CHAPTER 1

Basic Concepts of Differential Equations

1.1 Introduction

The differential equation is one of the most important entities in the study of applied sciences. It occurs in different disciplines of engineering as well as in most areas of science. Studies of differential equations by pure and applied mathematicians, theoretical and applied physicists, chemists, engineers and other scientists have established that there are certain definite methods by which many of the differential equations can be solved. Many interesting methods have been developed so for, however, there remain many unsolved equations, some of which are of great importance. Here some methods are discussed to solve some particular types of differential equations.

1.2 Definition and Terminology

Definition 1.2.1 (Differential equation). *An equation containing independent and dependent variables and the derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a* **differential equation**.

The following are some examples of differential equation:

$$m\frac{d^2x}{dt^2} = -kx \tag{1.1}$$

$$5y = x\frac{dy}{dx} + \frac{k}{\frac{dy}{dx}} \tag{1.2}$$

$$\frac{d^2y}{dx^2} = \frac{W}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{1.3}$$

$$r = \frac{[1 + (y')^2]^{\frac{3}{2}}}{y''} \tag{1.4}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.5}$$

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \tag{1.6}$$

Definition 1.2.2 (Ordinary differential equation (ODE)). *A differential equation which involves derivatives with respect to a single independent variable is known as an **ordinary differential equation (ODE)**.*

Equations (1.1) to (1.4) are examples of ordinary differential equations.

Definition 1.2.3 (Partial differential equation (PDE)). *A differential equation which contains two or more independent variables and partial derivatives with respect to them is called a **partial differential equation (PDE)**.*

Equations (1.5) and (1.6) are examples of partial differential equations.

Definition 1.2.4 (Order of differential equation). *The order of a differential equation (ODE or PDE) is the order of the highest derivative in the equation.*

The order of Eqs. (1.2) is one and the order of Eqs. (1.1), (1.3), (1.4), (1.5) and (1.6) is two.

Definition 1.2.5 (Degree of differential equation). *The degree of a differential equation is the degree of the highest-order derivative in the equation, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.*

The degree of Eqs. (1.1), (1.5) and (1.6) is one. The degree of Eqs. (1.2), (1.3) and (1.4) is two as they are written as

$$5y\left(\frac{dy}{dx}\right) = x\left(\frac{dy}{dx}\right)^2 + k$$

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{W}{H}\right)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

and

$$(y''r)^2 = [1 + (y')^2]^3$$

Definition 1.2.6 (Linear and non-linear differential equations). *An nth order differential equation is said to be **linear** in y if it can be written in the form*

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x)$$

*where $a_0(x)$, $a_1(x)$, $\ldots$, $a_n(x)$ and $f(x)$ are functions of x on some interval. The functions $a_0, a_1, \ldots, a_n$ are called **coefficient functions**.*

*A differential equation that is not linear is called **non-linear**, i.e. the equation other than the above form.*

The equations $y'' - 5y' + 6y = 0$, $x^2y'' - 2xy' + 3e^xy = 0$ and $\cos x\ y'' + 2e^xxy' - 3y + x^5 = 0$ are linear, while $yy'' + y' + 3 = 0$, $y'' + \sin y = 5$ and $yy'' + yy' + xy = 5$ are non-linear.

Note 1.2.1 An ordinary differential equation is linear if the following conditions are satisfied:

- The unknown function and its derivatives occur in the first degree only;
- There are no products involving either the unknown function and its derivatives or two or more derivatives;
- There are no transcendental functions involving in the unknown function or any of its derivatives.

1.3 Solution of Differential Equation

A **solution** of a differential equation is a relation between the dependent and independent variables, not involving the derivatives, such that this relation and its derivatives satisfy the given differential equation.

For example, $y = c_1e^x + c_2e^{-x}$ is the general solution of the differential equation $y'' - y = 0$.

A solution of a differential equation of order n will have n independent arbitrary constants. This solution is called a **general solution**. Any solution obtained by assigning particular values to some or all of the arbitrary constants is called a **particular solution**.

The solutions $y = c_1e^x + 5e^{-x}$ and $y = 2e^x - 3e^{-x}$ are particular solutions of $y'' - y = 0$.

A solution of a differential equation that is not obtained from a general solution by assigning particular values to the arbitrary constants is called a **singular solution**.

For example, $y = 0$ is a singular solution of $y' = 2x\sqrt{y}$.

If the solution of a differential equation can be expressed in the form $y = \phi(x)$, where x and y are respectively independent and dependent variables, then the solution is called an **explicit solution**.

Two solutions $y_1(x)$ and $y_2(x)$ of the differential equation $y'' + a_1(x)y' + a_2(x)y = 0$ are said to be **linearly independent** if $c_1y_1 + c_2y_2 = 0$ such that $c_1 = 0$ and $c_2 = 0$.

If c_1 and c_2 both are not zero, then the two solutions y_1 and y_2 are said to be **linearly dependent**. If y_1 and y_2 are two solutions, then their linear combination $c_1y_1 + c_2y_2$ is also a solution of that differential equation and this process is called the **principle of superposition**.

1.4 Initial-value and Boundary-value Problems

It is mentioned that an nth order differential equation has n arbitrary constants. To obtain a particular solution, the n conditions are required on dependent variables and its derivatives. There are two well-known methods for specifying these auxiliary conditions – **initial conditions** and **boundary conditions**.

Definition 1.4.1 (Initial-value problem). *If the auxiliary conditions for a given differential equations are assigned to a single value of x, the conditions are called* ***initial conditions****. The differential equation with its initial conditions is called an* ***initial-value problem*** **(IVP)**.

Definition 1.4.2 (Boundary-value problem). *If the auxiliary conditions for a given differential equations are assigned to two or more values of x, the conditions are called* ***boundary conditions*** *or* ***boundary values****. The differential equation with its boundary conditions is called a* ***boundary-value problem*** **(BVP)**.

The problem $y'' + 3y = x,\ y(0) = 1,\ y'(0) = 2$ is a second order initial-value problem, since the conditions $y(0) = 1,\ y'(0) = 2$ are specified at a single point, $x = 0$. But the problem $y'' - y' + 2y = 0,\ y(0) = -2,\ y(1) = 2$ is a second order boundary-value problem. Here the boundary conditions are specified at two distinct values of x viz., $x = 0$ and $x = 1$.

1.5 Formation of Differential Equation

A differential equation is formed in two different ways. One is by eliminating arbitrary constants from a relation in the variables (independent and dependent) and constants. The other way is by formulation of the geometrical or physical problem as per some mathematical models.

1.5.1 Differential Equation of a Family of Curves

Suppose there is a differential equation containing n arbitrary constants. Then by differentiating it successfully n times, we get n equations containing n arbitrary constants and derivatives. By eliminating n constants, we get a differential equation of order n.

EXAMPLE 1.5.1

(i) Find the differential equation of the family of curves $y = e^x\ (A\cos x + B\sin x)$, where A and B are arbitrary constants.

(ii) Form a differential equation by eliminating the parameters A and B from the equation $y = A\cos x + B\sin x + x\sin x$.

Solution

(i) Here $y = e^x(A\cos x + B\sin x)$.
Differentiating with respect to x, we get

$$\begin{aligned}\frac{dy}{dx} &= e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)\\ &= y - e^x(A\sin x - B\cos x)\end{aligned}$$

Again differentiating, we have

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{dy}{dx} - e^x(A\sin x - B\cos x) - e^x(A\cos x + B\sin x)\\ &= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y\end{aligned}$$

or

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

(ii) Differentiating the given equation twice with respect to x, we get

$$y_1 = -A\sin x + B\cos x + \sin x + x\cos x$$

and

$$\begin{aligned}y_2 &= -A\cos x - B\sin x + \cos x + \cos x - x\sin x\\ &= -(A\cos x + B\sin x + x\sin x) + 2\cos x = -y + 2\cos x\end{aligned}$$

or

$$y_2 + y = 2\cos x$$

EXAMPLE 1.5.2

(i) Construct a differential equation by the elimination of the constants a and b from the equation $ax^2 + by^2 = 1$.

(ii) If $v = A + \frac{B}{r}$, where A, B are arbitrary constants, prove that $\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$.

Solution

(i) Differentiating $ax^2 + by^2 = 1$ with respect to x, $2ax + 2byy_1 = 0$.

or
$$ax + byy_1 = 0 \tag{i}$$

Again differentiating, we get
$$a + by_1^2 + byy_2 = 0$$

or
$$a + b(y_1^2 + yy_2) = 0$$

or
$$a = -b(y_1^2 + yy_2)$$

Putting this value in (i), we get

$$-b(y_1^2 + yy_2)x + byy_1 = 0 \qquad \text{or} \qquad xyy_2 + xy_1^2 - yy_1 = 0$$

(ii) Differentiating $v = A + \frac{B}{r}$ with respect to r twice, we get

$$\frac{dv}{dr} = -\frac{B}{r^2} \qquad \text{and} \qquad \frac{d^2v}{dr^2} = \frac{2B}{r^3}$$

Now
$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = \frac{2B}{r^3} - \frac{2B}{r^3} = 0$$

EXAMPLE 1.5.3

(i) Obtain the differential equation whose general solution is $ax + by + c = 0$, where a, b, c are arbitrary constants.

(ii) Find the differential equation of the curve $y = A\cos(px - B)$, where A and B are the parameters and p is a constant.

Solution

(i) Differentiating $ax + by + c = 0$ with respect to x, we get $a + b\frac{dy}{dx} = 0$, or $\frac{dy}{dx} = -\frac{a}{b}$.

Again differentiating with respect to x, $\frac{d^2y}{dx^2} = 0$.

Thus the required differential equation is $\frac{d^2y}{dx^2} = 0$.

(ii) Differentiating $y = A\cos(px - B)$ twice, we get

$$y_1 = -A\sin(px - B)p$$

and

$$y_2 = -Ap^2\cos(px - B) = -p^2\{A\cos(px - B)\} = -p^2y$$

or

$$y_2 + p^2y = 0$$

EXAMPLE 1.5.4

(i) Find the differential equation of all the straight lines passing through the point $(-2, 3)$.
(ii) Find the differential equation of all circles having constant radius a.

Solution

(i) The equation of straight lines passing through $(-2, 3)$ are

$$y - 3 = m(x + 2) \tag{i}$$

Differentiating, we get $y_1 = m$. Putting the value of m in (i), we have $y - 3 = y_1(x + 2)$.

(ii) Let the equation of the circle of constant radius a be

$$x^2 + y^2 = a^2$$

Differentiating, we get $2x + 2yy_1 = 0$ or $x + yy_1 = 0$, which is the required differential equation.

EXAMPLE 1.5.5

(i) Find the differential equation of all circles touching the x-axis at the origin.
(ii) Find the differential equation of the system of circles having constant radii and centres lying on the x-axis.
(iii) Find the differential equation of all parabolas having their axes parallel to the y-axis.
(iv) Show that the differential equation of all parabolas with the foci at the origin and axis along the x-axis is given by $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$.

Solution

(i) Since the circles touch the x-axis at the origin, so their centre lies on the y-axis. Let the centre be $(0, a)$.

Then the equation is

$$x^2 + (y - a)^2 = a^2 \qquad \text{or} \qquad x^2 + y^2 - 2ay = 0 \tag{i}$$

Differentiating with respect to x, we get

$$2x + 2yy_1 - 2ay_1 = 0 \quad \text{or} \quad a = (x + yy_1)/y_1$$

Putting in (i), we have

$$x^2 + y^2 - 2y(x + yy_1)/y_1 = 0 \quad \text{or} \quad y_1(x^2 + y^2) - 2y(x + yy_1) = 0$$

(ii) Let the centre of the circles be $(a, 0)$ with constant radius r. Then the equations are

$$(x - a)^2 + y^2 = r^2 \tag{i}$$

Differentiating with respect to x, we get

$$2(x - a) + 2yy_1 = 0 \quad \text{or} \quad (x - a) = -yy_1$$

Putting the value of $(x - a)$ in (i), we get

$$y^2y_1^2 + y^2 = r^2 \quad \text{or} \quad y^2(1 + y_1^2) = r^2$$

(iii) The equations of the parabolas whose axes are parallel to y-axis are $y = ax^2 + bx + c$. Differentiating twice, we get $y_1 = 2ax + b$ and $y_2 = 2a$. Again differentiating, we finally have $y_3 = 0$. This is the required differential equation.

(iv) The equation of the parabola whose foci is at the origin and axis is along the x-axis is

$$y = 4a(x + a) \tag{i}$$

where a is the parameter.

Differentiating with respect to x, we get $2yy_1 = 4a$, or $2a = yy_1$.

Putting the value of a in (i), we get

$$y^2 = 2yy_1\left(x + \frac{1}{2}yy_1\right) = 2xyy_1 + (yy_1)^2$$

or $yy_1^2 + 2xy_1 - y = 0$, which is the required differential equation.

EXAMPLE 1.5.6 Find the differential equation of the family of circles touching the x-axis at the origin.

Solution Let the equation of circle passing through the origin be

$$x^2 + y^2 + 2gx + 2fy = 0 \tag{i}$$

If it touches the x-axis then $y = 0$. That is, $\dfrac{dy}{dx} = 0$ at (0,0).

Now, from (i), $2x + 2y\dfrac{dy}{dx} + 2g + 2f\dfrac{dy}{dx} = 0$, or $2g = 0$, or $g = 0$.

Then (i) becomes

$$x^2 + y^2 + 2fy = 0 \tag{ii}$$

Differentiating (ii) with respect to x,

$$2x + 2yy_1 + 2fy_1 = 0, \text{ or } x + yy_1 + fy_1 = 0, \text{ or } f = -(x + yy_1)/y_1$$

Putting this value in (ii), we get

$$x^2 + y^2 - 2y\left(\frac{x + yy_1}{y_1}\right) = 0 \qquad \text{or} \qquad y_1(x^2 + y^2) - 2xy - 2y^2y_1 = 0$$

or

$$y_1(x^2 - y^2) - 2xy = 0 \qquad \text{or} \qquad (x^2 - y^2)\,dy - 2xy\,dx = 0$$

which is the required differential equation.

EXAMPLE 1.5.7 Find the differential equation whose two independent solutions are $\cos x$ and $\sin x$.

Solution The general solution of the required differential equation is

$$y = c_1 \cos x + c_2 \sin x \tag{i}$$

Differentiating (i) with respect to x, we get

$$y' = -c_1 \sin x + c_2 \cos x \tag{ii}$$

Again differentiating (ii) with respect to x we obtain

$$y'' = -c_1 \cos x - c_2 \sin x \tag{iii}$$

Adding (iii) and (i), we get the required differential equation as

$$y'' + y = 0$$

1.5.2 Physical Origins of Differential Equations

Differential equations are used to model different type of problems which appear in engineering, physics, economics and many other fields. These equations are studied in different perspectives. For examples, engineers are tried to model a problem in terms of differential equation, and mathematicians are mainly involved to find out the solution of a differential equation.

EXAMPLE 1.5.8 Suppose a particle of mass m is falling freely under the influence of gravity only. If x is the distance travelled by the particle and if we assume that the upward direction is positive, then by Newton's law the equation is

$$m\frac{d^2x}{dt^2} = -mg \qquad \text{or} \qquad \frac{d^2x}{dt^2} = -g \tag{1.7}$$

The negative sign is used since the weight of the body is a force directed opposite to the positive direction.

EXAMPLE 1.5.9 The human population growth can also be modelled using differential equations. A very common assumption of the Malthusian model is that the rate at which a population grows at a certain time is proportional to the total population at that time. In mathematical form, if $N(t)$ denotes the total population at time t, then this assumption can be expressed as

$$\frac{dN}{dt} \propto N \qquad \text{or} \qquad \frac{dN}{dt} = kN(t) \tag{1.8}$$

where k is the constant of proportionality.

EXAMPLE 1.5.10 We know that Newton's law of cooling states that the rate at which a body cools is proportional to the difference between the temperature of the body and the temperature of the surrounding medium. Let $T(t)$ be the temperature of a body, and T_0 denote the constant temperature of the surrounding medium. Then the rate at which the body cools is $\frac{dT(t)}{dt}$, which is proportional to $T(t) - T_0$ according to Newton's law of cooling. That is

$$\frac{dT(t)}{dt} = \alpha[T(t) - T_0] \tag{1.9}$$

where α is the constant of proportionality.

EXAMPLE 1.5.11 Consider the single-loop series circuit containing an inductor, resistor and capacitor shown in Fig. 1.1. L, C and R denote inductance, capacitance and resistance, and they are all constants.

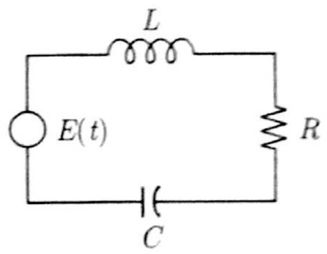

Figure 1.1: Example 1.5.11.

Let $i(t)$ denote the amount of current after the switch is closed, and let the charge on a capacitor at time t be denoted by $q(t)$. Let $E(t)$ denote the impressed voltage on a closed loop. $i(t)$ and $q(t)$ are related by $i = \frac{dq}{dt}$. The voltage drops across an inductor, a resistor and a capacitor are

$$L\frac{di}{dt} = L\frac{d^2q}{dt^2}$$

$$iR = R\frac{dq}{dt}$$

and

$$\frac{1}{C}q$$

respectively. According to Kirchhoffs' second law, the impressed voltage $E(t)$ on a closed loop must be equal to the sum of the voltage drops in the loop. Therefore,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \tag{1.10}$$

This is a second order differential equation.

EXERCISES

Section A Multiple Choice Questions

1. The degree and order of the differential equation $\left[1+\left(\frac{d^3y}{dx^3}\right)^2\right]^{\frac{1}{2}} = 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2$ are

   (a) 2, 3 (b) 2, 2 (c) 3, 3 (d) 3, 2.

2. The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y$ are

   (a) 2, 2 (b) 2, 1 (c) 1, 2 (d) 1, 1. (WBUT 2007)

3. The degree and order of the differential equation

$$\left(\frac{d^2y}{dx^2}+2\right)^{3/2} = x\frac{dy}{dx}$$

   are

   (a) degree $= \frac{3}{2}$, order $= 2$ (b) degree $= 2$, order $= 3$ (c) degree $= 3$, order $= 2$

   (d) degree $= 2$, order $= 1$. (WBUT 2006)

4. The order and degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2 + 5\left(\frac{dy}{dx}\right)^3 + 3y = x^2\log x + \sin^2 x$$

   are

   (a) 2, 2 (b) 2, 3 (c) 3, 2 (d) 3, 3.

5. The order and degree of the differential equation $\frac{d^3y}{dx^3} = \left[y+\left(\frac{d^2y}{dx^2}\right)^3\right]^{1/5}$ are

   (a) 3, 5 (b) 5, 3 (c) 3, 1 (d) 3, $\frac{1}{5}$.

6. The degree and order of the differential equation $\left(\frac{d^2y}{dx^2}\right)^{1/4} + 5\left(\frac{dy}{dx}\right)^{1/3} = 0$ are

   (a) $\frac{1}{4}$, 2 (b) $\frac{1}{3}$, 2 (c) 3, 2 (d) $\frac{1}{4}$, $\frac{1}{3}$.

7. The order of the differential equation whose general solution $x = a\sin(\omega t + b)$, where ω is a constant, is

   (a) 0 (b) 1 (c) 2 (d) 3.

8. The order of the differential equation whose general solution $y = ae^{2x} + be^{-3x} + ce^{2x}$ is

   (a) 0 (b) 1 (c) 2 (d) 3.

9. The general solution of the differential equation $y'' + 9y = 0$ is
(a) $Ae^{3x} + Be^{-3x}$ (b) $(A + Bx)e^{3x}$ (c) $A\cos 3x + B\sin 3x$
(d) $(A + Bx)\sin 3x$. (WBUT 2008)

10. The general solution of the ordinary differential equation $\frac{d^2y}{dx^2} + 4y = 0$ (where A and B are arbitrary constants) is
(a) $Ae^{2x} + Be^{-2x}$ (b) $(A + Bx)e^{2x}$ (c) $A\cos 2x + B\sin 2x$
(d) $(A + Bx)\cos 2x$. (WBUT 2006)

11. The differential equation of the family of curves $y = c_1/x + c_2$ is
(a) $y'' + \frac{2}{x}y' = 0$ (b) $y'' + 2y' = 0$ (c) $y' + \frac{2}{x}y = 0$ (d) $y'' - y = 0$.

12. The differential equation of the family of straight lines $y = ax + b$ is
(a) $y' - y = 0$ (b) $y'' + y = 0$ (c) $y' = 0$ (d) $y'' = 0$.

13. The differential equation of the family of straight lines $y = mx$ is
(a) $y' = 0$ (b) $y'' = 0$ (c) $xy' = y$ (d) $xy = y'$.

14. The differential equation of the family of curves $y = ce^{2x}$ is
(a) $y'' - 2y = 0$ (b) $y' - 2y = 0$ (c) $y' + 2y = 0$ (d) $y'' + y = 0$.

15. The family of curves $y = cx + c^2 + c^3$ is a solution of a differential equation of degree
(a) 1 (b) 2 (c) 3 (d) 4.

16. The differential equation whose two independent solutions e^x and e^{-x} is
(a) $y' - y = 0$ (b) $y'' - y = 0$ (c) $y''' - 2y'' + y' = 0$ (d) $y'' + y = 0$.

17. If the set of independent solutions of a differential equation is $\{e^x, xe^x, x^2e^x\}$, then its order is
(a) 1 (b) 2 (c) 3 (d) 4.

18. The differential equation $yy'' + xy' + 5y = 0$ is
(a) linear (b) non-linear.

19. The differential equation $x^2y'' + x^3y' + 3y = \cos x$ is
(a) linear (b) non-linear.

20. If the rate of change of momentum of a particle of mass m (when particle is moving with velocity v) is proportional to the applied force F, then the corresponding differential equation (for proportional constant k) is
(a) $m\frac{dv}{dt} = kF$ (b) $\frac{d}{dt}(mv) = kF$ (c) $\frac{d^2}{dt^2}(mv) = kF$ (d) $\frac{dF}{dt} = kmv$.

21. $y = e^x - 1$ is a solution of the differential equation
(a) $y'' - 2y + 1 = 0$ (b) $y' = e^x + 2$ (c) $y' - y = 1$ (d) $y' - 2y = 0$.

Section B Review Questions

1. State whether the following differential equations are linear or non-linear and write the order of each equation:

(a) $(1-x^2)y'' - 16xy' + 2y = \sin x$ (b) $x^2\frac{d^3y}{dx^3} - 5\left(\frac{dy}{dx}\right)^2 + 2y = 0$

(c) $yy' + 2y = 3 + x^3$ (d) $\frac{d^2y}{dx^2} + \frac{1}{y} = \cos y$

(e) $\frac{dy}{dx} = \left[1 + \left(\frac{d^2y}{dx^2}\right)^2\right]^{\frac{1}{2}}$ (f) $\frac{d^2r}{dt^2} = \frac{k}{r^2}$

2. Form the differential equations from the following equations:
   (a) $y = e^x(A\cos x + B\sin x)$ (b) $xy = Ae^x + Be^{-x}$
   (c) $c_1y^2 + 4y = 2x^2$ (d) $y^2 = 4c_1(x + c_1)$
   (e) $y = c_1 + c_2\log x$ (f) $r = c_1(1 + \cos\theta)$
   (g) $Ax^2 + By^2 = 1$ (h) $y = a\sin x + b\cos x + x\sin x$
3. Find the differential equation whose three independent solutions are e^x, xe^x, x^2e^x.
4. Find the differential equation of a family of circles passing through the origin.
5. Find the differential equation of a family of circles passing through the origin with their centres on the x-axis.
6. Find the differential equation of a family of circles whose centres are on the y-axis and which touch the x-axis.
7. Find the differential equations of all circles in the XOY plane which have their centres on the x-axis and have given radii.
8. Find the differential equation of all circles having constant radius a.
9. Find the differential equation of all parabolas whose axes are parallel to the y-axis.
10. Find the differential equation of a family of parabolas whose vertices and foci are on the x-axis.
11. Show that the differential equation of a general parabola $a^2x^2 + 2abxy + b^2y^2 + 2gx + 2fy + c = 0$ is

$$\frac{d^2}{dx^2}\left[\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right] = 0$$

12. Show that the differential equation corresponding to the family of curves $x^2 + y^2 + 2c_1x + 2c_2y + c_3 = 0$, where c_1, c_2, c_3 are arbitrary constants, is

$$\frac{d^3y}{dx^3}\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2$$

13. The equation to a system of confocal ellipses is $\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} = 1$, where k is an arbitrary constant. Find the corresponding differential equation.
14. Obtain the differential equation for the velocity v of a body of mass m falling vertically downward through a medium offering a resistance proportional to the square of the instantaneous velocity.

15. Find the differential equation of all rectangular hyperbolas which have the axes of coordinates as asymptotes.

16. If $y = y_1(x)$ and $y = y_2(x)$ are two independent solutions of the following differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

then show that $y = c_1y_1(x) + c_2y_2(x)$, where c_1, c_2 are arbitrary constants, is also a solution of the given differential equation.

17. A spherical rain drop evaporates at a rate proportional to its surface area. Write a differential equation which gives the formula for its volume V as a function of time.

18. The growth rate of a population of bacteria is directly proportional to the population. The number of bacteria in a culture grow from 100 to 400 in 24 hours. Write down the initial value problem which helps to determine the population after 12 hours.

19. If u and v are two particular solutions of $\dfrac{dy}{dx} + Py = Q$, where P and Q are functions of x, then show that $y = k(u - v) + v$, where k is any constants, is a solution of the given differential equation.

