{\varepsilon}{2},\ y^2 < \frac{\varepsilon}{2}$$

or $$\text{if } |x| < \frac{\sqrt{\varepsilon}}{2},\ |y| < \frac{\sqrt{\varepsilon}}{2} = \delta$$

$\therefore$ $$\left|xy\,\frac{x^2-y^2}{x^2+y^2} - 0\right| < \varepsilon \text{ for } |x| < \frac{\sqrt{\varepsilon}}{2} = \delta,\ |y| < \frac{\sqrt{\varepsilon}}{2} = \delta.$$

Thus $$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

Hence, f is continuous at (0, 0).

7.4 PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two independent variables x, y in a region R where the function is defined.

If y is constant, $f(x, y)$ becomes a function of x only. One variable x and its derivative (when exists) is called the partial derivative of $f(x, y)$ with respect to x. We denote it by

$$f_x(x, y) \text{ or } \frac{\partial f}{\partial x} \text{ or } z_x \text{ or } \frac{\partial z}{\partial x}.$$

Thus, $f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$, if the limit exists.

This formula gives the first order partial derivative of f w.r.t. x at any point (x, y).

In particular, the partial derivative of $f(x, y)$ w.r.t. x at the point (a, b) is given by

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Similarly, if x is constant $f(x, y)$ becomes a function of y alone, whose derivative (when exists) is called the *partial derivative* of $f(x, y)$ w.r.t. y, we denote it by

$$f_y(x, y) \text{ or } \frac{\partial f}{\partial y} \text{ or } z_y \text{ or } \frac{\partial z}{\partial y}.$$

Thus, $f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$, provided the limit exists.

By the statement "$f(x, y)$ is differentiable", we mean that both the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ exist.

The derivatives of f_x w.r.t. x and w.r.t. y are denoted by $\frac{\partial}{\partial x}(f_x)$ or $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ or $\frac{\partial^2 f}{\partial x^2}$ or f_{xx} and $\frac{\partial}{\partial y}(f_x)$ or $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ or $\frac{\partial^2 f}{\partial y\, \partial x}$ or f_{yx}.

Similarly, the derivatives of f_y are denoted by

$$\frac{\partial}{\partial x}(f_y) \text{ or } \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \text{ or } \frac{\partial^2 f}{\partial x\, \partial y} \text{ or } f_{xy} \text{ and } \frac{\partial}{\partial y}(f_y) \text{ or } \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}.$$

The derivates $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ are called *second order derivatives*. The second order derivatives f_{xy} and f_{yx} are called *mixed derivatives*.

The formulae for second order derivaties at the point (a, b) are given below.

(i) $f_{xx}(a,b) = \lim\limits_{h \to 0} \dfrac{f_x(a+h,b) - f_x(a,b)}{h}$

(ii) $f_{yy}(a,b) = \lim\limits_{k \to 0} \dfrac{f_y(a,b+k) - f_y(a,b)}{k}$

(iii) $f_{xy}(a,b) = \lim\limits_{h \to 0} \dfrac{f_y(a+h,b) - f_y(a,b)}{h}$

(iv) $f_{yx}(a,b) = \lim\limits_{k \to 0} \dfrac{f_x(a,b+k) - f_x(a,b)}{k}$.

EXAMPLE 7.8 If $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

Examine whether $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ exist and are not equal.

Solution

Now $$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$f_y(h,0) = \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k \to 0} \frac{hk(h^2-k^2)}{k(h^2+k^2)} = -h$$

and $$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0.$$

$\therefore$ $$f_{xy}(0,0) = \lim_{h \to 0} \frac{h-0}{h} = 1.$$

Again, $$f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}.$$

Now, $$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h \to 0} \frac{hk(h^2-k^2)}{h(h^2+k^2)} = -k$$

and $$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$

$\therefore$ $$f_{yx}(0,0) = \lim_{k \to 0} \frac{-k-0}{k} = -1.$$

$\therefore$ $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ exist but they are not equal.

EXAMPLE 7.9 If $f(x, y) = (x^2 + y^2)\tan^{-1}\dfrac{y}{x}$ when $x \neq 0$ and $f(0, y) = \dfrac{\pi y}{2}$, show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Solution

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(0+h, 0) - f_y(0,0)}{h}$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{\frac{\pi}{2}k^2 - 0}{k} = 0$$

$$f_y(h,0) = \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k \to 0} \frac{(h^2 + k^2)\tan^{-1}\frac{k}{h} - 0}{k}$$

$$= \lim_{k \to 0} \frac{(h^2 + k^2)\dfrac{1}{1 + k^2/h^2}\left(\dfrac{1}{h}\right) + 2k\tan^{-1}\dfrac{k}{h}}{1}$$

$$= \lim_{k \to 0}\left(h + 2k\tan^{-1}\frac{k}{h}\right) = h.$$

$$\therefore \qquad f_{xy}(0,0) = \lim_{h \to 0} \frac{h - 0}{h} = 1.$$

Again, $$f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}.$$

Now, $$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h \to 0} \frac{(h^2 + k^2)\tan^{-1}\frac{k}{h} - \frac{\pi k^2}{2}}{h}$$

$$= \lim_{h \to 0} \frac{(h^2 + k^2)\dfrac{1}{1 + \dfrac{k^2}{h^2}}\left(\dfrac{-k}{h^2}\right) + 2h\tan^{-1}\dfrac{k}{h} - 0}{1}$$

$$= -k.$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

$$\therefore \quad f_{yx}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}=\lim_{k\to 0}\frac{-k-0}{k}=-1.$$

Thus, $$f_{xy}(0,0)\neq f_{yx}(0,0).$$

The sufficient conditions for equality of mixed derivatives f_{xy} and f_{yx} are stated in the following two theorems.

Theorem 7.1 *Schwarz's Theorem:* If f_y exists in a certain neighbourhood of a point (a, b) of the domain of definition of a function f and f_{yx} is continuous at (a, b) then $f_{xy}(a, b)$ exists and is equal to $f_{yx}(a, b)$.

Note 1: If f_{xy} and f_{yx} are both continuous at (a, b), then $f_{xy}(a, b)=f_{yx}(a, b)$ for the assumption of continuity of both these derivatives is a wider assumption.

Note 2: If the conditions of Schwarz's theorem are satisfy then $f_{xy}=f_{yx}$ at a point (a, b). But if the condition are not satisfied, we cannot draw any conclusion regarding the equality of f_{xy} and f_{yx} they may or may not be equal. Thus the conditions are sufficient but not necessary.

Theorem 7.2 *Young's Theorem:* If f_x and f_y are both differentiable at a point (a, b) of the domain of definition of a function f then

$$f_{xy}(a, b)=f_{yx}(a, b).$$

Note: The conditions of this theorem are sufficient but not necessary.

7.4.1 Differentiability of a Function of Two Variables

Let (x, y) and $(x+\delta x, y+\delta y)$ be two neighbourhood points in the domain of definition of a function f. The change δf in the function as the point changes from (x, y) to $(x+\delta x, y+\delta y)$ is given by

$$\delta f = f(x+\delta x, y+\delta y)-f(x, y).$$

The function f is said to be differentiable at (x, y) if the change δf can be expressed in the form

$$\delta f = A\delta x + B\delta y + \delta x\,\phi(\delta x, \delta y) + \delta y\,\psi(\delta x, \delta y) \tag{7.1}$$

where A and B are constants independent of δx, δy and ϕ, ψ are functions of δx, δy tending to zero as δx, δy tend to 0 simultaneously.

If we replace δx, δy by h, k in Equation (7.1), we say that the function is differentiable at a point (a, b) of the domain of definition if it can be expressed as

$$\begin{aligned} df &= f(a+h, b+k)-f(a, b) \\ &= Ah + Bk + h\varphi(h, k) + k\psi(h, k) \end{aligned}$$

where $A=f_x$, $B=f_y$ and φ, ψ are functions of h, k tending to 0 as h, k tend to 0 simultaneously.

EXAMPLE 7.10 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous and possesses partial derivatives at (0, 0) but is not differentiable at (0, 0).

Solution

Differentiability:

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{-k}{k} = -1.$$

Thus the function possesses partial derivatives at (0, 0). If the function is differentiable at (0, 0) then by definition

$$df = f(h, k) - f(0, 0) = Ah + Bk + h\phi + k\psi \tag{1}$$

where A and B are constants.

i.e. $$A = f_x(0, 0) = 1, \quad B = f_y(0, 0) = -1$$

and ϕ, ψ tend to 0 as $(h, k) \to (0, 0)$.

Putting, $h = \rho \cos\theta, k = \rho \sin\theta$ in equation (1) and dividing by ρ and taking $\rho \to 0$, we get

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta$$

or $$\cos\theta \sin\theta = 0$$

which is impossible for arbitrary θ. Thus the function is not differentiable at the origin.

EXAMPLE 7.11 Prove that the function $f(x, y) = \sqrt{|xy|}$ is not differentiable at the point (0, 0) but that f_x and f_y both exist at the origin and have the value 0.

Solution

Now

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0.$$

If the function is differentiable at (0, 0) then by definition

$$f(h, k) - f(0, 0) = 0 \cdot h + 0 \cdot k + h\varphi + k\psi$$

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where φ and ψ are functions of h and k and tends to 0 as $(h, k) \to (0, 0)$.
Putting $h = \rho \cos \theta$, $k = \rho \sin \theta$ and dividing by ρ we get

$$|\cos \theta \sin \theta|^{1/2} = \phi \cos \theta + \psi \sin \theta.$$

For arbitrary θ, $\rho \to 0$ implies $(h, k) \to (0, 0)$.
Taking limit as $\rho \to 0$

we get $$|\cos \theta \sin \theta|^{1/2} = 0$$

which is impossible for arbitrary θ.
$\therefore f$ is not differentiable at (0, 0).

Sufficient condition for differentiability

Theorem 7.3 If (a, b) be a point of the domain of definition of a function f such that

(i) f_x is continuous at (a, b)

(ii) f_y exists at (a, b)

then f is differentiable at (a, b).

EXAMPLE 7.12 Show that the function

$$f(x, y) = \begin{cases} xy \dfrac{x^2 - y^2}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x = 0 = y \end{cases}$$

is differentiable at (0, 0).

Solution

It is obvious $f_x(0, 0) = 0 = f_y(0, 0)$.

Also when $x^2 + y^2 \neq 0$

$$|f_x| = \frac{|x^4 y + 4x^2 y^3 - y^5|}{(x^2 + y^2)} \leq \frac{6(x^2 + y^2)^{5/2}}{(x^2 + y^2)^2} = 6\sqrt{x^2 + y^2}.$$

Similarly, $$|f_y| \leq 6\sqrt{x^2 + y^2}.$$

Evidently, $$\lim_{(x, y) \to (0, 0)} f_x(x, y) = 0 = f_x(0, 0).$$

Thus, f_x is continuous at (0, 0) and f_y (0, 0) exists, i.e. f is differentiable at (0, 0).

EXAMPLE 7.13 Given that $$f(x, y) = xy, \quad \text{if } |y| \leq |x|$$
$$= -xy, \quad \text{if } |x| < |y|$$

show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Solution

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h \to 0} \frac{-hk-0}{h} = -k$$

and
$$f_y(h,0) = \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k \to 0} \frac{hk-0}{k} = h.$$

Now
$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h \to 0} \frac{h-0}{h} = 1$$

and
$$f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k \to 0} \frac{-k-0}{k} = -1.$$

Hence $f_{xy}(0,0) \neq f_{yx}(0,0)$.

The derivatives calculated in the previous examples, are determined from the definition. But, practically, it is difficult to obtain the derivatives of the first or the higher order from definition. So, the derivatives at any point can be determined by using the rules adopted for single variable, taken other variables as constants. For example, let $z = ax^2 + 2hxy + by^2$.

Then $\frac{\partial z}{\partial x}$ is to be determined by taking y as constant. That is, for this expression a, h, b and y are constants and hence

$$\frac{\partial z}{\partial x} = 2ax + 2hy.$$

Similarly, $\frac{\partial z}{\partial y}$ is to be calculated by taking x as constant.

Thus,
$$\frac{\partial z}{\partial y} = 2hx + 2by.$$

The other derivatives are obtained as

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(2ax + 2hy) = 2a$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(2hx + 2by) = 2b$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(2hx + 2by) = 2h$$

and $$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(2ax + 2hy) = 2h.$$

It may be noted that for this problem $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$.

EXAMPLE 7.14 Find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ when $u = \tan^{-1} \dfrac{x - y}{x^2 + y^2}$.

Solution

Since $$u = \tan^{-1} \frac{x - y}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \left(\frac{x - y}{x^2 + y^2}\right)^2} \frac{\partial}{\partial x}\left(\frac{x - y}{x^2 + y^2}\right)$$

$$= \frac{(x^2 + y^2)^2}{(x^2 + y^2)^2 + (x - y)^2} \frac{1 \cdot (x^2 + y^2) - (x - y)\, 2x}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2 + 2xy}{(x^2 + y^2)^2 + (x - y)^2}$$

and $$\frac{\partial u}{\partial y} = \frac{1}{1 + \left(\frac{x - y}{x^2 + y^2}\right)^2} \frac{\partial}{\partial y}\left(\frac{x - y}{x^2 + y^2}\right)$$

$$= \frac{(x^2 + y^2)^2}{(x^2 + y^2)^2 + (x - y)^2} \frac{(-1)\,(x^2 + y^2) - (x - y)\, 2y}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2 + (x - y)^2}.$$

EXAMPLE 7.15 If $U = \sqrt{xy}$, find the value of $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$. (WBUT 2001)

Solution

Here $$U = \sqrt{xy} = x^{1/2}\, y^{1/2}.$$

$$\frac{\partial U}{\partial x} = \frac{1}{2} x^{-1/2} \, y^{1/2}, \quad \frac{\partial U}{\partial y} = \frac{1}{2} x^{1/2} \, y^{-1/2}$$

$\therefore$

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{2}\left(\frac{-1}{2}\right) x^{-3/2} \, y^{1/2} \text{ and } \frac{\partial^2 U}{\partial y^2} = \frac{1}{2}\left(\frac{-1}{2}\right) x^{1/2} \, y^{-3/2}$$

Now,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{-1}{4} x^{-3/2} \, y^{1/2} - \frac{1}{4} x^{1/2} \, y^{-3/2}$$

$$= \frac{-1}{4}\left(\frac{y^{1/2}}{x^{3/2}} + \frac{x^{1/2}}{y^{3/2}}\right)$$

$$= -\frac{1}{4} \frac{x^2 + y^2}{(xy)^{3/2}}.$$

EXAMPLE 7.16 If $x^x \, y^y \, z^z = k$, show that at $x = y = z$

$$\frac{\partial^2 z}{\partial x \partial y} = -\,(x \log(ex))^{-1}.$$

Solution

Taking log on both sides of $x^x \, y^y \, z^z = k$, we get

$$x \log x + y \log y + z \log z = \log k \tag{1}$$

We assume that z is a function of two independent variables x and y. Differentiating equation (1) w.r.t. y treating x as constant, we get

$$\log y + 1 + \frac{\partial z}{\partial y} \log z + z \frac{1}{z} \frac{\partial z}{\partial y} = 0$$

or

$$\frac{\partial z}{\partial y}(1 + \log z) = -\,(1 + \log y)$$

or

$$\frac{\partial z}{\partial y} = -\left(\frac{1 + \log y}{1 + \log z}\right). \tag{2}$$

Similarly,

$$\frac{\partial z}{\partial x} = -\left(\frac{1 + \log x}{1 + \log z}\right). \tag{3}$$

Differentiating equation (1) partially w.r.t. y, we get

$$1 + \log y + \log z \frac{\partial z}{\partial y} + \frac{\partial z}{\partial y} = 0.$$

Differentiating w.r.t. x, we get

$$(1+\log z)\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{z}\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial x}.$$

From equations (2) and (3), we see that $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial x} = -1$ at $x = y = z$.

$$\therefore \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x(1+\log x)} \text{ at } x = y = z$$

$$= -(x \log ex)^{-1}.$$

EXAMPLE 7.17 If $\theta = t^n e^{-r^2/4t}$, find the value of n for which

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}.$$

Solution

Partially differentiating the given relation w.r.t. r, we get

$$\frac{\partial \theta}{\partial r} = t^n e^{-r^2/4t} \times \left(-\frac{1}{4t}\right) 2r$$

$$= -\frac{\theta \times 2r}{4t} = -\frac{\theta r}{2t}.$$

$$\therefore \quad r^2 \frac{\partial \theta}{\partial r} = -\frac{\theta r^3}{2t}.$$

Hence

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2t}\left\{\frac{\partial \theta}{\partial r} r^3 + \theta \cdot 3r^2\right\}$$

$$= -\frac{1}{2t}\left\{-\frac{\theta r}{2t}\cdot r^3 + 3\theta r^2\right\} \quad \left[\because \frac{\partial \theta}{\partial t} = -\frac{\theta r}{2t}\right]$$

$$= \frac{\theta r^4}{4t^2} - \frac{3}{2}\frac{\theta r^2}{t}$$

or

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\theta r^2}{4t^2} - \frac{3}{2}\frac{\theta}{t}. \quad (1)$$

Again,

$$\frac{\partial \theta}{\partial t} = nt^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t}\left(-\frac{r^2}{4}\right)\left(-\frac{1}{t^2}\right)$$

$$= \frac{n\theta}{t} + \frac{\theta r^2}{4t^2}. \quad (2)$$

From equations (1) and (2), we have

$$\frac{\theta r^2}{4t^2} - \frac{3}{2}\frac{\theta}{t} = \frac{n\theta}{t} + \frac{\theta r^2}{4t^2}$$

or

$$n = -\frac{3}{2}.$$

EXAMPLE 7.18 If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that

(i) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = \dfrac{3}{x+y+z}$

(ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right)u = -\dfrac{3}{(x+y+z)^2}$

(iii) $\left(\dfrac{\partial}{\partial x^2} + \dfrac{\partial}{\partial y^2} + \dfrac{\partial}{\partial z^2}\right)^2 u = -\dfrac{9}{(x+y+z)^2}$ (WBUT 2003)

Solution

We know
$$x^3 + y^3 + z^3 - 3xyz$$
$$= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega^2 z),$$

where ω^2 are to the cube roots of unity.

$\therefore$
$$u = \log(x+y+z) + \log(x+\omega y+\omega^2 z) + \log(x+\omega^2 y+\omega^2 z)$$

(i) $\therefore$
$$\frac{\partial u}{\partial x} = \frac{1}{x+y+z} + \frac{1}{x+\omega y+\omega^2 z} + \frac{1}{x+\omega^2 y+\omega z} \tag{1}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x+y+z} + \frac{\omega}{x+\omega y+\omega^2 z} + \frac{\omega^2}{x+\omega^2 y+\omega z} \tag{2}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x+y+z} + \frac{\omega^2}{x+\omega y+\omega^2 z} + \frac{\omega}{x+\omega^2 y+\omega z} \tag{3}$$

Adding equations (1), (2) and (3) we get

$\therefore$
$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z} \qquad [\because 1+\omega+\omega^2 = 0]$$

(ii) Now
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{1}{x+y+z} + \frac{1}{x+\omega y+\omega^2 z} + \frac{1}{x+\omega^2 y+\omega z}\right]$$

$$= -\frac{1}{(x+y+z)^2} - \frac{1}{(x+\omega y+\omega^2 z)^2} - \frac{1}{(x+\omega^2 y+\omega z)^2}.$$

Similarly, we have

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{(x+y+z)^2} - \frac{\omega^2}{(x+\omega y+\omega^2 z)^2} - \frac{\omega^4}{(x+\omega^2 y+\omega z)^2}$$

and $$\frac{\partial^2 u}{\partial z^2} = -\frac{1}{(x+y+z)^2} - \frac{\omega^4}{(x+\omega y+\omega^2 z)^2} - \frac{\omega^2}{(x+\omega^2 y+\omega z)^2}.$$

Adding above three equations, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2}.$$

(iii) Now, $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= -\frac{9}{(x+y+z)^2}.$$

EXAMPLE 7.19 If $x = e^{r\cos\theta}\cos(r\sin\theta)$ and $y = e^{r\cos\theta}\sin(r\sin\theta)$

prove that $$\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta} \text{ and } \frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}$$

hence deduce that $$\frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = 0.$$

Solution

Given $$x = e^{r\cos\theta}\cos(r\sin\theta) \quad (1)$$

$$y = e^{r\cos\theta}\sin(r\sin\theta). \quad (2)$$

Differentiating equation (1) partially w.r.t. r, we get

$$\frac{\partial x}{\partial r} = e^{r\cos\theta}\cos(r\sin\theta)\cos\theta - e^{r\cos\theta}\sin(r\sin\theta)\sin\theta \quad (3)$$

Differentiating equation (1) partially w.r.t. θ, we get

$$\frac{\partial x}{\partial \theta} = e^{r\cos\theta}(-\sin\theta\cdot r)\cos(r\sin\theta) - e^{r\cos\theta}\sin(r\sin\theta)\times r\cos\theta$$

$$= -r\,(e^{r\cos\theta})\{\sin\theta\cos(r\sin\theta) + \sin(r\sin\theta)\cos\theta\} \quad (4)$$

$$\frac{\partial y}{\partial r} = e^{r\cos\theta} (\cos\theta) \sin(r\sin\theta) + e^{r\cos\theta} \cos(r\sin\theta) \sin\theta. \qquad (5)$$

Differentiating equation (2) partially w.r.t. θ, we get

$$\frac{\partial y}{\partial \theta} = e^{r\cos\theta} (-r\sin\theta) \sin(r\sin\theta) + e^{r\cos\theta} \cos(r\sin\theta)(r\cos\theta)$$

$$= re^{r\cos\theta} \{-\sin\theta \sin(r\sin\theta) + \cos(r\sin\theta)\cos\theta\} \qquad (6)$$

∴ From equations (3) and (6), we have

$$\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta} \qquad (7)$$

and from equations (4) and (5), we have

$$\frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}. \qquad (8)$$

Differentiating equation (7) partially w.r.t. r, we get

$$\frac{\partial^2 x}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial y}{\partial \theta}\right)$$

$$= -\frac{1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial^2 y}{\partial r \partial \theta}$$

$$= -\frac{1}{r}\frac{\partial x}{\partial r} + \frac{1}{r}\frac{\partial^2 y}{\partial r \partial \theta}. \qquad (9)$$

[using equation (7)]

Equation (8) can be written as,

$$\frac{\partial x}{\partial \theta} = -r\frac{\partial y}{\partial r}.$$

Differentiating w.r.t. θ, we get

$$\frac{\partial^2 x}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(-r\frac{\partial y}{\partial r}\right) = -r\frac{\partial^2 y}{\partial \theta \partial r}$$

or

$$\frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = -\frac{1}{r}\frac{\partial^2 y}{\partial \theta \partial r}. \qquad (10)$$

Adding equations (9) and (10), we get

$$\frac{\partial^2 x}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = -\frac{1}{r}\frac{\partial x}{\partial r}$$

or

$$\frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2} = 0.$$

EXAMPLE 7.20 If $z = f(x+ay) + \phi(x-ay)$ show that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

Solution

We have $$z = f(x+ay) + \phi(x-ay). \quad (1)$$

Differentiating w.r.t. x, we get

$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay).$$

Again differentiating w.r.t. x, we get

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay). \quad (2)$$

Differentiating equation (1) twice w.r.t. y, we get

$$\frac{\partial z}{\partial y} = af'(x+ay) - a\phi'(x-ay)$$

and $$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2\phi''(x-ay) \quad (3)$$

From equations (2) and (3), we have

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

EXAMPLE 7.21 Suppose $v = f(u)$, where $u = (x^2 + y^2) \tan^{-1}(y/x)$.

Prove that $$x\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial y} = (x^2 + y^2) f'(u).$$

Solution

Differentiating $$v = f(u) \quad (1)$$
w.r.t. x, we get

$$\frac{\partial v}{\partial x} = f'(u)\frac{\partial u}{\partial x}$$

$$= f'(u)\left\{2x \tan^{-1}\left(\frac{y}{x}\right) + (x^2+y^2)\frac{\left(-\frac{y}{x^2}\right)}{1+\frac{y^2}{x^2}}\right\}$$

$$= f'(u)\left\{2x \tan^{-1}\left(\frac{y}{x}\right) - y\right\}$$

or
$$y\frac{\partial v}{\partial x} = f'(u)\left\{2xy\tan^{-1}\left(\frac{y}{x}\right) - y^2\right\} \qquad (2)$$

Differentiating equation (1) w.r.t. y, we get

$$\frac{\partial v}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

$$= f'(u)\left\{2y\tan^{-1}\frac{y}{x} + (x^2+y^2)\frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right\}$$

$$= f'(u)\left\{2y\tan^{-1}\frac{y}{x} + x\right\}$$

or
$$x\frac{\partial v}{\partial y} = f'(u)\left\{2yx\tan^{-1}\frac{y}{x} + x^2\right\}. \qquad (3)$$

From equations (2) and (3)

$$-x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = f'(u)\{x^2+y^2\}$$

or
$$y\frac{\partial v}{\partial y} - x\frac{\partial v}{\partial x} = (x^2+y^2)\,f'(u)$$

or
$$y\frac{\partial f}{\partial y} - x\frac{\partial f}{\partial x} = (x^2+y^2)\,f'(u). \qquad [\text{as } v = f(u)]$$

EXAMPLE 7.22 Let $z = f(u)$ where $u = x^m y^n$ show that

$$nx\frac{\partial z}{\partial x} - my\frac{\partial z}{\partial y} = 0$$

and hence deduce that

$$n^2x\frac{\partial z}{\partial x} - m^2y\frac{\partial z}{\partial y} + n^2x^2\frac{\partial^2 z}{\partial x^2} - m^2y^2\frac{\partial^2 z}{\partial y^2} = 0.$$

Solution

Differentiating w.r.t. x we get
$$z = f(u) \qquad (1)$$

$$\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x} = f'(u)\,mx^{m-1}\,y^n$$

or
$$nx\frac{\partial z}{\partial x} = f'(u)\,nm\,x^m y^n. \qquad (2)$$

Differentiating equation (1) w.r.t. y we get

$$\frac{\partial z}{\partial y} = f'(u)\, x^m\, n\, y^{n-1}$$

or

$$my\frac{\partial z}{\partial y} = f'(u)\, mn\, x^m y^n. \tag{3}$$

From equations (2) and (3), we get

$$nx\frac{\partial z}{\partial x} - my\frac{\partial z}{\partial y} = 0. \tag{4}$$

Differentiating equation (4) w.r.t. x, we get

$$nx\frac{\partial^2 z}{\partial x^2} + n\frac{\partial z}{\partial x} - my\frac{\partial^2 z}{\partial x \partial y} = 0.$$

Multiplying both sides by nx, we get

$$n^2x^2\frac{\partial^2 z}{\partial x^2} + n^2x\frac{\partial z}{\partial x} - mn\,xy\frac{\partial^2 z}{\partial x \partial y} = 0. \tag{5}$$

Differentiating equation (4) w.r.t. y, we get

$$nx\frac{\partial^2 z}{\partial y \partial x} - m\frac{\partial z}{\partial y} - my\frac{\partial^2 z}{\partial y^2} = 0.$$

Multiplying both sides by my, we get

$$mn\,xy\frac{\partial^2 z}{\partial x \partial y} - m^2y\frac{\partial z}{\partial y} - m^2y^2\frac{\partial^2 z}{\partial y^2} = 0. \tag{6}$$

Adding equations (5) and (6), we get the required result.

$$n^2x^2\frac{\partial^2 z}{\partial x^2} + n^2x\frac{\partial z}{\partial y} - m^2y\frac{\partial z}{\partial y} - m^2y^2\frac{\partial^2 z}{\partial y^2} = 0.$$

EXAMPLE 7.23 If $u = e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$

Solution

Since

$$u = e^{xyz}$$

$$\frac{\partial u}{\partial z} = xy\, e^{xyz},$$

$$\frac{\partial^2 u}{\partial y \partial z} = xe^{xyz} + xy\,(xze^{xyz}) = (x + x^2yz)\, e^{xyz}$$

and
$$\frac{\partial^2 u}{\partial x \partial y \partial z} = (1 + 2xyz)\, e^{xyz} + (x + x^2 yz)(yz)\, e^{xyz}$$

$$= (1 + 2xyz + xyz + x^2 y^2 z^2)\, e^{xyz}$$

$$= (1 + 3xyz + x^2 y^2 z^2)\, e^{xyz}.$$

7.5 HOMOGENEOUS FUNCTIONS AND EULER'S THEOREM

If every term of a function is of same degree, say, n, then the function is said to be a *homogeneous function* of degree n. If the function has two independent variables x and y then the degree of a term is the algebraic sum of the indices of x and y.