Answers

Section A Multiple Choice Questions

1. (a)	2. (b)	3. (c)	4. (a)	5. (a)	6. (c)	7. (c)	8. (d)	9. (c)
10. (c)	11. (a)	12. (d)	13. (c)	14. (b)	15. (a)	16. (b)	17. (c)	18. (b)
19. (a)	20. (b)	21. (c)						

Section B Review Questions

1. (a) linear, 2 (b) linear, 3 (c) non-linear, 1
   (d) non-linear, 2 (e) linear, 2 (f) linear, 2
2. (a) $y'' - 2y' + 2y = 0$
   (b) $xy'' + 2y' - xy = 0$
   (c) $(x^2 - y)y' = xy$
   (d) $y[1 - y'^2] = 2xy'$
   (e) $xy'' + y' = 0$
   (f) $(1 + \cos\theta)\frac{dr}{d\theta} + r\sin\theta = 0$
   (g) $x[yy'' + y'^2] = yy'$
   (h) $y'' + y = 2\cos x$
3. $y''' - 3y'' + 3y' - y = 0$
4. $xy' - y = 0$

5. $2xyy' = y^2 - x^2$
6. $(x^2 - y^2)y' = 2xy$
7. $\pm\sqrt{(a^2 - y^2)} + yy' = 0$, a is the given radius
8. $x + yy' = 0$
9. $y''' = 0$
10. $yy'' + (y')^2 = 0$
13. $(x^2 - y^2) + xy(p - \frac{1}{p}) = a^2 - b^2$, where $p = \frac{dy}{dx}$
14. $\frac{dv}{dt} + k\frac{v^2}{m} = g$
15. $xy' + y = 0$
17. $\frac{dV}{dt} = -cV^{2/3}$, c is constant
18. $\frac{dN}{dt} = kt$, $N(0) = 100$ (assumed)

Chapter 2

Differential Equations of First Order

2.1 Introduction

In the previous chapter, we have seen how a differential equation is constructed from geometrical problems as well as from real-life problems. It is also observed that the form of a differential equation is not simple; it may be linear or non-linear, and the degree and order are also one or more than one. It is not possible to solve a differential equation by a fixed method. The solution procedure of a first order and first degree differential equation is relatively easier than that of other differential equations. In this chapter, the differential equations of first order and first degree are considered.

2.2 Equations of First Order and of First Degree

The first order and first degree differential equation is of the form

$$\frac{dy}{dx} = f(x, y) \tag{2.1}$$

which is sometimes written as

$$M(x, y)dx + N(x, y)dy = 0 \tag{2.2}$$

There is no common method to solve the equation of the form (2.1) or (2.2). However, some special methods are applied to the following types of equations

(a) Equations in which variables are separable
(b) Homogeneous equations
(c) Linear equations
(d) Exact equations

The equations of the types (c) and (d) will only be discussed here. Other types of equations are not within the scope of this book.

2.3 Linear Equations

A differential equation is said to be **linear** if the dependent variable y and its differential coefficients occur only in the first degree and are not multiplied together.

Thus the standard form of a linear differential equation of first order is of the form

$$\frac{dy}{dx} + Py = Q \tag{2.3}$$

where P and Q are functions of x only. This equation is also known as **Leibnitz's linear equation**. This type of equation can be solved by making the left hand side a perfect differential after multiplying a suitable function $I(x)$, called the **integrating factor (I.F.)**.

Multiplying both sides of (2.3) by $I(x)$, so that

$$I(x)\frac{dy}{dx} + I(x)Py = I(x)Q \tag{2.4}$$

Let us assume that

$$I\frac{dy}{dx} + IPy = \frac{d}{dx}(Iy)$$

Then

$$I\frac{dy}{dx} + IPy = I\frac{dy}{dx} + y\frac{dI}{dx}; \text{ this gives } \frac{dI}{dx} = IP$$

That is, $I = e^{\int P dx}$.

Thus Eq. (2.4) becomes

$$e^{\int P dx}\frac{dy}{dx} + e^{\int P dx}Py = e^{\int P dx}Q$$

That is,

$$\frac{d}{dx}(ye^{\int P dx}) = e^{\int P dx}Q$$

After integration it becomes

$$ye^{\int P dx} = \int Qe^{\int P dx}dx + c \qquad \text{or} \qquad y = e^{-\int P dx}\int e^{\int P dx}Q\,dx + ce^{-\int P dx} \tag{2.5}$$

which is the required solution.

Note 2.3.1 The factor $e^{\int P dx}$ is called the **integrating factor**, which makes the LHS an exact differential.

Note 2.3.2 Sometimes a differential equation becomes linear if we take x as the dependent variable and y as the independent variable, i.e. it can be written as

$$\frac{dx}{dy} + P_1 x = Q_1 \tag{2.6}$$

where P_1 and Q_1 are constants or functions of y only. In this case the I.F. $= e^{\int P_1 dy}$ and the solution is given by

$$xe^{\int P_1\,dy} = \int Q_1\,e^{\int P_1\,dy}dy + c$$

Steps to solve linear differential equation:

Step 1: Express the first order and first degree differential equation of the form (2.3).

Step 2: Identify $P(x)$ and compute I.F. $I(x) = e^{\int P\,dx}$.

Step 3: Multiply Eq. (2.3) by $I(x)$.

Step 4: The solution is $yI(x) = \int Q\,I(x)\,dx + c$.

EXAMPLE 2.3.1 Solve $(1-x^2)\dfrac{dy}{dx} + 2xy = x(1-x^2)^{\frac{1}{2}}$.

Solution The given equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x\sqrt{1-x^2}}{1-x^2}$$

Here

$$P = \frac{2x}{1-x^2} \text{ and } Q = \frac{x\sqrt{1-x^2}}{1-x^2}$$

Therefore, the integrating factor is

$$e^{\int P\,dx} = e^{\int \frac{2x}{1-x^2}\,dx} = e^{-\log(1-x^2)} = \frac{1}{1-x^2}$$

Hence, the solution of the given equation is

$$ye^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$$

or

$$y\frac{1}{1-x^2} = \int \frac{x\sqrt{1-x^2}}{1-x^2}\,\frac{1}{1-x^2}\,dx + c$$

We substitute, $z = 1-x^2$ to the RHS. Then $dz = -2xdx$.

$$\therefore \quad \frac{y}{1-x^2} = -\int \frac{1}{2}\,\frac{\sqrt{z}}{z^2}\,dz + c = \frac{1}{\sqrt{z}} + c = \frac{1}{\sqrt{1-x^2}} + c$$

or $y = \sqrt{1-x^2} + c(1-x^2)$, where c is arbitrary constant.

EXAMPLE 2.3.2 Solve $(1+x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1}x}$.

Solution The given equation can be written as

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$$

The integrating factor (I.F.) is $e^{\int \frac{1}{1+x^2}\,dx} = e^{\tan^{-1}x}$.
Multiplying both sides by I.F., we get

$$\frac{d}{dx}\left(ye^{\tan^{-1}x}\right) = \frac{e^{\tan^{-1}x}}{1+x^2}e^{\tan^{-1}x}$$

Integrating

$$ye^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}}{1+x^2}\, e^{\tan^{-1}x}\, dx + c$$

Substituting $e^{\tan^{-1}x} = z$, we get

$$\frac{e^{\tan^{-1}x}}{1+x^2}\, dx = dz$$

Thus

$$ye^{\tan^{-1}x} = \int z\, dz = \frac{z^2}{2} + c \qquad \text{or} \qquad 2ye^{\tan^{-1}x} = e^{2\tan^{-1}x} + 2c$$

where c is arbitrary constant.

EXAMPLE 2.3.3 Solve $(1+x^2)\dfrac{dy}{dx} + 2xy - 4x^2 = 0$.

Solution The given equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$$

$$\text{I.F.} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$$

Multiplying both sides by I.F., we get

$$\frac{d}{dx}\{y(1+x^2)\} = 4x^2$$

Integrating

$$y(1+x^2) = \int 4x^2\, dx + c \qquad \text{or} \qquad y(1+x^2) = \frac{4}{3}x^3 + c$$

EXAMPLE 2.3.4 Solve $y^2 + \left(x - \dfrac{1}{y}\right)\dfrac{dy}{dx} = 0$.

Solution The given equation can be written as

$$y^2\frac{dx}{dy} + x = \frac{1}{y} \qquad \text{or} \qquad \frac{dx}{dy} + \frac{x}{y^2} = \frac{1}{y^3}$$

This is a linear equation in x and I.F. $= e^{\int \frac{1}{y^2}dy} = e^{-\frac{1}{y}}$
Multiplying the above equation by I.F. and integrating, we obtain

$$xe^{-\frac{1}{y}} = \int e^{-\frac{1}{y}}\frac{1}{y^3}\, dy = -\int z e^{-z}\, dz \qquad (\text{where } z = 1/y)$$

$$= -\left[-ze^{-z} + \int e^{-z}\, dz\right] = ze^{-z} + e^{-z} + c$$

$$= e^{-1/y}\left(\frac{1}{y} + 1\right) + c$$

or

$$x = \left(\frac{1}{y} + 1\right) + ce^{1/y}$$

EXAMPLE 2.3.5 Solve $(1+y^2)\,dx = (\tan^{-1} y - x)\,dy$.

Solution The given equation can be written as

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1} y}{1+y^2}$$

This is a linear equation in x and I.F. $= e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$.
Multiplying the given equation by I.F. and integrating, we get

$$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y}\,dy$$

Substituting $\tan^{-1} y = z$. Then $\frac{1}{1+y^2}\,dy = dz$.
Therefore

$$\begin{aligned} xe^{\tan^{-1} y} &= \int z\,e^z\,dz = (z-1)\,e^z + c \\ &= (\tan^{-1} y - 1)e^{\tan^{-1} y} + c \end{aligned}$$

or

$$x = (\tan^{-1} y - 1) + ce^{-\tan^{-1} y}$$

2.3.1 Bernoulli's Equation

The equation of the form

$$\frac{dy}{dx} + Py = Qy^n \tag{2.7}$$

where P and Q are constants or functions of x only and n is constant other than 0 and 1, is called **Bernoulli's equation.**

This equation can be reduced to the linear form by suitable substitution.

Divide (2.7) by y^n, so that $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$.

Let $y^{1-n} = z$, so that $(1-n)y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

Thus Eq. (2.7) reduces to

$$\frac{1}{-n+1}\frac{dz}{dx} + Pz = Q \qquad \text{or} \qquad \frac{dz}{dx} + (1-n)Pz = (1-n)Q$$

which is a linear equation in z and can be solved easily.

EXAMPLE 2.3.6 Solve $x\dfrac{dy}{dx} + y = y^2 \log x$. (WBUT 2008)

Solution This equation can be written as

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{\log x}{x}$$

Substituting $\dfrac{1}{y} = z$. Then $-\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.
Then the above equation reduces to

$$-\frac{dz}{dx} + \frac{1}{x}z = \frac{\log x}{x} \quad \text{or} \quad \frac{dz}{dx} - \frac{1}{x}z = -\frac{\log x}{x}$$

This is a linear equation in z and I.F. $= e^{-\int \frac{1}{x}\,dx} = e^{-\log x} = \dfrac{1}{x}$.
Multiplying the above equation by I.F. and integrating, we obtain

$$z\left(\frac{1}{x}\right) = \int -\frac{\log x}{x^2}\,dx = \frac{1}{x}\log x - \int \frac{1}{x}\frac{1}{x}\,dx \quad \text{(integration by parts)}$$
$$= \frac{1}{x}\log x + \frac{1}{x} + c$$

or

$$z = \log x + 1 + cx = \log(xe) + cx$$

or

$$\frac{1}{y} = cx + \log(ex)$$

or

$$y[cx + \log(ex)] = 1$$

2.3.2 Equations Reducible to Linear Form

If the equation is of the form

$$f'(y)\frac{dy}{dx} + Pf(y) = Q \tag{2.8}$$

where P, Q are functions of x, then it can be reduced to the linear form by substituting $f(y) = z$.

EXAMPLE 2.3.7 Solve $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y$ (WBUT 2007)

Solution This equation can be written as

$$\cos y\frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$$

Substituting $\sin y = z$. Then $\cos y\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

Then the above equation reduces to

$$\frac{dz}{dx} - \frac{1}{1+x}z = (1+x)e^x$$

which is linear equation in z and I.F. $= e^{-\int \frac{1}{1+x}dx} = \frac{1}{1+x}$.
Multiplying the above equation by I.F. and integrating, we get

$$z\frac{1}{1+x} = \int e^x \, dx = e^x + c$$

That is, $\sin y = (1+x)(e^x + c)$ is the required solution.

EXAMPLE 2.3.8 Solve $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

Solution Dividing this equation by $\cos^2 y$

$$\sec^2 y\frac{dy}{dx} + 2x\frac{\sin y \cos y}{\cos^2 y} = x^3$$

or

$$\sec^2 y\,\frac{dy}{dx} + 2x\tan y = x^3$$

Substituting, $\tan y = z$, so that $\sec^2 y\frac{dy}{dx} = \frac{dz}{dx}$.
Then the above equation reduces to

$$\frac{dz}{dx} + 2xz = x^3$$

which is linear equation in z and I.F. $= e^{\int 2x\,dx} = e^{x^2}$.
Multiplying this linear equation by I.F. and integrating, we obtain

$$ze^{x^2} = \int x^3 e^{x^2}\,dx + c = \frac{1}{2}(x^2-1)e^{x^2} + c$$

or

$$z = \frac{1}{2}(x^2-1) + ce^{-x^2} \quad \text{or} \quad \tan y = \frac{1}{2}(x^2-1) + ce^{-x^2}$$

EXAMPLE 2.3.9 Solve $\frac{dy}{dx} + \frac{y\log y}{x} = \frac{y(\log y)^2}{x^2}$. (WBUT 2007)

Solution Dividing by $y(\log y)^2$, the given equation reduces to

$$\frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x(\log y)} = \frac{1}{x^2}$$

Substituting $\frac{1}{\log y} = z$. Then $-\frac{1}{y(\log y)^2}\frac{dy}{dx} = \frac{dz}{dx}$.
Therefore, the above equation becomes

$$\frac{dz}{dx} - \frac{1}{x}z = -\frac{1}{x^2}$$

This is linear equation in z and I.F. $= e^{-\int \frac{1}{x} dx} = \frac{1}{x}$.
Multiplying the above equation by I.F. and integrating we obtain

$$z\frac{1}{x} = \int \frac{1}{x^3}\,dx = -\frac{1}{2}\frac{1}{x^2} + c$$

or

$$\frac{1}{x\log y} = \frac{1}{2x^2} + c \qquad \text{or} \qquad (1 + 2cx^2)\log y = 2x$$

EXAMPLE 2.3.10 Solve $(xy^2 - e^{1/x^3})dx - x^2 y dy = 0$.

Solution The given equation can be written as

$$x^2 y\frac{dy}{dx} = xy^2 - e^{1/x^3} \qquad \text{or} \qquad y\frac{dy}{dx} - \frac{1}{x}y^2 = -\frac{1}{x^2}e^{1/x^3}$$

Substituting $y^2 = z$. Then $2y\dfrac{dy}{dx} = \dfrac{dz}{dx}$. Therefore

$$\frac{dz}{dx} - \frac{2}{x}z = -\frac{2}{x^2}e^{1/x^3}$$

which is a linear equation in z and I.F. $= e^{-\int 2/x\,dx} = \dfrac{1}{x^2}$.
Multiplying by I.F. and integrating

$$z\frac{1}{x^2} = \int -\frac{2}{x^4}e^{1/x^3}\,dx$$

Substituting $1/x^3 = u$ to the RHS.
Then $-\dfrac{3}{x^4}dx = du$. Therefore

$$z\frac{1}{x^2} = \frac{2}{3}\int e^u\,du = \frac{2}{3}e^u + c \qquad \text{or} \qquad 3y^2 = 2x^2e^{1/x^3} + cx^2$$

2.3.3 Applications of First Order Linear Equations

EXAMPLE 2.3.11 Show that the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin is $y = 2(e^x - x - 1)$.

Solution Given that $\dfrac{dy}{dx} = y + 2x$ or $\dfrac{dy}{dx} - y = 2x$. This is a linear equation and I.F. $= e^{-\int dx} = e^{-x}$.
Therefore, its solution is $ye^{-x} = \int 2xe^{-x}\,dx = -2e^{-x}(x+1) + c$
It passes through (0,0), so that $c = 2$.
Hence the required curve is $y = 2(e^x - x - 1)$.

EXAMPLE 2.3.12 Show that the curve in which the portion of the tangent included between the coordinate axes is bisected by the point of contact is a rectangular hyperbola.

Solution Let (x, y) be a point on the curve. The equation of tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

or

$$Y - Xy_1 = y - xy_1$$

or

$$\frac{Y}{y - xy_1} + \frac{X}{-(y - xy_1)/y_1} = 1$$

The points of intersection between the tangent and coordinates axes are respectively $(\frac{y-xy_1}{-y_1}, 0)$ and $(0, y - xy_1)$.

Since (x, y) is the middle point of the line segment joining these two points, therefore, $(y - xy_1)/(-y_1) = 2x$ and $y - xy_1 = 2y$.
Dividing, we get

$$2y/(-y_1) = 2x, \quad \text{or} \quad y = -x\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{y} = -\frac{dx}{x}$$

Integrating, $\log y = -\log x + \log c$, or $xy = c$, which represents a rectangular hyperbola.

EXAMPLE 2.3.13 A particle P moves so that its component velocities parallel to the axes of x and y are respectively $-ky$ and kx, where k is a constant other than zero. Find the path of the particle if it passes through the point (3, 4).

Solution The differential equations of the path are

$$\frac{dx}{dt} = -ky \quad \text{and} \quad \frac{dy}{dt} = kx$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = -\frac{x}{y}$$

That is, $ydy + xdx = 0$. Integrating, we get $x^2 + y^2 = c$.
If it passes through the point (3, 4), therefore $c = 9 + 16 = 25$.
Hence the required solution is $x^2 + y^2 = 25$.

EXAMPLE 2.3.14 Radium decomposes at a rate proportional to the amount present. If 5% of the original amount disappears in 50 years, how much will remain at the end of 100 years?

Solution Let x be the amount of radium at time t. Then the differential equation is $\frac{dx}{dt} = -kx$, k is proportional constant, i.e. $\frac{dx}{x} = -k\,dt$. This gives $\log x = -kt + \log c$,

or

$$x = ce^{-kt} \tag{i}$$

Let A be the amount of radium when $t = 0$.
$\therefore$ $A = c$, (i) becomes $x = Ae^{-kt}$.

Given that 5% of the original amount disappears in 50 years, i.e. 95% of the original amount present in 50 years. Therefore

$$0.95A = Ae^{-50k} \quad \text{or} \quad e^{-50k} = 0.95 \tag{ii}$$

When $t = 100$, then let $x = X$.
$\therefore \quad X = Ae^{-100k}$, or $X = A(0.95)^2 = 0.9025A$.
Hence 90.25% of the original amount will remain at the end of 100 years.

2.4 Exact Differential Equations

A differential equation of the form $Mdx + Ndy = 0$, where M and N are functions of x and y, is said to be **exact** if its left hand member is the exact differential of some function $u(x, y)$. i.e. $du \equiv Mdx + Ndy = 0$ and the solution is $u(x, y) = c$.

Theorem 2.1 *The necessary and sufficient condition for the differential equation* $Mdx + Ndy = 0$ *to be exact is that* $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Proof. The condition is necessary:
Let $Mdx + Ndy = 0$ is exact. Then by definition

$$Mdx + Ndy = du \tag{2.9}$$

where u is a function of x and y.
Again, by Chain rule

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{2.10}$$

Therefore

$$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Equating the coefficients of dx and dy, we get $M = \dfrac{\partial u}{\partial x}$ and $N = \dfrac{\partial u}{\partial y}$. Now

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Hence $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, assuming $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

The condition is sufficient:

Let $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. We have to show that $Mdx + Ndy = 0$ is exact.

Let $P = \int M\,dx$, where y is taken as constant while performing the integration. Then $\dfrac{\partial P}{\partial x} = M$, so that $\dfrac{\partial^2 P}{\partial y \partial x} = \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ by assumption.

That is, $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial P}{\partial y}\right)$.

Integrating both sides with respect to x (taking y as constant), we get

$$N = \frac{\partial P}{\partial y} + \phi(y)$$

where $\phi(y)$ is a function of y only.

Thus

$$\begin{aligned}
M dx + N dy &= \frac{\partial P}{\partial x} dx + \left[\frac{\partial P}{\partial y} + \phi(y)\right] dy \\
&= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy\right) + \phi(y) dy \\
&= dP + d\psi(y) \qquad \text{where } d\psi(y) = \phi(y)dy \\
&= d[P + \psi(y)]
\end{aligned} \tag{2.11}$$

which shows that $Mdx + Ndy = 0$ is exact.

Note 2.4.1 From Eq. (2.11), $Mdx + Ndy = 0$ becomes $d[P + \psi(y)] = 0$. Integrating, $P + \psi(y) = c$, or $P + \int \phi(y) dy = c$.

But

$$P = \underset{y \text{ is constant}}{\int M\,dx} \qquad \text{and} \qquad \phi(y) = \text{terms of } N \text{ not containing } x.$$

Thus the solution of $Mdx + Ndy = 0$ is given by

$$\underset{y \text{ is constant}}{\int M\,dx} \quad + \quad \underset{\text{terms free from } x}{\int N\,dy = c}$$

That is, the solution of an exact differential equation can be obtained by performing the following steps:

Step 1: Integrate M with respect to x, taking y as constant.

Step 2: Integrate the terms of N which do not contain x, with respect to y

Step 3: Add the two expressions obtained in Steps 1 and 2 and equate the result to an arbitrary constant.

EXAMPLE 2.4.1 Solve $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0$.

Solution The given equation can be written as

$$(ax + hy + g)dx + (hx + by + f)dy = 0$$

Here $M = ax + hy + g$ and $N = hx + by + f$

Now, $\dfrac{\partial M}{\partial y} = h = \dfrac{\partial N}{\partial x}$, i.e. the given equation is exact.

$$\underset{y \text{ is constant}}{\int M dx} = \int (ax + hy + g)dx = a\frac{x^2}{2} + hxy + gx$$

$$\underset{\text{terms free from } x}{\int N dy} = \int (by + f)dy = b\frac{y^2}{2} + fy.$$

Hence the required solution is

$$a\frac{x^2}{2} + hxy + gx + b\frac{y^2}{2} + fy = \frac{k}{2}$$

or

$$ax^2 + by^2 + 2hxy + 2gx + 2fy = k, \ k \text{ is arbitrary constant}$$

EXAMPLE 2.4.2 Solve $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$.

Solution Here $M = y^2e^{xy^2} + 4x^3$ and $N = 2xye^{xy^2} - 3y^2$.

Now,

$$\frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 2xye^{xy^2} = e^{xy^2}(2y + 2xy^3)$$

and

$$\frac{\partial N}{\partial x} = y(e^{xy^2} + xe^{xy^2}y^2) = e^{xy^2}(2y + 2xy^3)$$

$\therefore \quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, i.e. the equation is exact.

Its solution is

$$\underset{y \text{ is constant}}{\int (y^2e^{xy^2} + 4x^3)\,dx} + \underset{\text{terms free from } x}{\int (2xye^{xy^2} - 3y^2)\,dy} = c$$

or

$$e^{xy^2} + x^4 + \int (-3y^2)\,dy = c$$

or

$$e^{xy^2} + x^4 - y^3 = c$$

EXAMPLE 2.4.3 Solve $(1+e^{x/y})dx + e^{x/y}(1-x/y)dy = 0$.

Solution Here $M = 1+e^{x/y}$, $N = e^{x/y}(1-x/y)$ and $\dfrac{\partial M}{\partial y} = \dfrac{1}{y}e^{x/y} = \dfrac{\partial N}{\partial x}$.
Therefore, the given equation is exact and the solution is given by

$$\underset{y \text{ is constant}}{\int M dx} + \underset{\text{terms free from } x}{\int N dy} = c$$

$$\underset{y \text{ is constant}}{\int (1+e^{x/y}dx + 0)} = c \quad \text{or} \quad x + ye^{x/y} = c$$

2.4.1 Equations Reducible to Exact Form

If the equation $Mdx + Ndy = 0$ is not exact, then it can be made exact by multiplying a suitable function of x and y. Such a function is called an **integrating factor.** But, there is no general method to find an integrating factor. Here some methods are discussed to find an I.F.

Rule I. If the differential equation $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$, then $\dfrac{1}{Mx+Ny}$ is an integrating factor.

Proof. Since the equation $M\,dx + N\,dy = 0$ is homogenous, so M and N are homogeneous. Let M and N be the homogeneous functions of x and y of degree n. Then by Euler's theorem

$$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = nM \quad \text{and} \quad x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = nN \tag{i}$$

Multiplying the given equation by $\frac{1}{Mx+Ny}$, we get

$$\frac{M}{Mx+Ny}dx + \frac{N}{Mx+Ny}dy = 0 \quad \text{or} \quad M'\,dx + N'\,dy = 0 \tag{ii}$$

where $M' = \dfrac{M}{Mx+Ny}$ and $N' = \dfrac{N}{Mx+Ny}$.

Now

$$\frac{\partial M'}{\partial y} = \frac{\frac{\partial M}{\partial y}(Mx+Ny) - M\left(x\frac{\partial M}{\partial y} + y\frac{\partial N}{\partial y} + N\right)}{(Mx+Ny)^2}$$

$$= \frac{Ny\frac{\partial M}{\partial y} - My\frac{\partial N}{\partial y} - MN}{(Mx+Ny)^2}$$

$$\frac{\partial N'}{\partial x} = \frac{\frac{\partial N}{\partial y}(Mx+Ny) - N\left(M + x\frac{\partial M}{\partial x} + y\frac{\partial N}{\partial x}\right)}{(Mx+Ny)^2}$$

$$= \frac{Mx\frac{\partial N}{\partial x} - Nx\frac{\partial M}{\partial x} - MN}{(Mx+Ny)^2}$$

Equation (ii) will be exact if $\dfrac{\partial M'}{\partial y} = \dfrac{\partial N'}{\partial x}$

i.e. if

$$Ny\frac{\partial M}{\partial y} - My\frac{\partial N}{\partial y} - MN = Mx\frac{\partial N}{\partial x} - Nx\frac{\partial M}{\partial x} - MN$$

or, if

$$N\left(x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y}\right) = M\left(x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y}\right)$$

or, if $N \cdot nM = M \cdot nN$, which is true.

Hence $\dfrac{1}{Mx + Ny}$ is an integrating factor.