That is the degree of $x^3 y^{1/2}$ is 5/2 and that of $\frac{x^3}{y^2}$ is one.

Alternatively, a function $f(x, y, z)$ is said to be homogeneous of degree n, if
$$f(tx, ty, tz) = t^n f(x, y, z),$$
for every positive value of t. In case of two variables, this expression is
$$f(tx, ty) = t^n f(x, y).$$

A homogeneous function of three variables can also be written in the following forms:

$$f(x, y, z) = x^n f_1\left(\frac{y}{x}, \frac{z}{x}\right)$$

or
$$f(x, y, z) = y^n f_2\left(\frac{x}{y}, \frac{z}{y}\right)$$

or
$$f(x, y, z) = z^n f_3\left(\frac{x}{z}, \frac{y}{z}\right)$$

where f_1, f_2, f_3 are some functions depending on f.

Similarly a homogeneous function of two variables can be written as
$$f(x, y) = x^n \phi\left(\frac{y}{x}\right)$$

or
$$f(x, y) = y^n \psi\left(\frac{x}{y}\right).$$

For example, the function
$$f(x, y) = x^2 + 2y^2 = x^2\left(1 + \frac{2y^2}{x^2}\right)$$

is a homogeneous function of degree two, while the function

$$f(x, y, z) = x^3 + \frac{y^4}{z} + z^3 = x^3 \left\{1 + \frac{y^4}{x^3 z} + \frac{z^3}{x^3}\right\}$$

is a homogeneous function of degree 3.

Theorem 7.4 *Euler's theorem (case of two variables)*

If $u = f(x, y)$ be a homogeneous function in x and y of degree n and if u has continuous partial derivatives, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Proof: Since $u = f(x, y)$ is a homogeneous function in x and y of degree n, we may write it as

$$u = x^n \phi\left(\frac{y}{x}\right) \tag{1}$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1} \phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

or
$$x\frac{\partial u}{\partial x} = nx^n \phi\left(\frac{y}{x}\right) - x^{n-1} y\phi'\left(\frac{y}{x}\right). \tag{2}$$

Again from equation (1),

$$\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right) \times \frac{1}{x}$$

or
$$y\frac{\partial u}{\partial y} = x^{n-1} y\phi'\left(\frac{y}{x}\right) \tag{3}$$

Adding equations (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \phi\left(\frac{y}{x}\right) = nf(x, y) = nu.$$

Hence proved.

Theorem 7.5 *Euler's theorem (case of three variables)*

If $f(x, y, z)$ be a homogeneous function in x, y, z of degree n, having continuous partial derivatives then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf.$$

The converse of this theorem is also valid. That is, $f(x, y, z)$ admits of continuous partial derivatives and satisfies the relation

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf$$

where n is any number. Therefore, f is a homogeneous function of degree n.

EXAMPLE 7.24 Verify Euler's theorem for the function

$$u=f(x,y)=ax^2+2hxy+by^2.$$

Solution

Here
$$u=f(x,y)=x^2\left\{a+2h\left(\frac{y}{x}\right)+b\left(\frac{y}{x}\right)^2\right\}$$

$$=x^2\,\phi\left(\frac{y}{x}\right)\text{(say)}$$

$\therefore$ u is a homogeneous function of degree two. So according to Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u. \tag{1}$$

Now
$$\frac{\partial u}{\partial x}=2ax+2hy \text{ and } \frac{\partial u}{\partial y}=2hx+2by$$

$\therefore$
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(2ax^2+2hxy)+(2hxy+2by^2)$$

$$=2\,(ax^2+2hxy+2by^2)$$

$$=2u.$$

Hence verified.

EXAMPLE 7.25 Let u be a homogeneous function in x and y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n\,(n-1)\,u,$$

where all partial derivatives of the first and the second order are continuous. The above result can also be written as

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 u=n\,(n-1)\,u.$$

Solution

Since u is a homogeneous function of x and y of degree n, we have by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu. \tag{1}$$

Differentiating equation (1) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = n\frac{\partial u}{\partial x}$$

or
$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}. \qquad (2)$$

Differentiating equation (1) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y}$$

or
$$x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}. \qquad (3)$$

Multiplying equation (2) by x and equation (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = (n-1)\left\{x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right\}$$

$$\left[\text{using } \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}\right]$$

$$= n(n-1)\,u \qquad \text{[using equation (1)]}$$

Hence proved.

EXAMPLE 7.26 Let $u = \sin^{-1}\sqrt{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$ show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13+\tan^2 u). \qquad \text{(WBUT 2001)}$$

Solution

Let
$$v(x, y) = \sin u = \sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}.$$

Putting $x = tx$ and $y = ty$

we get

$$v(tx, ty) = t^{\left(\frac{1}{3}-\frac{1}{2}\right)\frac{1}{2}}\, v(x, y) = t^{-\frac{1}{12}}\, v(x, y).$$

$\therefore$ v is a homogeneous function of degree $\left(-\dfrac{1}{12}\right)$.

∴ By Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=-\frac{1}{12}v$$

or

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=-\frac{1}{12}\sin u$$

or

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\frac{\sin u}{\cos u}=-\frac{1}{12}\tan u. \qquad (1)$$

Differentiating equation (1) partially w.r.t. x, we get

$$\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y^2}=-\frac{1}{12}\sec^2 u\frac{\partial u}{\partial x} \qquad (2)$$

Again differentiating equation (1) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=-\frac{1}{12}\sec^2 u\frac{\partial u}{\partial y}. \qquad (3)$$

Using equation (2) × x + equation (3) × y, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)-\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=-\frac{1}{12}\sec^2 u\left(-\frac{1}{12}\tan u\right)+\frac{1}{12}\tan u$$

$$=\frac{1}{144}(1+\tan^2 u)\ \tan u+\frac{1}{12}\tan u$$

$$=\left(\frac{1}{144}+\frac{1}{12}+\frac{1}{144}\tan^2 u\right)\tan u$$

$$=\left(\frac{13}{144}+\frac{1}{144}\tan^2 u\right)\tan u$$

$$=\frac{\tan u}{144}(13+\tan^2 u).$$

Hence proved.

EXAMPLE 7.27 If $u=\dfrac{(x^2+y^2)^n}{2n(2n-1)}+xf\left(\dfrac{y}{x}\right)+g\left(\dfrac{y}{x}\right)$, then prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(x^2+y^2)^n.$$

Solution

Let
$$u = P + Q + R$$
where
$$P = \frac{(x^2+y^2)^n}{2n(2n-1)} = \text{ a homogeneous function of degree } 2n$$

$$Q = xf\left(\frac{y}{x}\right) = \text{ a homogeneous function of degree } 1$$

$$R = g\left(\frac{y}{x}\right) = \text{ a homogeneous function of degree } 0.$$

∴ By Euler's theorem on P, Q, R we have

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = 2nP \tag{1}$$

$$x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} = 1\cdot Q \tag{2}$$

and
$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} = 0\cdot R \tag{3}$$

Adding, we get

$$x\frac{\partial}{\partial x}(P+Q+R) + y\frac{\partial}{\partial y}(P+Q+R)$$
$$= 2nP + Q$$

or
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2n(x^2+y^2)^n}{2n(2n-1)} + xf\left(\frac{y}{x}\right)$$
$$= 2nP + Q. \tag{4}$$

Differentiating equation (4) partially w.r.t. x and multiplying by x, we get

$$x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} = 2nx\frac{\partial P}{\partial x} + x\frac{\partial Q}{\partial x}. \tag{5}$$

Again, differentiating equation (4) partially w.r.t. y and multiplying by y, we get

$$xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 2ny\frac{\partial P}{\partial y} + y\frac{\partial Q}{\partial y}. \tag{6}$$

Adding equations (5) and (6), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2n\left(x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y}\right) + \left(x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y}\right) - \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= 2n\,(2nP) + Q - (2nP + Q)$$

using equations (1), (2) and (3)

$$= (2n-1)\,2nP$$

$$= 2n\,(2n-1)\,\frac{(x^2+y^2)^n}{2n\,(2n-1)} = (x^2+y^2)^n.$$

Hence proved.

EXAMPLE 7.28 If $u = xf\left(\dfrac{y}{x}\right) + g\left(\dfrac{y}{x}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xf\left(\frac{y}{x}\right)$$

and
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0. \qquad \text{(WBUT 2004)}$$

Solution

Let $u = v + w$, where $v = xf\left(\dfrac{y}{x}\right)$ and $w = g\left(\dfrac{y}{x}\right)$. Both v and w are homogeneous functions of degree 1 and degree 0 respectively.

Then by Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v \text{ and } x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0.w.$$

Adding these equations, we get

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = v$$

or
$$x\frac{\partial}{\partial x}(v+w) + y\frac{\partial}{\partial y}(v+w) = v$$

or
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xf\left(\frac{y}{x}\right). \qquad (1)$$

Differentiating equation (1) partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

or
$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} + x\frac{\partial u}{\partial x} = xf\left(\frac{y}{x}\right) - yf'\left(\frac{y}{x}\right). \qquad (2)$$

Again differentiating equation (2) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y\,\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

or
$$xy\frac{\partial^2 u}{\partial y\,\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=yf'\left(\frac{y}{x}\right).$$

Adding equations (2) and (3), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=xf\left(\frac{y}{x}\right)$$

or
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+xf\left(\frac{y}{x}\right)=xf\left(\frac{y}{x}\right)$$

[using equation (1)]

or
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

EXAMPLE 7.29 If $r^2=x^2+y^2+z^2$ and $v=r^m$, then show that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=m\,(m+1)\,r^{m-2}$$

More generally, if $v=f(r)$, then prove that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=f''(r)+\frac{2}{r}f'(r).$$

Solution

First Part:

$$\frac{\partial v}{\partial x}=mr^{m-1}\frac{\partial r}{\partial x}=\frac{m\,x\,r^{m-1}}{r}=m\,x\,r^{m-2}$$

$$\left[\text{as } r^2=x^2+y^2+z^2 \text{ or } 2r\frac{\partial r}{\partial x}=2x\right]$$

Again
$$\frac{\partial^2 v}{\partial x^2}=m\,r^{m-2}+mx\,(m-2)\,r^{m-3}\frac{\partial r}{\partial x}$$

$$=m\,r^{m-2}+m\,(m-2)\,x\,r^{m-3}\,\frac{x}{r}$$

$$=m\,r^{m-2}+m\,(m-2)\,r^{m-4}.$$

Similarly,
$$\frac{\partial^2 v}{\partial y^2}=m\,r^{m-2}+m\,(m-2)\,y^2\,r^{m-4}$$

and
$$\frac{\partial^2 v}{\partial z^2} = m\,r^{m-2} + m\,(m-2)\,z^2\,r^{m-4}.$$

Adding the above results, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 3m\,r^{m-2} + m\,(m-2)\,r^{m-4}\,(x^2 + y^2 + z^2)$$

$$= 3m\,r^{m-2} + m\,(m-2)\,r^{m-2}$$

$$= m\,(m+1)\,r^{m-2}.$$

If $v = f(r)$

then
$$\frac{\partial v}{\partial x} = f'(r)\frac{\partial r}{\partial x} = \frac{x}{r}\,f'(r)$$

and
$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{r}\,f'(r) - \frac{x}{r^2}\frac{\partial r}{\partial x}\,f'(r) + \frac{x}{r}\,f''(r)\frac{\partial r}{\partial x}$$

$$= \frac{1}{r}\,f'(r) - \frac{x^2}{r^3}\,f'(r) + \frac{x^2}{r^2}\,f''(r).$$

Similarly,
$$\frac{\partial^2 v}{\partial y^2} = \frac{1}{r}\,f'(r) - \frac{y^2}{r^3}\,f'(r) + \frac{y^2}{r^2}\,f''(r)$$

and
$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{r}\,f'(r) - \frac{z^2}{r^3}\,f'(r) + \frac{z^2}{r^2}\,f''(r).$$

Adding the above results, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{3}{r}\,f'(r) - \frac{1}{r^3}\,f'(r)(x^2 + y^2 + z) + \frac{1}{r^2}\,f''(r)(x^2 + y^2 + z^2)$$

$$= \frac{3}{r}\,f'(r) - \frac{1}{r}\,f'(r) + \frac{1}{r^2}\,r^2\,f''(r)$$

$$= \frac{2}{r}\,f'(r) + f''(r).$$

EXAMPLE 7.30 If $v = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ and $a^2 + b^2 + c^2 = 1$, then prove that $v_{xx} + v_{yy} + v_{zz} = 0$.

Solution

We have

$$v = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2). \tag{1}$$

Differentiating partially w.r.t. x, we get

$$\frac{\partial v}{\partial x} = 3 \cdot 2\,(ax + by + cz)\,a - 2x$$

and $$\frac{\partial^2 v}{\partial x^2} = 6a^2 - 2.$$

Similarly, $$\frac{\partial^2 v}{\partial y^2} = 6b^2 - 2 \text{ and } \frac{\partial^2 v}{\partial z^2} = 6c^2 - 2.$$

∴ $$v_{xx} + v_{yy} + v_{zz} = 6(a^2 + b^2 + c^2) - 6 = 6 - 6 = 0.$$

Hence proved.

EXAMPLE 7.31 If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

Solution

Let u be a function of x, y and z.

Differentiating partially,

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \tag{1}$$

w.r.t. x, we get

$$\frac{2x}{a^2+u} - \frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x} - \frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x} - \frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x} = 0$$

or $$\frac{\partial u}{\partial x} = \frac{2x}{a^2+u}\Big/ A$$

where $$A = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}.$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{2y}{(a^2+u)}\Big/ A \text{ and } \frac{\partial u}{\partial z} = \frac{2z}{(a^2+u)}\Big/ A.$$

∴ $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$$

$$= \frac{4}{A^2}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] = \frac{4}{A^2}\cdot A = \frac{4}{A}. \tag{2}$$

Again, $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{2}{A}\left(\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right)$$

$$= \frac{2}{A}. \tag{3}$$

[using equation (1)]

∴ From equations (2) and (3)

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right).$$

EXAMPLE 7.32 If $x^2 + y^2 + z^2 - 2xyz = 1$, show that

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

Solution

Differentiating $x^2 + y^2 + z^2 - 2xyz = 1$, we get

$$2x\,dx + 2y\,dy + 2z\,dz - 2\,(yz\,dx + xz\,dy + xy\,dx) = 0$$

or $$(x - yz)\,dx + (y - zx)\,dy + (z - xy)\,dz = 0. \tag{1}$$

Now $$x^2 + y^2 + z^2 - 2xyz = 1$$

or $$x^2 - 2xyz + y^2z^2 + y^2 + z^2 - y^2z^2 = 1$$

or $$(x - yz)^2 = 1 - y^2 - z^2 + y^2z^2$$

$$= (1 - y^2) - z^2\,(1 - y^2) = (1 - y^2)\,(1 - z^2)$$

or $$x - yz = \pm\sqrt{(1 - y^2)\,(1 - z^2)}.$$

Similarly, $$y - zx = \pm\sqrt{(1 - x^2)\,(1 - z^2)}$$

and $$z - xy = \pm\sqrt{(1 - x^2)\,(1 - y^2)}.$$

∴ Form equation (1)

$$\pm\sqrt{(1 - z^2)\,(1 - y^2)}\,dx \pm \sqrt{(1 - x^2)\,(1 - z^2)}\,dy \pm \sqrt{(1 - x^2)\,(1 - y^2)}\,dz = 0.$$

Dividing both sides by $-\sqrt{(1 - z^2)\,(1 - y^2)\,(1 - z^2)}$, we get

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

EXAMPLE 7.33 If v be a homogeneous function in x, y, z of degree n, prove that $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}$ are each a homogeneous function in x, y, z of degree $(n-1)$.

Solution

Since v is a homogeneouos function of degree n.

$\therefore$ By Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = nv.$$

Differentiating partially w.r.t. x, we get

$$\frac{\partial v}{\partial x} + x\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + y\frac{\partial^2 v}{\partial x \partial y} + z\frac{\partial^2 v}{\partial x \partial z} = n\frac{\partial v}{\partial x}$$

or
$$x\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}\right) = (n-1)\frac{\partial v}{\partial x}.$$

This shows that $\frac{\partial v}{\partial x}$ is a homogeneous function in x, y, z of degree $(n-1)$.

Similarly $\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial z}$ are homogeneous functions of degree $(n-1)$.

EXAMPLE 7.34 If V be a homogeneous function in x, y, z of degree n and if $V = f(X, Y, Z)$ where X, Y, Z are respectively $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$, show that

$$X\frac{\partial V}{\partial X} + Y\frac{\partial V}{\partial Y} + Z\frac{\partial V}{\partial Z} = \frac{n}{n-1}V.$$

Solution

Given $V = f(X, Y, Z)$

$\therefore$
$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial x}$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial y}$$

$$\frac{\partial V}{\partial z} = \frac{\partial V}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial V}{\partial Y}\frac{\partial Y}{\partial z} + \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial z}$$

$\therefore$
$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = \frac{\partial V}{\partial x}\left(x\frac{\partial X}{\partial x} + y\frac{\partial X}{\partial y} + z\frac{\partial X}{\partial z}\right)$$
$$+\frac{\partial V}{\partial y}\left(x\frac{\partial Y}{\partial x} + y\frac{\partial Y}{\partial y} + z\frac{\partial Y}{\partial z}\right) + \frac{\partial V}{\partial z}\left(x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} + z\frac{\partial Z}{\partial z}\right)$$

$$nV = \frac{\partial V}{\partial X}(n-1)X + \frac{\partial V}{\partial Y}(n-1)Y + \frac{\partial V}{\partial Z}(n-1)Z.$$

or

Since V is a homogeneous function of degree n and $\frac{\partial V}{\partial x}$ i.e. X, Y and Z are homogeneous functions of degree $(n-1)$.

$$X\frac{\partial V}{\partial X} + Y\frac{\partial V}{\partial Y} + Z\frac{\partial V}{\partial Z} = \frac{n}{n-1}V.$$

or

7.6 CHAIN RULES

If $z = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$; f, ϕ, ψ are all differentiable functions, then the composite function $z = f(\phi(t), \psi(t)) = F(t)$, say, is a differentiable function of t and the total derivative is given by

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{7.2}$$

also the total differential

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{7.3}$$

Again, if $z = f(x, y)$ be a differentiable function in x and y where $x = \phi(u, v)$, $y = \psi(u, v)$ are also differentiable functions of u and v, then the composite function $z = f(\phi, \psi) = F(u, v)$, say, is a differentiable function of u, v whose partial derivatives $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ are given by

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \tag{7.4}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \tag{7.5}$$

and the total differential dz of z is given by

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{7.6}$$

Formula of equations (7.2), (7.4) and (7.5) are known as chain rules.

From Equations (7.3) and (7.6) it may be observed that if $z = f(x, y) = 0$ then $dz = 0$ and hence

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

or
$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}.$$

This formula may be used to find $\frac{dy}{dx}$ of an implicit function.

EXAMPLE 7.35 Find $\frac{dy}{dx}$ when $x^3 - 3xy + 2y^3 = 0$.

Solution

Let $$f(x, y) = x^3 - 3xy + 2y^3.$$

Now, $$f_x = 3x^2 - 3y \text{ and } f_y = -3x + 6y^2$$

Therefore, $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2 - 3y}{-3x + 6y^2} = \frac{x^2 - y}{x - 2y^2}.$$

EXAMPLE 7.36 If $x^y y^x = c$, find the value of $\frac{dy}{dx}$.

Solution

Taking log of $x^y y^x = c$ both sides.
$$y \log x + x \log y = \log c.$$

Let $$f(x, y) = y \log x + x \log y - \log c.$$

$\therefore$ $$f_x = y\frac{1}{x} + \log y \text{ and } f_y = \log x + x\frac{1}{y}.$$

Hence, $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\frac{y}{x} + \log y}{\log x + \frac{x}{y}} = -\frac{y\,(y + x\log y)}{x\,(x + y\log x)}.$$

EXAMPLE 7.37 If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that
$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - yx)\frac{\partial u}{\partial z} = 0.$$ (WBUT 20

Solution

Let $$r = x^2 + 2yz, \quad s = y^2 + 2zx.$$

$\therefore$ $$u = f(r, s)$$

Now,
$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}=\frac{\partial u}{\partial r}(2x)+\frac{\partial u}{\partial s}(2z)$$
$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}=\frac{\partial u}{\partial r}(2z)+\frac{\partial u}{\partial s}(2y)$$
and
$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}=\frac{\partial u}{\partial r}(2y)+\frac{\partial u}{\partial s}(2x).$$
Now,
$$(y^2-zx)\frac{\partial u}{\partial x}+(x^2-yz)\frac{\partial u}{\partial y}+(z^2-xy)\frac{\partial u}{\partial z}$$
$$=\frac{\partial u}{\partial r}\{2x(y^2-zx)+2z(x^2-yz)+2y(z^2-xy)\}$$
$$+\frac{\partial u}{\partial s}\{2z(y^2-zx)+2y(x^2-yz)+2x(z^2-xy)\}$$
$$=0.$$

EXAMPLE 7.38 If $z=\sin uv$ where $u=3x^2$ and $v=\log x$, find $\frac{dz}{dx}$.

(WBUT 2004)

Solution

Since z is a function of u and v,
$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\frac{du}{dx}+\frac{\partial z}{\partial v}\frac{dv}{dx}$$
$$=v\cos uv\cdot 6x+u\cos uv\frac{1}{x}$$
$$=6xv\cos uv+\frac{u}{x}\cos uv$$
$$=(6x\log x+3x)\cos(3x^2\log x).$$

EXAMPLE 7.39 If $z=u^3+v^3$, where $u=\sin xy$ and $v=y^2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution

Since z is a function of u, v and u, v are functions of x and y.

Therefore,
$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$
$$=3u^2\cdot y\cos xy+3v^2\cdot 0$$
$$=3yu^2\cos xy$$
$$=3y\sin^2(xy)\cos xy$$

and
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$
$$= 3u^2 \cdot x\cos xy + 3v^2 \cdot 2y$$
$$= 3u^2 x\cos xy + 6v^2 y$$
$$= 3x\sin^2 xy\cos xy + 6y^5.$$

EXAMPLE 7.40 If $z = f(u, v)$ where $u = x^2 - 2xy - y^2$ and $v = y$, show that $(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0$ can be transformed into $\frac{\partial z}{\partial v} = 0$.

Solution

Since $z = f(u, v)$

$\therefore$
$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$
$$= \frac{\partial f}{\partial u}(2x - 2y) + 0$$
$$= -2(x-y)\frac{\partial f}{\partial u}$$

and
$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$
$$= \frac{\partial f}{\partial u}(-2x - 2y) + \frac{\partial f}{\partial v}\cdot 1$$
$$= -2(x+y)\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}.$$

Now, $(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0$ becomes
$$2(x^2 - y^2)\frac{\partial f}{\partial u} - 2(x^2 - y^2)\frac{\partial f}{\partial u} + (x-y)\frac{\partial f}{\partial u} = 0$$

or
$$(x-y)\frac{\partial f}{\partial v} = 0$$

or
$$\frac{\partial f}{\partial v} = 0 \text{ if } x \neq y$$

or
$$\frac{\partial z}{\partial v} = 0 \text{ since } z = f(u, v).$$

Hence proved.

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EXAMPLE 7.41 If $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ where u and v are functions of x and y, prove that

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial v}{\partial r}$$

where $x=r\cos\theta$, $y=r\sin\theta$.

Solution

Given
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \tag{1}$$

and
$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}. \tag{2}$$

Now,
$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}(\cos\theta)+\frac{\partial u}{\partial y}(\sin\theta)$$

$$=\cos\theta\frac{\partial u}{\partial x}-\sin\theta\frac{\partial v}{\partial x}. \qquad \text{[using equation (2)]}$$

Again
$$\frac{\partial v}{\partial \theta}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$

$$=\frac{\partial v}{\partial x}(-r\sin\theta)+\frac{\partial v}{\partial y}(r\cos\theta)$$

$$=r\left(-\sin\theta\frac{\partial v}{\partial x}+\cos\theta\frac{\partial u}{\partial x}\right) \qquad \text{[by equation (1)]}$$

∴ From the above two relations, we have

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Now,
$$\frac{\partial v}{\partial r}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial v}{\partial x}\cos\theta+\frac{\partial v}{\partial y}\sin\theta$$

$$=\frac{\partial v}{\partial x}\cos\theta+\frac{\partial u}{\partial x}\sin\theta$$

and
$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$=\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}(r\cos\theta)$$

$$=-r\left(\sin\theta\frac{\partial u}{\partial x}+\cos\theta\frac{\partial v}{\partial x}\right)$$

and hence $$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

or $$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

EXAMPLE 7.42 If $u = f(x, y)$ where $x = r\cos\theta$, $y = r\sin\theta$, then prove that

(i) $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

(ii) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$ (WUBT 2002)

Solution

(i) $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$

and $$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}.$$

$$\therefore \quad \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right)^2 + \left(-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

(ii) We use the operators $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ as

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

and $$\frac{\partial}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

$$\therefore \quad \frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right) \quad [r, \theta \text{ are independent}]$$

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$$= \cos\theta\left[\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right]\left(\frac{\partial u}{\partial x}\right)+\sin\theta\left[\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}\right]\left(\frac{\partial u}{\partial y}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 u}{\partial y^2}.$$

Similarly,

$$\frac{\partial^2 u}{\partial\theta^2}=\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial\theta}\right)=\frac{\partial}{\partial\theta}\left\{-r\sin\theta\frac{\partial u}{\partial x}+r\cos\theta\frac{\partial u}{\partial y}\right\}$$

$$=-r\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial u}{\partial x}-r\sin\theta\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)$$

$$+r\frac{\partial}{\partial\theta}(\cos\theta)\frac{\partial u}{\partial y}+r\cos\theta\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial y}\right)$$

$$=-r\cos\theta\frac{\partial u}{\partial x}-r\sin\theta\frac{\partial u}{\partial y}-r\sin\theta\left[-r\sin\theta\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y}\right]\left(\frac{\partial u}{\partial x}\right)$$

$$+r\cos\theta\left[-r\sin\theta\frac{\partial}{\partial x}+r\cos\theta\frac{\partial}{\partial y}\right]\left(\frac{\partial u}{\partial y}\right)$$

$$=-r\cos\theta\frac{\partial u}{\partial x}-r\sin\theta\frac{\partial u}{\partial y}+r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2}-r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial y\partial x}$$

$$-r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}+r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2}$$

$$=-r\frac{\partial u}{\partial r}+r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2}+r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2}-2r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial y\partial x}.$$

$$\therefore\quad \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\sin^2\theta\frac{\partial^2 u}{\partial x^2}+\cos^2\theta\frac{\partial^2 u}{\partial y^2}-2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}.$$

Hence by adding, we get

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

EXAMPLE 7.43 By the transfromation $\xi=a+\alpha x+\beta y, \eta=b-\beta x+\alpha y$ where a, b, α, β are constants and $\alpha^2+\beta^2=1$ the function $u(x, y)$ is transformed to $U(\xi,\eta)$. Prove that

$$U_{\xi\xi}U_{\eta\eta}-U_{\xi\eta}^2=U_{xx}U_{yy}-U_{xy}^2.$$

Solution

Since the function $u(x, y)$ transformed to $U(\xi, \eta)$, so we have

$$\frac{\partial u}{\partial x}=\frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x}=\alpha\frac{\partial U}{\partial \xi}-\beta\frac{\partial U}{\partial \eta}$$

and
$$\frac{\partial^2 u}{\partial x^2}=\left(\alpha\frac{\partial}{\partial \xi}-\beta\frac{\partial}{\partial \eta}\right)\left(\alpha\frac{\partial U}{\partial \xi}-\beta\frac{\partial U}{\partial \eta}\right)$$

$$=\alpha^2 U_{\xi\xi}-2\alpha\beta U_{\xi\eta}+\beta^2 U_{\eta\eta}$$

$$\frac{\partial u}{\partial y}=\frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial y}=\beta U_{\xi}+\alpha U_{\eta},$$

$$\frac{\partial^2 u}{\partial y^2}=\beta^2 U_{\xi\xi}+2\alpha\beta U_{\xi\eta}+\alpha^2 U_{\eta\eta},$$

and
$$\frac{\partial^2 u}{\partial x\partial y}=\left(\alpha\frac{\partial}{\partial \xi}-\beta\frac{\partial}{\partial \eta}\right)\left(\beta\frac{\partial u}{\partial \xi}+\alpha\frac{\partial u}{\partial \eta}\right)$$

$$=\alpha\beta(U_{\xi\xi}-U_{\eta\eta})+(\alpha^2-\beta^2)U_{\xi\eta}.$$

Now,

$$U_{xx}U_{yy}-U_{xy}^2=\alpha^2\beta^2U_{\xi\xi}^2-2\alpha\beta^3U_{\xi\eta}U_{\xi\xi}+\beta^4U_{\xi\xi}U_{\eta\eta}+\alpha^4U_{\xi\xi}U_{\eta\eta}$$

$$-2\alpha^3\beta U_{\xi\eta}U_{\eta\eta}+\alpha^2\beta^2U_{\eta\eta}^2+2\alpha^3\beta U_{\xi\xi}U_{\eta\eta}$$

$$-4\alpha^2\beta^2U_{\xi\eta}^2+2\alpha\beta^3U_{\eta\eta}U_{\xi\eta}-\alpha^2\beta^2(U_{\xi\xi}^2+U_{\eta\eta}^2-2U_{\xi\xi}U_{\eta\eta})$$

$$-(\alpha^2-\beta^2)U_{\xi\eta}^2-2\alpha\beta(\alpha^2-\beta^2)U_{\xi\eta}(U_{\xi\xi}-U_{\eta\eta})$$

$$=\alpha^2\beta^2(U_{\xi\xi}^2+U_{\eta\eta}^2)+(\alpha^4+\beta^4)(U_{\xi\xi}U_{\eta\eta})+2\alpha\beta(\alpha^2-\beta^2)$$

$$\times U_{\xi\eta}(U_{\xi\xi}-U_{\eta\eta})-4\alpha^2\beta^2U_{\xi\eta}^2-\alpha^2\beta^2(U_{\xi\xi}^2+U_{\eta\eta}^2)$$

$$+2\alpha^2\beta^2U_{\xi\xi}U_{\eta\eta}-(1-4\alpha^2\beta^2)U_{\xi\eta}^2$$

$$-2\alpha\beta(\alpha^2-\beta^2)U_{\xi\eta}(U_{\xi\xi}-U_{\eta\eta})$$

$$=(1-2\alpha^2\beta^2)(U_{\xi\xi}U_{\eta\eta})-4\alpha^2\beta^2U_{\xi\eta}^2+2\alpha^2\beta^2U_{\xi\xi}U_{\eta\eta}$$

$$-U_{\xi\eta}^2+4\alpha^2\beta^2U_{\xi\eta}^2$$

$$=U_{\xi\xi}U_{\eta\eta}-U_{\xi\eta}^2.$$

EXAMPLE 7.44 Let u be a function of x and y satisfying $x = \theta \cos \alpha - \phi \sin \alpha$, $y = \theta \sin \alpha + \phi \cos \alpha$ (α = constant). Prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2}.$$

Solution

Since u is a function of x and y therefore

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \cos \alpha \frac{\partial u}{\partial x} + \sin \alpha \frac{\partial u}{\partial y}$$

and
$$\frac{\partial}{\partial \theta} \equiv \cos \alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial y}.$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \left(\cos \alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial y}\right)\left(\cos \alpha \frac{\partial u}{\partial x} + \sin \alpha \frac{\partial u}{\partial y}\right)$$

$$= \cos^2 \alpha \frac{\partial^2 u}{\partial x^2} + 2\cos \alpha \sin \alpha \frac{\partial^2 u}{\partial x \partial y} + \sin^2 \alpha \frac{\partial^2 u}{\partial y^2}. \quad (1)$$

Again,
$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi} = \frac{\partial u}{\partial x}(-\sin \alpha) + \frac{\partial u}{\partial y}(\cos \alpha)$$

or
$$\frac{\partial}{\partial \phi} \equiv -\sin \alpha \frac{\partial}{\partial x} + \cos \alpha \frac{\partial}{\partial y}.$$

Now,
$$\frac{\partial^2 u}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial \phi}\right) = \left(-\sin \alpha \frac{\partial}{\partial x} + \cos \alpha \frac{\partial}{\partial y}\right)\left(-\sin \alpha \frac{\partial u}{\partial x} + \cos \alpha \frac{\partial u}{\partial y}\right)$$

$$= \sin^2 \alpha \frac{\partial^2 u}{\partial x^2} + \cos^2 \alpha \frac{\partial^2 u}{\partial y^2} - 2\sin \alpha \cos \alpha \frac{\partial^2 u}{\partial x \partial y}. \quad (2)$$

Adding equations (1) and (2), we get

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = \frac{\partial^2 u}{\partial x^2}(\sin^2 \alpha + \cos^2 \alpha) + \frac{\partial^2 u}{\partial y^2}(\cos^2 \alpha + \sin^2 \alpha)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

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Since dx, dy, dz are independent, so by equating we get

$$\frac{\partial f}{\partial x} = \mu P, \frac{\partial f}{\partial y} = \mu Q, \frac{\partial f}{\partial z} = \mu R.$$

Now, $$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial \mu}{\partial y} P + \mu \frac{\partial P}{\partial y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial \mu}{\partial x} Q + \mu \frac{\partial Q}{\partial x}$$

$\therefore$ $$\frac{\partial \mu}{\partial y} P + \mu \frac{\partial P}{\partial y} = \frac{\partial \mu}{\partial x} Q + \mu \frac{\partial Q}{\partial x}$$

or $$\frac{\partial \mu}{\partial y} P - \frac{\partial \mu}{\partial x} Q = \mu \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

Similarly, $$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial \mu}{\partial x} R + \mu \frac{\partial R}{\partial x}$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial \mu}{\partial z} P + \mu \frac{\partial P}{\partial z}$$

$\therefore$ $$\frac{\partial \mu}{\partial x} R + \mu \frac{\partial R}{\partial x} = \frac{\partial \mu}{\partial z} P + \mu \frac{\partial P}{\partial z}$$

or $$\frac{\partial \mu}{\partial x} R - \frac{\partial \mu}{\partial z} P = \mu \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)$$

and finally $$\frac{\partial \mu}{\partial y} R - \frac{\partial \mu}{\partial z} Q = \mu \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right).$$

$\therefore$ $$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial R}{\partial z} \right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$$

$$= \frac{P}{\mu}\left(\frac{\partial \mu}{\partial y} R - \frac{\partial \mu}{\partial z} Q \right) + \frac{Q}{\mu}\left(\frac{\partial \mu}{\partial z} P - \frac{\partial \mu}{\partial x} R \right) + \frac{R}{\mu}\left(\frac{\partial \mu}{\partial x} Q - \frac{\partial \mu}{\partial y} P \right)$$

$$= \frac{1}{\mu}\left[\frac{\partial \mu}{\partial y}(PR - RP) + \frac{\partial \mu}{\partial z}(-PQ + QP) + \frac{\partial \mu}{\partial x}(-RQ + QR) \right]$$

$$= 0.$$

CHAPTER 8

Maxima and Minima

8.1 INTRODUCTION

Finding out extrema (maxima and/or minima) of a function (with or without some constraints) is very important subject in every branch of engineering, science, social science, medicine, etc. Students have already solved problems on maxima and minima of functions of a single variable. Here, we shall discuss maxima and minima of functions containing two independent variables.

8.2 TOTAL DIFFERENTIAL

Let $u = f(x, y)$ be a function of two independent variables. Also, let Δx and Δy be their increment, then $\Delta u = f(x + \Delta x,\ y + \Delta y) - f(x, y)$ is the increment of u or f.

Now, by chain rule (Chapter 7), we have

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \tag{8.1}$$

The quantity du or df is called the total differential or simply differential (first order) of u or f.

The above equation (8.1) can also be expressed as

$$\Delta u = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y. \tag{8.2}$$

This result can be generalised for n variables as stated below:

If
$$u = f(x_1, x_2, \ldots, x_n)$$

then
$$du = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

The equation (8.1) can be written as

$$du = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)u \quad \text{or} \quad d = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)$$

Now, the second order differential d^2u is given by

$$d^2u = d(du) = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)\left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)u$$

$$= \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^2 u$$

$$= \frac{\partial^2 u}{\partial x^2}(dx)^2 + 2\frac{\partial^2 u}{\partial x \partial y}dx\,dy + \frac{\partial^2 u}{\partial y^2}(dy)^2$$

$\left(\text{Assuming } \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}\right)$.

EXAMPLE 8.1 Let $u = x^3 + xy^2$. Find du and d^2u.

Solution

Here

$$\frac{\partial u}{\partial x} = 3x^2 + y^2, \quad \frac{\partial u}{\partial y} = 2xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial^2 u}{\partial x \partial y} = 2y, \quad \frac{\partial^2 u}{\partial y^2} = 2x.$$

$\therefore$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = (3x^2 + y^2)dx + (2xy)dy$$

and

$$d^2u = \frac{\partial^2 u}{\partial x^2}(dx)^2 + 2\frac{\partial^2 u}{\partial x \partial y}dx\,dy + \frac{\partial^2 u}{\partial y^2}(dy)^2$$

$$= 6x(dx)^2 + 4y\,dx\,dy + 2x(dy)^2.$$

EXAMPLE 8.2 Let $f(x, y) = xy + x^2 + y^2$. Find the change in f when the increments of x and y are respectively 0.01 and 0.05 at $x = 2, y = 1$.

Solution

The change in f is given by

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

Here

$$f(x, y) = xy + x^2 + y^2.$$

Thus, $$\frac{\partial f}{\partial x} = y + 2x \text{ and } \frac{\partial f}{\partial y} = x + 2y.$$

$\therefore$ $$\Delta x = 0.01,\ \Delta y = 0.05 \text{ and at } (2, 1), \frac{\partial f}{\partial x} = 5,\ \frac{\partial f}{\partial y} = 4.$$

Hence change in f is

$$\Delta f = 5 \times 0.01 + 4 \times 0.05 = 0.25.$$

8.3 DEFINITION OF MAXIMA AND MINIMA

Let $f(x, y)$ be a function of two independent variables x and y. Let us further assume that f is *continuous* and *finite* for all values of x and y in the neighbourhood of their values a, b. Then the value of $f(a, b)$ is said to be a maximum or a minimum if

$$f(a+h, b+k) - f(a, b) \tag{8.3}$$

keeps some sign (positive or negative) for all positive or negative small values of h and k. The point (a, b) is called *extreme point*.

The values $f(a, b)$ is maximum if the difference shown in equation (8.3) is negative for all positive and negative small values of h and k; and minimum if the difference shown in equation (8.3) is positive for all positive and negative small values of h and k.

8.4 THE NECESSARY CONDITION FOR EXTREMA

Let $f(x, y)$ be *continuous* and *differentiable* function of two variables x and y. Then by Taylor's theorem for two variables,

$$f(x+h,\ y+k) = f(x,\ y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \text{ terms of second and higher order.}$$

$$f(x+h,\ y+k) - f(x,\ y) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \text{ terms of second and higher order.} \tag{8.4}$$

But, for extrema

$$f(x+h,\ y+k) - f(x,\ y) \tag{8.5}$$

must preserve same sign.

For sufficiently small values of h and k, the sign of equation (8.5) depends on the sign of $h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}$, and the sign of this first degree terms depends on the signs of h and k.

Hence the necessary condition for the existence of extrema is $h\dfrac{\partial f}{\partial x} + k\dfrac{\partial f}{\partial y} = 0$.

But, h and k are independent increments of x and y respectively, so the above equation must hold if

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0. \tag{8.6}$$

Solve these equations for x and y, we get the values of variable for which $f(x, y)$ is either a maximum or a minimum.

It may be noted that the condition of equation (8.6) is necessary and not sufficient for the continuous and differentiable function.

Note: 1. The function $f(x, y) = |x| + |y|$ has a minimum value at (0, 0) even though the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ do not exist at (0, 0).

2. If $f(x, y) = 0$, if $x = 0, y = 0$
$= 1$, elsewhere

Then both the partial derivatives exist at the origin but $f(0, 0)$ is not an extreme value. Thus the conditions are only necessary, but, not sufficient.

8.5 THE SUFFICIENT CONDITION FOR EXTREMA (THE LAGRANGE'S CONDITION)

Let $f(x, y)$ be a function of two independent variables x and y. We assume that f is continuous and differentiable at the neighbourhood of (a, b).

Let $$r = \frac{\partial^2 f}{\partial x^2},\quad s = \frac{\partial^2 f}{\partial x \partial y} \text{ and } t = \frac{\partial^2 f}{\partial y^2} \text{ at } x = a,\ y = b.$$

As a set of necessary conditions for a maximum or minimum at (a, b) we have $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ at (a, b).

Therefore, $f(a+h, b+k) - f(a, b) = \dfrac{1}{2!}(rh^2 + 2shk + tk^2) +$ terms of third and higher order. (8.7)

By taking h and k sufficiently small, the sign of left hand side of equation (8.7) is depends on $rh^2 + 2shk + tk^2$. Thus, f has a maximum if $I = rh^2 + 2shk + tk^2$ is negative and a minimum if I is positive.

Now,

$$I = rh^2 + 2shk + tk^2$$

$$=\frac{1}{r}\left[r^2h^2+2srhk+rtk^2\right]$$

$$=\frac{1}{r}[(rh+sk)^2+(rt-s^2)k^2].$$

Thus, we observed that, if $rt-s^2>0$ then sign of I is same as r. Hence, the expression I is positive, i.e. f is minimum if

$$rt-s^2>0 \text{ and } r>0 \tag{8.8}$$

and the expression I is negative, i.e. f is maximum if

$$rt-s^2>0 \text{ and } r<0. \tag{8.9}$$

Stationary point or critical point

A point (a, b) is said to be *stationary point* or *critical point* of a function $f(x, y)$ if $f_x(a, b)=0$ and $f_y(a, b)=0$.

Saddle point

A point (a, b) is saddle point of a function $f(x, y)$ if $f_x(a, b)=0$ and $f_y(a, b)=0$, but, $f(x, y)$ has neither a maximum nor a minimum at (a, b).

Thus, every saddle point is a stationary point, but converse is not true (see Figure 8.1)

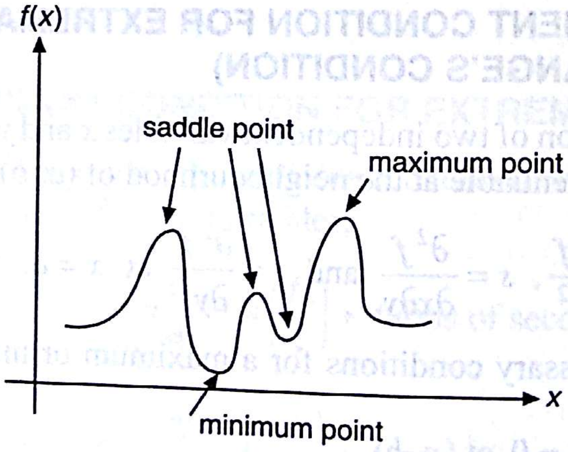

Figure 8.1 Stationary points for $y=f(x)$.

8.6 WORKED-OUT EXAMPLES

EXAMPLE 8.3 Find the maximum and minimum values of the function $f(x, y) = x^3+y^3-3axy$.

(WBUT 2002, 2008)

Solution

Here

$$f(x, y) = x^3+y^3-3axy.$$

Therefore, at stationary points

$$f_x = 3x^2 - 3ay = 0,\ f_y = 3y^2 - 3ax = 0.$$

That is,

$$x^2 = ay \text{ and } y^2 = ax,$$

or

$$x^4 = a^2y^2 = a^2\,ax = a^3x \text{ or } x(x^3 - a^3) = 0$$

or

$$x = 0,\ a.$$

Similarly,

$$y = 0,\ a.$$

Thus the stationary points are (0, a) and (0, 0).

Now,

$$f_{xx} = 6x, f_{yy} = 6y, f_{xy} = -3a.$$

At (a,a)

$$f_{xx} = 6a, f_{yy} = 6a \text{ and } f_{xy} = -3a.$$

Also

$$f_{xx} \cdot f_{yy} - (f_{xy})^2 = 36a^2 - 9a^2 > 0.$$

Again, $f_{xx} = 6a$ is positive or negative depending on whether a is positive or negative. Hence f has a maximum or a minimum value at $x = y = a$ depending on whether a is negative or positive.

The optimum value is $f(a, a) = -a^3$.

At (0, 0)

$$f_{xx} = 0,\ f_{yy} = 0,\ f_{xy} = -3a \text{ and } f_{xx} \cdot f_{yy} - (f_{xy})^2 = -9a^2 < 0.$$

Hence, (0, 0) is not an extreme point.

EXAMPLE 8.4 Find the maximum and minimum values of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$, and also find saddle points if any.

Solution

$$f_x(x, y) = 3x^2 - 3 = 0 \text{ when } x = \pm 1$$

$$f_y(x, y) = 3y^2 - 12 = 0 \text{ when } y = \pm 2.$$

Thus the function has four stationary points (1, 2); (–1, 2); (1, –2); (–1, –2).

Now,

$$f_{xx}\,(x, y) = 6x,\ f_{xy}\,(x, y) = 0,\ f_{yy}\,(x, y) = 6y.$$

At (1, 2)

$$f_{xx} = 6 > 0 \text{ and } f_{xx} \cdot f_{yy} - (f_{xy})^2 = 72 > 0.$$

Hence, (1, 2) is a point of minimum of the function.

At (–1, 2)

$$f_{xx} = -6 \text{ and } f_{xx} \cdot f_{yy} - (f_{xy})^2 = -72 < 0.$$

Hence, the function has neither maximum nor minimum at (–1, 2).

At (1, –2)

$$f_{xx} = 6 \text{ and } f_{xx} \cdot f_{yy} - (f_{xy})^2 = -72 < 0.$$

Hence, the function has neither maximum nor minimum at (1, –2).

At (–1, –2)

$$f_{xx} = -6 \text{ and } f_{xx} \cdot f_{yy} - (f_{xy})^2 = 72 > 0.$$

Hence the function has a maximum value at (–1, –2). Therefore, the maximum value is 38 and minimum value is 2 and the saddle points are (–1, 2) and (1, –2).

EXAMPLE 8.5 Prove that the function $f(x, y) = x^2 - 2xy + y^2 + x^4 + y^4$ is a minimum at the origin.

Solution

Now,

$$f_x(x, y) = 2x - 2y + 4x^3 = 0$$

if

$$x = 0, y = 0$$

$$f_y(x, y) = -2x + 2y + 4y^3 = 0$$

if

$$x = 0, y = 0$$

$$f_{xx}(x, y) = 2 + 12x^2, f_{xx}(0, 0) = 2$$

$$f_{yy}(x, y) = 2 + 12y^2, f_{yy}(0, 0) = 2$$

$$f_{xy}(x, y) = -2$$

Now,

$$f_{xx} \cdot f_{yy} - (f_{xy})^2 = 4 - 4 = 0.$$

Therefore, we cannot conclude about extremum at (0, 0). Thus we arrive at a doubtful case and it requires further investigation.

$f(x, y)$ can be written as

$$f(x, y) = (x - y)^2 + x^4 + y^4$$

and

$$f(x, y) - f(0, 0) = (x - y)^2 + x^4 + y^4$$

which is greater than zero for all values of (x, y). Thus f has a minimum value at the origin.

EXAMPLE 8.6 Prove that the function $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$ has neither a maximum nor a minimum at the origin.

Solution

Now,

$$f_x(x, y) = 2x - 2y + 3x^2 + 5x^4$$

and

$$f_y(x, y) = -2x + 2y - 3y^2$$

at extrema,

$$f_x = 0 \text{ and } f_y = 0.$$

$$\therefore \quad 2x - 2y + 3x^2 + 5x^4 = 0 \text{ and } -2x + 2y - 3y^2 = 0.$$

Solving these two equations, we get

$$x = 0, y = 0.$$

$\therefore$

$$f_{xx}(x, y) = 2 + 6x + 20x^3, f_{xx}(0, 0) = 2$$

$$f_{yy}(x, y) = 2 - 6y, f_{yy}(0, 0) = 2$$

$$f_{xy}(x, y) = -2, f_{xy}(0, 0) = -2$$

$\therefore$

$$f_{xx}(0, 0) f_{yy}(0, 0) - f_{yy}^2(0, 0) = 0.$$

So, there is a doubtful case. Thus we apply direct method. The function can be written as

$$f(x, y) = (x - y)^2 + x^5 + (x - y)(x^2 + xy + y^2)$$

or

$$f(x, y) - f(0, 0) = (x - y)^2 + x^5 + (x - y)(x^2 + xy + y^2).$$

Now, we put $x = 2$ and $y = 1$, then we have

$$f(2, 1) - f(0, 0) = 40 > 0$$

and

$$f(1, 2) - f(0, 0) = -5 < 0.$$

Therefore near (0, 0), the sign of $f(x, y) - f(0, 0)$ is not fixed, i.e. (0, 0) is not a minimum or maximum point.

EXAMPLE 8.7 Find the maximum value of the function

$f(x, y) = \sin\frac{x}{2}\ \sin\frac{y}{2}\sin\frac{x+y}{2}$ defined on the triangular area bounded by the coordinate axes and the line $x + y = 2\pi$.

Solution

We have for maximum and minimum

$$f_x(x, y) = \sin\frac{y}{2}\left[\frac{1}{2}\cos\frac{x}{2}\sin\frac{x+y}{2} + \frac{1}{2}\sin\frac{x}{2}\cos\frac{x+y}{2}\right]$$

which gives

$$\tan\frac{x+y}{2} = -\tan\frac{x}{2} \qquad (1)$$

and

$$f_y(x, y) = \sin\frac{x}{2}\left[\frac{1}{2}\cos\frac{y}{2}\sin\frac{x+y}{2} + \frac{1}{2}\sin\frac{y}{2}\cos\frac{x+y}{2}\right] = 0$$

or

$$\tan\frac{x+y}{2} = -\tan\frac{y}{2}. \qquad (2)$$

From equations (1) and (2), we get

$$\tan\frac{x}{2} = \tan\frac{y}{2} \text{ or } \frac{x}{2} = \frac{y}{2} \text{ or } x = y$$

Again from equation (1)

$$\tan\left(\frac{2x}{2}\right) = -\tan\frac{x}{2}$$

or

$$\tan x = \tan\left(\pi - \frac{x}{2}\right)$$

or

$$x = \pi - \frac{x}{2} \text{ or } \frac{3x}{2} = \pi \text{ or } x = \frac{2\pi}{3}$$

and

$$y = \frac{2\pi}{3}.$$

The point $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ lies inside the triangle, defined in the problem.