EXAMPLE 2.4.4 Solve the equation $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$. (WBUT 2008)

Solution Here $M = x^2y - 2xy^2$ and $N = -(x^3 - 3x^2y)$, both are homogenous and $Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$.

$\therefore$ $\frac{1}{Mx+Ny}$ is an integrating factor.

Thus

$$\frac{x^2y - 2xy^2}{x^2y^2}\,dx - \frac{x^3 - 3x^2y}{x^2y^2}\,dy = 0$$

or

$$\frac{x - 2y}{xy}\,dx - \frac{x - 3y}{y^2}\,dy = 0 \text{ is exact}$$

Now

$$\underbrace{\int (1/y - 2/x)\,dx}_{\text{taken } y \text{ constant}} - \underbrace{\int (x/y^2 - 3/y)\,dy}_{\text{terms free from } x} = 0$$

or

$$\frac{x}{y} - 2\log x + 3\log y = c$$

or

$$\frac{x}{y} + \log(y^3/x^2) = c \qquad c \text{ is arbitrary constant}$$

EXAMPLE 2.4.5 Show that $\{x(x^2 - y^2)\}^{-1}$ is an integrating factor of the differential equation $(x^2 + y^2)dx - 2xydy = 0$ and hence solve the equation.

Solution Here $M = x^2 + y^2$ and $N = -2xy$, both are homogeneous.
Also, $Mx + Ny = x(x^2 - y^2) \neq 0$. Hence $\dfrac{1}{Mx + Ny} = \dfrac{1}{x(x^2 - y^2)}$ is an integrating factor.
Divide the given equation by I.F. we get

$$\frac{x^2 + y^2}{x(x^2 - y^2)}dx - \frac{2xy}{x(x^2 - y^2)}dy = 0$$

or

$$\left(\frac{2x}{x^2 - y^2} - \frac{1}{x}\right)dx - \frac{2y}{x^2 - y^2}dy = 0$$

Integrating

$$\underbrace{\int \left(\frac{2x}{x^2-y^2} - \frac{1}{x}\right)dx}_{y \text{ is constant}} - \underbrace{\int \frac{2y}{x^2-y^2}\,dy}_{\text{terms free from } x} = 0$$

or

$$\log(x^2-y^2) - \log x - 0 = \log c$$

or

$$\log\left(\frac{x^2-y^2}{x}\right) = \log c \qquad \text{or} \qquad x^2 - y^2 = cx$$

which is the required solution.

Rule II. If the equation $Mdx + Ndy = 0$ is not exact but is of the form

$$yf_1(xy)dx + xf_2(xy)dy = 0 \tag{2.12}$$

then $\dfrac{1}{Mx - Ny}$ is an integrating factor provided $Mx - Ny \neq 0$.
The proof is similar to Rule I.

EXAMPLE 2.4.6 Solve $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$.

Solution The given equation is of the form

$$yf_1(xy)dx + xf_2(xy)dy = 0$$

Now

$$Mx - Ny = xy(xy + 2x^2y^2) - xy(xy - x^2y^2) = 3x^3y^3 \neq 0$$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying the given equation by I.F.

$$\frac{y(xy + 2x^2y^2)}{3x^3y^3}dx + \frac{x(xy - x^2y^2)}{3x^3y^3}dy = 0$$

or

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

Now

$$\frac{\partial M}{\partial y} = -\frac{1}{3x^2y^2} = \frac{\partial N}{\partial x}$$

Hence the given equation becomes exact and its solution is

$$\underbrace{\int \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx}_{y \text{ is constant}} + \underbrace{\int \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy}_{\text{terms free from } x} = 0$$

or

$$-\frac{1}{3xy} + \frac{2}{3}\log x + \int -\frac{1}{3y}\,dy = c$$

or

$$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c \qquad \text{or} \qquad \log\left(\frac{x^2}{y}\right) - \frac{1}{xy} = 3c$$

EXAMPLE 2.4.7 Solve $(x^3y^3 + x^2y^2 + xy + 1)ydx + (x^3y^3 - x^2y^2 - xy + 1)xdy = 0$.

Solution Here the equation is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$.
Now, $Mx - Ny = 2x^2y^2(xy + 1)$.

$$\text{I.F. is } \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2(xy+1)}$$

Multiplying the given equation by I.F.

$$\frac{(xy+1)(x^2y^2+1)}{2x^2y^2(xy+1)}ydx + \frac{(xy+1)(x^2y^2-2xy+1)}{2x^2y^2(xy+1)}xdy = 0$$

or

$$\frac{x^2y^2+1}{2x^2y^2}ydx + \frac{x^2y^2-2xy+1}{2x^2y^2}xdy = 0$$

or

$$\frac{1}{2}\left(y + \frac{1}{x^2y}\right)dx + \frac{1}{2}\left(x - \frac{2}{y} + \frac{1}{xy^2}\right)dy = 0$$

Also

$$\frac{\partial M}{\partial y} = \frac{1}{2}\left(1 - \frac{1}{x^2y^2}\right) = \frac{\partial N}{\partial x}$$

Thus the equation is now exact and its solution is

$$\underbrace{\int \frac{1}{2}\left(y + \frac{1}{x^2y}\right)dx}_{y \text{ is constant}} + \underbrace{\frac{1}{2}\int -\frac{2}{y}dy}_{\text{terms free from } x} = 0$$

or

$$\frac{1}{2}\left(xy - \frac{1}{xy}\right) - \log y = c$$

or

$$xy - \frac{1}{xy} - \log y^2 = 2c$$

Rule III. If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone, say $f(x)$, then the integrating factor of $Mdx + Ndy = 0$ is $e^{\int f(x)\,dx}$.

EXAMPLE 2.4.8 Solve $(x^3 + xy^4)dx + 2y^3dy = 0$.

Solution Here $M = x^3 + xy^4$ and $N = 2y^3$.

$$\text{Now, } \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2y^3}(4xy^3 - 0) = 2x = f(x), \text{ say.}$$

Then

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

Multiplying the given equation by I.F., we get $e^{x^2}(x^3 + xy^4)dx + 2e^{x^2}y^3dy = 0$.

Now,

$$\frac{\partial}{\partial y}\left[e^{x^2}(x^3+xy^4)\right] = 4xe^{x^2}y^3 = \frac{\partial}{\partial x}(2e^{x^2}y^3)$$

Thus the above equation is exact and its solution is

$$\underset{y \text{ is constant}}{\int e^{x^2}(x^3+xy^4)dx} + \underset{\text{terms free from } x}{\int 2e^{x^2}y^3 dy} = 0$$

That is

$$\frac{1}{2}\int e^z(z+y^4)dz + 0 = c \quad \text{where } z = x^2$$

or

$$\frac{1}{2}\{(z-1)+y^4\}e^z = c \qquad \text{or} \qquad (x^2-1+y^4)e^{x^2} = 2c$$

EXAMPLE 2.4.9 Solve $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$.

Solution Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2y$.

Now, $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{-x^2y}(2xy+2xy) = -\dfrac{4}{x} = f(x)$, say.

$\therefore$ I.F. $= e^{\int f(x)\,dx} = e^{-\int (4/x)\,dx} = \dfrac{1}{x^4}$

Multiplying the given equation by I.F.

$$\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{y}{x^2}dy = 0 \qquad \text{which is obviously exact}$$

Its solution is

$$\underset{y \text{ is constant}}{\int \left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx} - \underset{\text{terms free from } x}{\int \frac{y}{x^2}dx} = 0$$

or

$$-\frac{y^2}{2x^2} + \frac{1}{3}\int e^{1/x^3}\left(-\frac{3}{x^4}\right)dx + 0 = c \qquad \text{or} \qquad -\frac{y^2}{2x^2} + \frac{1}{3}e^{1/x^3} = c$$

which is the required solution.

Rule IV. If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is a function of y only, say $g(y)$, then an integrating factor of $M\,dx + N\,dy = 0$ is $e^{\int g(y)\,dy}$.

EXAMPLE 2.4.10 Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$.

Solution Here $M = 3x^2y^4 + 2xy$ and $N = 2x^3y^3 - x^2$.

Now

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x, \quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

and

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{xy(3xy^3 + 2)} = \frac{-2x(3xy^3 + 2)}{xy(3xy^3 + 2)} = -\frac{2}{y} = g(y)$$

Therefore

$$\text{I.F.} = e^{\int -\frac{2}{y}\,dy} = e^{\log\frac{1}{y^2}} = \frac{1}{y^2}$$

Multiplying the given equation by I.F., we get

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0 \qquad \text{which is obviously exact}$$

Integrating

$$\underbrace{\int \left(3x^2y^2 + \frac{2x}{y}\right)dx}_{y\text{ is constant}} + \underbrace{\int \left(2x^3y - \frac{x^2}{y^2}\right)dy}_{\text{terms free from } x} = 0$$

or

$$x^3y^2 + \frac{x^2}{y} + 0 = c$$

or

$$x^3y^3 + x^2 = cy$$

EXAMPLE 2.4.11 Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$.

Solution Here $M = y^4 + 2y$, $N = xy^3 + 2y^4 - 4x$ and also

$$\frac{\partial M}{\partial y} = 4y^3 + 2, \qquad \frac{\partial N}{\partial x} = y^3 - 4$$

Now

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y^4 + 2y}(y^3 - 4 - 4y^3 - 2) = -\frac{3}{y} = g(y) \qquad \text{say}$$

$$\therefore \qquad \text{I.F.} = e^{\int -\frac{3}{y}\,dy} = e^{-3\log y} = \frac{1}{y^3}$$

Multiplying the given equation by I.F. we get

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$$

which is obviously exact.

Integrating

$$\underset{y \text{ is constant}}{\int \left(y + \frac{2}{y^2}\right) dx} + \underset{\text{terms free from } x}{\int \left(x + 2y - \frac{4x}{y^3}\right) dy} = 0$$

or

$$\left(y + \frac{2}{y^2}\right) x + \int 2y\, dy = c \qquad \text{or} \qquad \left(y + \frac{2}{y^2}\right) x + y^2 = c$$

Rule V. If the equation $M\,dx + N\,dy = 0$ is of the form

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0 \tag{2.13}$$

where a, b, c, d, m, n, p and q are constants and $x^h y^k$ is an integrating factor, where h and k are constants and can be determined by applying that condition, then after multiplication by $x^h y^k$ the given equation becomes exact.

EXAMPLE 2.4.12 Solve $(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$.

Solution The given equation can be written as

$$(2y + 6xy^2)dx + (3x + 8x^2 y)dy = 0$$

Assume that the I.F. of this equation is of the form $x^h y^k$. Multiplying the above equation by $x^h y^k$, we get

$$(2x^h y^{k+1} + 6x^{h+1} y^{k+2})dx + (3x^{h+1} y^k + 8x^{h+2} y^{k+1})dy = 0$$

Let

$$M = 2x^h y^{k+1} + 6x^{h+1} y^{k+2},\ N = 3x^{h+1} y^k + 8x^{h+2} y^{k+1}$$

and

$$\frac{\partial M}{\partial y} = 2(k+1)x^h y^k + 6(k+2)x^{h+1} y^{k+1},\ \frac{\partial N}{\partial x} = 3(h+1)x^h y^k + 8(h+2)x^{h+1} y^{k+1}$$

If the equation $M\,dx + N\,dy = 0$ is exact, then we must have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, i.e.

$$2(k+1)x^h y^k + 6(k+2)x^{h+1} y^{k+1} = 3(h+1)x^h y^k + 8(h+2)x^{h+1} y^{k+1}$$

Equating the coefficients of $x^h y^k$ and $x^{h+1} y^{k+1}$, we get

$$2(k+1) = 3(h+1) \text{ and } 6(k+2) = 8(h+2)$$

The solution of these equations is $h = 1$, $k = 2$.

Thus the integrating factor is $x^h y^k = xy^2$.

Multiplying the given equation by xy^2, we get

$$(2xy^3 + 6x^2 y^4)dx + (3x^2 y^2 + 8x^3 y^3)dy = 0$$

and obviously it is an exact equation. Integrating

$$\underset{y \text{ is constant}}{\int (2xy^3 + 6x^2y^4)dx} + \underset{\text{terms free from } x}{\int (3x^2y^2 + 8x^3y^3)dy} = 0$$

or

$$x^2y^3 + 2x^3y^4 + 0 = c$$

or

$$x^2y^3 + 2x^3y^4 = c$$

EXAMPLE 2.4.13 Solve $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$.

Solution Multiplying both sides by x^hy^k, we get

$$(x^hy^{k+1}\,dx + 3x^{h+1}y^k\,dy) + x^{h+2}y^{k+1}(2x^hy^{k+1}\,dx - x^{h+1}y^k\,dy) = 0$$

or

$$(x^hy^{k+1} + 2x^{2h+2}y^{2k+2})\,dx + (3x^{h+1}y^k - x^{2h+3}y^{2k+1})\,dy = 0$$

Let

$$M = x^hy^{k+1} + 2x^{2h+2}y^{2k+2}$$

and

$$N = 3x^{h+1}y^k - x^{2h+3}y^{2k+1}$$

Then

$$\frac{\partial M}{\partial y} = x^h(k+1)y^k + 2x^{2h+2}(2k+2)y^{2k+1}$$

and

$$\frac{\partial N}{\partial x} = 3(h+1)x^hy^k - (2h+3)x^{2h+2}y^{2k+1}$$

If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then $k+1 = 3(h+1)$ and $2(2k+2) = -(2h+3)$.

Solving, we get $h = -11/7, k = -19/7$.
Therefore, (i) becomes

$$(x^{-11/7}y^{-12/7} + 2x^{3/7}y^{-5/7})\,dx - (x^{10/7}y^{-12/7} - 3x^{-4/7}y^{-19/7})\,dy = 0$$

Integrating, we get

$$\int (x^{-11/7}y^{-12/7} + 2x^{3/7}y^{-5/7})\,dx - \int (x^{10/7}y^{-12/7} - 3x^{-4/7}y^{-19/7})\,dy = 0$$

or

$$y^{-12/7}\frac{x^{-4/7}}{-4/7} + 2\frac{x^{10/7}}{10/7}y^{-5/7} = c \text{ or } -\frac{7}{4}y^{-12/7}x^{-4/7} + \frac{7}{5}x^{10/7}y^{-5/7} = c$$

EXERCISES

Section A Multiple Choice Questions

Linear Equations

1. An integrating factor of $\frac{dy}{dt}+y=1$ is (WBUT 2008)
   (a) e^t (b) e/t (c) et (d) t/e.

2. Integrating factor of $\frac{dy}{dx}+2xy=5$ is
   (a) e^x (b) e^{x^2} (c) e^{x^3} (d) e^{2x}.

3. The I.F. of $(1+x)\frac{dy}{dx}-y=e^{3x}(1+x)^2$ is
   (a) $1+x$ (b) $1+x^2$ (c) $(1+x)^{-1}$ (d) $(1+x)^{-2}$.

4. The I.F. of $y(\log y)dx+(x-\log y)dy=0$ is
   (a) y (b) $\log x$ (c) e^{xy} (d) $\log y$.

5. The I.F. of the differential equation $(x+2y^3)\frac{dy}{dx}=y$ is
   (a) y (b) $1/y$ (c) x (d) $1/x$.

6. The I.F. of the differential equation $\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$ is
   (a) x (b) y (c) $1/x$ (d) $1/y$.

7. The I.F. of $\frac{dy}{dx}+2y=3$ is
   (a) e^x (b) e^{x^2} (c) e^{-2x} (d) e^{2x}.

8. The value of $I(x)$ for which the LHS of the equation $I(x)\frac{dy}{dx}+I(x)y\tan x=x^2$ becomes exact is
   (a) $\cos x$ (b) $\sin x$ (c) $\operatorname{cosec} x$ (d) $\sec x$.

9. The solution of $\frac{dy}{dx}+y=1$ is
   (a) $y=1+ce^{-x}$ (b) $\log y+x=cx$ (c) $y=e^x+c$ (d) $ye^x=c$.

10. The solution of $\frac{dy}{dx}+\frac{y}{x}=\frac{1}{x}$ is
    (a) $xy=c$ (b) $xy=x+c$ (c) $\log(xy)=c$ (d) $e^{xy}=x+c$.

Exact Equations

11. The solution of the differential equation $\frac{dy}{dx}+\frac{ax+hy+g}{hx+by+f}=0$ is
    (a) $ax^2+by^2+c=0$ (b) $ax^2+2hxy+by^2=0$ (c) $ax^2+hxy+gx=0$
    (d) $ax^2+2hxy+by^2+2gx+2fy+c=0$.

12. The equation $(k+e^{x/y})dx+e^{x/y}(1-x/y)dy=0$ is exact if k is
    (a) 0 (b) 1 (c) 2 (d) 3.

13. The I.F. of the equation $a(x\,dy+2y\,dx)=xy\,dy$ is
    (a) $1/(xy)$ (b) $1/x$ (c) $1/y$ (d) xy.

14. The I.F. of $x^2y\,dx - (x^3 + y^3)dy = 0$ is
 (a) $-1/y$ (b) $-1/y^2$ (c) $-1/y^3$ (d) $-1/y^4$.
15. The I.F. of $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ is
 (a) $1/(xy)$ (b) xy (c) x^2y^2 (d) $1/x^2y^2$.
16. The I.F. of $(x^2 + y^2)\,dx - 2xy\,dy = 0$ is
 (a) $1/x^2$ (b) $1/x$ (c) $1/y^2$ (d) $1/y$.
17. The I.F. of $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$ is
 (a) y (b) $1/y$ (c) $1/x$ (d) x.
18. If x^hy^k is the integrating factor of $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$, then the values of h and k are respectively
 (a) 2, 1 (b) $-1/2$, $-1/2$ (c) $-5/2$, $-1/2$ (d) -2, -1.
19. The solution of $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$ is
 (a) $x^2y + xy - x\tan y + \tan y = c$ (b) $x^2y + xy + x^3 - x^2\tan^2 y = c$
 (c) $x^2 + y^2 = cxy\tan x\tan y$ (d) $xy^2 + x^2y + \tan x + \tan y = c$.
20. The solution of $(x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0$ is
 (a) $x^4 + y^4 = c$ (b) $x^4 + y^4 + 6x^2y^2 = c$ (c) $x^4 + 3x^2y^2 = c$
 (d) $x^3 + 3xy^2 + 3x^2y^2 + y^4 = c$.
21. The differential equation $(xe^{kxy} + 2y)\dfrac{dy}{dx} + ye^{xy} = 0$ is exact for c equal to
 (a) 0 (b) 1 (c) 2 (d) 3.

Section B Review Questions

Linear Equations

Solve the following differential equations:

1. $(1 + x)\frac{dy}{dx} - xy = 1$
2. $y\,dx - x\,dy + \log x\,dx = 0$
3. $(1 - x^2)\frac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$
4. $\cos^2 x\frac{dy}{dx} + y = \tan x$
5. $\frac{dy}{dx} - \frac{2y}{x+1} = (x + 1)^3$
6. $\frac{dy}{dx} + 2xy = 2e^{-x^2}$
7. $(x + 2y^3)\frac{dy}{dx} = y$
8. $\sin 2x\left(\frac{dy}{dx}\right) - y = \tan x$
9. $\frac{dx}{dy} + \frac{xy}{1-y^2} - y\sqrt{x} = 0$
10. $x\log x\frac{dy}{dx} + y = \log x^2$

11. $ye^y dx = (y^3 + 2xe^y)dy$

12. $e^{-y}\sec^2 y dy = dx + x\,dy$

13. $xy(1 + xy^2)\frac{dy}{dx} = 1$

14. $\frac{dy}{dx} + x\sin^2 y = x^3\cos^2 y$

15. $x\frac{dy}{dx} + y = x^3y^6$

16. $y(2xy + e^x)dx - e^x dy = 0$

17. $\frac{dy}{dx} = y\tan x - y^2\sec x$

18. $\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$

19. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1 + x)e^x\sec y$

20. $\tan y\frac{dy}{dx} + \tan x = \cos y\cos^2 x$

21. $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$

22. $(x + y + 1)\frac{dy}{dx} = 1$

23. $\frac{dy}{dx} - y\cot x = \operatorname{cosec} x$

24. $\frac{dy}{dx} + \frac{n}{x}y = \frac{a}{x^n}$

25. $(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$

26. $(1 + y^2)dx = (\tan^{-1} y - x)dy$

27. $(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$

28. $(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x$

29. $x\log x\frac{dy}{dx} + y = 2\log x$

30. $dx + x\,dy = e^{-y}\sec^2 y dy$

31. $\frac{dy}{dx} + \frac{y}{(1-x^2)^{3/2}} = \frac{x+\sqrt{(1-x^2)}}{(1-x^2)^2}$

32. $\frac{dy}{dx} + \frac{3x^2}{1+x^3}y = \frac{\sin^2 x}{1+x^3}$

33. $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

34. $\frac{dy}{dx} + \frac{1}{x}\tan y = \frac{1}{x^2}\tan y\sin y$

35. $\frac{dy}{dx} + y\cos x = y^n\sin 2x$

36. $\sin y\frac{dy}{dx} = \cos y(1 - x\cos y)$

37. Solve $\dfrac{dy}{dx} + 2y\tan x = \sin x$, given that $y = 0$ when $x = \pi/3$.

38. Solve $x\dfrac{dy}{dx} = -y + e^x$, given that $y = 2$ when $x = 1$.

Exact equations

Solve the following differential equations:

39. $x\,dy + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$

40. $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy = 0$

41. $(\sin x \cos y + e^{2x})dx + (\cos x \sin y + \tan y)dy = 0$

42. $\{y(1 + 1/x) + \cos y\}dx + (x + \log x - x \sin y)dy = 0$

43. $(1 + 2xy\cos x^2 - 2xy)dx + (\sin x^2 - x^2)dy = 0$

44. $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

45. $(x^2 + y^2 - a^2)x\,dx + (x^2 - y^2 - b^2)y\,dy = 0$

46. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

47. $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$

48. $ye^{xy}\,dx + (xe^{xy} + 2y)\,dy = 0$

49. $y\,dx - x\,dy + (1 + x^2)\,dx + x^2\sin y\,dy = 0$

50. $y(axy + e^x)\,dx - e^x\,dy = 0$

51. $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$

52. $y\sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$

53. $(\sec x \tan x \tan y - e^x)\,dx + \sec x \sec^2 y\,dy = 0$

54. $y(2xy + e^x)\,dx = e^x\,dy$

55. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

56. $x^2y\,dx - (x^3 + y^3)\,dy = 0$

57. $(x^3 + xy^4)\,dx + 2y^3\,dy = 0$

58. $(xy + 2x^2y^2)y\,dx + (xy - x^2y^2)x\,dy = 0$

59. $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$

60. $(1+xy)y\,dx + (1-xy)x\,dy = 0$

61. $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$

62. $(x^4y^4 + x^2y^2 + xy)y\,dx + (x^4y^4 - x^2y^2 + xy)x\,dy = 0$

63. $(x^2+y^2)\,dx - 2xy\,dy = 0$

64. $(y + \frac{1}{3}y^3 + \frac{1}{2}x^2)\,dx + \frac{1}{4}(x + xy^2)\,dy = 0$

65. $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

66. $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$

67. $(xy^3 + y)\,dx + 2(xy^2 + x + y^4)\,dy = 0$

68. $(2x^2y - 3y^4)\,dx + (3x^3 + 2xy^3)\,dy = 0$

69. $(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$

70. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$

71. $(3x + 2y^2)y\,dx + 2x(2x + 3y^2)\,dy = 0$

72. Prove that $\frac{1}{(x+y+1)^4}$ is an integrating factor of

$$(2xy - y^2 - y)dx + (2xy - x^2 - x)dy = 0$$

and hence solve it.

73. If $x^\alpha y^\beta$ is an integrating factor of $2y\,dx - 3xy^2\,dx - x\,dy = 0$, find α and β and use it to solve the equation.

74. Show that a constant k can be found so that $(x+y)^k$ is an integrating factor of $(4x^2 + 2xy + 6y)\,dx + (2x^2 + 9y + 3x)\,dy = 0$ and hence solve the equation.