Now, $$f_x(x, y) = \frac{1}{2}\sin\frac{y}{2}\sin\left(x + \frac{y}{2}\right)$$

and $$f_y(x, y) = \frac{1}{2}\sin\frac{x}{2}\sin\left(y + \frac{x}{2}\right).$$

$\therefore$ $$f_{xx}(x, y) = \frac{1}{2}\sin\frac{y}{2}\cos\left(x + \frac{y}{2}\right) = \frac{1}{2}\sin\frac{\pi}{3}\cos\pi$$

$$= \frac{-\sqrt{3}}{4} \text{ at } x = y = \frac{2\pi}{3}.$$

$$f_{yy}(x, y) = \frac{1}{2}\sin\frac{x}{2}\cos\left(y + \frac{x}{2}\right) = \frac{-\sqrt{3}}{4} \text{ at } x = y = \frac{2\pi}{3}$$

$$f_{yx}(x, y) = \frac{1}{4}\cos\frac{y}{2}\sin\left(x + \frac{y}{2}\right) + \frac{1}{4}\sin\frac{y}{2}\cos\left(x + \frac{y}{2}\right)$$

$\therefore$ $$f_{xy}\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = \frac{1}{4}\cos\frac{\pi}{3}\sin\pi + \frac{1}{4}\sin\frac{\pi}{3}\cos\pi$$

$$= -\frac{1}{4}\frac{\sqrt{3}}{2}.$$

$\therefore$ $$f_{xx}f_{yy} - (f_{xy})^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{\sqrt{3}}{8}\right)^2$$

$$= 3 - \frac{3}{64} = \frac{189}{64} > 0$$

and $$f_{xx} = -\sqrt{3} < 0.$$

Hence there is a maximum at $x = y = \frac{2\pi}{3}$ and the maximum value is

$$f_{\max} = \sin\frac{\pi}{3}\sin\frac{\pi}{3}\sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{8}.$$

8.7 LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Let $u = f(x, y)$ be a function of 2 variables x and y which are connected by the equation $\phi(x, y) = 0$.

We define a function

$$F = f + \lambda\phi$$

and consider all the variables x, y and λ are independent, λ is called *Lagrange's multiplier*.

At a stationary point of F, $dF = 0$.

Therefore,

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial \lambda}d\lambda = 0.$$

Since $dx, dy, d\lambda$ are arbitrary (independent), the above equation will satisfy only if

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial \lambda} = 0.$$

The solutions of these equations give the stationary points of F. It can be shown that these stationary points are the stationary points of f.

A stationary point will be an extreme point of f if d^2F or d^2f keeps the same sign and will be a maximum or minimum depending on whether d^2F or d^2f is negative or positive.

The above results can also be extended for the function of three or more variables. Suppose $u = f(x, y, z)$ and the independent variables x, y and z are connected by $\phi(x, y, z) = 0$ and $\psi(x, y, z) = 0$. Then we consider

$$F = f + \lambda_1\phi + \lambda_2\psi$$

where λ_1, λ_2 are the Lagrange's multipliers.

As in the previous case, the stationary points of F as well as f, are obtained by solving the following equations

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial \lambda_1} = 0, \frac{\partial F}{\partial \lambda_2} = 0.$$

The function f will be maximum or minimum according as d^2F or d^2f is negative or positive.

EXAMPLE 8.8 Show that the greatest value of x^3y^2 and $x + y = k$ (k is a constant) is $108\ k^5/3125$.

Solution

Let

$$u = x^3y^2$$

and

$$F = x^3\ y^2 + \lambda(x + y - k)$$

$$F = u + \lambda(x + y - k)$$

where λ is Lagrange's multiplier.

At stationary point $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial \lambda} = 0.$

That is,
$$F_x = 3x^2y^2 + \lambda = 0$$
$$F_y = 2x^3y + \lambda = 0$$
$$F_z = x + y - k = 0.$$

First two equations give $3y = 2x$ or $y = \dfrac{2x}{3}$.

Substitute this value in $x + y = k$. Then $x + \dfrac{2x}{3} = k$

or
$$\frac{5x}{3} = k \text{ or } x = \frac{5k}{3}.$$

Therefore,
$$x = \frac{3k}{5} \text{ and } y = \frac{2x}{3} = \frac{2k}{5}.$$

Now, $\quad F_x = 3x^2y^2,\ F_y = 2x^3y,\ F_{xx} = 6xy^2,\ F_{yy} = 2x^3,\ F_{xy} = 6x^2y.$

At $\left(\dfrac{3k}{5}, \dfrac{2k}{5}\right)$,

$$F_{xx} = 6\cdot\frac{3k}{5}\cdot\frac{4k^2}{25} = \frac{72k^3}{125},\ F_{yy} = 2\frac{27k^3}{125} = \frac{54k^3}{125}$$

and
$$F_{xy} = 6\cdot\frac{9k}{5}\cdot\frac{2k^2}{25} = \frac{108k^3}{125}.$$

Also,
$$dx + dy = 0 \text{ or } dy = -dx \text{ since } x + y = k.$$

Now,
$$d^2F = F_{xx}(dx)^2 + 2F_{xy}\,dxdy + F_{yy}(dy)^2$$
$$= \frac{72k^3}{125}dx + 2\frac{108k^3}{125}dx\,(-dx) + \frac{54k^3}{125}(-dx)^2$$
$$= -\frac{90}{125}k^3(dx)^2 < 0.$$

Hence F i.e. u is maximum at $x = \dfrac{3k}{5}$, $y = \dfrac{2k}{5}$ and the maximum value is

$$\left(\frac{3k}{5}\right)^3\left(\frac{2k}{5}\right)^2 = \frac{108k^5}{3125}.$$

EXAMPLE 8.9 **Find the maximum value of $2x + y$ $(x, y > 0)$ where x and y satisfy the equation $x^2 + xy + y^2 = 3$.**

Solution

Let $$u = 2x + y$$

and $$F = u + \lambda(x^2 + xy + y^2 - 3) = 2x + y + \lambda(x^2 + xy + y^2 - 3),$$

where λ is Lagrange's multiplier.

$$F_x = 2 + 2x\lambda + \lambda y$$

$$F_y = 1 + \lambda x + 2\lambda y$$

$$F_\lambda = x^2 + xy + y^2 - 3.$$

At stationary points,

$$F_x = 2 + 2x\lambda + \lambda y = 0$$

$$F_y = 1 + \lambda x + 2\lambda y = 0$$

$$F_\lambda = x^2 + xy + y^2 - 3 = 0.$$

Solving the first two equations for x and y, we get $x = -\dfrac{1}{\lambda}$, $y = 0$.

Substituting these values in the third equation, we get

$$\frac{1}{\lambda^2} - 3 = 0 \text{ or } \frac{1}{\lambda} = \pm\sqrt{3}.$$

Thus, $x = \pm\sqrt{3}$ and $y = 0$ are the stationary points.

Now, we calculate second order derivative of F.

$$F_{xx} = 2\lambda, \; F_{yy} = 2\lambda, \; F_{xy} = \lambda.$$

From the equation $x^2 + xy + y^2 = 3$, we have

$$2xdx + xdy + ydx + 2ydy = 0$$

or $$(2x + y)dx + (x + 2y)dy = 0$$

or $$\frac{dy}{dx} = -\frac{2x + y}{x + 2y} = -2 \text{ at } (\pm\sqrt{3}, 0).$$

Thus, $$dy = -2dx \text{ at } (\pm\sqrt{3}, 0).$$

Now, $$d^2F = F_{xx}(dx)^2 + 2F_{xy}dxdy + F_{yy}(dy)^2$$

$$= 2\lambda(dx)^2 + 2\lambda\, dx(-2dx) + 2\lambda(-2dx)^2$$

$$= 6\lambda(dx)^2 = -\frac{6}{x}(dx)^2.$$

Thus, $$d^2F = -\frac{6}{\sqrt{3}}(dx)^2 < 0 \text{ when } x = \sqrt{3}$$

and $$d^2F = \frac{6}{\sqrt{3}}(dx)^2 > 0 \text{ when } x = -\sqrt{3}.$$

$\therefore$ F, i.e. $u = 2x + y$ is maximum at $x = \sqrt{3}$ and the maximum value is $2\sqrt{3}$.

EXAMPLE 8.10 Find the maximum and minimum distances of the point (4, 5) from the circle $x^2 + y^2 = 4$.

Solution

Let (x, y) be any point on the given circle. Then the distance between (x, y) and $(4, 5)$ is $d = \sqrt{(x-4)^2 + (y-5)^2}$ or $d^2 = (x-4)^2 + (y-5)^2$, which is to be maximized or minimized subject to the condition $x^2 + y^2 = 4$.

Let $\quad F = d^2 + \lambda(x^2 + y^2 - 4) = (x-4)^2 + (y-5)^2 + \lambda(x^2 + y^2 - 4)$,

where λ is Lagrange's multiplier.

Now,
$$F_x = 2(x-4) + 2\lambda x$$
$$F_y = 2(y-5) + 2\lambda y$$
$$F_\lambda = x^2 + y^2 - 4.$$

At stationary points,
$$F_x = 2(x-4) + 2\lambda x = 0$$
$$F_y = 2(y-5) + 2\lambda y = 0$$
$$F_\lambda = x^2 + y^2 - 4 = 0.$$

Solving first two equations, we get

$$x = \frac{4}{1+\lambda} \text{ and } y = \frac{5}{1+\lambda}.$$

Substituting these values in the third equation, we get

$$\frac{16}{(1+\lambda)^2} + \frac{25}{(1+\lambda)^2} = 4 \text{ or } (1+\lambda)^2 = \frac{41}{4} \text{ or } 1 + \lambda = \pm\frac{\sqrt{41}}{2}.$$

Therefore,
$$x = \pm\frac{8}{\sqrt{41}} \text{ and } y = \pm\frac{10}{\sqrt{41}}.$$

Thus the stationary points are $\left(\frac{8}{\sqrt{41}}, \frac{10}{\sqrt{41}}\right)$ and $\left(-\frac{8}{\sqrt{41}}, -\frac{10}{\sqrt{41}}\right)$.

The second order derivatives are

$$F_{xx} = 2 + 2\lambda,\ F_{yy} = 2 + 2\lambda,\ F_{xy} = 0.$$

Again,
$$2x\,dx + 2y\,dy = 0 \text{ or } dy = -\frac{x}{y}dx.$$

Now,
$$d^2F = F_{xx}(dx)^2 + 2F_{xy}\,dxdy + F_{yy}(dy)^2$$
$$= 2(1+\lambda)(dx)^2 + 2(1+\lambda)\frac{x^2}{y^2}(dx)^2$$

$$= 2(1+\lambda)\left\{1+\frac{x^2}{y^2}\right\}(dx)^2.$$

The term $2\left\{1+\frac{x^2}{y^2}\right\}(dx)^2$ is positive.

Therefore, $d^2F < 0$ if $1+\lambda < 0$, i.e. if $1+\lambda = -\frac{\sqrt{41}}{2}$ and $d^2F > 0$ if $1+\lambda = \frac{\sqrt{41}}{2}$.

Hence, F or d^2 or d is maximum when $1+\lambda = -\frac{\sqrt{41}}{2}$ or $x = -\frac{8}{\sqrt{41}}$ and $y = -\frac{10}{\sqrt{41}}$ and is minimum when $1+\lambda = \frac{\sqrt{41}}{2}$ or $x = +\frac{8}{\sqrt{41}}$ and $y = +\frac{10}{\sqrt{41}}$.

Therefore, the maximum distance is

$$d = \sqrt{\left(\frac{8}{\sqrt{41}}+4\right)^2 + \left(\frac{10}{\sqrt{41}}+5\right)^2} = \sqrt{45+4\sqrt{41}}$$

and the minimum distance is

$$d = \sqrt{\left(\frac{8}{\sqrt{41}}-4\right)^2 + \left(\frac{10}{\sqrt{41}}-5\right)^2} = \sqrt{45-4\sqrt{41}}.$$

EXAMPLE 8.11 Find the area of the greatest rectangle that can be inscribed in an ellipse.

or

Find the maximum value of $4xy$ subject to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution

Let $2x$ and $2y$ be the length and breadth of the rectangle (see Figure 8.2)

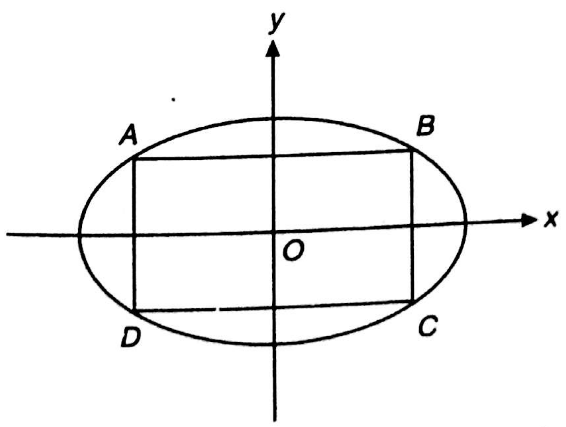

Figure 8.2 A rectangle inscribed inside the ellipse.

Then its area is $A = 4xy$, where x and y satisfy the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Now, the problem is to find the maximum value of A subject to the condition

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let $$F = A+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=4xy+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

where λ is Lagrange's multiplier.

$$F_x = 4y+2\lambda x/a^2,\quad F_y = 4x+2\lambda y/b^2,\quad F_\lambda = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1.$$

At stationary points,

$$F_x = 4y+2\lambda x/a^2 = 0$$
$$F_y = 4x+2\lambda y/b^2 = 0$$
$$F_\lambda = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1 = 0.$$

Multiplying the first equation by x and the second equation by y and adding we get

$$4xy+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0 \quad\text{or}\quad 4xy+\lambda=0 \left[\text{as } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right].$$

Therefore,

$$xy=-\lambda/4 \quad\text{or}\quad y=-\frac{\lambda}{4x}.$$

From first equation

$$-\frac{\lambda}{2x}+\frac{\lambda x}{a^2}=0$$

or $$2x^2=a^2 \quad\text{or}\quad x=\pm\frac{a}{\sqrt{2}}.$$

Similarly, $$y=\pm\frac{b}{\sqrt{2}}.$$

Thus, the stationary points are $\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$, $\left(\frac{a}{\sqrt{2}},\frac{-b}{\sqrt{2}}\right)$, $\left(\frac{-a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$, $\left(\frac{-a}{\sqrt{2}},\frac{-b}{\sqrt{2}}\right)$.

Now, $$F_{xz} = 2\lambda / a^2,\ F_{yy} = 2\lambda / b^2,\ F_{xy} = 4.$$

Also, $$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy = 0 \qquad \left[\text{since } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right]$$

or $$dy = -\frac{b^2}{a^2}\frac{x}{y}dx.$$

$$\therefore \quad d^2F = F_{xx}(dx)^2 + 2F_{xy}\,dxdy + F_{yy}(dy)^2$$

$$= \frac{2\lambda}{a^2}(dx)^2 + 2.4\,dx\left(-\frac{b^2}{a^2}\frac{x}{y}\right)dx + \frac{2\lambda}{b^2}\left(-\frac{b^2}{a^2}\frac{x}{y}\right)^2(dx)^2$$

$$= \left[\frac{2\lambda}{a^2} - \frac{8b^2}{a^2}\frac{x}{y} + 2\lambda\frac{b^2}{a^4}\left(\frac{x^2}{y^2}\right)\right](dx)^2.$$

If $\quad x = a/\sqrt{2},\ y = b/\sqrt{2} \quad$ then $\quad \lambda = -4xy = -2ab$ and hence

$$d^2F = \left[-\frac{4ab}{a^2} - \frac{8b^2}{a^2}\frac{a}{b} - 4ab\frac{b^2}{a^4}\frac{a^2}{b^2}\right](dx)^2$$

$$= -\frac{16b}{a}(dx)^2 < 0.$$

Thus F, i.e. A is maximum when $x = \frac{a}{\sqrt{2}},\ y = \frac{b}{\sqrt{2}}$.

Therefore, the dimensions of the rectangle is $a\sqrt{2}$ and $b\sqrt{2}$ and its maximum area is $2ab$.

EXAMPLE 8.12 Divide 10 into two parts such that the sum of whose squares is as small as possible. Method of Lagrange's multiplier may be used.

Solution

Let x and y be two parts. That is, $x + y = 10$.

The sum of squares of them is $x^2 + y^2$.

Now, the problem is to minimized $x^2 + y^2$ subject to the condition $x + y = 10$.

Let $$F = x^2 + y^2 + \lambda(x + y - 10).$$

$$\therefore \quad F_x = 2x + \lambda,\quad F_y = 2y + \lambda,\quad F_\lambda = x + y - 10.$$

At stationary points,

$$F_x = 2x + \lambda = 0$$
$$F_y = 2y + \lambda = 0$$
$$F_\lambda = x + y - 10 = 0.$$

The first two equations give $x = -\lambda/2,\ y = -\lambda/2$.

Putting these values in the third equation we get $\lambda = -10$. Therefore, $x = 5, y = 5$.

Again,

$$F_{xx} = 2,\ F_{yy} = 2,\ F_{xy} = 0 \text{ and } dx + dy = 0 \qquad [\because\ x + y = 10]$$

$$\therefore \quad d^2F = F_{xx}(dx)^2 + 2F_{xy}\,dx\,dy + F_{yy}(dy)^2$$
$$= 2(dx)^2 + 2(dx)^2 = 4(dx)^2 > 0.$$

Thus, F i.e. $x^2 + y^2$ is minimum when $x = 5, y = 5$. Hence the required parts are 5, 5.

EXAMPLE 8.13 Find a point in the plane $x + 2y + 3z = 13$ nearest to the point $(1, 1, 1)$ using the method of Lagrange's multiplier. (WBUT 2002)

Solution

Let (x, y, z) be any point on the plane. Then the distance between (x, y, z) and $(1, 1, 1)$ is $d = \sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}$.

Let
$$F = d^2 + \lambda(x + 2y + 3z - 13)$$
$$= (x-1)^2 + (y-1)^2 + (z-1)^2 + \lambda(x + 2y + 3z - 13).$$

Therefore, at stationary points,

$$F_x = 2(x-1) + \lambda = 0 \quad \text{or} \quad x = \frac{2-\lambda}{2}$$

$$F_y = 2(y-1) + 2\lambda = 0 \quad \text{or} \quad y = \frac{2-2\lambda}{2}$$

$$F_z = 2(z-1) + 3\lambda = 0 \quad \text{or} \quad z = \frac{2-3\lambda}{2}$$

$$F_\lambda = x + 2y + 3z - 13 = 0.$$

Putting the values of x, y, z in $x + 2y + 3z - 13 = 0$.

Then $\dfrac{2-\lambda}{2} + 2\dfrac{2-2\lambda}{2} + 3\dfrac{2-3\lambda}{2} = 13$ or $-14\lambda = 14$ or $\lambda = -1$.

$$\therefore \quad x = \frac{3}{2},\ y = 2,\ z = \frac{5}{2}.$$

Now,
$$d^2 = (x-1)^2 + (y-1)^2 + (z-1)^2$$

$$= (x-1)^2 + (y-1)^2 + \left(\frac{13 - x - 2y}{3} - 1\right)^2$$

$$= (x-1)^2 + (y-1)^2 + \left(\frac{10 - x - 2y}{3}\right)^2 = f(x, y) \quad \text{(say)}$$

$$f_x = 2(x-1) - \frac{2}{9}(10-x-2y), \quad f_{xx} = 2 + \frac{2}{9} = \frac{20}{9}$$

$$f_y = 2(y-1) - \frac{4}{9}(10-x-2y), \quad f_{yy} = 2 + \frac{8}{9} = \frac{26}{9}$$

$$f_{xy} = -\frac{2}{9}(-2) = \frac{4}{9}.$$

Therefore,
$$f_{xx} \cdot f_{yy} - (f_{xy})^2 = \frac{20}{9} \cdot \frac{26}{9} - \left(\frac{4}{9}\right)^2 > 0$$

and
$$f_{xx} = \frac{20}{9} > 0.$$

Thus, f, i.e. d is minimum when $x = 3/2$, $y = 2$ and $z = 5/2$, which is the required point on the plane.

EXAMPLE 8.14 Find the volume of the largest rectangular parallelepiped that can be inscribed in the ethipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, using Lagrange's method of multipliers.

(WBUT 2002)

Solution

Let $2x$, $2y$, $2z$ be the length, breath and height of the rectangular parallelepiped. Therefore, its volume is $V = 8xyz$.

Now, the problem is to find the greatest value of $V = 8xyz$ subject to the conditions

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \; x > 0, y > 0, z > 0. \tag{1}$$

Let
$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right).$$

Now, at stationary points

$$\left.\begin{aligned} F_x &= 8yz + \frac{2x\lambda}{a^2} = 0 \\ F_y &= 8zx + \frac{2y\lambda}{b^2} = 0 \\ F_z &= 8xy + \frac{2z\lambda}{c^2} = 0. \end{aligned}\right\} \tag{2}$$

Multiplying the above equations by x, y, z and adding, we get

$$24xyz + 2\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0$$

or $$24xyz + 2\lambda = 0 \qquad \text{[Using equation (1)]}$$

or $$\lambda = -12\,xyz. \qquad (3)$$

Hence from (2), $x=\frac{a}{\sqrt{3}},\ y=\frac{b}{\sqrt{3}},\ z=\frac{c}{\sqrt{3}}$ and $\lambda=-\frac{4abc}{\sqrt{3}}$.

Differentiating (1) w.r.t. x, taking x and y are independent variables and z is dependent variable, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \quad \text{or} \quad \frac{\partial z}{\partial x}=-\frac{x}{z}\frac{c^2}{a^2}.$$

Similarly, $\frac{\partial z}{\partial y}=-\frac{y}{z}\frac{c^2}{a^2}$.

Now, $$V_x = 8yz + 8xy\frac{\partial z}{\partial x} = 8yz + 8xy\left(-\frac{c^2}{a^2}\frac{x}{z}\right)$$

$$= 8yz - \frac{8x^2yc^2}{a^2z}.$$

$$V_{xx} = 8y\frac{\partial z}{\partial x} - \frac{16c^2xy}{a^2z} + \frac{8x^2yc^2}{a^2z^2}\frac{\partial z}{\partial x}$$

$$= -8y\frac{x}{z}\frac{c^2}{a^2} - \frac{16c^2xy}{a^2z} + \frac{8x^2yc^2}{a^2z^2}\left(-\frac{x}{z}\frac{c^2}{a^2}\right)$$

$$= -8\frac{b}{\sqrt{3}}\cdot\frac{c}{a} - 16\frac{c}{a}\cdot\frac{b}{\sqrt{3}} - 8\frac{b}{\sqrt{3}}\frac{c}{a} \qquad \left[\because \frac{x}{a}=\frac{1}{\sqrt{3}}=\frac{z}{c}\right]$$

$$= -32\frac{bc}{a\sqrt{3}} < 0.$$

Similarly, $$V_{yy} = -32\frac{ac}{a\sqrt{3}}.$$

$$V_{xy} = 8z + 8y\frac{\partial z}{\partial y} - \frac{8x^2c^2}{a^2z} - \frac{8x^2c^2y}{a^2}\left(-\frac{1}{z^2}\frac{\partial z}{\partial y}\right)$$

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$$= 8z - \frac{8x^2c^2}{a^2z} - 8y\frac{y}{z}\frac{c^2}{b^2} - \frac{8x^2c^2y}{a^2z^2}\frac{y}{z}\frac{c^2}{b^2}$$

$$= -\frac{16c}{\sqrt{3}}.$$

$$\therefore\ V_{xx}.V_{yy} - (V_{xy})^2 = 32\frac{bc}{a\sqrt{3}}\cdot 32\frac{ac}{b\sqrt{3}} - \frac{(16)^2c^2}{3}$$

$$= 256c^2 > 0.$$

Therefore, V is maximum when $x = \frac{a}{\sqrt{3}}$, $y = \frac{b}{\sqrt{3}}$ and $z = \frac{c}{\sqrt{3}}$ and the greatest volume of the rectangular parallelepiped is $8xyz = \frac{8abc}{3\sqrt{3}}$.

EXAMPLE 8.15 Finding the maximum of the function $u = xyz$ under the condition $x + y + z = S$, S is a given constant, prove that the inequality.

$$(x+y+z)^3 \geq 27xyz \text{ for } x \geq 0, y \geq 0, z \geq 0.$$

Solution

Let $F = u + \lambda(x+y+z-S) = xyz + \lambda(x+y+z-S)$.

Therefore, at stationary points

$$F_x = yz + \lambda = 0$$
$$F_y = xz + \lambda = 0$$
$$F_z = xy + \lambda = 0$$
$$F_\lambda = x + y + z - S = 0.$$

Multiplying the first, the second, and the third equations by x, y and z respectively and adding, we get

$$3xyz + \lambda(x+y+z) = 0$$

or $\quad 3u + \lambda S = 0$ or $\lambda = -3u/S$.

Therefore, $\quad xyz + \lambda x = 0$ or $u + \lambda x = 0$

or $\quad x = -\frac{u}{\lambda} = \frac{u}{3u/S} = S/3.$

Similarly, $\quad y = S/3, z = S/3.$

Here, $x + y + z = S$. Let x, y be the independent and z be the dependent variables. Then $1 + \frac{\partial z}{\partial x} = 0$ or $\frac{\partial z}{\partial x} = -1.$

Similarly, $\quad \frac{\partial z}{\partial y} = -1.$

Now, $$u_x = yz + xy\frac{\partial z}{\partial x} = yz - xy$$

$$u_{xx} = y\frac{\partial z}{\partial x} - y = -2y = 2S/3 < 0, \text{ at } (S/3, S/3, S/3).$$

Similarly, $$u_{yy} = -2S/3.$$

$$u_{xy} = z + y\frac{\partial z}{\partial y} - x = z - y - x = -S/3 .$$

$\therefore$ $$u_{xx}\cdot u_{yy} - (u_{xy})^2 = \frac{4S^2}{9} - \frac{S^2}{9} = \frac{S^2}{3} > 0.$$

Hence u is maximum at ($S/3$, $S/3$, $S/3$) and maximum value is $\frac{S^3}{27}$.