Answers

Section A Multiple Choice Questions

1. (a)	2. (b)	3. (c)	4. (d)	5. (b)	6. (c)	7. (d)	8. (d)	9. (a)
10. (b)	11. (d)	12. (b)	13. (a)	14. (d)	15. (d)	16. (a)	17. (a)	18. (c)
19. (a)	20. (b)	21. (b)						

Section B Review Questions

1. $y(1+x) = x + ce^x$
2. $y = cx - (1 + \log x)$
3. $y = \sqrt{1-x^2} + c(1-x^2)$
4. $y = ce^{-\tan x} + \tan x - 1$

5. $y = \frac{1}{2}(x+1)^4 + c(x+1)^2$

6. $ye^{x^2} = 2x + c$

7. $x = y^3 + cy$

8. $y = (c+1)\sqrt{\tan x}$

10. $y = \log x + \frac{c}{\log x}$

11. $x/y^2 = c - e^{-y}$

12. $xe^y = c + \tan y$

13. $1/x = (2 - y^2) + ce^{-y^2/2}$

14. $2\tan y = (x^2 - 1) + 2ce^{-x^2}$

15. $(5/2 + cx^2)x^3y^5 = 1$

16. $e^x = y(c - x^2)$

17. $\sec x = y(c + \tan x)$

18. $y^2 = x^2 + cx - 1$

19. $\sin y = (1+x)(e^x + c)$

20. $\cos y = \cos x(\sin x + c)$

21. $\sqrt{x} = \sqrt{y}(\log\sqrt{y} + c)$

22. $x = ce^y - (y+2)$

23. $y\operatorname{cosec} x = -\cot x + c$

24. $yx^n = ax + c$

25. $2ye^{\tan^{-1}x} = e^{2\tan^{-1}x} + 2c$

26. $x = \tan^{-1}y - 1 + ce^{\tan^{-1}y}$

27. $xe^{\tan^{-1}y} = \tan^{-1}y + c$

28. $y = \tan^{-1}x - 1 + ce^{\tan^{-1}x}$

29. $y\log x = (\log x)^2 + c$

30. $xe^y = \tan y + c$

31. $y = z + ce^{-z}$, where $z = \frac{x}{\sqrt{(1-x^2)}}$

32. $y(1+x^3) = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

33. $y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = x + \frac{2}{3}x^{3/2} + c$

34. $2x = \sin y(1 - 2cx^2)$

35. $\frac{1}{y^{n-1}} = 2\sin x - \frac{2}{1-n} + ce^{(n-1)\sin x}$

36. $\sec y\, e^{-x} = e^{-x}(1+x) + c$

37. $y = \cos x - 2\cos^2 x$

38. $y = \frac{1}{x}e^x + \frac{2-e}{x}$

39. $x^2 - 2\tan^{-1}(x/y) + y^2 = 2c$

40. $x + ye^{x/y} = c$

41. $\frac{1}{2}e^{2x} - \cos x\cos y + \log\sec y = c$

42. $(x + \log x)y + x\cos y = c$

43. $x + y\sin x^2 - x^2 y = c$

44. $y\sin x + (\sin y + y)x = c$

45. $x^4 + 2x^2y^2 - y^4 - 2a^2x^2 - 2b^2y^2 = c$

46. $x^3 - 6x^2y - 6xy^2 + y^3 = c$

47. $\frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$

48. $e^{xy} + y^2 = c$

49. $x^2 - y - 1 - x\cos y = cx$

50. $\frac{ax^2}{2} + e^{x/y} = c$

51. $x = cye^{\frac{1}{xy}}$

52. $3y\cos 2x + 6y + 2y^3 = c$

53. $e^x = \sec x\tan y + c$

54. $e^x + x^2y = cy$

55. $x/y - 2\log x + 3\log y = c$

56. $x^3 = 3y^3(\log y - c)$

57. $(x^2 - 1 + y^4)e^{x^2} = 2c$

58. $2\log x - \log y = \frac{1}{xy} + c$

59. $xy + \log(\frac{x}{y}) - \frac{1}{xy} = c$

60. $\log(x/y) - \frac{1}{xy} = c$

61. $x = cy\cos xy$

62. $\frac{1}{2}x^2y^2 - \frac{1}{xy} + \log(x/y) = c$

63. $x - y^2/x = c$

64. $x^6 + 3x^4y + x^4y^3 = c$

65. $x^3y^3 + x^2 = cy$

66. $e^{6y}\left(\frac{x^2y^2}{2} - \frac{x^3}{3} + \frac{y^2}{6} - \frac{y}{18} + \frac{1}{108}\right) = c$

67. $3x^2y^4 + 6xy^3 + 2y^6 = c$

68. $5x^{-36/13}y^{24/13} - 12x^{-10/13}y^{-15/13} = c$

69. $x^2y^3 + 2x^3y^4 = c$

70. $4x^{1/2}y^{1/2} + \frac{2}{3}x^{-3/2}y^{3/2} = c$

71. $x^2y^4(x + y^2) = c$

72. $xy = c(x + y + 1)^3$

73. $\alpha = 1,\ \beta = -2,\ x^2 - x^3y = cy$

74. $k = 1,\ x^4 + 2x^3y + x^2y^2 + 3x^2y + 6xy^2 + 3y^3 = c$

Chapter 3

Differential Equations of First Order and Higher Degree

3.1 Introduction

The most general form of a differential equation of the first order and of higher degree, say, nth degree is

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1} p + P_n = 0 \tag{3.1}$$

where $p = \frac{dy}{dx}$ and $P_1, P_2, \ldots, P_n$ are functions of x and y. This equation can also be written as $F(x, y, p) = 0$. But, this equation cannot be solved in its general form. In this chapter, the following three special types of such equations are discussed:

- Equations solvable for p
- Equations solvable for y
- Equations solvable for x

3.2 Equations Solvable for p

Suppose the differential Eq. (3.1) can be solved for p and is of the form

$$(p - f_1)(p - f_2) \cdots (p - f_n) = 0 \tag{3.2}$$

where each f is a function of x and y.

Equating each factor to zero and we get n equations of first order and first degree. Let their solutions be

$$\phi_1(x, y, c_1) = 0, \phi_2(x, y, c_2) = 0, \ldots, \phi_n(x, y, c_n) = 0$$

All possible solutions of Eq. (3.2) will then be included in the relation

$$\phi_1(x, y, c_1)\phi_2(x, y, c_2) \cdots \phi_n(x, y, c_n) = 0 \tag{3.3}$$

It may be observed that the general solution contains n arbitrary constants, whereas we expect only one constant, as the equation is only of the first order.

See that the generality will still be maintained if all the constants $c_1, c_2, \ldots, c_n$ be made the same c. Hence the complete solution of (3.2) is

$$\phi_1(x,y,c)\phi_2(x,y,c)\cdots\phi_n(x,y,c) = 0 \tag{3.4}$$

where c is arbitrary constant.

EXAMPLE 3.2.1 Solve $p^2 + px + py + xy = 0$.

Solution The equation may be written as $(p+x)(p+y) = 0$. Now, $p+x = 0$ gives $\dfrac{dy}{dx} + x = 0$ or $2y = -x^2 + c_1$ and $p + y = 0$ gives $\dfrac{dy}{dx} + y = 0$ or $\log y + x = c_2$.

Thus the complete solution is $(2y + x^2 + c)(x + \log y + c) = 0$.

EXAMPLE 3.2.2 Solve $p^2 - p(e^x + e^{-x}) + 1 = 0$.

Solution The given equation can be written as $(p - e^x)(p - e^{-x}) = 0$. Now, $p - e^x = 0$ gives $\dfrac{dy}{dx} = e^x$, or $y = e^x + c_1$ and $p - e^{-x} = 0$ gives $\dfrac{dy}{dx} = e^{-x}$, or $y = -e^{-x} + c_2$.

Hence the general solution is $(y - e^x - c)(y + e^{-x} - c) = 0$, where c is arbitrary constant.

EXAMPLE 3.2.3 Solve $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$, i.e. $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$. (WBUT 2006)

Solution This equation can be written as

$$p^2 + p\left(\frac{x}{y} - \frac{y}{x}\right) - 1 = 0$$

or

$$\left(p + \frac{y}{x}\right)\left(p - \frac{x}{y}\right) = 0$$

When $p + \dfrac{y}{x} = 0$, then $\dfrac{dy}{dx} + \dfrac{y}{x} = 0$, or $\dfrac{dy}{y} + \dfrac{dx}{x} = 0$, or $\log(xy) = \log c_1$, or $xy = c_1$.

When $p - \dfrac{x}{y} = 0$, then $\dfrac{dy}{dx} - \dfrac{x}{y} = 0$, or $ydy - xdx = 0$ or, $y^2 - x^2 = c_2$.

Hence the solution is $(xy - c)(x^2 - y^2 - c) = 0$, where c is arbitrary constant.

EXAMPLE 3.2.4 Solve $x^2p^2 - 2xyp + 2y^2 - x^2 = 0$.

Solution From the given equation

$$p = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2} = \frac{y \pm \sqrt{(x^2 - y^2)}}{x}$$

This is a homogeneous equation. Substituting $y = vx$. Then $p = \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Thus $v + x\dfrac{dv}{dx} = \dfrac{v \pm \sqrt{1 - v^2}}{1}$, or $x\dfrac{dv}{dx} = \pm\sqrt{1 - v^2}$ or, $\dfrac{dv}{\sqrt{1 - v^2}} = \pm\dfrac{dx}{x}$.

Integrating, we get $\sin^{-1} v = \pm \log x \pm \log c = \pm \log cx$.

That is, $\sin^{-1}(y/x) = \pm \log(cx)$ is the general solution.

3.3 Equations Solvable for y

If the equation is solvable for y then it can be written as

$$y = f(x, p) \tag{3.5}$$

Differentiating with respect to x, gives

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right) \tag{3.6}$$

which is a differential equation in two variables x and p. Let its solution be

$$F(x, p, c) = 0 \tag{3.7}$$

By eliminating p between Eqs. (3.5) and (3.7), we get the required solution. If the elimination of p is not possible, then we solve (3.5) and (3.7) for x and y and obtain

$$x = F_1(p, c) \qquad y = F_2(p, c) \tag{3.8}$$

as required solution, where p is taken as parameter.

EXAMPLE 3.3.1 Solve $y = 2px + p^4x^2$.

Solution Differentiating with respect to x,

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 4p^3x^2\frac{dp}{dx} + 2p^4x$$

or

$$\frac{dp}{dx}(2x + 4p^3x^2) + (p + 2p^4x) = 0$$

or

$$\left(p + 2x\frac{dp}{dx}\right)(1 + 2p^3x) = 0$$

Now, $p + 2x\dfrac{dp}{dx} = 0$, or $2\dfrac{dp}{p} + \dfrac{dx}{x} = 0$.

Integrating, $\log p^2 + \log x = \log c$, or $p^2x = c$, or $p^2 = c/x$.

To eliminate p, substituting $p^2 = c/x$ to the given equation, i.e. we obtain $y = 2px + c^2$, or $(y - c^2)^2 = 4p^2x^2 = 4x^2\dfrac{c}{x}$, or $(y - c^2)^2 = 4cx$, which is the required solution.

EXAMPLE 3.3.2 Solve $y = 2px + p^2$.

Solution Given

$$y = 2px + p^2 \tag{i}$$

Differentiating with respect to x,

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or

$$p = 2p + (2x + 2p)\frac{dp}{dx}$$

or

$$2(x + p)\frac{dp}{dx} + p = 0$$

or

$$\frac{dx}{dp} + \frac{2x}{p} = -2$$

which is a linear differential equation in x and I.F. is $e^{\int (2/p)dp} = p^2$.

Multiplying the above equation by p^2, we get $\frac{d}{dp}(xp^2) = -2p^2$.

Integrating, $$xp^2 = -\frac{2}{3}p^3 + \frac{c}{3}, \text{ or } x = -\frac{2}{3}p + \frac{c}{3p^2} \qquad \text{(ii)}$$

The Eqs. (i) and (ii) taken together, with parameter p, constitute the general solution.

EXAMPLE 3.3.3 Solve $y = \sin p - p\cos p$.

Solution Differentiating with respect to x, we get

$$p = \frac{dy}{dx} = (\cos p - \cos p + p\sin p)\frac{dp}{dx} \qquad \text{or} \qquad \sin p \, dp = dx$$

Integrating, $\cos p = x + c$. Therefore, $\sin p = \sqrt{1 - (x + c)^2}$.

Eliminating p, we get $y = \sqrt{1-(x+c)^2} - (x + c)\cos^{-1}(x + c)$, which is the required solution.

EXAMPLE 3.3.4 Solve $y = (p + p^2)x + p^{-1}$. (WBUT 2003)

Solution Differentiating with respect to x, we get

$$p = (p + p^2) + (1 + 2p)x\frac{dp}{dx} - \frac{1}{p^2}\frac{dp}{dx}$$

or

$$p^2 + \left\{(1 + 2p)x - \frac{1}{p^2}\right\}\frac{dp}{dx} = 0$$

or

$$\frac{dx}{dp} + \frac{1 + 2p}{p^2}x = \frac{1}{p^4}$$

This is a linear equation in p and its I.F. is

$$e^{\int (1+2p)/p^2 dp} = p^2 e^{-1/p}$$

Multiplying the above equation by I.F. and integrating, we get

$$xp^2e^{-1/p} = \int \frac{1}{p^2}e^{-1/p}dp = -\int e^{-1/p}d(1/p) = e^{-1/p} + c$$

or

$$x = \frac{1}{p^2} + \frac{c}{p^2}e^{1/p} \qquad \text{(i)}$$

Thus the given equation and (i) gives the required solution, where p is taken as parameter.

3.4 Equations Solvable for x

When the equation is solvable for x, then it can be expressed as

$$x = f(y, p) \tag{3.9}$$

Then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dp} = \phi\left(y, p, \frac{dp}{dy}\right)$$

It becomes an equation containing two variables y and p. Let its solution be

$$F(y, p, c) = 0 \tag{3.10}$$

By eliminating p between (3.9) and (3.10), the required solution is obtained. If the elimination is not possible, (3.9) and (3.10) may be expressed in terms of p, and p may be considered the parameter.

EXAMPLE 3.4.1 Solve $(1 + p^2)y - 2px = 0$.

Solution The given equation can be written as $2x = yp + y/p$.

Differentiating with respect to y, we get

$$\frac{2}{p} = p + y\frac{dp}{dx} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}$$

or

$$\frac{1}{p} - p = \left(y - \frac{y}{p^2}\right)\frac{dp}{dy}$$

or

$$\left(\frac{1-p^2}{p}\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

Thus

$$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

or

$$\frac{dy}{y} + \frac{dp}{p} = 0$$

Integrating, $\log y + \log p = \log c$, or $yp = c$, or $p = c/y$.

Substituting $p = c/y$ to the given equation, we get $2x = c + y^2/c$, or $y^2 + c^2 - 2cx = 0$, which is the required solution.

EXAMPLE 3.4.2 Solve $x - y\left(\frac{dy}{dx}\right) = 3\left(\frac{dy}{dx}\right)^2$.

Solution The given equation is

$$x - yp = 3p^2 \qquad \text{or} \qquad x = yp + 3p^2 \tag{i}$$

Differentiating with respect to y,

$$\frac{dx}{dy} = p + y\frac{dp}{dy} + 6p\frac{dp}{dy}$$

or

$$\frac{1}{p} = p + (y + 6p)\frac{dp}{dy}$$

or

$$\left(\frac{1}{p} - p\right)\frac{dy}{dp} = y + 6p$$

or

$$\frac{dy}{dp} - y\frac{p}{1-p^2} = \frac{6p^2}{1-p^2} \tag{ii}$$

This is a linear equation and its I.F.

$$= e^{-\int \frac{p}{1-p^2}dp} = e^{\frac{1}{2}\int \frac{-2p}{1-p^2}dp} = e^{\frac{1}{2}\log(1-p^2)} = \sqrt{1-p^2}$$

Multiplying (ii) by $\sqrt{1-p^2}$ and integrating

$$y\sqrt{1-p^2} = \int \frac{6p^2}{\sqrt{1-p^2}}dp = -6\int \sqrt{1-p^2}dp + 6\int \frac{1}{\sqrt{1-p^2}}dp$$

$$= -6\left[\frac{p}{2}\sqrt{1-p^2} + \frac{1}{2}\sin^{-1}p\right] + 6\sin^{-1}p + c$$

$$= -3p\sqrt{1-p^2} + 3\sin^{-1}p + c$$

Therefore

$$y = -3p + \frac{3}{\sqrt{1-p^2}}\sin^{-1}p + \frac{c}{\sqrt{1-p^2}} \tag{iii}$$

Equations (i) and (iii) give the required solution where p is treated as parameter.

3.5 Clairaut's Equation

A differential equation of the form

$$y = px + f(p) \tag{3.11}$$

is called **Clairaut's equation**. The general solution of this equation can be determined easily.

Differentiating with respect to x, we have

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or

$$\frac{dp}{dx}\left\{x + f'(p)\right\} = 0$$

Therefore, either $\dfrac{dp}{dx} = 0$ or $x + f'(p) = 0$.

The equation $\dfrac{dp}{dx} = 0$ gives $p = c$.

Substituting $p = c$ in (3.11), we get

if $\dfrac{dp}{dx} = 0$, then $p = c$
and singular sol is $y = cx + f(c)$

$$y = cx + f(c) \tag{3.12}$$

as the general solution.

Thus, the solution of Clairaut's equation is obtained on replacing p by c.

By eliminating p between $x + f'(p) = 0$ and (3.11) we get an equation without any constant. This is also a solution of the given equation and this solution is known as a **singular solution** of (3.11). This solution gives the envelope of the family of the straight line (3.12).

EXAMPLE 3.5.1 Find the general solution of $p = \cos(y - px)$. (WBUT 2007)

Solution The given equation can be written as $y = px + \cos^{-1} p$, which is the Clairaut's equation.

Differentiating with respect to x, we get

$$p = p + x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

or

$$\left(x - \frac{1}{\sqrt{1-p^2}}\right)\frac{dp}{dx} = 0$$

Now, $\dfrac{dp}{dx} = 0$ gives $p = c$.

Therefore, the general solutions is $y = cx + \cos^{-1} c$, where c is arbitrary constant.

EXAMPLE 3.5.2 Solve the differential equation $y = px + \sqrt{a^2p^2 + b^2}$. Also, find its singular solution. (WBUT 2005)

Solution The given equation is $y = px + \sqrt{a^2p^2 + b^2}$. (i)

Differentiating (i) with respect to x, we get

$$p = p + x\frac{dp}{dx} + \frac{1}{\sqrt{a^2p^2 + b^2}}a^2p\frac{dp}{dx} \quad \text{or} \quad \frac{dp}{dx}\left(x + \frac{a^2p}{\sqrt{a^2p^2 + b^2}}\right) = 0$$

Either $\dfrac{dp}{dx} = 0$ or $x + \dfrac{a^2p}{\sqrt{a^2p^2 + b^2}} = 0$.

When $\dfrac{dp}{dx} = 0$, then $p = c$, a constant.

Then the general solution is

$$y = cx + \sqrt{a^2c^2 + b^2} \qquad \text{[obtain by replacing } p \text{ by } c \text{ in (i)]}$$

When $x + \dfrac{a^2p}{\sqrt{a^2p^2 + b^2}} = 0$, then

$$x = -\frac{a^2p}{\sqrt{a^2p^2 + b^2}} \qquad \text{(ii)}$$

From (i)

$$y = -\frac{a^2p^2}{\sqrt{a^2p^2 + b^2}} + \sqrt{a^2p^2 + b^2} = \frac{b^2}{\sqrt{a^2p^2 + b^2}} \qquad \text{(iii)}$$

Squaring and adding (ii) and (iii)

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \frac{(-ap)^2 + b^2}{a^2p^2 + b^2} = 1 \qquad \text{or} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is the required singular solution.

EXAMPLE 3.5.3 Solve $y = 2px + y^2p^3$.

Solution Multiplying the given equation by y. Then $y^2 = 2pxy + y^3p^3$.

Substituting $y^2 = v$ and $2y\dfrac{dy}{dx} = \dfrac{dv}{dx}$, i.e. $2yp = P$, where $P = \dfrac{dv}{dx}$.

Therefore, the above equation reduces to $v = xP + \dfrac{1}{8}P^3$.

This is Clairaut's form and its general solution is

$$v = xc + \frac{c^3}{8} \qquad \text{or} \qquad y^2 = xc + \frac{c^3}{8}$$

EXAMPLE 3.5.4 Find the general solution of $(px - y)(x - py) = 2p$.

Solution Putting $x^2 = u$ and $y^2 = v$ and differentiating, we get $2x\,dx = du$ and $2y\,dy = dv$

$\therefore \quad \dfrac{y\,dy}{x\,dx} = \dfrac{dv}{du}$ or, $\dfrac{yp}{x} = q$ where $q = \dfrac{dv}{du}$. Therefore, $p = \dfrac{xq}{y}$.

Putting this value in $(px - y)(x - py) = 2p$, we get

$$\left(\frac{x^2q}{y} - y\right)(x - xq) = \frac{2xq}{y}$$

or

$$(x^2q - y^2)x(1 - q) = 2xq$$

or

$$(uq - v)(1 - q) = 2q$$

or

$$v = uq - \frac{2q}{1 - q}$$

This is the Clairaut's form.

The general solution is

$$v = uc - \frac{2c}{1-c} \qquad \text{or} \qquad y^2 = cx^2 - \frac{2c}{1-c}$$

where c is any arbitrary constant.

EXAMPLE 3.5.5 Reduce the differential equation

$$(x^2 + y^2 - 1)\frac{dy}{dx} = xy\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}$$

into Clairaut's form and find its general solution.

Solution The given equation is

$$(x^2 + y^2 - 1)p = xy(1 + p^2) \tag{i}$$

Substituting $x^2 = u$ and $y^2 = v$, $2xdx = du$ and $2ydy = dv$.

Now, $\dfrac{dy}{dx}\dfrac{y}{x} = \dfrac{dv}{du}$, or $p\dfrac{y}{x} = q$, where $p = \dfrac{dy}{dx}$ and $q = \dfrac{dv}{du}$.

Using $p = xq/y$, (i) becomes

$$(u + v - 1)\frac{xq}{y} = xy\left(1 + \frac{x^2q^2}{y^2}\right)$$

or

$$(u + v - 1)q = y^2 + x^2q^2 = v + uq^2$$

or

$$v(q-1) + uq(1-q) = q$$

or

$$v = uq + \frac{q}{q-1}$$

This is Clairaut's form and its general solution is

$$v = uc + \frac{c}{c-1} \qquad \text{or} \qquad y^2 = cx^2 + \frac{c}{c-1}$$

where c is arbitrary constant.

EXAMPLE 3.5.6 Reduce the equation $x^2\left(\dfrac{dy}{dx}\right)^2 + y(2x+y)\dfrac{dy}{dx} + y^2 = 0$ to Clairaut's form by the substitution $y = u, xy = v$. Hence find its general solution.

Solution Here $y = u$ and $xy = v$, or $x = v/u$.

$\therefore dy = du$ and $dx = \dfrac{udv - vdu}{u^2}$.

Thus, $\dfrac{dy}{dx} = \dfrac{u^2du}{udv - vdu} = \dfrac{u}{p - v/u}$ where $p = \dfrac{dv}{du}$.

Using this value, the given equation reduces to

$$\frac{v^2}{u^2}\frac{u^2}{(p-v/u)^2}+v\left(2\frac{v}{u}+u\right)\frac{u}{p-v/u}+u^2=0$$

After simplification this equation reduces to $v=up+p^2$.

This equation is clearly of Clairaut's form and its general solution is $v=uc+c^2$ or, $xy=cy+c^2$, c is arbitrary constant.

EXAMPLE 3.5.7 Reduce the equation $(px^2+y^2)(px+y)=(p+1)^2$ where $p=\frac{dy}{dx}$, to Clairaut's form by the substitution $xy=u$ and $x+y=v$ and hence find its singular solutions.

Solution Putting $u=xy, v=x+y$.

Differentiating with respect to x, we get $\dfrac{du}{dx}=y+xp$, $\dfrac{dv}{dx}=1+p$.

Now

$$\frac{dv}{du}=\frac{1+p}{y+xp}$$

or

$$q=\frac{1+p}{y+xp}$$

where $q=\frac{dv}{du}$, or

$$q(y+xp)=1+p \quad \text{or} \quad p=\frac{1-yq}{xq-1}$$

Now

$$px^2+y^2=\frac{x^2-x^2yq}{xq-1}+y^2=\frac{x^2-y^2-xy(x-y)q}{xq-1}$$

$$=\frac{(x-y)\{(x+y)-xyq\}}{xq-1}=\frac{(x-y)(v-uq)}{xq-1}$$

Putting this value into the given equation, we get

$$\frac{(x-y)(v-uq)}{xq-1}\cdot\frac{1+p}{q}=(1+p)^2$$

or

$$\frac{(x-y)(v-uq)}{xq-1}=q(1+p)=q\left\{1+\frac{1-yq}{xq-1}\right\}=\frac{q^2(x-y)}{xq-1}$$

or $v-uq=q^2$, or $v=uq+q^2$, which is Clairaut's form and its general solution is $v=cu+c^2$, or $x+y=cxy+c^2$, where c is arbitrary constant.

To find singular solution, differentiate $v=uq+q^2$ with respect to u.

$$\frac{dv}{du}=q+u\frac{dq}{du}+2q\frac{dq}{du}$$

or

$$\frac{dq}{du}(u+2q)=0$$

Now, $u+2q=0$ gives $q=-u/2$.

Substituting $q = -u/2$ to $v = uq + q^2$, we get the singular solution as $v = -u^2/2 + u^2/4$, or $4v + u^2 = 0$, or $4(x + y) + (xy)^2 = 0$.

(A singular solution may also be obtained by equating the c-discriminant to the zero of the general solution)

EXERCISES

Section A Multiple Choice Questions

1. The general solution of the equation $p^2 - 5p + 6 = 0$ is
   (a) $(y - 2x + c)(y - 3x + c) = 0$ (b) $(y - 2x + c_1)(y - 3x + c_2) = 0$
   (c) $(y - 2x + c)^2(y - 3x + c)^2 = 0$ (d) $p = 2, p = 3$.
2. The general solution of the equation $\left(\frac{dy}{dx}\right)^2 - (x + y)\frac{dy}{dx} + xy = 0$ is
   (a) $(y - x^2)(y - \log x + c) = 0$ (b) $(2y - x^2 + c)(\log y - x + c) = 0$
   (c) $(2y - x^2)(y - \log x) = 0$ (d) $(2y - x^2 + c_1)(\log y - x - c_2) = 0$.
3. The general solution of $x^2p^2 + xyp - 6y^2 = 0$ is
   (a) $(y - x^2)(xy - c) = 0$ (b) $(y - cx^2)(yx^3 - c) = 0$
   (c) $(y - x^2 - c)(yx^2 - c) = 0$ (d) $(y - x^2)(xy^2 - c) = 0$.
4. The general solution of $y = x^4p^2 - px$ is
   (a) $(x^2 - c)(y^2 - c) = 0$ (b) $(xy - c)(x^2 + y^2 - c) = 0$
   (c) $xy + c = c^2x$ (d) none of these.
5. The general solution of $x = y + a\log p$ is
   (a) $y = c - a\log(1 - p)$ (b) $(y - x)(y - x^2 - c) = 0$
   (c) $x = c - a\log(1 - p)$ (d) $x^2 + y^2 = cp$.
6. The general solution of $(xp - y)(p - xy) = 0$ is
   (a) $(xy - y^2 - c)(y - x^2y - c) = 0$ (b) $(xy - c)(x^2 - y^2 - c) = 0$
   (c) $(y - cx)(2y - x^2 - c) = 0$ (d) $(y - xc)(2\log y - x^2 - c) = 0$.
7. The general solution of $y = x\frac{dy}{dx} + \sqrt{a\left(\frac{dy}{dx}\right)^2 + 5}$ is
   (a) $y = xc + ac$ (b) $y = x^2 + ay^2$
   (c) $y^2 = x^2 + \sqrt{ac^2 + 5}$ (d) $y = xc + \sqrt{ac^2 + 5}$.
8. The general solution of $(y - px)(p - 1) = p$ is
   (a) $(y - cx)(c - 1) = c$ (b) $(y^2 - x^2 - c)(y - x) = y$
   (c) $(y - cx)(c - 1) = cx$ (d) $(y^2 - x^2 - c_1)(y^2 - x^2 - c_2) = 0$.
9. The general solution of $p = \log(y - xp)$ is
   (a) $y = \log(y - cx)$ (b) $c = \log(y - cx)$
   (c) $e^c = y^2 - x^2$ (d) $y = \log(y^2 - x^2 - c)$.
10. The general solution of $(y - px)^2 = a^2p^2 + b^2$ is
    (a) $(y^2 - cx)(x^2 - cy) = 0$ (b) $(y - x)^2 = a^2c^2 + b^2$
    (c) $(y - cx)^2 = a^2c^2 + b^2$ (d) none of these.
11. The singular solution of $y = px + e^p$ is
    (a) $y = x(\log x + 1)$ (b) $y = cx + e^c$
    (c) $y = x^2 + e^c$ (d) $y^2 = x(\log x + y)$.

12. The solution of the equation $(p - xy)(p - x^2)(p - y^2) = 0$ is
(a) $(xy - c)(x^2 - c)(y^2 - c) = 0$ (b) $(xy - x^2)(y - cx)(x - cy) = 0$
(c) $(x + 1/y + c)(y - x^3/3 - c)(\log y - x^2/2 - c) = 0$ (d) none of these.

Section B Review Questions

Find the general solution of the following differential equations:

1. $xy\left(\frac{dy}{dx}\right)^2 + (x^2 + y^2)\frac{dy}{dx} + xy = 0$
2. $y(y - 2)p^2 - (y - 2x + xy)p + x = 0$
3. $p^2 + 2py \cot x = y^2$
4. $yp^2 + (x - y)p - x = 0$
5. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$
6. $p(p + y) = x(x + y)$
7. $y = x[p + \sqrt{1 + p^2}]$
8. $(p + y + x)(xp + y + x)(p + 2x) = 0$
9. $xy^2(p^2 + 2) = 2py^3 + x^3$
10. $x^2p^3 + y(1 + x^2y)p^2 + y^3p = 0$
11. $y = yp^2 + 2px$
12. $y = 2px + p^n$
13. $y = x + a \tan^{-1} p$
14. $y - 2px - \tan^{-1}(xp^2) = 0$
15. $y + px = x^4p^2$
16. $x^2p^4 + 2xp - y = 0$
17. $y = 2px + y^2p^3$
18. $p^3 - 4xyp + 8y^2 = 0$
19. $p = \tan(x - \frac{p}{1+p^2})$
20. $y = (1 + p)x + ap^2$
21. $e^{p-y} = p^2 - 1$
22. $y = 2px + f(xp^2)$

23. $p^3 - 4xyp + 8y^2 = 0$

24. $y = p\tan p + \log\cos p$

25. $y = \log(p^3 + p)$

26. $x(1 + p^2) = 1$

27. $(y-1)p - xp^2 + 2 = 0$

28. $(y - px)(p - 1) = p$ (WBUT 2009)

29. $(x-a)p^2 + (x-y)p - y = 0$

30. $p = \log(px - y)$

31. $\sin px\ \cos y = \cos px\ \sin y + p$

32. Solve $x^2p^2 + yp(2x+y) + y^2 = 0$ by reducing it to Clairaut's form by using the substitution $y = u$ and $xy = v$.

33. Use the transformation $x^2 = u$ and $y^2 = v$ solve the equation

$$axyp^2 + (x^2 - ay^2 - b)p - xy = 0$$

34. By the substitution $x^2 = u, y^2 = v$, reduce the equation $x^2 + y^2 - (p + p^{-1})xy = c^2$ to Clairaut's form and find the general integral and the singular solution.

35. Solve the equation $(px + y)^2 = py^2$ using the transformation $u = y, v = xy$.

36. Solve the equation $y^2(y - xp) = x^4p^2$ using the substitution $x = 1/u, y = 1/v$.

Answers

Section A Multiple Choice Questions

1. (a) 2. (b) 3. (b) 4. (c) 5. (a) 6. (d) 7. (d) 8. (a) 9. (b)
10. (c) 11. (a) 12. (c)

Section B Review Questions

1. $(x^2 + y^2 - c)(xy - c) = 0$
2. $(y^2 - 2y - x - c)(y^2 - x^2 - c) = 0$
3. $y(1 \pm \cos x) = c$
4. $(x - y + c)(x^2 + y^2 + c) = 0$
5. $(y - c)(y + x^2 - c)(xy + cy + 1) = 0$
6. $(2y - x^2 + c)(y + x + ce^{-x} - 1) = 0$
7. $x^2 + y^2 = cx$
8. $(1 - x - y - ce^{-x})(2xy + x^2 - c)(y + x^2 - c) = 0$

9. $(y^2 - x^2 - c)(y^2 - cx^4 - x^2) = 0$
10. $(y - c)(ye^{1/x} - c)(xy + cy - 1) = 0$
11. $cxy + 2x\sqrt{1 + cx} = 0$
12. $y = \frac{2c}{p} + \frac{1-n}{1+n}p^n$
13. $x = c + \frac{a}{2}\left[\log \frac{p-1}{\sqrt{1+p^2}} - \tan^{-1} p\right]$
14. $y = 2\sqrt{cx} + \tan^{-1} c$
15. $xy = c^2x + c$
16. $y = 2\sqrt{cx} + c^2$
17. $y^2 = 2cx + c^3$
18. $y = c(x - c)^2$
19. $y + (1 + p^2)^{-1} = c$
20. $x = 2a(1 - p) + ce^{-p}$
21. $p(p + 1) = c(p - 1)e^x$
22. $y = 2c\sqrt{x} + f(c^2)$
23. $y = c(x - c)^2$
24. $x = \tan p + c$
25. $x + (1/p) + c = 2\tan^{-1} p$
26. $y - c = \sqrt{y^2 - x^2} - \tan^{-1} \sqrt{(1 - x)/x}$
27. $y = cx + 1 - 2/c$
28. $y = cx + c/(c - 1)$
29. $y = cx - ac^2/(c + 1)$
30. $y = cx - e^c$
31. $y = cx - \sin^{-1} c$
32. $xy = cy + e^2$
33. $y^2 = cx^2 - bc/(ac + 1)$
34. $y^2 = Ax^2 + \{c^2A/(A - 1)\}$
35. $xy = cy - c^2$
36. $x = yc + xyc^2$

Chapter 4

Linear Differential Equations of Second and Higher Order

4.1 Introduction

An nth order differential equation is called **linear** if it is of **first degree** in the dependent variable y and its derivatives $y', y'', \ldots, y^{(n)}$. That is, the general form of an nth order linear differential equation is

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = g(x) \tag{4.1}$$

where a_1, $a_2, \ldots,$ a_n and $g(x)$ are functions of x.

If $g(x) \neq 0$, then the equation is called a **non-homogeneous linear equation**. If $g(x) = 0$, then the equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{4.2}$$

is called a **homogeneous linear equation**.

Two solutions y_1 and y_2 of (4.2) are called **dependent** if $c_1 y_1 + c_2 y_2 = 0$ for $c_1 \neq 0$ and $c_2 \neq 0$, i.e. if the ratio $\frac{y_1}{y_2} = -\frac{c_2}{c_1} =$ a constant. Otherwise, the two solutions are called **linearly independent**.

In general, a set of solutions y_1, $y_2, \ldots,$ y_n of an nth order differential equation is said to be linearly independent if $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$ implies that $c_i = 0$ for all $i = 1, 2, \ldots, n$.

The dependence of a set of functions can be tested by computing the value of the **Wronskian**, which is defined below.

Wronskian

Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two functions. Then $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$ is called the **Wronskian** of two functions y_1 and y_2 and it is denoted by $W(y_1, y_2)$.

If y_1 and y_2 are linearly dependent, then the Wronskian is identically 0, for if $y_1 = c y_2$, c is a constant, then $y_1' = c y_2'$ and hence

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = y_1 c y_1' - y_1' c y_1 = 0$$

The Wronskian of n functions $y_1, y_2, \dots, y_n$ is

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

It can be shown by the previous method that if $y_1, y_2, \dots, y_n$ are dependent, then

$$W(y_1, y_2, \dots, y_n) = 0$$

Theorem 4.1 *If y_1 and y_2 are two linearly independently solutions of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \tag{4.3}$$

on $[a, b]$ in which $p(x)$ and $q(x)$ are continuous, then their Wronskian is not zero at any point on $[a, b]$.

It can be shown that every nth order linear homogeneous differential equation, i.e. the equation of the form (4.2) has n independent solutions.

Theorem 4.2 *If y_1, y_2 are two solutions of the homogeneous Eq. (4.2), then $u = c_1y_1 + c_2y_2$ is also its solution.*

Proof. Let $y = y_1$ and $y = y_2$ be two solutions of the homogeneous Eq. (4.2). Then

$$\frac{d^n y_1}{dx^n} + a_1\frac{d^{n-1}y_1}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy_1}{dx} + a_n y_1 = 0 \tag{4.4}$$

and

$$\frac{d^n y_2}{dx^n} + a_1\frac{d^{n-1}y_2}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy_2}{dx} + a_n y_2 = 0 \tag{4.5}$$

Let $u = c_1y_1 + c_2y_2$.

Therefore

$$\begin{aligned}
&\frac{d^n u}{dx^n} + a_1\frac{d^{n-1}u}{dx^{n-1}} + \cdots + a_{n-1}\frac{du}{dx} + a_n u \\
&= \frac{d^n(c_1y_1 + c_2y_2)}{dx^n} + a_1\frac{d^{n-1}(c_1y_1 + c_2y_2)}{dx^{n-1}} + \cdots + a_{n-1}\frac{d(c_1y_1 + c_2y_2)}{dx} + a_n(c_1y_1 + c_2y_2) \\
&= c_1\left(\frac{d^n y_1}{dx^n} + a_1\frac{d^{n-1}y_1}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy_1}{dx} + a_n y_1\right) \\
&\quad + c_2\left(\frac{d^n y_2}{dx^n} + a_1\frac{d^{n-1}y_2}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy_2}{dx} + a_n y_2\right) \\
&= c_1.0 + c_2.0 \qquad \text{[using (4.4) and (4.5)]} \\
&= 0
\end{aligned}$$

Hence the result.

In general, if $y_1, y_2, \dots, y_n$ are n independent solutions of (4.2), then $c_1y_1 + c_2y_2 + \cdots + c_ny_n (= u)$ where $c_1, c_2, \dots, c_n$ are arbitrary constants, is the complete solution of (4.2). That is

$$\frac{d^n u}{dx^n} + a_1 \frac{d^{n-1}u}{dx^{n-1}} + \cdots + a_{n-1}\frac{du}{dx} + a_n u = 0 \tag{4.6}$$

If $y = v$ be any particular solution of (4.1), then

$$\frac{d^n v}{dx^n} + a_1 \frac{d^{n-1}v}{dx^{n-1}} + \cdots + a_{n-1}\frac{dv}{dx} + a_n v = g(x) \tag{4.7}$$

Adding (4.6) and (4.7), we have

$$\frac{d^n(u+v)}{dx^n} + a_1 \frac{d^{n-1}(u+v)}{dx^{n-1}} + \cdots + a_{n-1}\frac{d(u+v)}{dx} + a_n(u+v) = g(x) \tag{4.8}$$

This shows that $y = u + v$ is also a solution of (4.1), and this solution is the complete solution of (4.1).

The part u is called the **complementary function (C.F.)** and the part v is called the **particular integral (P.I.)** of (4.1). Thus the complete solution is

$$y = u + v = \text{ C.F. } + \text{ P.I.} \tag{4.9}$$

Introducing the operators D for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$, etc. We write the Eq. (4.1) in the form

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = g(x)$$

The expression $D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n$ is called the **linear differential operator of order** n and is denoted by $F(D)$.

4.2 Complementary Function

Let us consider Eq. (4.1), i.e.

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = 0 \tag{4.10}$$

where the coefficients $a_1, a_2, \dots, a_n$ are constants.

Let $y = ce^{mx}$, where c is arbitrary constant, be a possible solution of (4.10). Then $y' = Dy = mce^{mx}, y'' = D^2y = m^2ce^{mx}, \dots, y^{(n)} = D^ny = m^nce^{mx}$. Substituting these values, (4.10) reduces to

$$cm^ne^{mx} + a_1cm^{n-1}e^{mx} + a_2cm^{n-2}e^{mx} + \cdots + a_{n-1}cme^{mx} + a_nce^{mx} = 0$$

Since $ce^{mx} \neq 0$ for all m and x, we have

$$m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_{n-1}m + a_n = 0 \tag{4.11}$$

This equation is known as the **auxiliary equation (A.E.)** of (4.10) and this equation can easily be obtained from (4.11) by replacing y' with m, y'' with $m^2, \ldots, y^{(n)}$ by m^n.

Equation (4.11) is an algebraic equation in m of degree n and has exactly n roots. Let $m_1, m_2, \ldots, m_n$ be its roots. These roots may be distinct, equal or complex.

Case I. *Let all the roots $m_1, m_2, \ldots, m_n$ be real and distinct.*

In this case, the Eq. (4.10) can be expressed as

$$(D - m_1)(D - m_2)\cdots(D - m_n)y = 0 \tag{4.12}$$

This equation is satisfied by $(D - m_n)y = 0$, that is, $\dfrac{dy}{dx} - m_n y = 0$, or $\dfrac{dy}{y} = m_n dx$. Integrating, we get $y = c_n e^{m_n x}$, where c_n is arbitrary constant.

Since the factor of the above equation can be taken in any order, $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$, etc. are also the solutions. Hence the complete solution of (4.10) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} \tag{4.13}$$

Case II. *Let two roots m_1, m_2 be equal (i.e. $m_1 = m_2$).*

In this case, the solution (4.13) reduces to

$$\begin{aligned} y &= (c_1 + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \\ &= c e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x} \end{aligned}$$

which contains $(n - 1)$ arbitrary constants, so it is not the complete solution.

The part of (4.12) corresponding to the roots m_1, m_1 is $(D - m_1)(D - m_1) = 0$. Let $(D - m_1)y = z$. Then $(D - m_1)z = 0$, i.e. $z = c_1 e^{m_1 x}$ (as in the previous case).

Therefore, $\dfrac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$, which is linear in y and I.F. is $e^{-m_1 x}$.

Thus, $\dfrac{d}{dx}(y e^{-m_1 x}) = c_1$. Integrating, $y e^{-m_1 x} = c_1 x + c_2$, or $y = (c_1 x + c_2)e^{m_1 x}$.

Thus the complete solution of (4.10) is

$$y = (c_1 x + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

In case of k equal roots, the complete solution is

$$y = (c_1 x + c_2 x^2 + \cdots + c_k x^k)e^{m_1 x} + c_{k+1} e^{m_{k+1} x} + \cdots + c_n e^{m_n x}$$

Case III. *Let a pair of roots be complex.*

Let the roots be $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. Then the part of the solution is

$$
\begin{aligned}
y &= c_1 e^{m_1 x} + c_2 e^{m_2 x} \\
&= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \\
&= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) \\
&= e^{\alpha x}\{c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\} \\
&= e^{\alpha x}\{(c_1 + c_2)\cos\beta x + (c_1 i - c_2 i)\sin\beta x\} \\
&= e^{\alpha x}(A\cos\beta x + B\sin\beta x)
\end{aligned}
$$

where $A = c_1 + c_2$ and $B = (c_1 - c_2)i$ are arbitrary constants.

Thus the complete solution of (4.10) is

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

If two pair of complex roots be equal, i.e. $m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta$, then the part of complete solution is

$$y = e^{\alpha x}[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x]$$

EXAMPLE 4.2.1 Find the complete solution of $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 0.$

Solution Let $y = ce^{mx}$ be a solution. Then $y' = cme^{mx}$ and $y'' = cm^2e^{mx}$. The given equation reduces to

$$cm^2e^{mx} - cme^{mx} - 6ce^{mx} = 0$$

or

$$m^2 - m - 6 = 0$$

or

$$(m-3)(m+2) = 0$$

or

$$m = 3, -2$$

Thus the complete solution is $y = c_1e^{3x} + c_2e^{-2x}$.

EXAMPLE 4.2.2 Solve $(D^3 - 5D^2 + 8D - 4)y = 0$.

Solution Let $y = ce^{mx}$ be a solution. Then $A.E.$ is

$$m^3 - 5m^2 + 8m - 4 = 0$$

or

$$(m-1)(m^2 - 4m + 4) = 0$$

or

$$(m-1)(m-2)^2 = 0$$

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or

$$m = 1, 2, 2$$

Therefore, the complete solution is $y = c_1e^x + (c_2x + c_3)e^{2x}$

EXAMPLE 4.2.3 Solve $D^2y + 9y = 0$. (WBUT 200

Solution Let $y = ce^{mx}$ be a solution. Then $A.E.$ is

$$m^2 + 9 = 0$$

or

$$m = \pm 3i = 0 \pm 3i$$

Therefore, the complete solution is

$$y = e^{0x}(c_1 \cos 3x + c_2 \sin 3x) = c_1 \cos 3x + c_2 \sin 3x$$

EXAMPLE 4.2.4 Solve $(D^2 + 1)^3y = 0$.

Solution In this case, the $A.E.$ is $(m^2 + 1)^3 = 0$, or $m = \pm i, \pm i, \pm i$.
Therefore, the complete solution is

$$y = (c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5x + c_6x^2)\sin x$$

4.3 Particular Integral (P.I.)

To obtain a P.I. of $F(D)y = g(x)$, we apply the operator $[F(D)]^{-1}$ or $\frac{1}{F(D)}$, called the **inverse operator**, we have

$$\frac{1}{F(D)}F(D)y = y = \frac{1}{F(D)}g(x)$$

Suppose that $F(D) = (D - m_1)(D - m_2)\cdots(D - m_n)$ has no repeated factor. So, $\frac{1}{F(D)}$ can be expressed as

$$\frac{1}{F(D)} = \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \cdots + \frac{A_n}{D - m_n} = \sum_{i=1}^{n} \frac{A_i}{D - m_i}$$

Then

$$y = \frac{1}{F(D)}g(x) = \sum_{i=1}^{n} \frac{A_i}{D - m_i}g(x) \tag{4.14}$$

Let $\frac{A_i}{D - m_i}g(x) = z$, i.e. $(D - m_i)z = A_ig(x)$, or $\frac{dz}{dx} - m_iz = A_ig(x)$.

This is a linear differential equation and its I.F. is $e^{-m_i x}$. Multiplying this equation by I.F. and integrating, we get

$$ze^{-m_i x} = \int A_i g(x) e^{-m_i x} dx \qquad \text{or} \qquad z = e^{m_i x} \int A_i g(x) e^{-m_i x} dx$$

Equation (4.14) reduces to

$$y = \sum_{i=1}^{n} A_i e^{m_i x} \int g(x) e^{-m_i x} dx \tag{4.15}$$

We note that to find the P.I. the arbitrary constant which arises in integration need not be written down. The appropriate number of constants are included in the C.F.

Inverse Operators

Case I. $\frac{1}{D} g(x) = \int g(x) dx$

Let $\frac{1}{D} g(x) = z$. That is, $Dz = g(x)$, or $\frac{dz}{dx} = g(x)$ or $z = \int g(x) dx$.
Hence

$$\frac{1}{D} g(x) = \int g(x) dx \tag{4.16}$$

Case II. $\frac{1}{D-a} g(x) = e^{ax} \int g(x) e^{-ax} dx$

Let

$$\frac{1}{D-a} g(x) = z \qquad \text{or} \qquad \frac{dz}{dx} - az = g(x)$$

This is a linear equation and I.F. $= e^{-ax}$.
Its solution is

$$ze^{-ax} = \int g(x) e^{-ax} dx \qquad \text{or} \qquad z = e^{ax} \int g(x) e^{-ax} dx$$

Hence

$$\frac{1}{D-a} g(x) = e^{ax} \int g(x) e^{-ax} dx$$

4.3.1 Rules for Finding Particular Integral

In general, the P.I. of the Eq. (4.10) is

$$\text{P.I.} = \frac{1}{D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n} g(x) = \frac{1}{F(D)} g(x)$$

Case I. Let $g(x) = e^{ax}$.
Then

$$\begin{aligned}F(D)e^{ax} &= (D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n)e^{ax}\\ &= a^n e^{ax} + a_1 a^{n-1} e^{ax} + \cdots + a_{n-1} a e^{ax} + a_n e^{ax}\\ &= (a^n + a_1 a^{n-1} + a_2 a^{n-2} + \cdots + a_{n-1}a + a_n)e^{ax}\\ &= F(a)e^{ax}\end{aligned}$$

Operating both side by $\frac{1}{F(D)}$, we get

$$e^{ax} = \frac{1}{F(D)}F(a)e^{ax}$$

i.e.

$$\frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}, \qquad \text{provided } F(a) \neq 0$$

If $F(a) = 0$, then there should be a factor $(D - a)$ in $F(D)$. Let $F(D) = (D - a)F_1(D)$, s and $F_1(a) \neq 0$.

Then

$$\begin{aligned}\frac{1}{F(D)}e^{ax} &= \frac{1}{(D-a)F_1(D)}e^{ax} = \frac{1}{D-a}\frac{1}{F_1(a)}e^{ax}\\ &= \frac{1}{F_1(a)}\frac{1}{D-a}e^{ax} = \frac{1}{F_1(a)}e^{ax}\int dx\\ &= \frac{1}{F_1(a)}xe^{ax}\end{aligned}$$

i.e.

$$\frac{1}{F(D)}e^{ax} = x\frac{1}{F'(a)}e^{ax}$$

$[$as $F'(D) = F_1(D) + (D-a)F_1'(D)$, or $F'(a) = F_1(a)]$ provi

$F'(a) \neq 0$

Again, if $F'(a) = 0$, then by the same method, $\frac{1}{F(D)}e^{ax} = x^2\frac{1}{F''(a)}e^{ax}$, provided $F''(a) \neq ($

EXAMPLE 4.3.1 Find the particular integral of $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 9e^x$.

Solution The given equation is $(D^2 - 3D + 2)y = 9e^x$.
The P.I. is

$$\begin{aligned}\frac{1}{D^2 - 3D + 2}9e^x &= \frac{1}{(D-1)(D-2)}9e^x = 9\frac{1}{D-1}\frac{1}{D-2}e^x\\ &= 9\frac{1}{D-1}\frac{1}{1-2}e^x\\ &= -9\frac{1}{D-1}e^x = -9xe^x\end{aligned}$$

Case II. Let $g(x) = \sin ax$ or $\cos ax$.

We know that

$$\begin{aligned} D(\sin ax) &= a\cos ax & D^2(\sin ax) &= -a^2 \sin ax \\ D^3(\sin ax) &= -a^3 \cos ax & D^4(\sin ax) &= a^4 \sin ax \\ D^5(\sin ax) &= a^5 \cos ax & D^6(\sin ax) &= -a^6 \sin ax \end{aligned}$$

Thus we observed that

$$\begin{aligned} D^2(\sin ax) &= -a^2 \sin ax & (D^2)^2(\sin ax) &= (-a^2)^2 \sin ax \\ (D^2)^3(\sin ax) &= (-a^2)^3 \sin ax & (D^2)^4(\sin ax) &= (-a^2)^4 \sin ax \end{aligned}$$

and so on, and

$$(D^2)^n(\sin ax) = (-a^2)^n \sin ax$$

Hence $F(D^2)(\sin ax) = F(-a^2)\sin ax$.

That is

$$\frac{1}{F(D^2)} F(D^2)\sin ax = \frac{1}{F(D^2)} F(-a^2)\sin ax$$

or

$$\sin ax = F(-a^2)\frac{1}{F(D^2)} \sin ax$$

Thus

$$\frac{1}{F(D^2)} \sin ax = \frac{\sin ax}{F(-a^2)} \tag{4.17}$$

Similarly

$$\frac{1}{F(D^2)} \cos ax = \frac{\cos ax}{F(-a^2)} \tag{4.18}$$

In both cases, $F(-a^2) \neq 0$. In general,

$$\frac{1}{F(D^2)} \sin(ax+b) = \frac{1}{F(-a^2)} \sin(ax+b)$$

and

$$\frac{1}{F(D^2)} \cos(ax+b) = \frac{1}{F(-a^2)} \cos(ax+b)$$

provided $F(-a^2) \neq 0$.

If $F(-a^2) = 0$, then the above formula is not applicable. We proceed as follows: Since $\cos ax + i\sin ax = e^{iax}$, so the imaginary part (I.P.) of e^{iax} is equal to $\sin ax$. Therefore

$$\begin{aligned} \frac{1}{F(D^2)} \sin ax &= \text{I.P. of } \frac{1}{F(D^2)} e^{iax} \\ &= \text{I.P. of } x\frac{1}{F'(D^2)} e^{iax} \qquad [\because F(-a^2) = 0 \text{ as } D^2 = -a^2] \end{aligned}$$

Thus

$$\frac{1}{F(D^2)}\sin ax = x\frac{1}{F'(-a^2)}\sin ax \qquad \text{provided } F'(-a^2) \neq 0 \tag{4.1}$$

Again, if $F'(-a^2) = 0$, then

$$\frac{1}{F(D^2)}\sin ax = x^2\frac{1}{F''(-a^2)}\sin ax \qquad \text{provided } F''(-a^2) \neq 0 \tag{4.2}$$

EXAMPLE 4.3.2 Find the P.I. of $\dfrac{d^3y}{dx^3} + 9\dfrac{dy}{dx} = \cos 3x$.

Solution This equation can be written as $(D^3 + 9D)y = \cos 3x$. Therefore

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D(D^2+9)}\cos 3x && [\text{Here } D^2 = -9, F(D^2) = 0]\\
&= x\frac{1}{3D^2+9}\cos 3x && [\text{Using the rule } (4.19)]\\
&= x\frac{1}{3(-9)+9}\cos 3x && (\text{Putting } D^2 = -9)\\
&= -\frac{x}{18}\cos 3x
\end{aligned}$$

Case III. Let $g(x) = x^m$. Here

$$\text{P.I.} = \frac{1}{F(D)}x^m = [F(D)]^{-1}x^m$$

From $F(D)$, take out the lowest degree term and the remaining factor be of the form $[1 \pm F_1(D)]^{-1}$. Then expand $[1 \pm F_1(D)]^{-1}$ in ascending power of D as far as the term in D^m and operate on x^m term by term. Since the $(m+1)th$ and higher order derivatives of x^m is zero we need not consider terms beyond D^m.