Thus, $$xyz \le \frac{S^3}{27} \quad \text{or} \quad S^3 \ge 27xyz$$

$$(x+y+z)^3 \ge 27xyz.$$

Hence proved.

EXAMPLE 8.16 Show that the maximum and minimum values of $r^2 = x^2 + y^2$ where $ax^2 + 2hxy^2 + by^2 = 1$ are given by the roots of the quadratic

$$\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) = h^2.$$

Solution

Let $$F = r^2 + \lambda(ax^2 + 2hxy + by^2 - 1)$$
$$= x^2 + y^2 + \lambda(ax^2 + 2hxy + by^2 - 1),$$
where λ is Lagrange's multiplier.
At stationary points

$$F_x = 2x + 2\lambda ax + 2\lambda hy = 0 \qquad (1)$$
$$F_y = 2y + 2\lambda hx + 2\lambda by = 0 \qquad (2)$$
$$F_\lambda = ax^2 + 2hxy + by^2 - 1 = 0. \qquad (3)$$

Multiplying equations (1) and (2) by x and y and adding, we get

$$(x^2 + y^2) + \lambda(ax^2 + 2hxy + by^2) = 0$$

or $$r^2 + \lambda.1 = 0 \quad \text{or} \quad \lambda = -r^2. \qquad \text{[Using equation (3)]}$$

From equation (1),

$$x(1 + a\lambda) + hy\lambda = 0$$

$$x(1-ar^2) = hyr^2 \tag{4}$$

or

and from equation (2),

$$y(1+b\lambda) + hx\lambda = 0$$

or

$$y(1-br^2) = hxr^2. \tag{5}$$

Multiplying equations (4) and (5), we get

$$xy(1-ar^2)(1-br^2) = xyh^2r^4$$

or

$$(1-ar^2)(1-br^2) = h^2r^4$$

or

$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right) = h^2.$$

The roots of this equation give the maximum and the minimum values.

CHAPTER 9

Jacobians

9.1 DEFINITION

If $u(x, y)$ and $v(x, y)$ are two functions of two variables x and y then the function determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \text{ or } \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

is called the *Jacobian* of u and v with respect to x and y. The Jacobian of u, v with respect to x and y is denoted by $\dfrac{\partial(u, v)}{\partial(x, y)}$. The Jacobian is also denoted by $J(u, v)$.

The Jacobian is used to evaluate multiple integral when transformation is required. Also, one can test the dependence of functional relations, using the concept of Jacobian.

EXAMPLE 9.1 If $x = r\cos\theta,\ y = r\sin\theta,$ find $\dfrac{\partial(x, y)}{\partial(r, \theta)}$. (WBUT 2007)

Solution

By definition
$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

EXAMPLE 9.2 If $x = a\cos\theta\cosh\phi$, $y = a\sin\theta\sinh\phi$, prove that

$$\frac{\partial(x, y)}{\partial(\phi, \theta)} = \frac{1}{2}a^2(\cosh 2\phi - \cos 2\theta).$$

Solution

We have

$$\frac{\partial(x, y)}{\partial(\phi, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta\sinh\phi & -a\sin\theta\cosh\phi \\ a\sin\theta\cosh\phi & a\cos\theta\sinh\phi \end{vmatrix}$$

$$= a^2(\cos^2\theta\sinh^2\phi + \sin^2\theta\cosh^2\phi)$$

$$= a^2\left[\cos^2\theta(\cosh^2\phi - 1) + (1 - \cos^2\theta)\cosh^2\phi\right]$$

$$= a^2\left[\cosh^2\phi - \cos^2\theta\right]$$

$$= a^2\left[\frac{1}{2}(1 + \cosh 2\phi) - \frac{1}{2}(1 + \cos 2\theta)\right]$$

$$= \frac{a^2}{2}[\cosh 2\phi - \cos 2\theta].$$

9.2 PROPERTIES OF JACOBIANS

Theorem 9.1 If u_1, u_2 are functions of the variables y_1, y_2 and y_1, y_2 are the functions of x_1, x_2, then

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \times \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}.$$

Proof: By chain rule of partial derivative

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_1}$$

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_2}$$

$$\frac{\partial u_2}{\partial x_1} = \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_1}$$

and

$$\frac{\partial u_2}{\partial x_2} = \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_2}.$$

Now, right hand side

$$\frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \times \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \dfrac{\partial u_1}{\partial y_1} & \dfrac{\partial u_1}{\partial y_2} \\ \dfrac{\partial u_2}{\partial y_1} & \dfrac{\partial u_2}{\partial y_2} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u_1}{\partial y_1}\dfrac{\partial y_1}{\partial x_1} + \dfrac{\partial u_1}{\partial y_2}\dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial u_1}{\partial y_1}\dfrac{\partial y_1}{\partial x_2} + \dfrac{\partial u_1}{\partial y_2}\dfrac{\partial y_2}{\partial x_2} \\ \dfrac{\partial u_2}{\partial y_1}\dfrac{\partial y_1}{\partial x_1} + \dfrac{\partial u_2}{\partial y_2}\dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial u_2}{\partial y_1}\dfrac{\partial y_1}{\partial x_2} + \dfrac{\partial u_2}{\partial y_2}\dfrac{\partial y_2}{\partial x_2} \end{vmatrix}$$

[row by column multiplication]

$$= \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} \end{vmatrix} = \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}.$$

Hence proved.

In general, if $u_1, u_2, ..., u_n$ are functions of the set of the variables $y_1, y_2, ..., y_n$ and $y_1, y_2, ..., y_n$ are themselves functions of $x_1, x_2, ..., x_n$, then

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial(u_1, u_2, ..., u_n)}{\partial(y_1, y_2, ..., y_n)} \times \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)}.$$

Theorem 9.2 If J be the Jacobian of the system u, v with regard to x, y and J' the Jacobian of x, y with regard to u, v, then $JJ' = 1$, i.e.

$$\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1.$$

Proof: Let $u = f_1(x, y)$, $v = f_2(x, y)$, then differentiating these w.r.t. u and v partially, we get

$$1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}, \qquad 0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u}$$

$$0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}, \qquad 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}.$$

Now,
$$JJ' = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial v} \\ \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Thus, $JJ' = 1$. From this relation, we can write

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{\dfrac{\partial(x, y)}{\partial(u, v)}}.$$

In general, $$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} \times \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} = 1.$$

Theorem 9.3 (Jacobian of implicit functions). If 4 variables u_1, u_2 and x_1, x_2 are connected implicitly by two independent relations

$$f_1(u_1, u_2, x_1, x_2) = 0$$

$$f_2(u_1, u_2, x_1, x_2) = 0$$

then $$\frac{\partial(f_1, f_2)}{\partial(u_1, u_2)} \times \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}.$$

Proof: Differentiating $f_1(u_1, u_2, x_1, x_2) = 0$ w.r.t. x_1, x_2 we get

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_1} = 0 \text{ or } \frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_1} = -\frac{\partial f_1}{\partial x_1}$$

$$\frac{\partial f_1}{\partial x_2} + \frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_2} = 0 \text{ or } \frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_2} = -\frac{\partial f_1}{\partial x_2}.$$

Similarly, from $f_2(u_1, u_2, x_1, x_2) = 0,$ we get

$$\frac{\partial f_2}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial f_2}{\partial u_2}\frac{\partial u_2}{\partial x_1} = -\frac{\partial f_2}{\partial x_1}$$

$$\frac{\partial f_2}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial f_2}{\partial u_2}\frac{\partial u_2}{\partial x_2}=-\frac{\partial f_2}{\partial x_2}.$$

and

Now,
$$\frac{\partial(f_1, f_2)}{\partial(u_1, u_2)}\cdot\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}$$

$$=\begin{vmatrix}\frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2}\\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2}\end{vmatrix}\times\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_1} & \frac{\partial f_1}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial f_1}{\partial u_2}\frac{\partial u_2}{\partial x_2}\\ \frac{\partial f_2}{\partial u_1}\frac{\partial u_1}{\partial x_1}+\frac{\partial f_2}{\partial u_2}\frac{\partial u_2}{\partial x_1} & \frac{\partial f_2}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial f_2}{\partial u_2}\frac{\partial u_2}{\partial x_2}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{vmatrix}=\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}.$$

9.3 WORKED-OUT EXAMPLES

EXAMPLE 9.3 If $f(u, v) = 3uv^2$, $g(u, v) = u^2 - v^2$, find the Jacobian $\frac{\partial(f, g)}{\partial(u, v)}$. (WBUT 2004)

Solution

$$\frac{\partial f}{\partial u}=3v^2,\ \frac{\partial f}{\partial v}=6uv,\ \frac{\partial g}{\partial u}=2u,\ \frac{\partial g}{\partial v}=-2v.$$

Therefore,
$$\frac{\partial(f, g)}{\partial(u, v)}=\begin{vmatrix}\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v}\\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}\end{vmatrix}=\begin{vmatrix}3v^2 & 6uv\\ 2u & -2v\end{vmatrix}$$

$$=-6v^3-12u^2v=-6v(v^2+2u^2).$$

EXAMPLE 9.4 If $f(x, y)=\frac{x+y}{1-xy}$ and $g(x, y) = \tan^{-1}x + \tan^{-1}y$ find the Jacobian $\frac{\partial(f, g)}{\partial(x, y)}$.

Solution

$$\frac{\partial f}{\partial x}=\frac{1.(1-xy)-(x+y)(-y)}{(1-xy)^2}=\frac{1+y^2}{(1-xy)^2}.$$

Similarly,

$$\frac{\partial f}{\partial y}=\frac{1+x^2}{(1-xy)^2},\ \frac{\partial g}{\partial x}=\frac{1}{1+x^2} \text{ and } \frac{\partial g}{\partial y}=\frac{1}{1+y^2}.$$

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2}\\ \frac{1}{1+x^2} & \frac{1}{1+y^2}\end{vmatrix}$$

$$=\frac{1}{(1-xy^2)}-\frac{1}{(1-xy^2)}=0.$$

EXAMPLE 9.5 If $x+y=u$ and $x=uv$, then show that $\frac{\partial(x,y)}{\partial(u,v)}=-u$. Also, find the value of $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution

Here $x=uv$, $y=u-x=u-uv=u(1-v)$.

Therefore, $$\frac{\partial x}{\partial u}=v,\ \frac{\partial x}{\partial v}=u,\ \frac{\partial y}{\partial u}=1-v,\ \frac{\partial y}{\partial v}=-u.$$

$$\therefore \quad \frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}v & u\\ 1-v & -u\end{vmatrix}$$

$$=-u.$$

We know, $$\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(u,v)}=1.$$

$$\therefore \quad \frac{\partial(u,v)}{\partial(x,y)}=\frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}=\frac{1}{-u}=-\frac{1}{u}.$$

EXAMPLE 9.6 If $u^3+v^3=x+y$, and $u^2+v^2=x^3+y^3$, show that $\frac{\partial(u,v)}{\partial(x,y)}=\frac{1}{2}\frac{y^2-x^2}{uv(u-v)}$.

Solution

Here the variables u, v and x, y are implicitly connected by two relations, viz.

$$f_1 \equiv u^3 + v^3 - x - y = 0$$

$$f_2 \equiv u^2 + v^2 - x^3 - y^3 = 0.$$

Now,
$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(f_1, f_2)}{\partial(x, y)} + \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

$$= \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} \div \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} \div \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix}$$

$$= (3y^2 - 3x^2) \div (6u^2v - 6uv^2)$$

$$= \frac{1}{2}\,\frac{y^2 - x^2}{uv\,(u - v)}.$$

EXAMPLE 9.7 The roots of the equation $(\lambda - x)^2 + (\lambda - y^2) = 0$ in λ are u, v, prove that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{y - x}{u - v}.$$

Solution

The equation $(\lambda - x)^2 + (\lambda - y)^2 = 0$ can be written as

$$\lambda^2 - \lambda(x + y) + \frac{1}{2}(x^2 + y^2) = 0.$$

If u, v are the roots, then

$$u + v = -\text{ coefficient of } \lambda / \text{coefficient of } \lambda^2$$

$$= x + y$$

$$uv = \text{constant term/coefficient of } \lambda^2$$

$$= \frac{1}{2}(x^2 + y^2).$$

These relations can be written as

$$f_1 \equiv u + v - x - y = 0 \text{ and } f_2 \equiv uv - \frac{1}{2}(x^2 + y^2).$$

Now,
$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(f_1, f_2)}{\partial(x, y)} + \frac{\partial(f_1, f_2)}{\partial(u, v)}$$

$$= \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} \div \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 \\ -x & -y \end{vmatrix} \div \begin{vmatrix} 1 & 1 \\ v & u \end{vmatrix}$$

$$= (y-x) \div (u-v) = \frac{y-x}{u-v}.$$

EXAMPLE 9.8 If $u = x(1-r^2)^{-1/2}$ and $v = y(1-r^2)^{-1/2}$, where $r^2 = x^2 + y^2$, find the value of $\dfrac{\partial(u, v)}{\partial(x, y)}$.

Solution

Given $\quad u = x(1-x^2)^{-1/2}$

i.e. $\quad u^2(1-r^2) = x^2$

or $\quad x^2 - u^2(1-x^2-y^2) = 0.$

Similarly, $\quad y^2 - v^2(1-x^2-y^2) = 0$

Let $\quad f_1 \equiv x^2 - u^2(1-x^2-y^2) = 0$

and $\quad f_2 \equiv y^2 - v^2(1-x^2-y^2) = 0.$

Now,
$$\frac{\partial(f_1, f_2)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x + 2xu^2 & 2yu^2 \\ 2xv^2 & 2y + 2yv^2 \end{vmatrix}$$

$$= 4xy \begin{vmatrix} 1+u^2 & u^2 \\ v^2 & 1+v^2 \end{vmatrix}$$

$$= 4xy(1 + u^2 + v^2),$$

and
$$\frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -2u(1-x^2-y^2) & 0 \\ 0 & -2v(1-x^2-y^2) \end{vmatrix} = 4uv(1-x^2-y^2)^2.$$

$$\therefore \quad \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(f_1,f_2)}{\partial(x,y)} \div \frac{\partial(f_1,f_2)}{\partial(u,v)}$$

$$= \frac{4xy(1+u^2+v^2)}{4uv(1-x^2-y^2)^2} = \frac{xy(1+u^2+v^2)}{uv(1-r^2)^2}.$$

Now,

$$uv = xy(1-r^2)^{-1}$$

and

$$1+u^2+v^2 = 1+x^2(1-r^2)^{-1} + y^2(1-r^2)^{-1}$$

$$= 1 + \frac{x^2+y^2}{1-r^2} = 1 + \frac{r^2}{1-r^2} = \frac{1}{1-r^2}.$$

Hence,

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{xy(1-r^2)^{-1}}{xy(1-r^2)^{-1}(1-r^2)^2} = (1-r^2)^{-2}.$$

EXAMPLE 9.9 If $u = x^2+y^2$, $v = x^2-y^2$ and $x = r\theta$, $y = r+\theta$ then find the value of the Jacobian $\dfrac{\partial(u,v)}{\partial(r,\theta)}$.

Solution

We know,

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} \theta & r \\ 1 & 1 \end{vmatrix}$$

$$= (-4xy - 4xy) \times (\theta - r)$$

$$= 8x(r-\theta).$$

CHAPTER 16

Three-Dimensional Geometry

16.1 INTRODUCTION

It is well known that the coordinate geometry is the analytic method to study the geometric shape. The students are familiar with two-dimensional geometry and this geometry deals only with the shapes those are drawn on a plane. But, three-dimensional geometry deals with the solid shapes which frequently occur in scientific and engineering problems. In this chapter, we study the very common three-dimensional geometrical objects, such as plane, straight line, sphere, cone and cylinder.

16.2 THREE-DIMENSIONAL COORDINATES

In two dimensions, a point can be represented by two quantities x and y, but in three dimensions, to represent a single point needs three quantities x, y and z. These quantities are measure with respect to a fixed point O called the origin and from three mutually perpendicular lines XOX', YOY' and ZOZ' called *rectangular coordinates axes*, namely x, y and z-axes. OX, OY, OZ whose directions are right-handed are taken as positive directions, opposite directions as negative. The planes XOY, XOZ and ZOX are called xy, yz and zx coordinates planes respectively.

Let P be any point in space (the three-dimensional coordinate system). PN is drawn perpendicular to XOY plane, NM is perpendicular to OX, i.e. MN is parallel to OY. If $OM = x$, $MN = y$, $PN = z$, the coordinates of P are (x, y, z). These coordinates are positive or negative according to their measurement from the origin—along the positive or negative direction of the axes (see Figure 16.1).

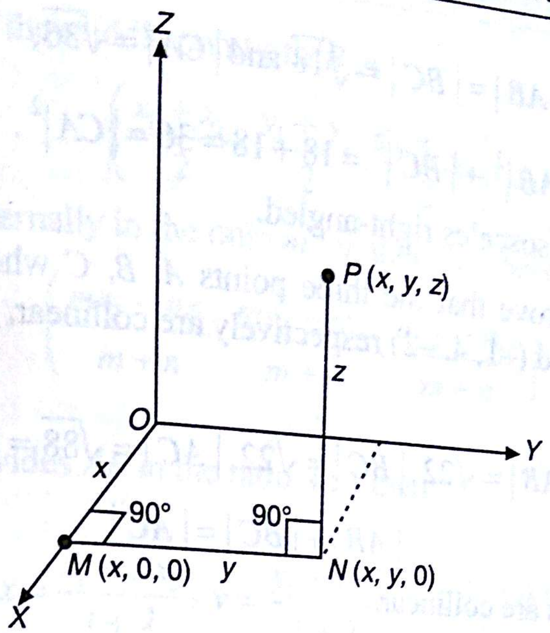

Figure 16.1 Point in space.

The projections of the point $P(x, y, z)$ on the xy, yz and zx-planes are respectively $(x, y, 0)$, $(0, y, z)$ and $(x, 0, z)$.

The three coordinate planes divide the whole of the space into eight parts which are called *octants*. The octant $OXYZ$ in which the point P is situated is called first octant. Any point in this octant has each of its coordinates positive.

16.2.1 Distance between Two Points

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space. The distance between these two points is denoted by $|AB|$ and is defined by

$$|AB| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

From this definition, we can test the collinearity between three points in space. If A, B, C be three points in space and if they are collinear then one of the following conditions hold:

(a) $|AB| + |BC| = |AC|$

(b) $|AB| + |AC| = |BC|$

(c) $|AC| + |BC| = |AB|$.

EXAMPLE 16.1 Show that the points (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) form an isosceles right angled triangle.

Solution

Let $A(0, 7, 10)$, $B(-1, 6, 6)$, $C(-4, 9, 6)$ be three points then

$$|AB| = \sqrt{(0+1)^2 + (7-6)^2 + (10-6)^2} = \sqrt{18}.$$

Similarly, $$|AB| = \sqrt{36} \text{ and } |BC| = \sqrt{18}.$$

Here $$|AB| = |BC| = \sqrt{18} \text{ and } |CA| = \sqrt{36},$$

i.e. $$|AB|^2 + |BC|^2 = 18 + 18 = 36 = |CA|^2.$$

Hence the triangle is isosceles right-angled.

EXAMPLE 16.2 Prove that the three points A, B, C whose coordinates are $(3, -2, 4)$, $(1, 1, 1)$ and $(-1, 4, -2)$ respectively are collinear.

Solution

Here $$|AB| = \sqrt{22}, |BC| = \sqrt{22}, |AC| = \sqrt{88} = 2\sqrt{22}.$$

Hence, $$|AB| + |BC| = |AC|$$

Therefore, the points are collinear.

EXAMPLE 16.3 Find the locus of a point the sum of whose distances from $(4, 0, 0)$ and $(-4, 0, 0)$ is 10.

Solution

Let $P(x, y, z)$ be any point in the space. The distance between P and $(4, 0, 0)$ is $\sqrt{(x-4)^2 + y^2 + z^2}$ and that of between P and $(-4, 0, 0)$ is $\sqrt{(x+4)^2 + y^2 + z^2}$.

Since the sum of distances is 10,

$$\sqrt{(x-4)^2 + y^2 + z^2} + \sqrt{(x+4)^2 + y^2 + z^2} = 10.$$

or $$\sqrt{(x-4)^2 + y^2 + z^2} = 10 - \sqrt{(x+4)^2 + y^2 + z^2}.$$

Squaring both sides, we get

$$(x-4)^2 + y^2 + z^2 = 100 + (x+4)^2 + y^2 + z^2 - 20\sqrt{(x+4)^2 + y^2 + z^2}$$

or $$4x - 25 = -5\sqrt{(x+4)^2 + y^2 + z^2}.$$

Again squaring both sides, we get

$$16x^2 - 200x + 625 = 25(x^2 + y^2 + z^2 + 8x + 16)$$

or $$9x^2 + 25(y^2 + z^2) = 225.$$

16.2.2 Division of the Line Joining Two Points

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points in space and P be a point which divides the line AB in the ratio $m : n$. Then the coordinates of P is

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right).$$

The coordinates of the middle point of AB is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

If P divides AB externally in the ratio $m : n$, then the coordinates of P are

$$\left(\frac{mx_2-nx_1}{m+n}, \frac{my_2-ny_1}{m+n}, \frac{mz_2-nz_1}{m+n}\right).$$

If $\frac{m}{n}=\lambda$, i.e. P divides AB in the ratio $\lambda : 1$ then

$$x=\frac{x_1+\lambda x_2}{1+\lambda}, \; y=\frac{y_1+\lambda y_2}{1+\lambda}, \; z=\frac{z_1+\lambda z_2}{1+\lambda}.$$

EXAMPLE 16.4 Find the ratio in which the *yz*-plane divides the join of the points (–2, 4, 7) and (3, –5, 8), and also find the coordinates of the point of intersection of the line with the *yz*-plane.

Solution

The coordinates of any point on the line joining the two points are

$$\left(\frac{3\lambda-2}{\lambda+1}, \frac{-5\lambda+4}{\lambda+1}, \frac{8\lambda+7}{\lambda+1}\right).$$

If the point is in the *yz*-plane, then its *x*-coordinate should be zero.

$$\therefore \quad \frac{3\lambda-2}{\lambda+1}=0 \text{ or } 3\lambda-2=0 \text{ or } \lambda=\frac{2}{3}.$$

Hence the required ratio is 2 : 3. The required point is $\left(0, \frac{2}{5}, \frac{37}{5}\right)$.

16.2.3 Area of a Triangle in Space

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of the triangle ABC in space. Then its area is given by

$$\Delta=\sqrt{\Delta_x^2+\Delta_y^2+\Delta_z^2}$$

where Δ_x, Δ_y and Δ_z are the areas of the projection of the triangle ABC on the *yz*-, *xz*- and *xy*-planes. That is

$$\Delta_x=\frac{1}{2}\{y_1(z_2-z_3)+y_2(z_3-z_1)+y_3(z_1-z_2)\}$$

$$\Delta_y=\frac{1}{2}\{z_1(x_2-x_3)+z_2(x_3-x_1)+z_3(x_1-x_2)\}$$

$$\Delta_z = \frac{1}{2}\{x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)\}.$$

EXAMPLE 16.5 Find the area of the triangle ABC whose vertices are $A(1, 2, 1)$, $B(0, 1, -1)$, $C(2, 1, 1)$.

Solution

Here
$$\Delta_x = \frac{1}{2}\{2(-1-1) + 1(1-1) + 1(1+1)\} = -1,$$

$$\Delta_y = \frac{1}{2}\{1(0-2) + (-1)(2-1) + 1(1-0)\} = -1,$$

$$\Delta_z = \frac{1}{2}\{1(1-1) + 0 + 2(2-1)\} = 1.$$

Therefore, the required area is $\sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2} = \sqrt{3}$.

16.3 DIRECTION COSINES AND DIRECTION RATIOS

If a directed line makes angles α, β, γ with the positive directions of x, y and z axes respectively, then $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* (DCs) of this line. There are generally denoted by l, m, n or $\{l, m, n\}$.

The direction cosines of x, y and z axes are respectively $\{1, 0, 0\}$, $\{0, 1, 0\}$ and $\{0, 0, 1\}$.

Let $P(x, y, z)$ be any point with respect to the origin O. Let OP makes angles α, β, γ with the coordinate axes (see Figure 16.2).

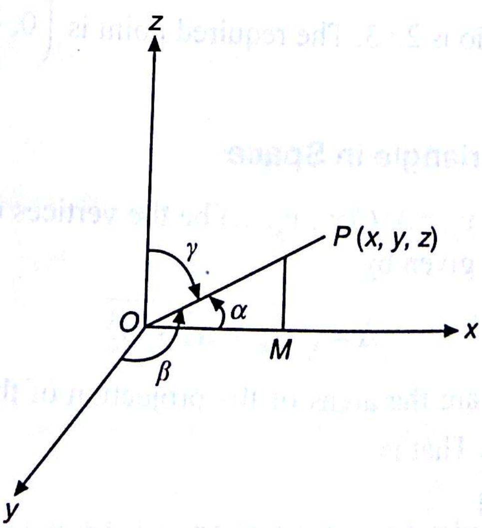

Figure 16.2 Direction cosines.

...should be perpendicular to x-axis

$$\cos\alpha = \frac{|OM|}{|OP|} = \frac{x}{r}$$

...but

$$|OP| = \sqrt{x^2+y^2+z^2} = r$$

...fore

similarly,

$$\cos\beta = \frac{y}{r}, \cos\gamma = \frac{z}{r}.$$

Now,

$$l^2+m^2+n^2 = \cos^2\alpha+\cos^2\beta+\cos^2\gamma = \frac{x^2+y^2+z^2}{r^2} = 1.$$

Again

$$l = \cos\alpha = \frac{x}{r}, m = \frac{y}{r}, n = \frac{z}{r}.$$

That is,

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r.$$

This implies that x, y, z are proportional to the direction cosines. Thus, any three numbers which are proportional to the direction cosines of a line are called the direction ratios (DRs) of the line. Therefore, if l, m, n are the DCs of a line, then for any r, lr, mr, nr are the DRs of this line.