EXAMPLE 4.3.3 Find the P.I. of $(2D^2 + 5D + 2)y = 5 + 2x$.

Solution

$$\begin{aligned}
\text{P.I.} &= \frac{1}{2D^2+5D+2}(5+2x)\\
&= \frac{1}{2}\frac{1}{1+\frac{5D+2D^2}{2}}(5+2x)\\
&= \frac{1}{2}\left(1+\frac{5D+2D^2}{2}\right)^{-1}(5+2x)
\end{aligned}$$

$$= \frac{1}{2}\left(1 - \frac{5D + 2D^2}{2} + \cdots\right)(5 + 2x)$$

$$= \frac{1}{2}\left[5 + 2x - \frac{5}{2}(2)\right]$$

$$= \frac{1}{2} \cdot 2x = x$$

Case IV. When $g(x) = e^{ax} f(x)$, $f(x)$ is a function of x.

Let v is a function of x, then

$$D(e^{ax}v) = e^{ax}Dv + ae^{ax}v = e^{ax}(D + a)v$$

$$D^2(e^{ax}v) = e^{ax}D^2v + 2ae^{ax}Dv + a^2e^{ax}v = e^{ax}(D + a)^2v$$

In this way

$$D^n(e^{ax}v) = e^{ax}(D + a)^n v$$

Therefore,

$$F(D)(e^{ax}v) = e^{ax}F(D + a)v$$

Operating both sides by $\dfrac{1}{F(D)}$

$$\frac{1}{F(D)}F(D)(e^{ax}v) = \frac{1}{F(D)}[e^{ax}F(D + a)v]$$

or

$$e^{ax}v = \frac{1}{F(D)}[e^{ax}F(D + a)v]$$

Let $F(D + a)v = f(x)$. That is, $v = \dfrac{1}{F(D + a)}f(x)$.

Thus, $e^{ax}\dfrac{1}{F(D + a)}f(x) = \dfrac{1}{F(D)}[e^{ax}f(x)]$.

Finally

$$\frac{1}{F(D)}[e^{ax}f(x)] = e^{ax}\frac{1}{F(D + a)}f(x) \tag{4.21}$$

EXAMPLE 4.3.4 Solve $(D^2 - 2D)y = e^x \sin x$. (WBUT 2007)

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 2D)y = 0$. Therefore, A.E. is

$$m^2 - 2m = 0 \qquad \text{or} \qquad m = 0, 2$$

Therefore C.F. is $c_1 + c_2e^{2x}$.

$$\text{P.I.} = \frac{1}{D(D-2)}e^{x}\sin x = e^x\frac{1}{(D+1)(D+1-2)}\sin x$$
$$= e^x\frac{1}{(D^2-1)}\sin x = e^x\frac{1}{(-1^2-1)}\sin x$$
$$= -\frac{1}{2}e^x\sin x$$

Hence the complete solution is $y = c_1 + c_2e^{2x} - \frac{1}{2}e^x\sin x$.

4.4 Additional Worked-Out Examples

EXAMPLE 4.4.1 Solve $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x + \sin x$.

Solution Let the equation be $(D^2 + D - 2)y = x + \sin x$ and $y = ce^{mx}$ be a solution of $(D^2 + D - 2)y = 0$.

$\therefore$ A.E. is $m^2 + m - 2 = 0$, or $m = -2, 1$

$\therefore$ C.F. is $c_1e^{-2x} + c_2e^x$.

$$\text{P.I. for } x = \frac{1}{D^2+D-2}x = -\frac{1}{2}\left(1 - \frac{D^2+D}{2}\right)^{-1}x$$
$$= -\frac{1}{2}\left(1 + \frac{D^2+D}{2} + \cdots\right)x = -\frac{1}{2}\left(x + \frac{1}{2}\right)$$
$$= -\frac{1}{4}(2x+1)$$

$$\text{P.I. for } \sin x = \frac{1}{D^2+D-2}\sin x$$
$$= \frac{1}{-1+D-2}\sin x \qquad (\text{Putting } D^2 = -1)$$
$$= \frac{D+3}{D^2-9}\sin x = \frac{1}{-1-9}[D(\sin x) + 3\sin x]$$
$$= -\frac{1}{10}(\cos x + 3\sin x)$$

Therefore, the complete solution is $y = c_1e^{-2x} + c_2e^x - \frac{1}{10}(\cos x + 3\sin x)$.

EXAMPLE 4.4.2 Solve $(D^2 - 4D + 3)y = e^{2x}\sin x$.

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 4D + 3)y = 0$.

Then A.E. is $m^2 - 4m + 3 = 0$ or $m = 3, 1$. The C.F. is $c_1 e^x + c_2 e^{3x}$.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - 4D + 3} e^{2x} \sin x \\
&= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 3} \sin x \\
&= e^{2x} \frac{1}{D^2 - 1} \sin x \\
&= e^{2x} \frac{1}{-1-1} \sin x \qquad (\text{Putting } D^2 = -a^2 = -1) \\
&= -\frac{1}{2} e^{2x} \sin x
\end{aligned}$$

Therefore, the general solution is $y = c_1 e^x + c_2 e^{3x} - \frac{1}{2} e^{2x} \sin x$.

EXAMPLE 4.4.3 Solve $(D^2 - 2D + 1)y = xe^x \sin x$.

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 2D + 1)y = 0$. Then A.E. is $m^2 - 2m + 1 = 0$, or $m = 1, 1$. Therefore, C.F. is $(c_1 + c_2 x)e^x$.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D-1)^2} xe^x \sin x = e^x \frac{1}{(D+1-1)^2} x \sin x \\
&= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D}\frac{1}{D} x \sin x \\
&= e^x \frac{1}{D} \int x \sin x dx = e^x \frac{1}{D}(-x\cos x + \sin x) \\
&= e^x \Big[-\int x \cos x dx + \int \sin x dx \Big] \\
&= e^x \Big[-x \sin x + \int \sin x dx + \int \sin x dx \Big] \\
&= e^x(-x \sin x - 2\cos x)
\end{aligned}$$

Therefore, the general solution is

$$y = (c_1 + c_2 x)e^x - e^x(x \sin x + 2\cos x)$$

EXAMPLE 4.4.4 Solve $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{2x} + x$.

Solution The given equation is $(D^3 - 2D^2 + D)y = e^{2x} + x$.
Let $y = ce^{mx} (\neq 0)$ be a trial solution of $(D^3 - 2D^2 + D)y = 0$.
The A.E. is $m^3 - 2m^2 + m = 0$, or $m = 0, 1, 1$.
$\therefore$ C.F. is $c_1 + (c_2 + c_3 x)e^x$, where c_1, c_2, c_3 are arbitrary constants.

$\frac{e^{2x}}{D(D-1)^2} = \frac{1}{D(2-1)^2}e^{2x} = \frac{1}{D}e^{2x} = \frac{e^{2x}}{2}$

P.I. of $e^{2x} + x$

$$= \frac{1}{D^3 - 2D^2 + D}(e^{2x} + x) = \frac{1}{2}e^{2x} + \frac{1}{D(1 + D^2 - 2D)}x$$

$$= \frac{e^{2x}}{2} + \frac{1}{D}(1 + D^2 - 2D)^{-1}x = \frac{e^{2x}}{2} + \frac{1}{D}(1 - D^2 + 2D - \cdots)x$$

$$= \frac{e^{2x}}{2} + \frac{1}{D}(x - 0 + 2) = \frac{e^{2x}}{2} + \frac{x^2}{2} + 2x$$

Therefore, the general solution is

$$y = c_1 + (c_2 + c_3x)e^x + \frac{e^{2x}}{2} + \frac{x^2}{2} + 2x$$

where c_1, c_2, c_3 are arbitrary constants.

EXAMPLE 4.4.5 Solve $\frac{d^2x}{dt^2} + 2x = t^2e^{3t} + e^t\cos 2t$.

Solution The given equation is $(D^2 + 2)x = t^2e^{3t} + e^t\cos 2t$, where $D \equiv \frac{d}{dt}$.

Let $x = ce^{mt}$ be a trial solution.

$\therefore$ A.E. is $m^2 + 2 = 0$, or $m = \pm\sqrt{2}i$.

$\therefore$ C.F. is $c_1\cos\sqrt{2}t + c_2\sin\sqrt{2}t$.

$$\text{P.I.} = \frac{1}{D^2 + 2}(t^2e^{3t} + e^t\cos 2t) = e^{3t}\frac{1}{(D+3)^2 + 2}t^2 + e^t\frac{1}{(D+1)^2 + 2}\cos 2t$$

$$= e^{3t}\frac{1}{D^2 + 6D + 11}t^2 + e^t\frac{1}{D^2 + 2D + 3}\cos 2t$$

$$= \frac{e^{3t}}{11}\left(1 + \frac{D^2 + 6D}{11}\right)^{-1}t^2 + e^t\frac{1}{-4 + 2D + 3}\cos 2t$$

$$= \frac{e^{3t}}{11}\left(1 - \frac{D^2 + 6D}{11} + \left(\frac{D^2 + 6D}{11}\right)^2 + \cdots\right)t^2 + e^t\frac{1}{2D - 1}\cos 2t$$

$$= \frac{e^{3t}}{11}\left(t^2 - \frac{12t}{11} + \frac{50}{121}\right) + e^t\frac{2D + 1}{4D^2 - 1}\cos 2t$$

$$= \frac{e^{3t}}{11}\left(t^2 - \frac{12t}{11} + \frac{50}{121}\right) + e^t\frac{2D + 1}{4(-4) - 1}\cos 2t$$

$$= \frac{e^{3t}}{11}\left(t^2 - \frac{12t}{11} + \frac{50}{121}\right) - \frac{e^t}{17}(-4\sin 2t + \cos 2t)$$

Therefore, the general solution is

$$x = c_1\cos\sqrt{2}t + c_2\sin\sqrt{2}t + \frac{e^{3t}}{11}\left(t^2 - \frac{12t}{11} + \frac{50}{121}\right) - \frac{e^t}{17}(-4\sin 2t + \cos 2t)$$

where c_1, c_2 are arbitrary constants.

EXAMPLE 4.4.6 Solve $(D^2+4)y = x\sin^2 x$. (WBUT 2008)

Solution Let $y = ce^{mx}$ be a solution of $(D^2+4)y = 0$

A.E. is $m^2 + 4 = 0$, or $m = \pm 2i$.

Thus, C.F. is $c_1 \cos 2x + c_2 \sin 2x$.

$$
\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+4} x\sin^2 x = \frac{1}{D^2+4}\cdot\frac{1}{2}x(1-\cos 2x) \\
&= \frac{1}{2}\cdot\frac{1}{D^2+4}x - \frac{1}{2}\cdot\frac{1}{D^2+4}x\cos 2x \\
&= \frac{1}{8}\left(1+\frac{D^2}{4}\right)^{-1}x - \frac{1}{2}\text{ R.P. of }\frac{1}{D^2+4}xe^{2ix} \qquad (\because \cos x = \text{Real part of } e^{2ix}) \\
&= \frac{1}{8}\left(1-\frac{D^2}{4}+\cdots\right)x - \frac{1}{2}\text{ R.P. of } e^{2ix}\frac{1}{(D+2i)^2+4}x \\
&= \frac{1}{8}x - \frac{1}{2}\text{ R.P. of } e^{2ix}\frac{1}{D^2+4Di}x \\
&= \frac{1}{8}x - \frac{1}{2}\text{ R.P. of } e^{2ix}\frac{1}{4Di}\left(1+\frac{D}{4i}\right)^{-1}x \\
&= \frac{1}{8}x - \frac{1}{8}\text{ R.P. of } e^{2ix}\frac{1}{Di}\left(x-\frac{1}{4i}\right) \\
&= \frac{1}{8}x - \frac{1}{8}\text{ R.P. of } e^{2ix}\frac{1}{i}\left(\frac{x^2}{2}-\frac{x}{4i}\right) \\
&= \frac{1}{8}x - \frac{1}{8}\text{ R.P. of } (\cos 2x + i\sin 2x)\left(-\frac{ix^2}{2}+\frac{x}{4}\right) \\
&= \frac{1}{8}x - \frac{1}{8}\left(\frac{x}{4}\cos 2x + \frac{x^2}{2}\sin 2x\right) \\
&= \frac{1}{8}x - \frac{1}{32}(x\cos 2x + 2x^2\sin 2x)
\end{aligned}
$$

Therefore, the general solution is

$$
y = c_1\cos 2x + c_2\sin 2x + \frac{1}{8}x - \frac{1}{32}(x\cos 2x + 2x^2\sin 2x)
$$

EXAMPLE 4.4.7 Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^x\cos x$.

Solution Let $y = ce^{mx}$ be a trial solution.

$\therefore$ A.E. is $m^2 - 2m + 2 = 0$, or $m = \dfrac{2\pm\sqrt{4i}}{2} = 1 \pm i$.

$\therefore$ C.F. is $(c_1 \cos x + c_2 \sin x)e^x$, where c_1, c_2 are arbitrary constants.

$$\text{P.I} = \frac{1}{D^2 - 2D + 2} e^x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1) + 2} \cos x$$

$$= e^x \frac{1}{D^2 + 1} \cos x = e^x \cdot \text{ R.P. of } \frac{1}{D^2+1} e^{ix}$$

$$= e^x \cdot \text{ R.P. of } e^{ix} \frac{1}{(D+i)^2 + 1} 1 = e^x \cdot \text{ R.P. of } e^{ix} \frac{1}{D^2 + 2Di} 1$$

$$= e^x \cdot \text{ R.P. of } e^{ix} \frac{1}{2Di}\left(1 + \frac{D}{2i}\right)^{-1} 1 = e^x \cdot \text{ R.P. of } e^{ix} \frac{1}{2Di} 1$$

$$= e^x \cdot \text{ R.P. of } e^{ix} \frac{1}{2i} x = xe^x \cdot \text{ R.P. of } (\cos x + i \sin x)\frac{1}{2i}$$

$$= xe^x \cdot \text{ R.P. of } \frac{1}{2}(-i\cos x + \sin x) = \frac{1}{2} xe^x \sin x$$

Therefore, the general solution is $y = (c_1 \cos x + c_2 \sin x)e^x + \frac{1}{2} xe^x \sin x$, where c_1, c_2 are arbitrary constants.

EXAMPLE 4.4.8 Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2 \cos x$.

Solution We have $(D^4 + 2D^2 + 1)y = x^2 \cos x$, or $(D^2+1)^2 y = x^2 \cos x$.

Let $y = ce^{mx}$ be a trial solution.

$\therefore$ A.E. is $(m^2+1)^2 = 0$, or $m^2 + 1 = 0$ (twice), or $m = \pm i$ (twice).

$\therefore$ C.F. is $(c_1 + xc_2)\cos x + (c_3 + xc_4)\sin x$, where c_1, c_2, c_3 and c_4 are arbitrary constants

P.I. $\dfrac{1}{(D^2+1)^2} x^2 \cos x$

$$= \text{R.P. of } \frac{1}{(D^2+1)^2} x^2 e^{ix} = \text{R.P. of } e^{ix} \frac{1}{\{(D+i)^2 + 1\}^2} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{(D^2 + 2Di)^2} x^2 = \text{R.P. of } e^{ix} \frac{1}{-4D^2\left(1 + \frac{D}{2i}\right)^2} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{-4D^2}\left(1 - \frac{2D}{2i} - \frac{3D^2}{4} - \cdots\right) x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{-4D^2}\left(x^2 + i \cdot 2x - \frac{3}{4} \cdot 2\right)$$

$$= \text{R.P. of } e^{ix} \frac{1}{-4D}\left(\frac{x^3}{3} + i \cdot x^2 - \frac{3}{2}x\right)$$

$$= \text{R.P. of } e^{ix}\frac{1}{-4}\left(\frac{x^4}{12}+i\cdot\frac{x^3}{3}-\frac{3x^2}{4}\right)$$

$$= -\frac{1}{48}\text{R.P. of } (\cos x + i\sin x)(x^4+4x^3i-9x^2)$$

$$= -\frac{1}{48}\{(x^4-9x^2)\cos x - 4x^3\sin x\}$$

Therefore, the general solution is

$$y = (c_1 + xc_2)\cos x + (c_3 + xc_4)\sin x - \frac{1}{48}\{(x^4-9x^2)\cos x - 4x^3\sin x\}$$

EXAMPLE 4.4.9 Solve $\dfrac{d^2y}{dx^2} + 2n\cos\alpha\dfrac{dy}{dx} + n^2y = a\cos nx$, where n, α are constants.

Solution The given equation is $(D^2 + 2n\cos\alpha D + n^2)y = a\cos nx$.
Let $y = ce^{mx}$ be a trial solution. Then A.E. is $m^2 + 2n\cos\alpha m + n^2 = 0$, or

$$m = \frac{-2n\cos\alpha \pm \sqrt{4n^2\cos^2\alpha - 4n^2}}{2} = -n\cos\alpha \pm ni\sin\alpha$$

$\therefore$ C.F. is $[c_1\cos\{(n\sin\alpha)x\} + c_2\sin\{(n\sin\alpha)x\}]e^{-nx\cos\alpha}$.

P.I. is

$$\frac{1}{D^2+2n\cos\alpha D+n^2}a\cos nx$$

$$= \frac{(D^2+n^2)-2n\cos\alpha D}{(D^2+n^2)^2-4n^2\cos^2\alpha D^2}a\cos nx$$

$$= \frac{(D^2+n^2)-2n\cos\alpha D}{-4n^2\cos^2\alpha(-n^2)}a\cos nx$$

$$= a\frac{-n^2\cos nx + n^2\cos nx - 2n\cos\alpha(-n\sin nx)}{4n^4\cos^2\alpha}$$

$$= \frac{a\sin nx}{2n^2\cos\alpha}$$

Therefore, the general solution is

$$y = [c_1\cos\{(n\sin\alpha)x\} + c_2\sin\{(n\sin\alpha)x\}]e^{-nx\cos\alpha} + \frac{a\sin nx}{2n^2\cos\alpha}$$

where c_1 and c_2 are arbitrary constants.

EXAMPLE 4.4.10 Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 3x^2e^{2x}\sin 2x$.

Solution Let $y = ce^{mx}$ be a trial solution of $(D^2 - 4D + 4)y = 0$.
$\therefore$ A.E. is $m^2 - 4m + 4 = 0$, or $(m-2)^2 = 0$, or $m = 2, 2$.

Then C.F. is $(c_1 + c_2 x)e^{2x}$.

$$\text{P.I.} = \frac{1}{(D-2)^2} 3x^2 e^{2x} \sin 2x = e^{2x} \frac{1}{(D+2-2)^2} 3x^2 \sin 2x$$

$$= e^{2x} \frac{1}{D^2} 3x^2 \sin 2x = \text{I.P. of } e^{2x} \frac{1}{D^2} 3x^2 e^{2ix}$$

$$= \text{I.P. of } 3e^{2x} e^{2ix} \frac{1}{(D+2i)^2} x^2$$

$$= \text{I.P. of } 3e^{2x} e^{2ix} \frac{-1}{4} \left(1 + \frac{D}{2i}\right)^{-2} x^2$$

$$= \text{I.P. of } 3e^{2x} e^{2ix} \frac{-1}{4} \left(1 - \frac{D}{i} - \frac{3D^2}{4} + \cdots\right) x^2$$

$$= \text{I.P. of } 3e^{2x} e^{2ix} \frac{-1}{4} \left(x^2 + i \cdot 2x - \frac{3}{4} 2\right)$$

$$= \text{I.P. of } 3e^{2x} (\cos 2x + i \sin 2x) \frac{-1}{4} \left\{(x^2 - 3/2) + i \cdot 2x\right\}$$

$$= -\frac{3}{4} e^{2x} \{2x \cos 2x + (x^2 - 3/2) \sin 2x\}$$

Therefore, the general solution is

$$y = (c_1 + c_2 x)e^{2x} - \frac{3}{4} e^{2x} \{2x \cos 2x + (x^2 - 3/2) \sin 2x\}$$

EXAMPLE 4.4.11 Solve $\dfrac{d^2y}{dx^2} + a^2 y = \sec ax + x \sin ax$.

Solution Let $y = ce^{mx}$ be a trial solution $(D^2 + a^2)y = 0$.
A.E. is $m^2 + a^2 = 0$, or $m = \pm ai$.
C.F. is $c_1 \cos ax + c_2 \sin ax$.
P.I. of $x \sin ax$

$$= \frac{1}{D^2 + a^2} x \sin ax = \text{I.P. of } \frac{1}{D^2 + a^2} x e^{iax}$$

$$= \text{I.P. of } e^{iax} \frac{1}{(D + ia)^2 + a^2} x$$

$$= \text{I.P. of } e^{iax} \frac{1}{D^2 + 2iaD} x$$

$$= \text{I.P. of } e^{iax} \frac{1}{2iaD} \left(1 + \frac{D}{2ia}\right)^{-1} x$$

$$= \text{I.P. of } e^{iax} \frac{1}{2iaD} \left(1 - \frac{D}{2ia} + \cdots\right) x$$

$$= \text{I.P. of } e^{iax} \frac{1}{2iaD}\left(x - \frac{1}{2ia}\right)$$

$$= \text{I.P. of } e^{iax} \frac{-i}{2a}\left(\frac{x^2}{2} + \frac{xi}{2a}\right)$$

$$= \text{I.P. of } (\cos ax + i \sin ax)\left(-\frac{x^2 i}{4a} + \frac{x}{4a^2}\right)$$

$$= -\frac{x^2}{4a} \cos ax + \frac{x}{4a^2} \sin ax$$

P.I. of $\sec ax$

$$= \frac{1}{D^2 + a^2} \sec ax = \frac{1}{(D + ia)(D - ia)} \sec ax = \frac{1}{2ia}\left(\frac{1}{D - ia} - \frac{1}{D + ia}\right) \sec ax$$

Now

$$\frac{1}{D - ia} \sec ax$$

$$= \frac{1}{D - ia} e^{iax} \frac{e^{-iax}}{\cos ax} = e^{iax} \frac{1}{D + ia - ia} \frac{e^{-iax}}{\cos ax}$$

$$= e^{iax} \frac{1}{D} \frac{\cos ax - i \sin ax}{\cos ax}$$

$$= e^{iax} \int (1 - i \tan ax)\, dx = e^{iax}\left(x - \frac{i}{a} \log \sec ax\right)$$

Similarly, $\frac{1}{D + ia} \sec ax = e^{-iax}\left(x + \frac{i}{a} \log \sec ax\right)$.

Hence P.I. of $\sec ax$

$$= \frac{1}{2ia}\left[e^{iax}\left(x - \frac{i}{a} \log \sec ax\right) - e^{-iax}\left(x + \frac{i}{a} \log \sec ax\right)\right]$$

$$= \frac{x}{a} \frac{e^{iax} - e^{-iax}}{2i} - \frac{1}{a^2} \log \sec ax \frac{e^{iax} + e^{-iax}}{2}$$

$$= \frac{x}{a} \sin ax - \frac{1}{a^2} \cos ax \log \sec ax$$

Therefore, the general solution is

$$y = c_1 \cos ax + c_2 \sin ax - \frac{x^2}{4a} \cos ax + \frac{x}{4a^2} \sin ax + \frac{x}{a} \sin ax - \frac{1}{a^2} \cos ax \log \sec ax$$

EXAMPLE 4.4.12 Solve $(D^2 - 3D + 2)y = e^x$, if $y = 3$ and $Dy = 3$ when $x = 0$.

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 3D + 2)y = 0$.

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Then A.E. is $m^2 - 3m + 2 = 0$, or $m = 1, 2$ and C.F. is $c_1 e^x + c_2 e^{2x}$.

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 3D + 2} e^x = \frac{1}{(D-1)(D-2)} e^x \\ &= \frac{1}{D-1} e^x \frac{1}{1-2} = -\frac{1}{D-1} e^x \\ &= -e^x \frac{1}{D+1-1}(1) = -e^x x \end{aligned}$$

Therefore, the general solution is $y = c_1 e^x + c_2 e^{2x} - xe^x$.
We are given that when $x = 0$, then $y = 3$ and $Dy = 3$.
$\therefore \quad 3 = c_1 + c_2$ and $3 = c_1 + 2c_2 - 1$ (as $Dy = c_1 e^x + 2c_2 e^{2x} - e^x - xe^x$).
The solution of these equations is $c_1 = 2, c_2 = 1$.
Hence the required solution is $y = 2e^x + e^{2x} - xe^x$.

EXAMPLE 4.4.13 Solve $\dfrac{d^2y}{dx^2} - y = 1$, given that $y = 0$ when $x = 0$ and y approaches a finite limit when x approaches $-\infty$.

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 1)y = 0$.
$\therefore$ A.E. is $m^2 - 1 = 0$, or $m = \pm 1$.
C.F. is $c_1 e^x + c_2 e^{-x}$. Now

$$\text{P.I.} = \frac{1}{D^2 - 1}(1) = -(1 - D^2)^{-1}(1) = -1$$

Thus, the general solution is $y = c_1 e^x + c_2 e^{-x} - 1$.
When $x = 0$, then $y = 0$, which gives $c_1 + c_2 - 1 = 0$
Since when x approaches $-\infty$ and y approaches a finite limit, then c_2 should be equal to zero otherwise y becomes infinite.
$\therefore \quad c_1 = 1$.
Hence the required solution is $y = e^x - 1$.

4.5 Cauchy–Euler and Legendre Differential Equations

In this section, two special types of ordinary differential equations with variable coefficients are considered. Both types of equations can be transformed to linear equations with constant coefficients by a suitable transformation of the independent variable. These reduced equations can then be solved by the methods discussed earlier.

4.5.1 Cauchy–Euler Equations

The equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = f(x) \quad (4.\ldots$$

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where $a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is known as the **Cauchy–Euler equation** and it is homogeneous in the sense that the coefficient of the nth derivative is x^n.

For this type of equation the suitable transformation is $x = e^z$ or $z = \log x$.

Therefore

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} = \frac{1}{x}Dy \qquad \text{where } D \equiv \frac{d}{dz}$$

That is, $x\dfrac{dy}{dx} = Dy$.

Again

$$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx} = \frac{d}{dz}\left(\frac{1}{x}Dy\right)\frac{1}{x}$$

$$= \frac{1}{x}\left(-\frac{1}{x^2}\frac{dx}{dz}Dy + \frac{1}{x}D^2y\right)$$

$$= \frac{1}{x^2}(D^2y - Dy)$$

Thus, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$.

Similarly, $x^3\dfrac{d^3y}{dx^3} = D(D-1)(D-2)y$, etc.

By these substitution, Eq. (4.22) becomes an equation in y with z as the independent variable with constant coefficients.

EXAMPLE 4.5.1 Solve $x^3\dfrac{d^3y}{dx^3} + x\dfrac{dy}{dx} - y = x^2$.

Solution Substitute $z = \log x$, or $x = e^z$

Now

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x} \qquad \text{or} \qquad x\frac{dy}{dx} = \frac{dy}{dz} = Dy$$

where $\equiv \dfrac{d}{dz}$

Differentiating with respect to x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dz^2}\frac{dz}{dx} = \frac{1}{x}\frac{d^2y}{dz^2}$$

or

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - x\frac{dy}{dx} = D^2y - Dy = D(D-1)y$$

Similarly, $x^3\dfrac{d^3y}{dx^3} = D(D-1)(D-2)y$.

Using these results, the given equation becomes

$$\{D(D-1)(D-2) + D - 1\}y = e^{2z} \qquad \text{or} \qquad (D^3 - 3D^2 + 3D - 1)y = e^{2z}$$

Let $y = ce^{mz}$ be a trial solution of $(D^3 - 3D^2 + 3D - 1)y = 0$.

Then A.E. is $m^3 - 3m^2 + 3m - 1 = 0$, or $(m-1)^3 = 0$, or $m = 1, 1, 1$.

$\therefore$ C.F. is $(c_1 + c_2 z + c_3 z^2)e^z$.

$$\text{P.I.} = \frac{1}{(D-1)^3} e^{2z} = \frac{e^{2z}}{(2-1)^3} = e^{2z}$$

Therefore, the general solution is

$$y = (c_1 + c_2 z + c_3 z^2)e^z + e^{2z} = \{c_1 + c_2 \cdot \log x + c_3 (\log x)^2\}x + x^2$$

where c_1, c_2, c_3 are arbitrary constants.

EXAMPLE 4.5.2 Solve $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$.

Solution Substitute $\log x = z$, or $x = e^z$. Therefore

$$x\frac{dy}{dx} = Dy, \quad x^2 \frac{d^2y}{dx^2} = D(D-1)y, \quad x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

The given equation becomes

$$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10(e^z + e^{-z})$$

or

$$(D^3 - D^2 + 2)y = 10(e^z + e^{-z})$$

Let $y = ce^{mz}$ be a trial solution of $(D^3 - D^2 + 2)y = 0$.

Thus the A.E. is $m^3 - m^2 + 2 = 0$, or $(m+1)(m^2 - 2m + 2) = 0$, or $m = -1, 1 \pm i$.
C.F. is $c_1 e^{-z} + (c_2 \cos z + c_3 \sin z)e^z$.

P.I. is

$$\frac{1}{D^3 - D^2 + 2} 10(e^z + e^{-z})$$

$$= 10\frac{1}{2}e^z + 10e^{-z} \frac{1}{(D-1)^3 - (D-1)^2 + 2} 1$$

$$= 5e^z + 10e^{-z} \frac{1}{D^3 - 4D^2 + 5D} 1$$

$$= 5e^z + 10e^{-z} \frac{1}{5D}\left(1 + \frac{D^2 - 4D}{5}\right)^{-1} 1$$

$$= 5e^z + 2e^{-z} \frac{1}{D} 1 = 5e^z + 2e^{-z} z$$

Therefore, the general solution is

$$y = c_1 e^{-z} + (c_2 \cos z + c_3 \sin z)e^z + 5e^z + 2e^{-z} z$$

$$= \frac{c_1}{x} + \{c_2 \cos(\log x) + c_3 \sin(\log x)\}x + 5x + \frac{2 \log x}{x}$$

where c_1, c_2 are arbitrary constants.

EXAMPLE 4.5.3 Solve $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x^2e^{2x}$.

Solution Put $\log x = z$, or $x = e^z$. Therefore

$$x\frac{dy}{dx}=Dy,\quad x^2\frac{d^2y}{dx^2}=D(D-1)y$$

where $D\equiv\dfrac{d}{dz}$.

The given equation becomes

$$[D(D-1)+D-1]y=e^{2z}\cdot e^{2e^z}$$

or

$$(D^2-1)y=e^{2z}\cdot e^{2e^z}$$

Let $y=ce^{mz}$ be a trial solution of $(D^2-1)y=0$.

$\therefore$ A.E. is $m^2-1=0$, or $m=\pm 1$.

C.F. is $c_1e^z+c_2e^{-z}$.

$$\text{P.I.} = \frac{1}{D^2-1}e^{2z}\cdot e^{2e^z} = e^{2z}\frac{1}{(D+2)^2-1}e^{2e^z} = e^{2z}\frac{1}{D^2+4D+3}e^{2e^z}$$

$$=e^{2z}\frac{1}{(D+3)(D+1)}e^{2e^z}$$

$$=e^{2z}\frac{1}{D+3}u,\quad \text{where } u=\frac{1}{D+1}e^{2e^z}$$

That is, $(D+1)u=e^{2e^z}$, or $\dfrac{du}{dz}+u=e^{2e^z}$, which is a linear differential equation and its I.F. is e^z.

Multiplying the above equation by e^z, we get $\dfrac{d}{dz}(ye^z)=e^ze^{2e^z}$.

Integrating, $ue^z=\int e^ze^{2e^z}\,dz$.

Putting $e^z=t$. Then $e^z\,dz=dt$.

$\therefore\quad ue^z=\displaystyle\int e^{2t}\,dt=\frac{e^{2t}}{2}=\frac{e^{2e^z}}{2}$, or $u=\dfrac{e^{-z}e^{2e^z}}{2}$.

$\therefore\quad$ P.I. is $e^{2z}\dfrac{1}{D+3}u=e^{2z}v$, where $v=\dfrac{1}{D+3}u$, or $\dfrac{dv}{dz}+3v=u=\dfrac{e^{-z}e^{2e^z}}{2}$, which is linear in v and its I.F. is e^{3z}.

Multiplying the above equation by e^{3z} and integrating, it becomes $ve^{3z}=\displaystyle\int\frac{e^{2z}e^{2e^z}}{2}\,dz$.

Putting $e^z=t$, $e^z\,dz=dt$.

$$ve^{3z}=\frac{1}{2}\int te^{2t}\,dt=\frac{1}{2}\left[t\frac{e^{2t}}{2}-\int\frac{e^{2t}}{2}\,dt\right]$$

$$=\frac{1}{4}\left[te^{2t}-\frac{e^{2t}}{2}\right]=\frac{1}{8}(2e^z-1)e^{2e^z}$$

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or

$$v = \frac{1}{8}(2e^z - 1)e^{2e^z} \cdot e^{-3z}$$

$\therefore$ P.I. is $e^{2z}v = e^{2z}\frac{1}{8}(2e^z - 1)e^{2e^z} \cdot e^{-3z} = \frac{1}{8}e^{-z}(2e^z - 1)e^{2e^z}$.

Therefore, the general solution is

$$\begin{aligned} y &= c_1 e^z + c_2 e^{-z} + \frac{1}{8}e^{-z}(2e^z - 1)e^{2e^z} \\ &= c_1 x + \frac{c_2}{x} + \frac{1}{8}\left(2 - \frac{1}{x}\right)e^{2x} \end{aligned}$$

where c_1, c_2 are arbitrary constants.

4.5.2 Legendre Equations

An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + c_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + c_{n-1}(ax+b)\frac{dy}{dx} + c_n y = f(x) \qquad (4.23)$$

where $c_1, c_2, \ldots, c_{n-1}, c_n, a, b$ are constants, is called **Legendre's equation**.

For this type of problem the suitable transformation is $ax + b = e^z$ or $z = \log(ax+b)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx} = \dfrac{a}{ax+b}Dy$, where $D \equiv \dfrac{d}{dz}$.

That is, $(ax+b)\dfrac{dy}{dx} = aDy$.

Also

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx} = \frac{d}{dz}\left(\frac{a}{ax+b}Dy\right)\frac{dz}{dx} \\ &= -\frac{a^2}{(ax+b)^2}Dy + \frac{a}{ax+b}D^2y\frac{dz}{dx} \\ &= \frac{a^2}{(ax+b)^2}(D^2y - Dy) \end{aligned}$$

Thus, $(ax+b)^2\dfrac{d^2y}{dx^2} = a^2D(D-1)y$.

Similarly, $(ax+b)^3\dfrac{d^3y}{dx^3} = a^3D(D-1)(D-2)y$ and so on.

By these substitutions too the given equation reduces to a linear equation with constant coefficients whose independent variable is z.

EXAMPLE 4.5.4 Solve

$$(2+x)^2\frac{d^2y}{dx^2} + (2+x)\frac{dy}{dx} + 4y = \sin\{2\log(2+x)\}$$

Solution. Let $z = \log(2 + x)$, or $2 + x = e^z$. Therefore

or
$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{2+x}$$

$$(2+x)\frac{dy}{dx} = \frac{dy}{dz}$$

Again, on differentiating we get

or
$$(2+x)\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dz^2}\frac{dz}{dx} = \frac{1}{2+x}\frac{d^2y}{dz^2}$$

$$(2+x)^2\frac{d^2y}{dx^2} + (2+x)\frac{dy}{dx} = \frac{d^2y}{dz^2}$$

Substituting these values into the given equation, we get

$$\frac{d^2y}{dz^2} + 4y = \sin 2z$$

Let $y = ce^{mz}$ be the trial solution.

The A.E. is $m^2 + 4 = 0$, or $m = \pm 2i$.

The C.F. is $c_1 \cos 2z + c_2 \sin 2z$.

$$\text{P.I.} = \frac{1}{D^2+4}\sin 2z = \text{I.P. of } \frac{1}{D^2+4}e^{2iz}$$

$$= \text{I.P. of } e^{2iz}\frac{1}{(D+2i)^2+4}1 = \text{I.P. of } e^{2iz}\frac{1}{D^2+4Di}1$$

$$= \text{I.P. of } e^{2iz}\frac{1}{4Di}\left(1+\frac{D}{4i}\right)^{-1}1 = \text{I.P. of } e^{2iz}\frac{1}{4Di}\left(1-\frac{D}{4i}+\cdots\right)1$$

$$= \text{I.P. of } e^{2iz}\frac{1}{4Di}(1) = \text{I.P. of } e^{2iz}\frac{1}{4i}z$$

$$= \text{I.P. of } \frac{z}{4}(\cos 2z + i\sin 2z)(-i) = -\frac{z}{4}\cos 2z$$

Hence the general solution is

$$y = c_1\cos 2z + c_2\sin 2z - \frac{z}{4}\cos 2z$$

$$= c_1\cos\{2\log(2+x)\} + c_2\sin\{2\log(2+x)\} - \frac{\log(2+x)}{4}\cos\{2\log(2+x)\}$$

where c_1, c_2 are arbitrary constants.

EXAMPLE 4.5.6 Solve $(1+2x)^2\dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$.

Solution. Let $z = \log(1+2x)$, i.e., $e^z = 1+2x$.

Now $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx} = \dfrac{2}{1+2x}\dfrac{dy}{dz}$ or $(1+2x)\dfrac{dy}{dx} = 2Dy$, where $D = \dfrac{d}{dz}$.

Again differentiating with respect to x, we get

$$(1+2x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2\frac{d^2y}{dz^2}\frac{dz}{dx} = 2\cdot\frac{2}{1+2x}\frac{d^2y}{dz^2}$$

or

$$(1+2x)^2\frac{d^2y}{dx^2} + 2(1+2x)\frac{dy}{dx} = 4\frac{d^2y}{dz^2}$$

or

$$(1+2x)^2\frac{d^2y}{dx^2} = 4\frac{d^2y}{dz^2} - 2\cdot 2Dy = 4D(D-1)y$$

Thus the given equation reduces to

$$\{4D(D-1) - 6\cdot 2D + 16\}y = 8e^{2z} \qquad \text{or} \qquad (D^2 - 4D + 4)y = 2e^{2z}$$

The A.E. is $m^2 - 4m + 4 = 0$, or $m = 2, 2$. Thus C.F. is $(c_1 + c_2 z)e^{2z}$.

$$\text{P.I.} = \frac{1}{(D-2)^2}2e^{2z} = 2e^{2z}\frac{1}{(D+2-2)^2}1 = 2e^{2z}\frac{1}{D^2}1 = z^2e^{2z}$$

Therefore, the general solution is

$$y = (c_1 + c_2 z)e^{2z} + z^2e^{2z} = \{c_1 + c_2\log(1+2x)\}(1+2x)^2 + (1+2x)^2\{\log(1+2x)\}^2$$

4.6 Method of Variation of Parameters

Let us consider the second order linear differential equation of the form

$$y'' + p(x)y' + q(x)y = r(x) \tag{4.24}$$

Again, we suppose that $y = y_1(x)$ and $y = y_2(x)$ be two independent solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{4.25}$$

are known.

Then C.F. is $c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are arbitrary constants.

The method of variation of parameters actually determines the particular solution based on the C.F. by taking c_1 and c_2 as functions of x. That is, we assume that

$$y = Ay_1(x) + By_2(x)$$

is the general solution of (4.24), where A and B are functions of x. Differentiating, we get

$$y' = Ay_1' + By_2' + A'y_1 + B'y_2$$

We choose A and B in such a way that

$$A'y_1 + B'y_2 = 0 \tag{4.27}$$

Then y' reduces to $y' = Ay_1' + By_2'$.

Again we differentiate and obtain

$$y'' = Ay_1'' + By_2'' + A'y_1' + B'y_2'$$

The values of y, y' and y'' are substituted in (4.24). Therefore

$$Ay_1'' + By_2'' + A'y_1' + B'y_2' + p(x)[Ay_1' + By_2'] + q(x)[Ay_1 + By_2] = r(x) \tag{4.28}$$

This gives

$$A[y_1'' + p(x)y_1' + q(x)y_1] + B[y_2'' + p(x)y_2' + q(x)y_2] + A'y_1' + B'y_2' = r(x) \tag{4.29}$$

In view of the fact that y_1 and y_2 are the solutions of (4.25), then the first two terms of above equation become zero; hence

$$A'y_1' + B'y_2' = r(x) \tag{4.30}$$

The solutions of (4.27) and (4.30) for A' and B' is

$$A' = -\frac{r(x)y_2(x)}{W(x)} \quad \text{and} \quad B' = \frac{r(x)y_1(x)}{W(x)}$$

where $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$

Thus

$$A = -\int \frac{r(x)y_2(x)}{W(x)}dx + c_1 \qquad \text{and} \qquad B = \int \frac{r(x)y_1(x)}{W(x)}dx + c_2$$

Hence the required general solution is

$$y = -y_1 \int \frac{r(x)y_2(x)}{W(x)}dx + y_2 \int \frac{r(x)y_1(x)}{W(x)}dx + c_1y_1 + c_2y_2$$

EXAMPLE 4.6.1 Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 9e^x \qquad \text{(WBUT 2004)}$$

Solution Let $y = ce^{mx}$ be a solution of $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

The A.E. is $m^2 - 3m + 2 = 0$, or $m = 1, 2$.

$\therefore$ C.F. is $c_1e^x + c_2e^{2x}$.

Let the general solution be $y = Ae^x + Be^{2x}$, where A and B are functions of x.

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Let $y_1 = e^x$ and $y_2 = e^{2x}$. Then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$$

The values of A and B are given by

$$A = -\int \frac{y_2(x)r(x)}{W}\,dx + c_1 \qquad \text{and} \qquad B = \int \frac{y_1(x)r(x)}{W}\,dx + c_2$$

That is

$$A = -\int \frac{e^{2x}\cdot 9e^x}{e^{3x}}\,dx + c_1 = -\int 9dx + c_1 = -9x + c_1$$

and

$$B = \int \frac{e^x \cdot 9e^x}{e^{3x}}\,dx + c_2 = \int 9e^{-x}dx + c_2 = -9e^{-x} + c_2$$

Hence the general solution is

$$y = (-9x + c_1)e^x + (-9e^{-x} + c_2)e^{2x} = -9e^x(x+1) + c_1e^x + c_2e^{2x}$$

EXAMPLE 4.6.2 Apply the method of variation of parameter to solve the equation

$$\frac{d^2y}{dx^2} + y = \sec^3 x \cdot \tan x \qquad \text{(WBUT 2007)}$$

Solution Let $y = ce^{mx}$ be a solution of $(D^2 + 1)y = 0$.

$\therefore$ A.E. is $m^2 + 1 = 0$, or $m = \pm i$.

$\therefore$ C.F. is $c_1 \cos x + c_2 \sin x$.

Let the general solution be $y = A\cos x + B\sin x$, where A and B are functions of x.

Let $y_1 = \cos x$ and $y_2 = \sin x$. Then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

The values of A and B are given by

$$A = -\int \frac{y_2(x)r(x)}{W}\,dx + c_1 = -\int \frac{\sin x \cdot \sec^3 x \tan x}{1}\,dx + c_1$$

$$= -\int \frac{\sin^2 x}{\cos^4 x}dx + c_1 = -\int \tan^2 x \sec^2 x\, dx + c_1$$

(Putting $\tan x = z, \sec^2 x\, dx = dz$)

$$= -\int z^2\, dz + c_1 = -\frac{z^3}{3} + c_1 = -\frac{1}{3}\tan^3 x + c_1$$

and

$$B = \int \frac{y_1(x)r(x)}{W}\, dx + c_2$$

$$= -\int \frac{\cos x \cdot \sec^3 x \tan x}{1}\, dx + c_2$$

$$= \int \tan x \sec^2 x\, dx + c_2 = \int \tan x\, d(\tan x)\, dx + c_2$$

$$= \frac{1}{2}\tan^2 x + c_2$$

Hence the required solution is

$$y = -\frac{1}{3}\tan^3 x \cos x + c_1 \cos x + \frac{1}{2}\tan^2 x \sin x + c_2 \sin x$$

$$= \frac{1}{6}\tan^2 x \sin x + c_1 \cos x + c_2 \sin x$$

EXAMPLE 4.6.3 Solve $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$. The method of variation of parameters may be used.

Solution Let $y = ce^{mx}$ be a trial solution of

$$\frac{d^2y}{dx^2} + a^2y = 0 \qquad \text{(i)}$$

$\therefore$ A.E. is $m^2 + a^2 = 0$, or $m = \pm ia$.

$\therefore$ $y = \cos ax$ and $y = \sin ax$ are two independent solutions of (i). (*)

Let $y = A\cos ax + B\sin ax$, where A and B are functions of x, be the general solution of the given differential equation.

Then

$$\frac{dy}{dx} = A'\cos ax - Aa\sin ax + B'\sin ax + Ba\cos ax$$

We choose A and B in such a way that

$$A'\cos ax + B'\sin ax = 0 \qquad \text{(ii)}$$

Therefore

$$\frac{dy}{dx} = -Aa\sin ax + Ba\cos ax$$

Again differentiating

$$\frac{d^2y}{dx^2} = -A'a\sin ax - Aa^2\cos ax + B'a\cos ax - Ba^2\sin ax$$

Using the values of y and $\dfrac{d^2y}{dx^2}$, the given equation becomes

$$-Aa'\sin ax + B'a\cos ax = \sec ax \tag{iii}$$

The solution of (ii) and (iii) is $A' = -\dfrac{1}{a}\tan ax$ and $B' = \dfrac{1}{a}$.

Integrating, $A = \dfrac{1}{a^2}\log\cos ax + c_1$ and $B = \dfrac{x}{a} + c_2$.

Hence

$$\begin{aligned} y &= A\cos ax + B\sin ax \\ &= \left(\frac{1}{a^2}\log\cos ax + c_1\right)\cos ax + \left(\frac{x}{a} + c_2\right)\sin ax \\ &= c_1\cos ax + c_2\sin ax + \frac{\cos ax}{a^2}\log\cos ax + \frac{x}{a}\sin ax \end{aligned}$$

where c_1, c_2 are arbitrary constants.

EXAMPLE 4.6.4 Solve the equation $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = \dfrac{e^{2x}}{x^2}$ by the method of variation of parameters.

Solution Let $y = ce^{mx}$ be a solution of $(D^2 - 4D + 4)y = 0$.

$\therefore$ A.E. is $m^2 - 4m + 4 = 0$, or $m = 2, 2$.

$\therefore$ C.F. is $(c_1 + c_2x)e^{2x}$.

Let the general solution be $y = (A + Bx)e^{2x}$, where A, B are functions of x.

$$\frac{dy}{dx} = 2Ae^{2x} + A'e^{2x} + Be^{2x} + 2Bxe^{2x} + B'xe^{2x}$$

We choose A, B such that

$$A' + B'x = 0 \tag{i}$$

Now, $\dfrac{dy}{dx} = 2Ae^{2x} + Be^{2x} + 2Bxe^{2x}$.

Again, differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 2A'e^{2x} + 2Be^{2x} + B'e^{2x} + 2Be^{2x} + 2B'xe^{2x} + 4Bxe^{2x}$$

Putting these values into the given equation, we obtain

$$(4A + 2A' + 4B + B' + 2B'x + 4Bx)e^{2x} - 4(2A + B + 2Bx)e^{2x} + 4(A + Bx)e^{2x} = \frac{e^{2x}}{x^2}$$

or

$$2A' + B' + 2B'x = \frac{1}{x^2} \tag{ii}$$

Solving (i) and (ii) we obtain $A' = -1/x$, $B' = 1/x^2$. Therefore

$$A = -\log x + c_3,\ B = -\frac{1}{x} + c_4$$

Hence the general solution is $y = (-\log x + c_3 - 1 + c_4x)e^{2x}$, where c_3, c_4 are arbitrary constants.

EXAMPLE 4.6.5 Solve the equation $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ by using the method of variation of parameters.

Solution Let $y = ce^{mx}$ be a trial solution of $(D^2+1)y = 0$.

$\therefore$ A.E. is $m^2 + 1 = 0$, or $m = \pm i$.

C.F. is $c_1 \cos x + c_2 \sin x$, where c_1 and c_2 are arbitrary constants.

Let $y = A\cos x + B\sin x$ be the general solution of the given equation, where A and B are functions of x.

Now, $\frac{dy}{dx} = A'\cos x - A\sin x + B'\sin x + B\cos x$.

We choose A and B in such a way that

$$A'\cos x + B'\sin x = 0 \tag{i}$$

Then $\frac{dy}{dx} = -A\sin x + B\cos x$.

Again, on differentiating we get

$$\frac{d^2y}{dx^2} = -A\cos x - A'\sin x - B\sin x + B'\cos x$$

On putting the values of y and $\frac{d^2y}{dx^2}$ into the given equation, it reduces to

$$-A\cos x - A'\sin x - B\sin x + B'\cos x + A\cos x + B\sin x = \operatorname{cosec} x$$

or

$$-A'\sin x + B'\cos x = \operatorname{cosec} x \tag{ii}$$

The solution of (i) and (ii) is $A' = -1, B' = \cot x$.

$\therefore$ $A = -x + c_3$ and $B = \log \sin x + c_4$.

Hence the general solution is $y = (-x+c_3)\cos x + (\log\sin x + c_4)\sin x = c_3\cos x + c_4 \sin x - x\cos x + \sin x \log \sin x$, where c_3, c_4 are arbitrary constants.

EXAMPLE 4.6.6 Find the general solution of $(1+x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1+x)^2$ by the method of variation of parameters. It is given that $y = x$ and $y = e^{-x}$ are two independent solutions of the corresponding homogeneous equation.

Solution Let $y = x$ and $y = e^{-x}$ be two solutions. To test their independence, we consider the Wronskian

$$W(x) = \begin{vmatrix} x & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = -xe^{-x} - e^{-x} \neq 0$$

Thus the two solutions x and e^{-x} are independent.

Let $y = Ax + Be^{-x}$, where A and B are functions of x, be the general solution.

$\frac{dy}{dx} = A'x + A + B'e^{-x} - Be^{-x}$.

We choose A, B in such a way that

$$A'x + B'e^{-x} = 0 \tag{i}$$

$\therefore \quad \dfrac{dy}{dx} = A - Be^{-x}.$

Again, on differentiating we get $\dfrac{d^2y}{dx^2} = A' - B'e^{-x} + Be^{-x}$.

Putting the value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into the given equation, we get

$$(1+x)(A' - B'e^{-x} + Be^{-x}) + x(A - Be^{-x}) - (Ax + Be^{-x}) = (1+x)^2$$

or

$$(1+x)(A' - B'e^{-x}) = (1+x)^2 \qquad \text{or} \qquad A' - B'e^{-x} = 1 + x \tag{ii}$$

The solution of (i) and (ii) is $A' = 1$ and $B' = -xe^{-x}$.

Integrating, we get $A = x + c_1$ and $B = -\int xe^x\, dx = -xe^x + \int e^x\, dx = -xe^x + e^x + c_2$.

Hence the general solution is

$$y = (x + c_1)x + (-xe^x + e^x + c_2)e^{-x} = c_1 x + c_2 e^{-x} + x^2 + (1 - x)$$

where c_1, c_2 are arbitrary constants.

EXAMPLE 4.6.7 Solve $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 9y = 0$, given that $y = x^3$ is a solution.

Solution Since $y = x^3$ is a solution, let the solution of the given equation be $y = ux^3$, where u is a function of x.

Therefore, $\dfrac{dy}{dx} = 3x^2u + \dfrac{du}{dx}x^3$ and $\dfrac{d^2y}{dx^2} = 6xu + 6x^2\dfrac{du}{dx} + x^3\dfrac{d^2u}{dx^2}$.