Again, if a, b, c be the DRs of a line, then its DCs are

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

EXAMPLE 16.6 If α, β, γ be the direction angles of a line, then show that (WBUT 2005)

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma = 2.$$

Solution

If α, β, γ be the direction angles of a line

then

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$$

i.e.

$$(1-\sin^2\alpha)+(1-\sin^2\beta)+(1-\sin^2\gamma) = 1$$

or

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma = 2.$$

16.3.1 Direction Cosine of a Line Joining Two Points

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points in space then $x_2-x_1, y_2-y_1, z_2-z_1$ be the direction ratios of the line AB and $\frac{x_2-x_1}{|AB|}, \frac{y_2-y_1}{|AB|}, \frac{z_2-z_1}{|AB|}$ are the direction cosines of the line AB.

Angle between two lines: Let $\{a_1, b_1, c_1\}$ and $\{a_2, b_2, c_2\}$ be the direction ratios of two lines then the angle θ between them is

$$\theta = \cos^{-1} \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

If $\{l_1, m_1, n_1\}$ and $\{l_2, m_2, n_2\}$ be the DCs of two lines then angle θ is given by

$$\theta = \cos^{-1} (l_1\, l_2 + m_1\, m_2 + n_1\, n_2).$$

The above two expressions for θ can also be written as

$$\theta = \sin^{-1} \frac{\{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2\}^{1/2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and $\qquad \theta = \sin^{-1} \{(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2\}^{1/2}.$

Conditions for perpendicularity

If the lines are perpendicular to each other, then $\theta = \pi/2$, and hence $\cos \theta = 0$.

Therefore, $\quad a_1a_2 + b_1b_2 + c_1c_2 = 0$ or $l_1l_2 + m_1m_2 + n_1n_2 = 0$.

These are the conditions for perpendicularity.

Conditions for parallel lines: If the lines are parallel, then $\theta = 0$, i.e. $\sin \theta = 0$.

Therefore, $\quad (a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1) = 0.$

This implies,

$$a_1b_2 - a_2b_1 = 0,\;\; b_1c_2 - b_2c_1 = 0 \text{ and } c_1a_2 - c_2a_1 = 0$$

or
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2},$$

i.e. the DRs are proportional.

In terms of DCs the result is similar, i.e.,

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}.$$

These are the conditions for parallel lines.

EXAMPLE 16.7 Find the direction cosines of the line which are equally inclined to the axes.

Solution

If the line makes angles α, β, γ with the axes, then

$$\cos\alpha = \cos\beta = \cos\gamma.$$

Therefore,
$$\alpha = \beta = \gamma.$$

If the edges of a parallel[illegible]

Hence,
$$\frac{l}{1}=\frac{m}{1}=\frac{n}{1}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+1+1}}=\frac{1}{\sqrt{3}}.$$

Thus, the DCs of the line are $\left\{\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\}$.

EXAMPLE 16.8 A, B, C and D are points of $(\alpha, 3, -1)$, $(3, 5, -3)$, $(1, 2, 3)$ and $(3, 5, 7)$ respectively. If AB is perpendicular to CD then find the value of α.

(WBUT 2006)

Solution

The DRs of the lines AB and CD are respectively $\{\alpha - 3, 3 - 5, -1 + 3\}$ and $\{1 - 3, 2 - 5, 3 - 7\}$, i.e. $\{\alpha - 3, -2, 2\}$ and $\{-2, -3, -4\}$.

If AB and CD are perpendicular, then
$$(\alpha-3)\times(-2)+(-2)\times(-3)+2\times(-4)=0$$
or
$$2\alpha-4=0 \text{ or } \alpha=2.$$

EXAMPLE 16.9 If the edges of a rectangular parallelopiped be a, b, c, show that the angles between the four diagonals are given by
$$\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right).$$

Solution

Let $ABC\ LMN\ PO$ be the rectangular parallelopiped shown in Figure 16.3.

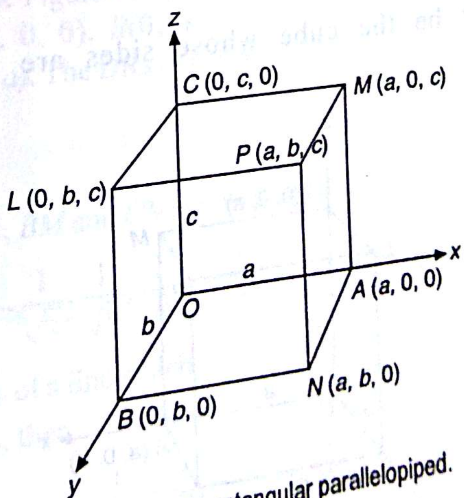

Figure 16.3 A rectangular parallelopiped.

Its four diagonals are AL, BM, CN and OP. The corner points are $O(0, 0, 0)$ $P(a, b, c)$, $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$, $L(0, b, c)$, $M(a, 0, c)$ and $N(a, b, 0)$.

The DRs of OP are $a - 0$, $b - 0$, $c - 0$ or a, b, c.

Therefore, DCs are

$$\frac{a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}}, \text{ where } \Sigma a^2 = a^2 + b^2 + c^2.$$

The DRs of AL are $0 - a$, $b - 0$, $c - 0$ or $-a$, b, c and hence DCs are

$$\frac{-a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}}.$$

Similarly, the DCs of BM and CN are

$$\frac{a}{\sqrt{\Sigma a^2}}, \frac{-b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}} \text{ and } \frac{a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{-c}{\sqrt{\Sigma a^2}} \text{ respectively.}$$

If θ be the angle between OP and AL then

$$\cos\theta = \frac{a \times (-a) + b \times b + c \times c}{\sqrt{a^2 + b^2 + c^2}\sqrt{a^2 + b^2 + c^2}} = \frac{-a^2 + b^2 + c^2}{a^2 + b^2 + c^2}.$$

Similarly, we can determine the angle between other pairs and hence the angles between the six pairs are given by

$$\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right).$$

EXAMPLE 16.10 Prove that the angle between two diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

Solution

Let $OABCLMNP$ be the cube whose sides are of length a (see Figure 16.4).

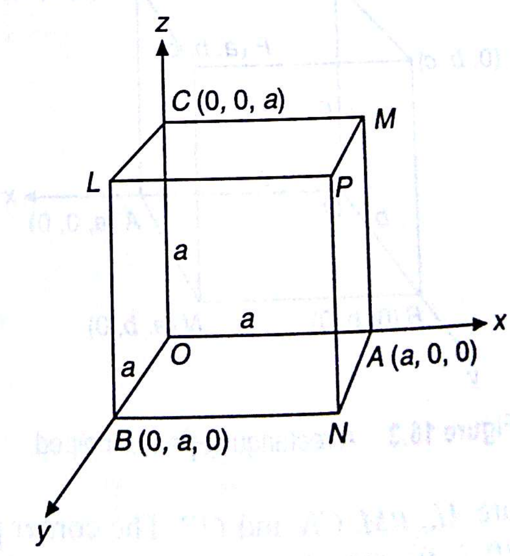

Figure 16.4 A cube.

The corner points of the cube are $O(0, 0, 0)$, $P(a, a, a)$, $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$, $L(0, a, a)$, $M(a, 0, a)$ and $N(a, a, 0)$.

The DRs of OP are $a-0, a-0, a-0$ or a, a, a.

Therefore, the DCs of OP are $\dfrac{a}{\sqrt{a^2+a^2+a^2}}, \dfrac{a}{\sqrt{a^2+a^2+a^2}}, \dfrac{a}{\sqrt{a^2+a^2+a^2}}$

or

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

Similarly, DCs of AL are $-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$.

Hence the required angle between the diagonals is

$$\cos^{-1}\left(\frac{\frac{1}{\sqrt{3}}\times\left(-\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{3}}\times\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\times\frac{1}{\sqrt{3}}}{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}\right)=\cos^{-1}\frac{1}{3}.$$

EXAMPLE 16.11 A line angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, show that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}.$$

Solution

Let $OABC\ LMNP$ (see Figure 16.4) be a cube of side a. The corners of the cube are $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$, $L(0, a, a)$, $M(a, 0, a)$, $N(a, a, 0)$ and $P(a, a, a)$. The DRs of OP are $a-0, a-0, a-0$ and hence DCs are

$\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$.

Similarly, DCs of AL, BM and CN are respectively

$$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}.$$

Let l, m, n be the DCs of a line which is inclined at angles $\alpha, \beta, \gamma, \delta$ respectively to the four diagonals, then

$$\cos\alpha = l\cdot\frac{1}{\sqrt{3}}+m\cdot\frac{1}{\sqrt{3}}+n\cdot\frac{1}{\sqrt{3}}=\frac{l+m+n}{\sqrt{3}}.$$

Similarly, $\cos\beta=\dfrac{-l+m+n}{\sqrt{3}}, \cos\gamma=\dfrac{l-m+n}{\sqrt{3}}, \cos\delta=\dfrac{l+m-n}{\sqrt{3}}$.

Hence $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2\ (l+m-n)^2]$$

$$= \frac{1}{3}\cdot 4\,(l^2+m^2+n^2) = \frac{4}{3}.$$

EXAMPLE 16.12 Find the acute angle between the lines whose DCs are given by the relations

$$l+m+n=0 \text{ and } l^2+m^2-n^2=0.$$

Solution

Putting $n=-(l+m)$ to $l^2+m^2-n^2=0$. We get

$$l^2+m^2-(l+m)^2=0 \text{ or } 2lm=0.$$

When $\quad l=0$ then $m+n=0$ or $m=-n.$

$$\therefore \quad \frac{l}{0}=\frac{m}{1}=\frac{n}{-1}.$$

When $\quad m=0$ then $l+n=0$ or $l=-n.$

$$\therefore \quad \frac{l}{1}=\frac{m}{0}=\frac{n}{-1}.$$

Hence the DRs of the lines are 0, 1, –1 and 1, 0, –1.
The angle θ between the lines is

$$\cos\theta = \frac{0\cdot 1+1\cdot 0+(-1)(-1)}{\sqrt{0+1+1}\,\sqrt{0+1+1}} = \frac{1}{2}.$$

Hence, $\quad \theta = \frac{\pi}{3}.$

EXAMPLE 16.13 Find the direction cosines of the straight line passing through the points (1, 2, 4) and (3, 1, 3).

Solution

The DRs of the line joining the points (1, 2, 4) and (3, 1, 3) are $1-3, 2-1, 4-3$, or $-2, 1, 1$.

The DCs are $\frac{-2}{\sqrt{4+1+1}}, \frac{1}{\sqrt{4+1+1}}, \frac{-1}{\sqrt{4+1+1}}$, i.e. $\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$.

16.4 PLANE

A *plane* is a surface such that if any two points are taken on it, the straight line joining them lies wholly in the surface, i.e., every point on the line joining the two points will be on the plane.

The most general equation of a plane is $ax + by + cz + d = 0$, where a, b, c, d are constants and a, b, c are not all zero. If the plane passes through the origin then its equation is $ax + by + cz = 0$. It can be shown that a, b, c represent the DRs of the plane and is directed perpendicular to the plane.

Equation of the plane passing through a point: Let $ax + by + cz + d = 0$ be the plane and it passes through the point (x_1, y_1, z_1). Then $ax_1 + by_1 + cz_1 + d = 0$. Subtracting these two equations, we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This is the required equation of the plane.

Intercept form of a plane: Let the plane cuts the coordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ then its equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. This form is known as intercept form.

Normal form of a plane: If l, m, n are the DCs of a plane and p be the perpendicular distance from the origin to the plane, then $lx + my + nz = p$ represents the normal form of a plane.

Let $ax + by + cz + d = 0$ be the equation of a plane we divide this equation by $\sqrt{a^2 + b^2 + c^2}$, we get

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}x + \frac{b}{\sqrt{a^2 + b^2 + c^2}}y + \frac{c}{\sqrt{a^2 + b^2 + c^2}}z = -\frac{d}{\sqrt{a^2 + b^2 + c^2}}.$$

The perperpendicular distance from the origin to the plane $ax + by + cz + d = 0$ is $\frac{d}{\sqrt{a^2 + b^2 + c^2}}$. Also, the sum of squares of the coefficients of x, y, z is 1. Hence the above equation is the normal form of the plane.

Equations of the coordinates planes: The planes XOY, YOZ, ZOX are called the coordinates planes the equations of these planes are respectively $z = 0, x = 0, y = 0$.

Equations of planes parallel to the axes: The equations of the planes parallel to x, y and z axes are respectively, $by + cz + d = 0$, $ax + cz + d = 0$ and $ax + by + d = 0$.

16.4.1 Distance from a Point to the Plane

Let $ax + by + cz + d = 0$ be the equation of the plane and $A(x_1, y_1, z_1)$ be the given point. Then the perpendicular distance from A to the plane is

$$\left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|.$$

EXAMPLE 16.14 Find the distance between the planes

$$x - 2y + z = 4 \quad \text{and} \quad 2x - 4y + 2z = 5.$$

Solution

The distance d_1 from (0, 0, 0) to the plane $x - 2y + z = 4$ is

$$d_1 = \frac{-4}{\sqrt{1+4+1}} = \frac{-4}{\sqrt{6}}.$$

Again, the distance d_2 from (0, 0, 0) to the plane $2x - 4y + 2z = 5$ is

$$d_2 = \frac{-5}{\sqrt{4+16+4}} = \frac{-5}{2\sqrt{6}}.$$

Hence the required distance between the planes is

$$|d_1 - d_2| = \left|\frac{-4}{\sqrt{6}} + \frac{5}{2\sqrt{6}}\right| = \frac{3}{2\sqrt{6}}.$$

16.4.2 Angle between Two Planes

The angle between planes is the angle between the normals to the plane drawn through a point. Let

$$a_1 x + b_1 y + c_1 z + d_1 = 0 \text{ and } a_2 x + b_2 y + c_2 z + d_2 = 0$$

be the equations of the planes.

The DRs of the normals to the planes are a_1, b_1, c_1 and a_2, b_2, c_2.

Hence the angle θ between the planes is given by

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Condition for perpendicular planes: If the planes are perpendicular then

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0.$$

Condition for parallel planes: If the above planes are parallel, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

EXAMPLE 16.15 Determine the value of h for which the planes

$$3x - 2y + hz - 1 = 0 \text{ and } x + hy + 5z + 2 = 0$$

may be perpendicular to each other.

Solution

The DRs of the planes are $1, h, 5$ and $3, -2, h$. If they are perpendicular, then

$$1 \times 3 + h \times (-2) + 5 \times h = 0 \quad \text{or} \quad 3h + 3 = 0$$

or

$$h = -1.$$

16.4.3 Planes Bisecting the Angles between Two Planes

Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ be the equations of two planes. If $P(\alpha, \beta, \gamma)$ be any point on either of the two bisecting planes, then P is equidistant from the two planes. Therefore,

$$\frac{a_1\alpha + b_1\beta + c_1\gamma + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2\alpha + b_2\beta + c_2\gamma + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Hence the equations of the planes are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

To distinguish the two bisecting planes follow the rule:

First write the equations of the planes with the same sign of the constants. Then calculate the angle between the planes. If $\cos\theta < 0$, i.e. $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then the origin is within the acute angle between the planes. If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then the origin is within the obtuse angle between the planes.

16.4.4 Worked-Out Examples on Plane

EXAMPLE 16.16 Show that the four points (0, –1, 0), (2, 1, –1), (1, 1, 1) and (3, 3, 0) are coplanar and obtain the equation of the plane.

Solution

Let the equation of the plane passes through the point (0, –1, 0) be

$$a(x-0) + b(y+1) + c(z-0) = 0. \tag{1}$$

Since it pass through the points (2, 1, –1) and (1, 1, 1).

$$2a + 2b - c = 0 \tag{2}$$

and

$$a + 2b + c = 0. \tag{3}$$

Solving equations (2) and (3), we get

$$\frac{a}{2+2} = \frac{b}{-1-2} = \frac{c}{4-2} \quad \text{or} \quad \frac{a}{4} = \frac{b}{-3} = \frac{c}{2} = k \text{ (say)}$$

i.e.

$$a = 4k, b = -3k, c = 2k.$$

Putting these values in equation (1), we get

$$4x - 3y + 2z = 3. \tag{4}$$

Again,

$$\text{LHS} = 4\times 3 - 3\times 3 + 2\times 0 = 3 = \text{RHS}$$

i.e. (3, 3, 0) satisfies equation (4). Hence the given four points are coplanar and the required plane is $4x - 3y + 2z = 3$.

EXAMPLE 16.17 Find the equation of the plane which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$ and which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$.

Solution

Let the equation of the plane be

$$x + 2y + 3z - 4 + \lambda\,(2x + y - z + 5) = 0$$

or $$x\,(1 + 2\lambda) + y\,(2 + \lambda) + z\,(3 - \lambda) + 5\lambda - 4 = 0. \tag{1}$$

Since it is perpendicular to the plane

$$5x + 3y + 6z + 8 = 0$$

therefore, $$5\,(1 + 2\lambda) + 3\,(2 + \lambda) + 6\,(3 - \lambda) = 0$$

or $$7\lambda + 29 = 0 \quad \text{or} \quad \lambda = -\frac{29}{7}.$$

Hence the required equation of the plane is

$$x\left(1 - \frac{58}{7}\right) + y\left(2 - \frac{29}{7}\right) + z\left(3 + \frac{29}{7}\right) - \frac{145}{7} - 4 = 0$$

or $$51x + 15y - 50z + 173 = 0.$$

EXAMPLE 16.18 Find the equation of the plane passing through the points (2, 3, –4), (1, –1, 3) and perpendicular to the plane $2x + 6y + 9z = 9$.

Solution

Let the equation of the plane passing through the point (2, 3, –4) be

$$a\,(x - 2) + b\,(y - 3) + c\,(z - 4) = 0. \tag{1}$$

Since it passes through the point (1, –1, 3),

$$-a - 4b + 7c = 0. \tag{2}$$

Also, (1) is perpendicular to the plane

$$2x + 6y + 9z = 9$$

$\therefore$ $$2a + 6b + 9c = 0. \tag{3}$$

Solving equations (2) and (3) we get

$$a = 78k,\ b = -23k,\ c = -2k.$$

Putting these values in equation (1) we get the required equation of the plane

$$78\,(x - 2) - 23\,(y - 3) - 2\,(z + 4) = 0$$

or $$78x - 23y - 2z + 217 = 0.$$

EXAMPLE 16.19 Find the equation of the plane which passes through the point (1, 2, 1) and parallel to the plane $x + 2y - 3z = 5$.

Solution

The plane parallel to the given plane is

$$x + 2y - 3z = k.$$

Since it passes through the point (1, 2, 1),

$$1 + 2 \times 2 - 3 \times 1 = k \quad \text{or} \quad k = 2.$$

Hence the required equation of the plane is $x + 2y - 3z = 2$.

EXAMPLE 16.20 Find the equation of the plane passing through the point (2, 5, –8) and perpendicular to each of the planes $2x - 3y + 4z + 7 = 0$ and $4x + y - 2z + 16 = 0$.

Solution

Let the equation of the plane passing through the point (2, 5, –8) be

$$a(x-2)+b(y-5)+c(z+8)=0.$$

Since it is perpendicular to both the given planes,

$$2a-3b+4c=0 \quad \text{and} \quad 4a+b-2c=0.$$

Solving these equations, we get

$$\frac{a}{1}=\frac{b}{10}=\frac{c}{7}.$$

Hence the required equation of the plane is

$$(x-2)+10\,(y-5)+7\,(z+8)=0$$

or

$$x+10y+7z+4=0.$$

EXAMPLE 16.21 A variable plane passes through a fixed point (a, b, c) and meets the axes of reference in A, B and C. Show that the locus of the points of intersection of the planes through A, B and C parallel to the coordinate planes is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1.$$

(WBUT 2005)

Solution

Let the equation of the plane be

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1. \tag{1}$$

Since it passes through the point (a, b, c),

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1.$$

It meets the axes at $A(\alpha, 0, 0)$, $B(0, \beta, 0)$, $C(0, 0, \gamma)$.

Also planes through A, B and C and parallel to the coordinates planes are

$$a=\alpha,\ y=\beta,\ z=\gamma.$$

The locus of the point of intersection is obtained by eliminating the variables α, β, γ.

Putting the value of α, β, γ in equation (1) we get the required locus as

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1.$$

EXAMPLE 16.22 A variable plane at a constant distance p from the origin meets the axes in A, B and C. Through A, B, C, planes are drawn parallel to the coordinates planes. Show that the locus of their point of intersection is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}. \qquad \text{(WBUT 2006)}$$

Solution

Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

It meets the axes in $A\,(a, 0, 0)$, $B\,(0, b, 0)$, $C\,(0, 0, c)$. The distance from origin to the plane is p (given).

$$\therefore \quad \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=p \quad \text{or} \quad \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}. \qquad (1)$$

Planes passing through A, B, C and parallel to the coordinates planes are $x=a, y=b, z=c$.

Putting the values of a, b, c in equation (1), we get the required locus as

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}.$$

EXAMPLE 16.23 A variable plane is at a constant distance $3p$ from the origin and meets the axes in A, B, C. Show that the locus of the centroid of the triangle ABC is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}.$$

Solution

Let the coordinates of A, B and C be respectively $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. Then the equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{x}{c}=1.$$

Its distance from origin is

$$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=3p$$

or

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{9p^2}. \qquad (1)$$

If (α, β, γ) be the centroid of the triangle ABC, then

$$\alpha = \frac{a+0+0}{3} = \frac{a}{3},\ \beta = \frac{0+b+0}{3} = \frac{b}{3},\ \gamma = \frac{0+0+c}{3} = \frac{c}{3}.$$

Putting the values of $a = 3\alpha, b = 3\beta, c = 3\gamma$ to equation (1), we get

$$\frac{1}{9\alpha^2} + \frac{1}{9\beta^2} + \frac{1}{9\gamma^2} = \frac{1}{9p^2}.$$

Hence the required locus is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}.$$

EXAMPLE 16.24 A variable plane is at a constant distance p from the origin and meets the axes in A, B, and C. Show that the locus of the centroid of the tetrahedron $OABC$ is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}.$$

Solution

Let the coordinates of A, B, C be $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

Then the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

Since its distance from origin is p,

$$\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = p \quad \text{or} \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2} \qquad (1)$$

Let (α, β, γ) be the centroid of the tetrahedron $OABC$. Therefore,

$$\alpha = \frac{0+a+0+0}{4} = \frac{a}{4},\ \beta = \frac{0+0+b+0}{4} = \frac{b}{4},\ \gamma = \frac{0+0+0+c}{4} = \frac{c}{4}.$$

Putting the values of $a = 4\alpha, b = 4\beta, c = 4\gamma$ to equation (1), we get,

$$\frac{1}{16\alpha^2} + \frac{1}{16\beta^2} + \frac{1}{16\gamma^2} = \frac{1}{p^2}.$$

Hence the required locus is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}.$

EXAMPLE 16.25 Find the equations of the planes bisecting the angle between the planes $x + 2y + 2z = 9$ and $4x - 3y + 12z + 13 = 0$ and distinguish them.

Solution

The equations of the planes bisecting the angle between the given planes are

$$\frac{x+2y+2z-9}{\sqrt{1+4+4}} = \pm\frac{4x-3y+12z+13}{\sqrt{16+9+144}}$$

or
$$25x+17y+62z-78=0$$

and
$$x+35y-10z-156=0.$$

Acute or obtuse

The angle between the planes $x+2y+2z=9$ and the bisecting plane $x+35y-10z-156=0$ is given by

$$\cos\theta = \frac{1\times 1+2\times 35+2\times(-10)}{\sqrt{1+35^2+10^2}\sqrt{1+2^2+2^2}} = \frac{17}{36}.$$

$\therefore$
$$\tan\theta = \frac{\sqrt{1007}}{17} > 1.$$

Thus,
$$\theta > 45^\circ.$$

Hence the plane $x-35y-10z-156=0$ bisects the obtuse angle between the planes and hence the other plane $25x+17y+62z-78=0$ bisects the acute angle.

With respect to origin: We write the equations of the planes so that the constant terms in both are positive, i.e.

$$-x-2y-2z+9=0 \quad\text{and}\quad 4x-3y+12z+13=0.$$

Then the plane
$$\frac{-x-2y-2z+9}{\sqrt{1^2+2^2+2^2}} = +\frac{4x-3y+12z+13}{\sqrt{4^2+3^3+12^2}}$$

or
$$25x+17y+62z-78=0$$

bisects the angle between the planes that contains the origin.

16.5 STRAIGHT LINE

Let (x_1, y_1, z_1) be a given point on the line and l, m, n be the DCs or DRs of the line. Let (x, y, z) be any point on the line, then the DCs are proportional to $x-x_1$, $y-y_1$, $z-z_1$.

Hence the equations of the line passing through the point (x_1, y_1, z_1) are

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}.$$

This equation is called the *standard form* or *canonical form* or *symmetric form* of a line.

In particular, the equations of straight line passing through the origin are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

...ay be noted that the direction of l, m, n is parallel to the line.

...quations of the straight line passing through two points: Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points in space. Then the equations of the line AB are

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2}$$

...s $x_1-x_2, y_1-y_2, z_1-z_2$ represents DRs of the line.

...rametric form: If the line passes through the point (x_1, y_1, z_1) and DRs or ...'s are l, m, n then its equations are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \text{ (say)}.$$

...hese equation can be written as

$$x=x_1+lr,\ y=y_1+mr,\ z=z_1+nr.$$

This form of a straight line is called parametric form and r is called the parameter.

Plane intercept form: If two planes intersect, then the intersection between them generates a straight line. Thus the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ represent a straight line.

Equation of coordinates axes: We know any point on x-axis is of the form $(a, 0, 0)$. Thus, the equation of x-axis is $y=0, z=0$.

Similarly, the equations of y-axis and z-axis are respectively $z=0, x=0$ and $x=0, y=0$.

16.5.1 Perpendicular Distance of a Point from a Straight Line

Let $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ be the equation of the line and $P(\alpha,\beta,\gamma)$ be the given point. Then the perpendicular distance from P to the line is given by

$$[(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2-\{(x_1-\alpha)l+(y-\beta)m+(z_1-\gamma)n\}^2]^{1/2}.$$

16.5.2 Condition of Coplanarity of Two Straight Lines

Let $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ be two lines. If these two lines are coplanar then the plane should pass through the points (x_1, y_1, z_1) and (x_2, y_2, z_2). Therefore, the equation of the plane can be taken as $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$. Since it passes through the point (x_2, y_2, z_2),

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \quad (16.1)$$

Also, the DRs of the lines are perpendicular to the *DRs* of the plane, hence

$$al_1 + bm_1 + cn_1 = 0 \quad (16.2)$$

and

$$al_2 + bm_2 + cn_2 = 0. \quad (16.3)$$

Eliminating *a, b, c* between equations (16.1), (16.2) and (16.3), we get the required conditions of the co-planarity

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

The equation of the plane where the straight lines lie (co-planar) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

EXAMPLE 16.26 Show that the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

are co-planar. Find also the equation of the plane.

Solution

Here
$$\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= 1(16 - 15) - 1(12 - 10) + (9 - 8) = 0.$$

Hence the given lines are co-planar.

The equation of the plane is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

or
$$(x-1)(15-16) - (y-2)(10-12) + (z-3)(8-9) = 0$$

or
$$x - 2y + z = 0.$$

EXAMPLE 16.27 Prove that the following lines:

$$x + 2y - 5z + 9 = 0 = 3x - y + 2z - 5$$

$$2x + 3y - z - 3 = 0 = 4x - 5y + z + 3$$

are coplanar and also find the equation of the plane containing them.

Solution

Let l, m, n be the DRs of the first line, then

$$l + 2m - 5n = 0 \quad \text{and} \quad 3l - m + 2n = 0.$$

Solving these equations, we get $\frac{l}{1} = \frac{m}{17} = \frac{n}{7}$.

Let $(0, \beta, \gamma)$ be any point on the first line, then

$$2\beta - 5\gamma + 9 = 0 \text{ and } -\beta + 2\gamma - 5 = 0.$$

The solution of these equations is $\beta = -7, \gamma = -1$.

Therefore, (0, –7, –1) is a point on the first line. Hence the symmetrical form of the first line is

$$\frac{x}{1} = \frac{y+7}{17} = \frac{z+1}{7}.$$

Similarly, if l', m', n' be the DRs of the second line, then

$$2l' + 3m' - n' = 0 \text{ and } 4l' - 5m' + n' = 0.$$

Solving we get, $$\frac{l'}{1} = \frac{m'}{3} = \frac{n'}{11}.$$

The point (0, 0, –3) lie on the second line. Hence the equation of the second line in symmetrical form is

$$\frac{x}{1} = \frac{y}{3} = \frac{z+3}{11}.$$

Test of coplanarity: Now, $\begin{vmatrix} 0-0 & -7-0 & -1+3 \\ 1 & 17 & 7 \\ 1 & 3 & 11 \end{vmatrix} = 0.$

Hence the lines are coplanar.

Equation of the plane: The equation of the required plane is

$$\begin{vmatrix} x & y+7 & z+1 \\ 1 & 17 & 7 \\ 1 & 3 & 11 \end{vmatrix} = 0$$

or $$x(187 - 21) - y(11 - 7) + (z + 3)(3 - 17) = 0$$

or $$83x - 2y - 7z - 51 = 0.$$

16.5.3 Shortest Distance between Two Skew Lines

The non-parallel non-intersecting lines are called *skew lines* (see Figure 16.5).

Let the equations of the skew lines LA and MB be

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} = r_1 \text{ (say)}$$

and $$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}=r_2 \text{ (say).}$$

Let LM be the shortest distance. Therefore, LM is perpendicular to both the lines.

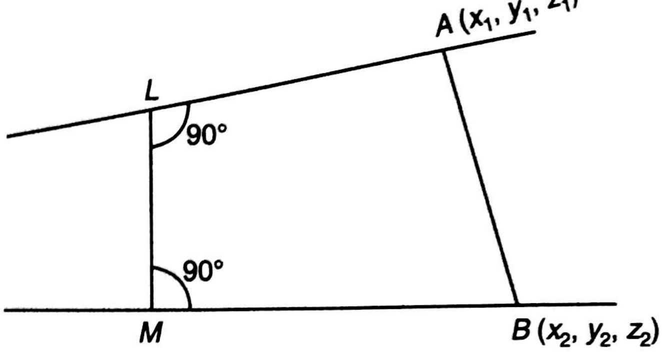

Figure 16.5 Skew lines.

Any point on first and second lines are respectively

$$(x_1+l_1r_1,\ y_1+m_1r_1,\ z_1+n_1r_1) \text{ and } (x_2+l_2r_2,\ y_2+m_2r_2,\ z_2+n_2r_2).$$

For particular values of r_1 and r_2 these points will be L and M. So, the DRs of the line LM are

$$x_1-x_2+l_1r_1-l_2r_2,\ y_1-y_2+m_1r_1-m_2r_2,\ z_1-z_2+n_1r_1-n_2r_2.$$

Since LM is perpendicular to both LA and MB, then

$$l_1(x_1-x_2+l_1r_1-l_2r_2)+m_1(y_1-y_2+m_1r_1-m_2r_2)+n_1(z_1-z_2+n_1r_1-n_2r_2)=0$$

and $$l_2(x_1-x_2+l_1r_1-l_2r_2)+m_2(y_1-y_2+m_1r_1-m_2r_2)+n_2(z_1-z_2+n_1r_1-n_2r_2)=0.$$

There are linear equations of r_1 and r_2 and solving these two equations, we get the values of r_1 and r_2 and hence, we get the coordinates of L, M. Then the distance between L and M is the required shortest distance and the equation of LM is the equation of shortest distance line.

The following example illustrates the method completely.

EXAMPLE 16.28 Find the shortest distance and its equation between the lines

$$\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5},\quad \frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}.$$

Solution

Given $$\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}=r_1 \text{ (say)}$$

and $$\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}=r_2 \text{ (say).}$$

Any points on the first and the second lines are respectively,

$$L(2r_1 + 3, -7r_1 - 15, 5r_1 + 9) \text{ and } M(2r_2 - 1, r_2 + 1, -3r_2 + 9).$$

The DRs of the line LM are

$$2r_1 - 2r_2 + 4, -7r_1 - r_2 - 16, 5r_1 + 3r_2.$$

If LM is the shortest distance then LM is perpendicular to both the lines, and hence

$$2(2r_1 - 2r_2 + 4) - 7(-7r_1 - r_2 - 16) + 5(5r_1 + 3r_2) = 0$$

and

$$2(2r_1 - 2r_2 + 4) + (-7r_1 - r_2 - 16) - 3(5r_1 + 3r_2) = 0.$$

That is,

$$13r_1 + 3r_2 + 20 = 0 \text{ and } 9r_1 + 7r_2 + 4 = 0.$$

Solving these equations, we get

$$r_1 = -2, \quad r_2 = 2.$$

Hence the coordinates of L and M are respectively

$$(-1, -1, -1) \text{ and } (3, 3, 3).$$

Therefore, the required shortest distance is

$$\sqrt{(-1-3)^2 + (-1-3)^2 + (-1-3)^2} = 4\sqrt{3}$$

and the equation of shortest distance line is

$$\frac{x+1}{-1-3} = \frac{y+1}{-1-3} = \frac{z+1}{-1-3}$$

or

$$\frac{x+1}{1} = \frac{y+1}{1} = \frac{z+1}{1}$$

or

$$x = y = z.$$

EXAMPLE 16.29 Find the magnitude of the shortest distance between the lines

$$\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2}$$

and

$$5x - 2y - 3z + 6 = 0, x - 3y + 2z - 3 = 0.$$

Solution

Any plane through the second line is

$$(5x - 2y - 3z + 6) + \lambda(x - 3y + 2z - 3) = 0$$

or

$$(5 + \lambda)x + (-2 - 3\lambda)y + (-3 + 2\lambda)z + (6 - 3\lambda) = 0.$$

If it is parallel to the first line, then

$$4(5 + \lambda) + 3(-2 - 3\lambda) + 2(-3 + 2\lambda) = 0 \text{ or } \lambda = 8.$$

Hence the equation of the plane containing the second line and parallel to the first line is

$$13x - 26y + 13z - 18 = 0. \qquad (1)$$

Let (0, −1, 2) be a point on the first line. Then the required shortest distance between the given lines is equal to the shortest distance from the point (0, −1, −2) to the plane of equation (1), which is

$$\frac{26 + 26 - 18}{\sqrt{13^2 + 26^2 + 13^2}} = \frac{34}{13\sqrt{6}} = \frac{17}{39}\sqrt{6}.$$

16.5.4 Worked-Out Example on Straight Lines

EXAMPLE 16.30 Find the value of k, so that the lines

$$\frac{x+4}{k} = \frac{y+6}{5} = \frac{z-1}{-2} \text{ and } 3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$$

may intersect. What is the point of intersection ?

Solution

Let $\frac{x+4}{k} = \frac{y+6}{5} = \frac{z-1}{-2} = r$ (say). Any point on the line is

$$(kr - 4, 5r - 6, -2r + 1).$$

If the lines intersect then this point lies on the second line. Therefore,

$$3(kr - 4) - 2(5r - 6) + (-2r + 1) + 5 = 0$$

and $$2(kr - 4) + 3(5r - 6) + 4(-2r + 1) - 4 = 0$$

or $$kr - 4r + 2 = 0 \text{ and } 2kr + 7r - 26 = 0.$$

Solving these equations, we get

$$r = 2, \quad k = 3.$$

Therefore, the required value of k is 3 and the point of intersection is (2, 4, −3).

EXAMPLE 16.31 Find the equations of the image of the line

$$\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{4}$$

in the plane $3x + y - 4z + 21 = 0$.

Solution

The find point of intersection between the line and the plane (see Figure 16.6).

Let $$\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{4} = r \text{ (say)}.$$

Any point on this line is

$$(2r + 2, 3r + 3, 4r + 4).$$

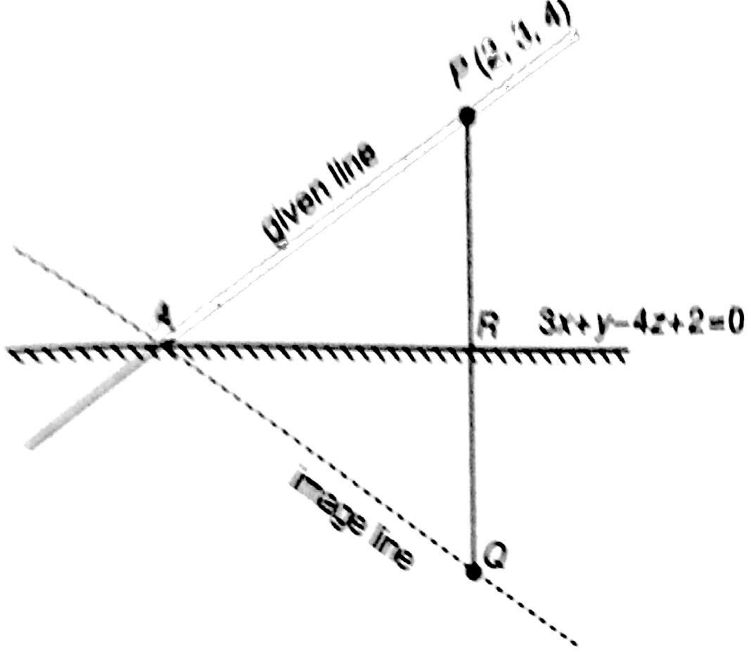

Figure 16.6 AQ is the image of AP.

If it lies on the plane, then

$$3\,(2r+2)+(3r+3)-4\,(4r+4)+21=0$$

or

$$r=2.$$

Hence the point of intersection is $A\,(6, 9, 12)$.

To find the image of the point $P(2, 3, 4)$: Here $P\,(2, 3, 4)$ is a point on the given line. Let $Q\,(a, b, c)$ be the image of P. Let R be the point of intersection between the given plane and the line PQ.

$\therefore$ $PR = RQ$, i.e. R is the middle point of PQ.

Therefore, the coordinate of R is $\left(\frac{a+2}{2}, \frac{b+3}{2}, \frac{c+4}{2}\right)$. Again, PQ is perpendicular to the given plane and passing through the point P, so its equation is

$$\frac{x-2}{3}=\frac{y-3}{1}=\frac{z-4}{-4}=r_1 \text{ (say)}.$$

Any point on this line (PQ) is $(3r_1+2, r_1+3, -4r_1+4)$.

If it lies on the plane, $3\,(3r_1+2)+(r_1+3)-4\,(-4r_1+4)+21=0$.

$$r_1=-\frac{7}{13}$$

i.e.

Therefore, the coordinates of R are $\left(\frac{5}{13}, \frac{32}{13}, \frac{80}{13}\right)$

Therefore,

$$\frac{a+2}{2}=\frac{5}{13}, \frac{b+3}{2}=\frac{32}{13}, \frac{c+4}{2}=\frac{80}{13}$$

Thus,

or $$a=-\frac{16}{13},\ b=\frac{25}{13},\ c=\frac{108}{13}.$$

Hence the image of P is $Q\left(-\frac{16}{13},\frac{25}{13},\frac{108}{13}\right)$.

The image of the line AP is the line AQ and its equation is

$$\frac{x-6}{6+\frac{16}{13}}=\frac{y-9}{9-\frac{25}{13}}=\frac{z-12}{12-\frac{108}{13}}$$

or $$\frac{x-6}{47}=\frac{y-9}{46}=\frac{z-12}{24}.$$

EXAMPLE 16.32 Prove that the equation of the perpendicular from the point $(3, -1, 11)$ to the line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ is $\frac{x-3}{1}=\frac{y+1}{-6}=\frac{z-11}{4}$ and that the coordinates of its foot are $(2, 5, 7)$.

Solution

Any point on the line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r$ is $A(2r, 3r+2, 4r+3)$. Let B be the point whose coordinates are $(3, -1, 11)$ (see Figure 16.7).

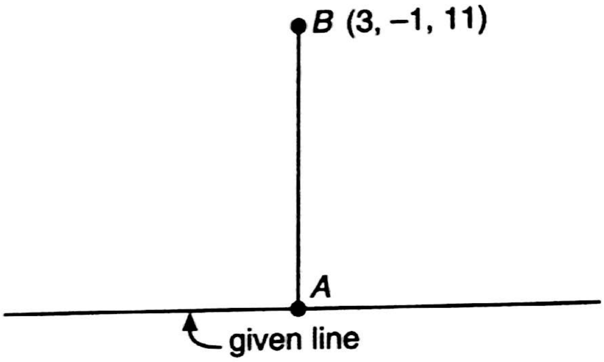

Figure 16.7

The DRs of AB are

$$2r-3,\ 3r+3,\ 4r-8.$$

If AB is perpendicular to the line, then

$$2(2r-3)+3(3r+3)+4(4r-8)=0$$

or $$r=1.$$

Therefore, the coordinates of A are $(2, 5, 7)$. Hence the equation of the line perpendicular to AB is

$$\frac{x-3}{3-2}=\frac{y+1}{-1-5}=\frac{z-11}{11-7} \quad \text{or} \quad \frac{x-3}{1}=\frac{y+1}{-6}=\frac{z-11}{4}$$

and the coordinates of foot of the perpendicular are $(2, 5, 7)$.

XAMPLE 16.33 Obtain the equation of the plane through the straight line $x - 4y + 5z - 10 = 0,\ 2x + 2y - 3z - 4 = 0$ and parallel to the line $x = 2y = 3z$.

(WBUT 2006)

olution

et the equation of the plane passing through the line $3x - 4y + 5z - 10 = 0,\ 2x + y - 3z - 4 = 0$ be

$$3x - 4y + 5z - 10 + \lambda(2x + 2y - 3z - 4) = 0$$

r $$(3 + 2\lambda)x + (-4 + 2\lambda)y + (5 - 3\lambda)z - 10 - 4\lambda = 0.$$

If this plane is parallel to the line $x = 2y = 3z$

r $$\frac{x}{1} = \frac{y}{1/2} = \frac{z}{1/3} \text{ then}$$

$$1(3 + 2\lambda) + \frac{1}{2}(-4 + 2\lambda) + \frac{1}{3}(5 - 3\lambda) = 0$$

r $$\lambda = -\frac{4}{3}.$$

Hence the required equation of the plane is

$$\left(3 - \frac{8}{3}\right)x + \left(-4 - \frac{8}{3}\right)y + (5 + 4)z - 10 + \frac{16}{3} = 0$$

or $$x - 20y + 27z - 14 = 0.$$

EXAMPLE 16.34 A straight line with direction ratios $2, 7, -5$ is drawn to intersect the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$ and $\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$.

Find the coordinates of the points of intersection and length intercepted on it.

(WBUT 2005)

Solution

Let AB be the line whose DRs are $2, 7, -5$ and it meets the given lines at

$$(3r_1 + 5, -r_1 + 7, r_1 - 2) \text{ and } (-3r_2 - 3, 2r_2 + 3, 4r_2 + 6).$$

Then the DRs of the line AB are $-3r_2 - 3r_1 - 8,\ 2r_2 + r_1 - 4,\ 4r_2 - r_1 + 8$.

These are proportional to $2, 7, -5$.

$$\therefore \quad \frac{-3r_2 - 3r_1 - 8}{2} = \frac{2r_2 + r_1 - 4}{7} = \frac{4r_2 - r_1 + 8}{-5}.$$

Form the first and the second ratio, we have

$$-21r_2 - 21r_1 - 56 = 4r_2 + 2r_1 - 8$$

or $$23r_1 + 25r_2 + 48 = 0$$

$$10r_2 - 5r_1 + 20 = 28r_2 - 7r_1 + 56$$

or $$r_1 - 19\, r_2 - 18 = 0.$$

Solving these equations, we get $r_1 = r_2 = -1$.

Thus the points of intersection are (2, 8, –3) and (0, 1, 2).

The length of intercept between the lines is

$$\sqrt{(2-0)^2 + (8-1)^2 + (-3-2)^2} = \sqrt{78}.$$

EXAMPLE 16.35 Find the equation of the projection of the line

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-4}{4}$$

on the plane $x + 3y + z + 5 = 0$.

Solution

The projection of a line on a plane is the line of intersection of the given plane and the plane passing through the given line and perpendicular to the given plane.

The given line and the plane are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-4}{4} \qquad (1)$$

and $$x + 3y + z + 5 = 0. \qquad (2)$$

The equation of a plane through the line of equation (1) is

$$a\,(x-1) + b\,(y-2) + c\,(z-4) = 0 \qquad (3)$$

where $$2a + 3b + 4c = 0 \qquad (4)$$

If the plane of equation (3) perpendicular to the plane of equation (2), then

$$a + 3b + c = 0 \qquad (5)$$

Solving equations (4) and (5), we get

$$\frac{a}{9} = \frac{b}{2} = \frac{c}{3}$$

Thus the equation of the plane containing the line of equation (1) and perpendicular to the plane of equation (2) is

$$-9\,(x-1) + 2\,(y-2) + 3\,(z-4) = 0$$

or $$9x - 2y - 3z + 7 = 0$$

Hence the equations of the projection are

$$x + 3y + z + 5 = 0, 9x - 2y - 3z + 7 = 0.$$

EXAMPLE 16.36 Show that the equation to the plane containing the line

$$\frac{y}{b} + \frac{z}{c} = 1,\ x = 0$$

and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1,\ y = 0$ is $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$

and if $2d$ is the shortest distance prove that

$$\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Solution

Let the equation of the plane containing first line be

$$\frac{y}{b} + \frac{z}{c} - 1 + \lambda x = 0. \tag{1}$$

It will be parallel to the second line, if it is parallel to the plane

$$\frac{x}{a} - \frac{z}{c} - 1 + \mu y = 0 \tag{2}$$

through the second line.

Since these two planes are parallel, therefore,

$$\frac{\lambda}{1/a} = \frac{1/b}{\mu} = \frac{1/c}{-1/c} \text{ or } \lambda = \frac{1}{a} \text{ and } \mu = -\frac{1}{b}.$$

Substituting these values of λ in equation (1), we get

$$-\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0 \text{ or } \frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0, \tag{3}$$

which is the required equation of the plane.

A point on the second line is $(a, 0, 0)$. The shortest distance is the distance of the plane of equation (3) from the point $(a, 0, 0)$.

$$\therefore \quad 2d = \frac{2}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} \text{ or } \frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

EXAMPLE 16.37 Find the distance of the point (3, 8, 2) from the line

$$\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$$

measured parallel to the plane $3x + 2y - 2z + 15 = 0$.

Solution

Let P be the given point (3, 8, 2) and Q be the point on the given line whose coordinates are $(2r+1, 4r+3, 3r+2)$. The DRs of PQ are

$$2r+1-3,\ 4r+3-8,\ 3r+2-2 \text{ or } 2r-2,\ 4r-5,\ 3r$$

Now, PQ is parallel to the plane

$$3x + 2y - 2z + 15 = 0$$

and hence it is perpendicular to normal 3, 2, –2.

$$\therefore \quad 3(2r-2) + 2(4r-5) - 2 \times 3r = 0 \text{ or } r = 2.$$

Hence the point Q is (5, 11, 8) and the required distance is

$$PQ = \sqrt{(5-3)^2 + (11-8)^2 + (8-2)^2} = 7.$$

EXAMPLE 16.38 Prove that the straight lines

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}, \frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma}, \frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

will lie in one plane if

$$\frac{l}{\alpha}(b-c) + \frac{m}{\beta}(c-a) + \frac{n}{\gamma}(a-b) = 0.$$

Solution

Given lines lie on a plane. Let L, M, N be the DRs of such plane. Then L, M, N are perpendicular to the DRs of the straight lines. Hence

$$L\alpha + M\beta + N\gamma = 0 \quad (1)$$

$$La\alpha + Mb\beta + Nc\gamma = 0 \quad (2)$$

$$Ll + Mm + Nn = 0 \quad (3)$$

Solving equations (1) and (3), we get

$$\frac{L}{\beta n - m\gamma} = \frac{M}{l\gamma - n\alpha} = \frac{N}{\alpha m - l\beta} = \lambda \text{ (say)}.$$

Substituting the values of L, M, N to equation (2), we get

$$a\alpha(\beta n - m\gamma) + b\beta(l\gamma - n\alpha) + c\gamma(\alpha m - l\beta) = 0$$

or $$l(b-c)\beta\gamma + m(c-a)\alpha\gamma + n(a-b)\alpha\beta = 0$$

or $$\frac{l}{\alpha}(b-c) + \frac{m}{\beta}(c-a) + \frac{n}{\gamma}(a-b) = 0.$$

EXAMPLE 16.39 A variable line intersects the lines $x = b$, $y + c = 0$; $y = c$, $z + a = 0$; $z = a$, $x + b = 0$. Show that the locus of the line is

$$axy + byz + czx + abc = 0.$$

Solution

Any line intersecting the first two lines is

$$x - b + k_1(y + c) = 0, \; y - c + k_2(z + a) = 0. \quad (1)$$

It meets the third line. Therefore, putting $z = a$ and $x = -b$, we get

$$-2b + k_1(y + c) = 0 \text{ and } y - c + 2ak_2 = 0.$$

Eliminating y, we get

$$ak_1k_2 - ck_1 + b = 0. \quad (2)$$

Putting the value of k_1 and k_2 from equations (1) in (2), we get

$$a\frac{x-\beta}{y-?}\cdot\frac{y-?}{z+?}+?\frac{x-\gamma}{?}+\gamma=0$$

or

$$axy+\beta yz+\gamma zx+abc=0.$$

EXAMPLE 16.40 Show that the surface generated by a straight line which intersects the lines $y=0, z=c; x=0, z=-c$ and the curve $z=0, xy+c^2=0$ is $z^2-c^2=xy$.

Solution

Any line intersecting the first two lines is

$$y+k_1(z-c)=0 \text{ and } x+k_2(z+c)=0. \tag{1}$$

It meets the curve $z=0, xy+c^2=0$.

Putting $z=0$ in equation (1), we get

$$y-k_1c=0 \text{ and } x+k_2c=0 \text{ i.e. } xy=-k_1k_2c^2$$

$$-c^2=-k_1k_2c^2 \qquad [\because xy+c^2=0] \tag{2}$$

or

$$k_1k_2=1.$$

or

Putting the values of k_1, k_2 from equation (1) to (2), we get

$$\frac{y}{z-c}\times\frac{x}{z+c}=1 \quad \text{or} \quad xy=z^2-c^2,$$

which is the required surface.

EXAMPLE 16.41 Prove that the locus of the point which is equidistant from the lines $y=mx, z=c$ and $y=-mx, z=-c$ is the surface $mxy+(1+m^2)cz=0$.

Solution

Let (α,β,γ) be the variable point. The gives equations of the given lines can be written as

$$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0} \text{ and } \frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0}.$$

The distance from (α,β,γ) to the straight lines are

$$\left\{\alpha^2+\beta^2+(\gamma-c)^2-\frac{(\alpha+m\beta+0)^2}{1^2+m^2}\right\}^{1/2}$$

$$\left\{\alpha^2+\beta^2+(\gamma+c)^2-\frac{(\alpha-m\beta+0)^2}{1^2+m^2}\right\}^{1/2}.$$

and

Since these are equal,

$$\alpha^2 + \beta^2 + (\gamma - c)^2 - \frac{(\alpha + m\beta)^2}{1+m^2} = \alpha^2 + \beta^2 + (\gamma + c)^2 - \frac{(\alpha - m\beta)^2}{1+m^2}$$

or
$$4\gamma c = \frac{(\alpha - m\beta)^2 - (\alpha + m\beta)^2}{1+m^2}$$

or
$$\gamma c\,(1+m^2) = -\,m\alpha\beta.$$

Hence the required locus of (α, β, γ) is

$$mxy + (1+m^2)\,cz = 0.$$

16.6 QUADRIC SURFACES

A surface defined in space by an equation of the second degree in x, y, z is called a *quadric surface* or a quadric or conicoid. The general equation of a conicoid is $ax^2 + by^2 + cz^2 + 2gzx + 2fyz + 2hxy + 2ux + 2vy + 2wz + d = 0$. For different values of its coefficients this equation represents different quadric, viz. sphere, cone, cylinder, ellipsoid, hyperboloid, paraboloid, etc. In this, section we discuss the quadric sphere, right circular cone and right circular cylinder. The equations of some standard conicoids are give below:

Ellipsoid : $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

M

Hyperboloid of one sheet : $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Hyperboloid of two sheets : $$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Elliptic paraboloid : $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$$

Hyperbolic paraboloid : $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}.$$

16.7 SPHERE

A sphere is the locus of a point which moves in a space so that it is always at a constant distance from a fixed point. The fixed point is called the *centre* of the sphere and the distance is called the *radius* of the sphere.

Equation of a sphere

General form: The very general equation of a sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

It may be noted that the coefficient of x^2, y^2, z^2 are same, there is no term of the form yz, zx, xy. If $u^2 + v^2 + w^2 - d > 0$ then the above quadric represents a sphere. The centre and radius of this sphere are respectively $(-u, -v, -w)$ and $\sqrt{(u^2 + v^2 + w^2 - d)}$.

Centre and radius being given: If (a, b, c) be the coordinates of centre and r be the radius, then the equation of the sphere is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Diameter form: If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the end points of a diameter then the equation of the sphere is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

and its centre is $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

We have seen that a straight line in space can be drawn uniquely if two points are known, a plane can be drawn if three points are given. Similarly, if four non-coplanar points are given then we can drawn a sphere uniquely.

In the following example, we illustrate how a sphere can be obtained if four non-coplanar points are given.

EXAMPLE 16.42 Find the equation of the sphere through the four points $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

Solution

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2zw + d = 0.$$

Since it passes through the points $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$,

$$d = 0$$

$$a^2 + 2ua = 0 \text{ or } 2u = -a$$

$$b^2 + 2vb = 0 \text{ or } 2v = -b$$

$$c^2 + 2wc = 0 \text{ or } 2w = -c.$$

Hence the required equation of the sphere is

$$x^2 + y^2 + z^2 - ax - by - cz = 0.$$

16.7.1 Plane Section of a Sphere

If the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ cuts by the plane $ax + by + cz + d' = 0$ then the section generates a circle in space. In Figure 16.8, O is the centre of the sphere, C is the centre of the circle PQ, OP is the radius of the sphere and CP is the radius of the circle. CO is perpendicular to the plane. It may be noted that the radius of the circle is $CP = \sqrt{OP^2 - OC^2}$. The equation of a circle is the combination of the sphere and the plane, i.e.,

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0, ax + by + cz + d' = 0$$

together represent a circle.

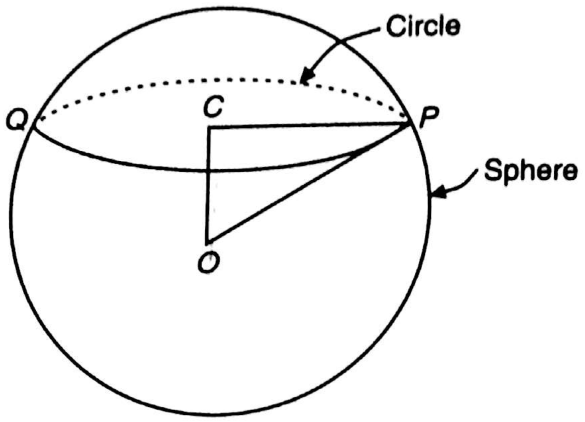

Figure 16.8 Circle in space.

If the circle passes through the centre of the sphere, then it is called *great circle*.

If $S = 0$, $P = 0$ ($S = 0$ is a sphere and $P = 0$ is a plane) represents a circle, then the equation of a sphere passing through this circle is $S + \lambda P = 0$, for a suitable value of λ.

EXAMPLE 16.43 Find the centre and radius of the circle

$$x^2 + y^2 + z^2 - 2x + 6y + 4z - 35 = 0, \quad x - 2y - 2z + 7 = 0.$$

Solution

The centre and the radius of the sphere are respectively $(1, -3, -2)$ and $\sqrt{1 + 9 + 4 + 35} = 7$.

The equation of the line perpendicular to the plane and passing through the centre of the sphere are

$$\frac{x-1}{1} = \frac{y+3}{-2} = \frac{z+2}{-2}.$$

Any point on the line is $(r + 1, -2r - 3, -2r - 2)$. If this point lies on the plane, then

$$1 \cdot (r + 1) + 2 \cdot (2r + 3) + 2\,(2r + 2) + 7 = 0 \text{ or } 9r = -18 \text{ or } r = -2.$$

Therefore, the centre of the circle is $(-1, 1, 2)$.

Now, the distance between C and O i.e. $(1, -3, -2)$ and $(-1, 1, 2)$

$$\sqrt{(1+1)^2 + (-3-1)^2 + (-2-2)^2} = 6.$$

Therefore, the radius of the circle is $\sqrt{7^2 - 6^2} = \sqrt{13}.$

EXAMPLE 16.44 Show that the equation to the sphere through the circle

$$x^2 + y^2 + z^2 = 9, \; 2x + 3y + 4z = 5$$

and the point (1, 2, 3) is $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$.

Solution

Let the equation of the sphere be

$$x^2 + y^2 + z^2 - 9 + \lambda (2x + 3y + 4z - 5) = 0.$$

Since it passes through the point (1, 2, 3),

$$1 + 4 + 9 - 9 + \lambda (2 + 6 + 12 - 5) = 0 \text{ or } \lambda = -\frac{1}{3}.$$

Hence the required equation of the sphere is

$$x^2 + y^2 + z^2 - 9 - \frac{1}{3}(2x + 3y + 4z - 5) = 0$$

or $$3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0.$$

EXAMPLE 16.45 Show that the equation of the sphere for which the circle

$$x^2 + y^2 + z^2 + 7y - 2z + 2 = 0, \; 2x + 3y + 4z - 8 = 0$$

is a great circle is $x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$.

Solution

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda (2x + 3y + 4z - 8) = 0$$

or $$x^2 + y^2 + z^2 + 2\lambda x + (7 + 3\lambda) y + (-2 + 4\lambda) z + (2 - 8\lambda) = 0.$$

Its centre is $\left(-\lambda, -\frac{(7 + 3\lambda)}{2}, (1 - 2\lambda)\right)$.

If the circle is a great circle, this centre lies on the given plane

$\therefore$ $$-2\lambda - \frac{3}{2}(7 + 3\lambda) + 4(1 - 2\lambda) - 8 = 0 \text{ or } \lambda = -1.$$

Hence the required equation of the sphere be

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0.$$

EXAMPLE 16.46 A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the locus of the centroid of the triangle ABC is the sphere $9(x^2 + y^2 + z^2) = 4k^2$.

Solution

Let the coordinates of A, B and C be $(a, 0, 0)$, $(0, b, 0)$ $(0, 0, c)$ and the fourth point be $(0, 0, 0)$. The equation of the sphere through these points is

$$x^2 + y^2 + z^2 - ax - by - cz = 0.$$

Its radius is
$$\sqrt{\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}} = k$$

or
$$a^2 + b^2 + c^2 = 4k^2. \tag{1}$$

Let (α, β, γ) be the coordinates of the centroid of the triangle ABC then

$$\alpha = \frac{a+0+0}{3} = \frac{a}{3}, \beta = \frac{0+b+0}{3} = \frac{b}{3}, \gamma = \frac{0+0+c}{3} = \frac{c}{3}.$$

Putting the values of a, b, c in equation (1), we get

$$9(\alpha^2 + \beta^2 + \gamma^2) = 4k^2$$

Hence the required locus is $9(x^2 + y^2 + z^2) = 4k^2$.

EXAMPLE 16.47 A plane passes through a fixed point (p, q, r) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere $OABC$ is

$$\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2.$$

Solution

Let the equation of the plane passing through A, B, C be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{1}$$

where the coordinates of A, B, C are respectively $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. The equation of the sphere $OABC$ is

$$x^2 + y^2 + z^2 - ax - by - cz = 0.$$

If (α, β, γ) be the centre then,

$$\alpha = \frac{a}{2}, \beta = \frac{b}{2}, \gamma = \frac{c}{2}.$$

Since the plane of equation (1) passes through (p, q, r),

$$\frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1.$$

Putting the values of $a = 2\alpha$, $b = 2\beta$, $c = 2\gamma$, we get

$$\frac{p}{2\alpha} + \frac{q}{2\beta} + \frac{r}{2\gamma} = 1.$$

Hence the required locus is $\dfrac{p}{x} + \dfrac{q}{y} + \dfrac{r}{z} = 2.$

EXAMPLE [illegible] Obtain the equations of the tangent planes to the sphere

$$[illegible]$$

[illegible] through the line [illegible]

Solution

[illegible] of the line can be written as

$$[illegible] \text{ and } [illegible]$$

[illegible] plane through this line is

$$[illegible]$$

$$[illegible] \quad (1)$$

If (1) is a tangent plane then perpendicular from centre ([illegible], 0, 1) of the sphere should be equal to the radius $\sqrt{9+0+1-1}=3$.

therefore

$$\frac{3+(1+3\lambda)-(16+10\lambda)}{\sqrt{1+4\lambda^2+(1+3\lambda)^2}}=3$$

$$9(2+3\lambda)=3\sqrt{(2+6\lambda+13\lambda^2)}$$

$$9(4+12\lambda+9\lambda^2)=2+6\lambda+13\lambda^2$$

$$2\lambda^2+3\lambda+1=0 \text{ or } \lambda=-1, -\frac{1}{2}$$

Hence the equations of the planes are

$$2x+2y-z-2=0 \text{ and } x+2y-2z+14=0.$$

Obtained by putting $\lambda=-1, -\frac{1}{2}$.

EXAMPLE 16.49 Find the value of k for which the plane $x+y+z=k$ touches the sphere $x^2+y^2+z^2-2x-2y-2z-6=0$.

Solution

The centre and radius of the sphere are respectively (1, 1, 1) and $\sqrt{1+1+1+6}=3=r$ (say).

The perpendicular distance d from (1, 1, 1) to the plane is $\frac{1+1+1-k}{\sqrt{3}}$.

If the plane touches the sphere then $d=r$, i.e.

$$\frac{3-k}{\sqrt{3}}=3 \text{ or } k=3-3\sqrt{3}=3(1-\sqrt{3}),$$

which is the required value of k.

16.8 RIGHT CIRCULAR CYLINDER

A *cylinder* is a surface generated by a variable straight line parallel to a fixed straight line called axis and satisfying one or more conditions, i.e. intersecting a given curve or touching a given surface is called a cylinder (see Figure 16.9).

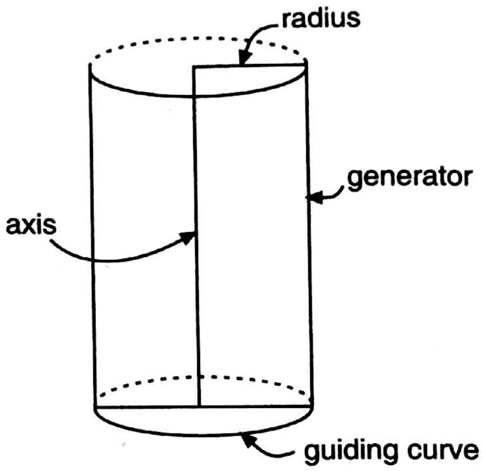

Figure 16.9 A right circular cylinder.

The given curve is called the *guiding curve* or *base* or *directrix* and the variable line is called as *generator* (see Figure 16.9). If the guiding curve is a circle and the fixed line is perpendicular to the plane of the circle through the centre of the circle, then the cylinder is called *right circular cylinder* and the fixed line is called the *axis* of the cylinder. The distance between the axis and any generator is the *radius* of the right circular cylinder and it is equal to the radius of the guiding circle.

EXAMPLE 16.50 Find the equation of the cylinder whose guiding curve is $x^2 + y^2 = 9, z = 1$ and the axis is $\frac{x}{2} = \frac{y}{3} = \frac{z}{-1}$.

Solution

Let (α, β, γ) be any point on the cylinder. The generator through this point is

$$\frac{x-\alpha}{2} = \frac{y-\beta}{3} = \frac{z-\gamma}{-1} \text{ (since axis and generator are parallel).}$$

It cuts the guiding curve at $z = 1$, then

$$\frac{x-\alpha}{2} = \frac{y-\beta}{3} = \frac{1-\gamma}{-1}$$

or

$$x = \alpha + 2\gamma - 2,\ y = \beta + 3\gamma - 3,\ z = 1.$$

This point lies on the curve $x^2 + y^2 = 9$.

$\therefore$

$$(\alpha + 2\gamma - 2)^2 + (\beta + 3\gamma - 3)^2 = 9.$$

Hence the locus of (α, β, γ), i.e. the equation of the cylinder is

$$(x + 2z - 2)^2 + (y + 3z - 3)^2 = 9.$$

EXAMPLE 16.31 Find the equation of the right circular cylinder of radius 3 and which passes through the point (1, 3, 4) and has 1, −2, 3 as its direction ratios.

Solution

The equation of the axis is

$$\frac{x-1}{1}=\frac{y-3}{-2}=\frac{z-4}{3}.$$

Let $P(x, y, z)$ be any point on the cylinder. Then perpendicular distance from P to the axis is

$$\left\{(x-1)^2+(y-3)^2+(z-4)^2-\frac{[1\cdot(x-1)-2(y-3)+3(z-4)]^2}{1^2+(-2)^2+3^2}\right\}^{1/2}$$

which is equal to the radius of the cylinder.

Thus $$(x-1)^2+(y-3)^2+(z-4)^2-\frac{(x-2y+3z-7)^2}{14}=3^2$$

$$\Rightarrow \quad 13x^2+10y^2+5z^2+4xy+12yz-6xz-14x-112y-70z+189=0.$$

This is the required equation of the cylinder.

EXAMPLE 16.32 Find the equation of the right circular cylinder whose guiding curve is $x^2+y^2+z^2=9$, $x-y+z=3$.

Solution

The equation of the sphere is

$$x^2+y^2+z^2=9 \qquad (1)$$

whose centre is (0, 0, 0) and radius is 3.

The given plane is

$$x-y+z=3. \qquad (2)$$

The axes of the cylinder is normal to the plane equation (2) and passes through the centre of the sphere (see Figure 16.10).

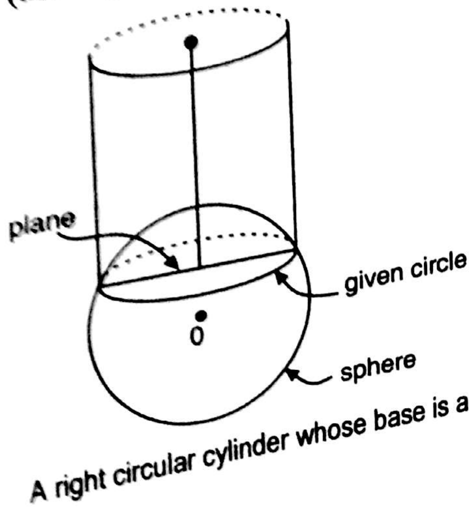

Figure 16.10 A right circular cylinder whose base is a circle in space.

Therefore, the equation of the axes of the cylinder are

$$\frac{x-0}{1}=\frac{y-0}{-1}=\frac{z-0}{1}.$$

Length of the perpendicular from the centre of the sphere on the plane equation (2) is

$$\left|\frac{0+0+0-3}{\sqrt{1+1+1}}\right|=\sqrt{3}.$$

Therefore, the radius of the circle is $\sqrt{3^2-3}=\sqrt{6}$. This is the radius of the cylinder. Let (x, y, z) be any point on the cylinder. Then the perpendicular distance from this point to the axis of the cylinder is

$$\left[x^2+y^2+z^2-\frac{(1\cdot x-1\cdot y+1\cdot z)^2}{1+1+1}\right]^{1/2}$$

which is equal to the radius of the cylinder $\sqrt{6}$.

$$\therefore \qquad x^2+y^2+z^2-\frac{(x-y+z)^2}{3}=6$$

or

$$x^2+y^2+z^2-xz+xy+yz-9=0,$$

which is the required equation of the cylinder.

EXAMPLE 16.53 Find the equation of the cylinder whose generator touches the following two spheres $x^2+y^2+z^2=36$ and $(x-3)^2+(y-1)^2+z^2=36$.

Solution

Since radii of both the spheres are equal, the generators touch both the sphere (see Figure 16.11). So the radius of the cylinder is 6, and equal for both the spheres.

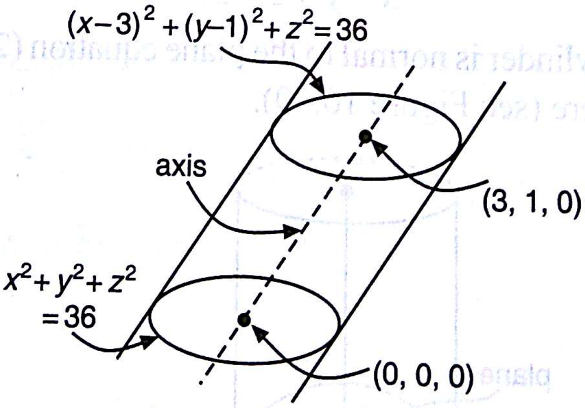

Figure 16.11 Cylinder touches two spheres.

The centres of the spheres are (0, 0, 0) and (3, 1, 0), and the axis of the cylinder passes through the centres. Thus, the equation of the axis is

$$\frac{x}{3}=\frac{y}{1}=\frac{z}{0}.$$

Let ... be any point on the generator. The perpendicular distance from P to the axis is

$$\left[x^2+y^2+z^2-\frac{(3x+y+z\cdot 0)^2}{9+1+0}\right]^{1/2}$$

and this is equal to the radius of the cylinder.

$$\therefore \quad x^2+y^2+z^2-\frac{(3x+y)^2}{10}=36$$

or

$$x^2+9y^2+10z^2-6xy-360=0.$$

This is the required equation of the cylinder.

16.9 RIGHT CIRCULAR CONE

A *cone* is a surface generated by a straight line passing through a fixed point and intersecting a curve or touching a given surface (see Figure 16.12).

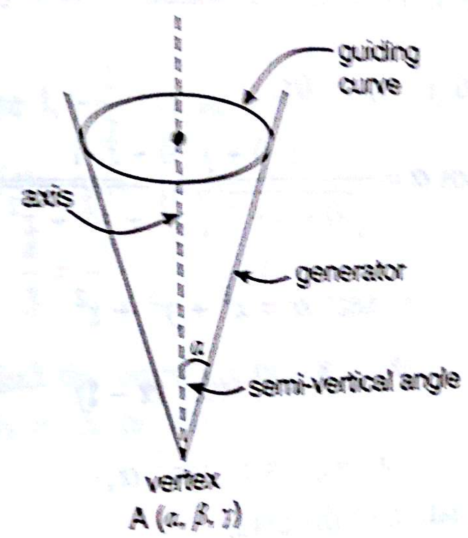

Figure 16.12 A right circular cone.

The fixed point is known as the vertex and the given curve is called the guiding curve or *base* or *directrix*. Any line lying on the cone is called its generator.

A *right circular cone* is the surface generated by a line passing through a fixed point called the *vertex* and which is inclined at a constant angle α to a fixed line through the vertex.

The constant angle is called the *semi-vertical angle* of the cone and the fixed line is called the *axis* of the cone.

It may be noted that the equation of a cone with its vertex as origin is homogeneous in x, y, z and conversely.

EXAMPLE 16.54 Find the equation to a right circular cone whose vertex is origin and axis is z-axis and semi-vertical angle α.

Solution

The equation of the axis is $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$, since the DCs of z-axis are 0, 0, 1 (see Figure 16.13).

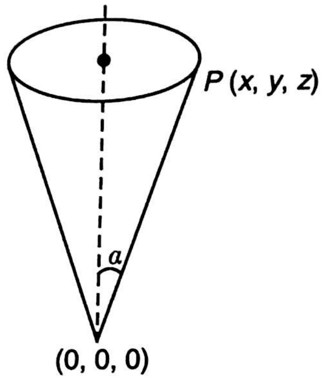

Figure 16.13

Let $P(x, y, z)$ be any point on the cone. The DRs of the generator OP are x, y, z. The semi-vertical angle of the cone is α, which is the angle between OP and the axis.

DRs of OP is $\{x-0, y-0, z-0\}$.

$\therefore$
$$\cos\alpha = \frac{x\cdot 0 + y\cdot 0 + z\cdot 1}{\sqrt{0+0+1}\sqrt{x^2+y^2+z^2}}$$

or
$$z^2\sec^2\alpha = x^2+y^2+z^2$$

or
$$x^2+y^2 = z^2(\sec^2\alpha - 1)$$

or
$$x^2+y^2 = z^2\tan^2\alpha,$$

which is the required equation of the cone.

EXAMPLE 16.55 Find the equations of the lines of intersection of the plane $3x+4y+z=0$ and the cone $15x^2-32y^2-7z^2=0$.

Solution

Let the equation of the generating line in which the plane cuts the cone be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

The line lies on the cone and also on the plane.

$\therefore$
$$3l+4m+n=0$$

and
$$15l^2-32m^2-7n^2=0.$$

$$15l^2 - 32m^2 - 7(3l+4m)^2 = 0$$

or
$$2l^2 + 7ml + 6m^2 = 0$$
$$(2l+3m)(l+2m) = 0$$

or
$$m = -\frac{2}{3}l, -\frac{1}{2}l.$$

When
$$m = -\frac{2}{3}l,$$

then
$$n = -3l - 4m = -\left(3l - \frac{8}{3}l\right) = -\frac{l}{3}$$

Therefore, the DRs are $1, -\frac{2}{3}, -\frac{1}{3}$ or $3, -2, -1$.

Again, when $m = -\frac{1}{2}l$, $n = -(3l - 2l) = -l$.

In this case, the DRs are $1, -\frac{1}{2}, -1$ or $2, -1, -2$.

Hence the equations of the required generators are

$$\frac{x}{3} = \frac{y}{-2} = \frac{z}{-1} \text{ and } \frac{x}{2} = \frac{y}{-1} = \frac{z}{-2}.$$

EXAMPLE 16.56 Find the equation of the right circular cone which passes through the line $2x = 3y = -5z$ and has the line $x = y = z$ as its axis.

Solution

The given line is $2x = 3y = -5z$

or
$$\frac{x}{1/2} = \frac{y}{1/3} = \frac{z}{-1/5}. \quad (1)$$

The equation of the axis is $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$. (2)

Let the angle between the lines represented by equations (1) and (2) be θ, then

$$\cos\theta = \frac{\frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 1 + \left(-\frac{1}{5}\right)\cdot 1}{\sqrt{1+1+1}\sqrt{\frac{1}{4}+\frac{1}{9}+\frac{1}{25}}} = \frac{1}{\sqrt{3}}. \quad (3)$$

Let (x, y, z) be a point on the surface of the cone.

Therefore, the DRs of the generator are $x-0, y-0, z-0$.

$$\therefore \qquad \cos\theta = \frac{x\cdot 1 + y\cdot 1 + z\cdot 1}{\sqrt{3}\sqrt{x^2+y^2+z^2}}. \qquad (4)$$

From equations (3) and (4),

$$\frac{1}{\sqrt{3}} = \frac{x+y+z}{\sqrt{3}\sqrt{(x^2+y^2+z^2)}}$$

or $$(x^2+y^2+z^2) = (x+y+z)^2$$

or $$xy+yz+zx=0$$

which is the required equation of the cone.

EXAMPLE 16.57 Find the equation of the right circular cone whose vertex lies in the yz-plane, axis is the line $\frac{x-2}{2}=\frac{y+1}{-2}=\frac{z+1}{-1}$ and which passes through the point $\left(1, 1, -\frac{1}{2}\right)$.

Solution

Let $(0, \alpha, \beta)$ be the vertex on the yz-plane.

Let

$$\frac{x-0}{l}=\frac{y-\alpha}{m}=\frac{z-\beta}{n} \qquad (1)$$

be any generator of the cone.

Since axis passes through the point $(0, \alpha, \beta)$.

$$\therefore \qquad \frac{0-2}{2}=\frac{\alpha+1}{-2}=\frac{\beta+1}{-1}$$

or $$\alpha = 1 \text{ and } \beta = 0.$$

Therefore, the vertex is (0, 1, 0) and equation (1) becomes

$$\frac{x}{l}=\frac{y-1}{m}=\frac{z-0}{n}. \qquad (2)$$

Direction ratios of the line joining (0, 1, 0) and $\left(1, 1, -\frac{1}{2}\right)$ are $1, 0, -\frac{1}{2}$.

The semi-vertical angle is

$$\cos\theta = \frac{2\cdot 1 + (-2)\cdot 0 + (-1)\cdot\left(-\frac{1}{2}\right)}{\sqrt{4+4+1}\sqrt{1+0+\frac{1}{4}}} = \frac{\sqrt{5}}{3}.$$

Also, the angle between generator and the axis is

$$\cos\theta = \frac{2l - 2m - n}{\sqrt{9}\sqrt{l^2 + m^2 + n^2}} = \frac{2l - 2m - n}{3\sqrt{l^2 + m^2 + n^2}}$$

$\therefore$
$$\frac{\sqrt{5}}{3} = \frac{2l - 2m - n}{3\sqrt{l^2 + m^2 + n^2}}$$

or
$$(2l - 2m - n)^2 = 5\,(l^2 + m^2 + n^2). \qquad (3)$$

Eliminating l, m, n between equation (2) and (3) we get the required cone

$$\{2x - 2\,(y - 1) - z\}^2 = 5\,\{x^2 + (y - 1)^2 + z^2\}$$

or
$$x^2 + y^2 + 4z^2 + 8xy + 4xz - 4yz - 8x - 2y + 4z + 1 = 0.$$

EXERCISES

Short Answer Questions

(Section A)

1. The direction cosines of the straight line passing through the points (1, 2, 4) and (3, 1, 3) are ......$\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$
2. A directed straight line makes angles 60°, 45° with the axes of x and y respectively. The angle it makes with the z-axis is .....60.....
3. The coordinates of the middle point of the line joining the points (2, 3, 4) and (–2, 1, 4) are .(0, 2, 4)..
4. The square of the distance between the points (0, 1, 2) and (1, 0, 1) is ....
5. The equation of the plane which passes through the points (1, –1, 2) and is parallel to the plane $2x + 3y + 4z + 1 = 0$, is ....
6. The equation of the plane passing through the point (4, –3, 5) and containing the y-axis is ....
7. The equation of the plane passing through the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) is ....
8. The distance of the point (1, 2, –3) from the plane $5x - 3y + z + 5 = 0$ is ....
9. The distance between the planes $x + 2y - 3z + 5 = 0$ and $x + 2y - 3z - 7 = 0$ is ....
10. The angle between the planes $x - y + 2z = 9$ and $2x + y + z = 7$ is ....
11. The equation of the plane through the point (1, 2, –3) and normal to the straight line joining the points (–1, 3, 4) and (5, 2, –1) is ....