By these substitutions, the given equation reduces to

$$x^2\left[x^3\frac{d^2u}{dx^2} + 6x^2\frac{du}{dx} + 6xu\right] + x\left[3x^2u + x^3\frac{du}{dx}\right] - 9x^3u = 0$$

or

$$x^5\frac{d^2u}{dx^2} + 7x^4\frac{du}{dx} + 6x^3u + 3x^3u - 9x^3u = 0$$

or

$$x^4\left[x\frac{d^2u}{dx^2} + 7\frac{du}{dx}\right] = 0$$

or

$$x\frac{d^2u}{dx^2} + 7\frac{du}{dx} = 0$$

Let $\dfrac{du}{dx} = p$. Then the above equation becomes

$$x\frac{dp}{dx} + 7p = 0 \quad \text{or} \quad \frac{dp}{p} = -7\frac{dx}{x}$$

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Integrating, we get $\log p = -7\log x + \log c_1$, where c_1 is arbitrary constant, or $p = c_1/x^7$. That is, $\dfrac{du}{dx} = \dfrac{c_1}{x^7}$, or $du = \dfrac{c_1}{x^7}dx$.

Now, integrating it, we get $u = -\dfrac{c_1}{6x^6} + c_2$, where c_2 is arbitrary constant.

Hence the solution is $y = ux^3 = -\dfrac{c_1}{6x^3} + c_2x^3$.

EXAMPLE 4.6.8 If $y = y_1$ and $y = y_2$ be two solutions of $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = 0$, then show that $y_1\dfrac{dy_2}{dx} - y_2\dfrac{dy_1}{dx} = c \cdot e^{-\int P dx}$ where c is a constant.

Solution Given that $y = y_1$ is a solution, let $y = vy_1$. Therefore,

$$\frac{dy}{dx} = v\frac{dy_1}{dx} + y_1\frac{dv}{dx}$$

and

$$\frac{d^2y}{dx^2} = v\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2}$$

Putting these values to the given equation, we have

$$\left(v\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2}\right) + P\left(v\frac{dy_1}{dx} + y_1\frac{dv}{dx}\right) + Qvy_1 = 0$$

or

$$v\left(\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1\right) + y_1\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{dv}{dx}\frac{dy_1}{dx} = 0 \qquad \text{(i)}$$

Since $y = y_1$ is a solution of the given equation

$$\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1 = 0$$

Therefore, (i) becomes

$$y_1\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{dv}{dx}\frac{dy_1}{dx} = 0$$

or

$$\frac{d^2v}{dx^2} + \frac{dv}{dx}\left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right) = 0$$

Now we put, $\dfrac{dv}{dx} = p$, so that $\dfrac{d^2v}{dx^2} = \dfrac{dp}{dx}$. Therefore,

$$\frac{dp}{dx} + p\left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right) = 0, \quad \text{or} \quad \frac{dp}{p} + \left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right)dx = 0$$

Integrating, we get

$$\log p + \int P\,dx + \int \frac{2}{y_1}\,dy_1 = 0$$

or

$$\log p + \int P\,dx + 2\log y_1 = \log c \qquad \text{or} \qquad \log\left(\frac{py_1^2}{c}\right) = -\int P\,dx$$

or

$$py_1^2 = ce^{-\int P\,dx} \qquad \text{or} \qquad \frac{dv}{dx}y_1^2 = ce^{-\int P\,dx}$$

Now, $y = vy_1$. But $y = y_2$ is also a solution.

$\therefore$ $y_2 = vy_1$, or $v = \frac{y_2}{y_1}$. So $\frac{dv}{dx} = \frac{y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}}{y_1^2}$.

Hence from (ii), $y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx} = ce^{-\int P\,dx}$.

EXERCISES

Section A Multiple Choice Questions

1. The general solution of the differential equation $D^2y + 9y = 0$ is
   (a) $Ae^{3x} + Be^{-3x}$ (b) $(A + Bx)e^{3x}$ (c) $A\cos 3x + B\sin 3x$
   (d) $(A + Bx)\sin 3x$.
   (WBUT 20[illegible]
2. The general solution of $(D^2 - 1)^2y = 0$ is
   (a) $c_1e^x + c_2e^{-x}$ (b) $c_1e^x + c_2e^x + c_3e^{-x} + c_4e^{-x}$
   (c) $(c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x}$ (d) $(c_1\cos x + c_2\sin x)^2$.
3. The general solution of $\frac{d^2y}{dx^2} + 4y = 0$ is
   (a) $Ae^{2x} + Be^{-2x}$ (b) $(A + Bx)e^{2x}$ (c) $A\cos 2x + B\sin 2x$
   (d) $(A + Bx)\cos 2x$.
   (WBUT 20[illegible]
4. The general solution of $(D^2 + 5D + 6)y = 0$ is
   (a) $c_1e^{-3x} + c_2e^{-2x}$ (b) $(c_1 + c_2x)e^{-2x}$ (c) $c_1\cos 3x + c_2\sin 3x$
   (d) $e^{-3x} + e^{-2x}$.
5. The C.F. of the differential equation $(D^2 + 16)y = \cos x$ is
   (a) $c_1e^{4x} + c_2e^{-4x}$ (b) $c_1e^{4x} + c_2e^{-4x} + \frac{1}{15}\cos x$
   (c) $c_1\cos 4x + c_2\sin 4x + \frac{1}{15}\cos x$ (d) $c_1\cos 4x + c_2\sin 4x$.
6. A particular solution of $(D^2 - 1)y = 0$ when $x = 0, y = 0$ and $x = 0, Dy = 1$ is
   (a) $c_1e^x + c_2e^{-x}$ (b) $\frac{1}{2}(e^x - e^{-x})$ (c) $c_1\cos x + c_2\sin x$ (d) $\frac{1}{2}(e^x + e^{-x})$
7. The P.I. of $(D^2 - 2D + 4)y = e^{2x}$ is
   (a) e^{2x} (b) $\frac{1}{2}e^{2x}$ (c) $\frac{x^2}{2}$ (d) $\frac{1}{4}e^{2x}$.
8. The P.I. of $(D^2 + 4)y = \sin 3x$ is
   (a) $\frac{1}{5}\sin 3x$ (b) $-\frac{1}{5}\sin 3x$ (c) $\frac{1}{5}\cos 3x$ (d) $-\frac{1}{5}\cos 3x$.
9. The value of $\frac{1}{D^3}x$ is
   (a) $x^2/2$ (b) $x^3/6$ (c) $x^4/24$ (d) 0.

10. $\frac{1}{D^2}\cos x$ is equal to
(a) $-\sin x$ (b) $-\cos x$ (c) $\cos x$ (d) $\sin x$.

11. The value of $\frac{1}{D^2-1}4xe^x$ is
(a) $e^x(x^2+x)$ (b) x^2e^x (c) x^3e^x (d) $e^x(x^2-x)$.

12. $\frac{1}{D+2}e^{-2x}\sin 3x$ is equal to
(a) $-\frac{1}{3}e^{-2x}\cos 3x$ (b) $e^{-2x}\cos 3x$ (c) $-\frac{1}{3}e^{-2x}\sin 3x$ (d) $\frac{1}{3}e^{-2x}\sin 3x$.

13. $\frac{1}{D-1}x^2$ is equal to
(a) x^2+2x+2 (b) $-(x^2+2x+2)$ (c) $2x-x^2$ (d) $-(2x-x^2)$.

14. $\frac{1}{D^2+4}\sin 2x$ is equal to
(a) $\frac{x}{4}\cos 2x$ (b) $-\frac{x}{4}\cos 2x$ (c) $\frac{x}{4}\sin 2x$ (d) $x\sin 2x$.

15. The P.I. of $(D^4-8D)y=x^2$ is
(a) $-\frac{x^4}{32}$ (b) $\frac{x^4}{32}$ (c) $\frac{x^3}{8}$ (d) $-\frac{1}{8}x^3$.

16. Using the transformation $x=e^z$ the equation $x\dfrac{dy}{dx}+y=(\log x)^2$ reduces to
(a) $\dfrac{d^2y}{dz^2}+y=z$ (b) $2\dfrac{d^2y}{dz^2}+2y=z^2$ (c) $\dfrac{d^2y}{dz^2}+y=z^2$
(d) none of these.

17. Using the substitution $z=\log x$ the equation $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+4y=2x^2$ reduces to
(a) $\dfrac{d^2y}{dz^2}-4\dfrac{dy}{dz}+4y=2e^{2z}$ (b) $\dfrac{d^2y}{dz^2}+4y=2e^{2z}$ (c) $\dfrac{d^2y}{dz^2}-3\dfrac{dy}{dz}+4y=2e^{2z}$
(d) none of these.

18. The C.F. of the equation $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-4y=x^2$ is
(a) $c_1e^{-x}+c_2e^{2x}$ (b) $c_1x^2+c_2/x$ (c) $c_1\cos 2x+c_2\sin 2x$
(d) none of these.

19. The C.F. of the equation $x^2\dfrac{d^2y}{dx^2}+4y=\sin(\log x)$ is
(a) $c_1\cos 2x+c_2\sin 2x$ (b) $c_1e^{2x}+c_2e^{-2x}$
(c) $\{c_1\cos(\frac{\sqrt{3}}{2}\log x)+c_2\sin(\frac{\sqrt{3}}{2}\log x)\}\sqrt{x}$ (d) $c_1\cos\sqrt{3}x+\sin\sqrt{3}x$.

20. The C.F. of the equation $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=x^2$ is
(a) $(c_1+c_2x)e^{-x}$ (b) $(c_1+c_2x)/x$ (c) $(c_1+c_2\log x)/x$ (d) $c_1e^x+c_2\frac{1}{x}$.

21. The C.F. of $x^2\dfrac{d^2y}{dx^2}+5x\dfrac{dy}{dx}+4y=x^4$ is
(a) $(c_1+c_2\log x)/x^2$ (b) $c_1e^x+c_2e^{-2x}$ (c) $c_1x+c_2x^{-1}$
(d) none of these.

22. The P.I. of $x^2\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+6y=x$ is
(a) $x^2/2$ (b) $x/2$ (c) $x^3/2$ (d) $x^3/4$.

23. The general solution of $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=\log x$ is

(a) $c_1+c_2x+\frac{x^3}{6}$ (b) $c_1+c_2\log x+\frac{(\log x)^3}{6}$ (c) $c_1+c_2\log x$ (d) [illegible]

24. Using the transformation $z=\log(x+a)$, the equation $(x+a)^2\dfrac{d^2y}{dx^2}-4(x+a)\dfrac{dy}{dx}$ [illegible] is transformed to

(a) $\dfrac{d^2y}{dz^2}-5\dfrac{dy}{dz}+6y=0$ (b) $\dfrac{d^2y}{dz^2}-6\dfrac{dy}{dz}=0$ (c) $\dfrac{d^2y}{dz^2}-4\dfrac{dy}{dz}+6y=0$

(d) none of these.

25. By the transformation $z=\log(5+2x)$ the equation $(5+2x)^2\dfrac{d^2y}{dx^2}-6(5+2x)\dfrac{dy}{dx}$ [illegible] is transformed to

(a) $\dfrac{d^2y}{dz^2}-4\dfrac{dy}{dz}+2y=0$ (b) $\dfrac{d^2y}{dz^2}-6\dfrac{dy}{dz}+8y=0$ (c) $\dfrac{d^2y}{dz^2}+4y=0$

(d) none of these.

Section B Review Questions

Solve:

1. $\dfrac{d^3y}{dx^3}+2\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=e^{2x}$
2. $(D^3+D^2-D-1)y=\cosh x$
3. $(D^2-D-2)y=4x$
4. $(D^4-8D)y=x^2+e^{2x}$
5. $(D^3-3D^2+3D-1)y=xe^x+e^x$
6. $(D^5-D)y=e^x+\sin x-x$
7. $(D^2-2D+1)y=e^{-2x}\sin 2x$
8. $(D^3-D^2+3D+5)y=e^x\cos 3x$
9. $(D^2+D+1)y=(1-e^x)^2$
10. $(D-2)^2y=8(e^{2x}+\sin 2x+x^2)$
11. $(D^2-3D+2)y=xe^{3x}+\sin 2x$
12. $(D^2-4)y=x\sinh x$
13. $(D^2-1)y=x\sin 3x+\cos x$
14. $(D^2-2D+1)y=xe^x\sin x$
15. $(D^4+2D^2+1)y=x^2\cos x$
16. $(D^3+2D^2+D)y=e^{-x}+\sin 2x$

17. $(D^2+1)^2 y = x^4 + 2\sin x \cos 3x$

18. $(D^4-1)y = e^x \cos x$

19. $(D^2+2)y = x^2 e^{3x} + e^x \cos 2x$

20. $(D^4-1)y = \cos x \cosh x$

21. $(D^2+4)y = x \sin x$

22. $(D^3+2D^2+D)y = x^2 e^{2x} + \sin^2 x$

23. $(D^2-1)y = x\sin x + (1+x^2)e^x$

24. $(D^3+3D+2)y = e^{e^x}$

25. $(D^4-1)y = x^2 \sin x$

26. $(D^2+a^2)y = \sec ax$

27. $(D^2+4)y = \sec 2x$

28. $(D^3-3D^2+4D-2)y = e^x + \cos x$

29. $(D^2-4D+3)y = e^x \cos 2x + \cos 3x$

30. $(D^2-2D+2)y = e^x \sin 2x$

31. $(D^4+2D^3+3D^2+2D+1)y = xe^x$

32. $(D^3-1)y = x\sin x$

33. $(D^2-1)y = e^{2x}\sin x + e^{x/2}\sin \frac{x\sqrt{3}}{2}$

34. $(D^4+D^3+D^2)y = x^2(a+bx)$

35. $(D^2+a^2)y = \cos ax$

36. $(D^2-5D+6)y = e^x \cos x$ (WBUT 2002)

37. $(D^2+4)y = x\sin^2 x$ (WBUT 2008)

38. $(D^2-2D+1)y = xe^x$

39. $(D^2+a^2)y = \tan ax$

40. $(D^4+D^2+16)y = 16x^2 + 256$

41. $(D^2-1)y = e^{-x} + \cos x + x^3 + e^x \cos x$

42. $(D^2-4D+4)y = x^2 + e^x + \sin 2x$

43. $(D^5 - D)y = 12e^x + 8\sin x - 2x$

44. $(D^2 + 1)y = \cos x + xe^{2x} + e^x \sin x$

45. $(D^2 + 2D + 1)y = e^x + x^2 - \sin x$

46. $(D^2 - 4D + 1)y = 73\sin 2x + x + 13e^{-x/2}$

47. $(D^2 - 4D + 3)y = e^{2x}\sin 3x$

48. $(D^2 - 5D + 6)y = x(x + e^x)$

49. $(D^2 + 4)(D^2 + 1)y = \cos 2x + \sin x$

50. $(D^3 - 3D - 2)y = 540x^3e^{-x}$

51. $(D^2 + a^2)y = x\cos ax$

52. $(D^4 - 1)y = x\sin x$

53. If $\dfrac{d^2x}{dt^2} + \dfrac{g}{b}(x - a) = 0$ and $x = a'$ and $\frac{dx}{dt} = 0$ when $t = 0$, show that

$$x = a + (a' - a)\cos\sqrt{\frac{g}{b}}t$$

54. Solve the equation $\dfrac{d^2x}{dt^2} + 2n\cos\alpha\dfrac{dx}{dt} + n^2x = a\cos nt$, given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

55. Solve $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y + 37\sin 3x = 0$, given that $y = 3$, $Dy = 0$ when $x = 0$.

56. Find the value of u which satisfies the equation $\frac{d^2u}{d\theta^2} + u = 2k\cos\theta$ and also the following conditions:

(i) u has the same value when $\theta = \pm\pi/2$

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} u\,d\theta = 0.$

57. Show that the transformation $x = \sinh z$ transforms $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 4y$ to $\dfrac{d^2y}{dz^2} = 4y$ and hence solve it.

58. Solve $(D^4 - n^4)y = 0$ completely. If $Dy = y = 0$ when $x = 0$ and $x = l$, then prove that $y = c_1(\cos nx - \cosh nx) + c_2(\sin nx - \sinh nx)$ and $\cos nl \cosh nl = 1$.

59. Solve the equation $\dfrac{d^2x}{dt^2} + 20\dfrac{dx}{dt} + 64x = 0$, given that $x = 1/3$ and $\frac{dx}{dt} = 0$ at $t = 0$.

60. Solve the equation $\dfrac{1}{5}\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 2x = 5\cos 4t$, given that $x(0) = \frac{1}{2}$ and $\left(\dfrac{dx}{dt}\right)_{t=0} = 0.$

61. $x^3\dfrac{d^3y}{dx^3}+6x^2\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}-4y=0$

62. $x^2\dfrac{d^2y}{dx^2}+y=3x^2$

63. $x^2\dfrac{d^2y}{dx^2}+5x\dfrac{dy}{dx}+4y=x^4$

64. $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-y=3x^3\cos(\log x)$

65. $x^4\dfrac{d^3y}{dx^3}+2x^3\dfrac{d^2y}{dx^2}-x^2\dfrac{dy}{dx}+xy=1$

66. $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+2y=x\log x$

67. $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-2y=(x+1)^2$

68. $x\dfrac{d^2y}{dx^2}-\dfrac{2y}{x}=x+\dfrac{1}{x^2}$

69. $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}-4y=x^4$

70. $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}=\dfrac{12\log x}{x^2}$

71. $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}-4y=x^2+2\log x$

72. $x^3\dfrac{d^3y}{dx^3}+2x^2\dfrac{d^2y}{dx^2}+2y=10(x+1/x)$

73. $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-12y=x^3\log x$

74. $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x^m$

75. $\dfrac{d^3y}{dx^3}-\dfrac{4}{x}\dfrac{d^2y}{dx^2}+\dfrac{5}{x^2}\dfrac{dy}{dx}-\dfrac{2y}{x^3}=1$

76. $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+y=\dfrac{\log x\cdot\sin(\log x)+1}{x}$

77. $\dfrac{d^2V}{dr^2}+\dfrac{1}{r}\dfrac{dV}{dr}=4\pi\rho, \rho=\text{constant}$

78. $(x^4D^4+6x^3D^3+9x^2D^2+3xD+1)y=(1+\log x)^2$

79. $(x^2D^2-xD+4)y=\cos(\log x)+x\sin(\log x)$

80. $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 10$

81. $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$

82. $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + x + 1$

83. $(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$

84. $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\log(1+x)$

85. $[(1+2x)^2D^2 - 6(1+2x)D + 16]y = 8(1+2x)^2$

86. $(x+1)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} = (2x+3)(2x+4)$

87. $(x+3)^2\frac{d^2y}{dx^2} - 4(x+3)\frac{dy}{dx} + 6y = \log(x+3)$

88. $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Using the method of variation of parameter, solve the following equations:

89. $y'' + 3y' + 2y = 12e^x$

90. $y'' + 2y' + y = x^2e^{-x}$

91. $y'' + y = 4x\sin x$

92. $y'' - 2y' + y = x^2\log x$

93. $y'' - y = \frac{2}{1+e^x}$

94. $y'' - 3y' + 2y = \frac{e^x}{1+e^x}$

95. $y'' + 2y' + y = \frac{1}{e^x x^2}$

96. $y'' + 4y = 4\tan 2x$ (WBUT 20

97. $y'' - 3y' + 2y = 9e^x$ (WBUT 20

98. $y'' + 9y = \sec 3x$ (WBUT 2005, 20

99. $y'' + y = \operatorname{cosec} x$

100. $y'' - 2y' + y = e^x/x$

101. $y'' - 2y' + 2y = e^x\tan x$

102. $y'' - 5y' + 6y = e^{3x} + \sin x$

103. $(D^2 - 2D)y = e^x \sin x$

104. $y'' + 4y = 4\sec^2 2x$ (WBUT 2006)

105. $y'' + y = \sec^3 x \tan x$ (WBUT 2007)

106. Solve $x^2y'' - xy' + y = 0$, given that $y_1 = x$ is a solution.

107. Given that $y = x$ is a solution of $x^2y'' + xy' - y = 0$, find the general solution of $x^2y'' + xy' - y = x$.

Answers

Section A Multiple Choice Questions

1. (c) 2. (c) 3. (c) 4. (a) 5. (d) 6. (b) 7. (d) 8. (b) 9. (c)
10. (b) 11. (d) 12. (a) 13. (b) 14. (b) 15. (a) 16. (c) 17. (a) 18. (b)
19. (c) 20. (c) 21. (a) 22. (b) 23. (b) 24. (a) 25. (a)

Section B Review Questions

1. $c_1 + (c_2 + c_3x)e^{-x} + \frac{1}{18}e^{2x}$
2. $c_1e^x + (c_2 + c_3x)e^{-x} + \frac{1}{8}xe^x - \frac{1}{8}x^2e^{-x}$
3. $c_1e^{-x} + c_2e^{2x} + 1 - 2x$
4. $c_1 + c_2e^{2x} + e^{-x}(c_3\cos\sqrt{3}x + c_4\sin\sqrt{3}x) - \frac{x^3}{24} + \frac{xe^{2x}}{24}$
5. $(c_1 + c_2x + c_3x^2)e^x + \frac{1}{24}e^x(4x^3 + x^4)$
6. $c_1 + c_2e^x + c_3e^{-x} + c_4\cos x + c_5\sin x + \frac{1}{4}xe^x + \frac{1}{8}x^2\cos x - \frac{1}{2}x^2$
7. $c_1e^{-2x} + c_2e^{-3x} - \frac{1}{10}e^{-2x}(\cos 2x + 2\sin 2x)$
8. $c_1e^{-x} + e^x(c_2\cos 2x + c_3\sin 2x) - \frac{1}{65}e^x(3\sin 3x + 2\cos 3x)$
9. $e^{-x/2}(c_1\cos\frac{\sqrt{3}x}{2} + c_2\sin\frac{\sqrt{3}x}{2}) + 1 - \frac{2}{3}e^x + \frac{e^{2x}}{7}$
10. $(c_1 + c_2x)e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$
11. $c_1e^x + c_2e^{2x} + e^{3x}(\frac{x}{2} - \frac{3}{4}) + \frac{1}{20}(3\cos 2x - \sin 2x)$
12. $c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$
13. $c_1e^x + c_2e^{-x} - \frac{1}{50}(5x\sin 3x + 3\cos 3x + 25\cos x)$
14. $(c_1 + c_2x)e^x - e^x(x\sin x + 2\cos x)$
15. $(c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x + \frac{1}{48}(4x^3\sin x - x^2(x^2 - 9)\cos x)$
16. $c_1 + (c_2 + c_3x)e^x - \frac{x^2}{2}e^{-x} + \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$
17. $(c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x + x^4 - 24x^2 + 72 + \frac{1}{225}\sin 4x - \frac{1}{9}\sin 2x$
18. $c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x - \frac{1}{5}e^x\cos x$
19. $c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{e^{3x}}{11}(x^2 - \frac{12}{11}x + \frac{50}{121}) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$

20. $c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x - \frac{1}{5}\cos x\cosh x$

21. $c_1\cos 2x + c_2\sin 2x + \frac{1}{9}(3x\sin x - 2\cos x)$

22. $c_1 + (c_2 + c_3x)e^{-x} + \frac{e^{2x}}{18}(x^2 - \frac{7x}{8} + \frac{11}{6}) + \frac{1}{100}(3\sin 2x + 4\cos 2x)$

23. $c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{xe^x}{12}(2x^2 - 3x + 9)$

24. $c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x} + e^{-2x}e^{e^x}$

25. $c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x + \frac{1}{24}(2x^3 - 15x)\cos x - \frac{3x^2}{8}\sin x$

26. $c_1\cos ax + c_2\sin ax + \frac{1}{a^2}\cos ax\log\cos ax + \frac{x}{a}\sin ax$

27. $c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\cos 2x\log\cos 2x + \frac{x}{2}\sin 2x$

28. $c_1e^x + (c_2\cos x + c_3\sin x)e^x + xe^x + \frac{1}{10}(\cos x + 3\sin x)$

29. $c_1e^x + c_2e^{3x} - \frac{e^x}{8}(\cos 2x + \sin 2x) - \frac{1}{30}(2\cos 3x + \sin 3x)$

30. $e^x(c_1\cos x + c_2\sin x) - \frac{1}{3}e^x\sin 2x$

31. $e^{-x/2}(c_1 + c_2x)\cos\frac{\sqrt{3}x}{2} + (c_3 + c_4x)\sin\frac{\sqrt{3}x}{2} + \frac{e^x}{9}(x - 2)$

32. $c_1e^x + e^{-\frac{x}{2}}\{c_2x\cos\frac{\sqrt{3}x}{2} + c_3x\sin\frac{\sqrt{3}x}{2}\} + \frac{1}{2}(x\cos x - x\sin x - 3\cos x)$

33. $c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^x(2x^2 - 3x + 9)$

34. $c_1 + c_2x + e^{-x/2}(c_3\sin\frac{\sqrt{3}x}{2} + c_4\cos\frac{\sqrt{3}x}{2}) + 3bx^2 - \frac{1}{3}cx^3$

35. $c_1\cos ax + c_2\sin ax + \frac{x}{2a}\sin ax$

36. $c_1e^{2x} + c_2e^{3x} + \frac{e^x}{10}(\cos x - 3\sin x)$

37. $c_1\cos 2x + c_2\sin 2x + \frac{1}{2}\{\frac{x}{4} - \frac{1}{8}\sin 2x(x^2 - \frac{1}{8}) + \frac{1}{2}x\cos 2x\}$

38. $(c_1 + c_2x)e^x + \frac{1}{6}e^xx^3$

39. $c_1\cos ax + c_2\sin ax - \frac{1}{a^2}\cos ax\log\tan(\frac{\pi}{4} + \frac{ax}{2})$

40. $c_1e^{-\frac{\sqrt{7}}{2}x}\cos(\frac{3x}{2} + c_2) + c_3e^{\frac{\sqrt{7}}{2}x}\sin(\frac{3x}{2} + c_4) + x^2 + \frac{127}{8}$

41. $c_1\cos x + c_2\sin x + \frac{1}{2}e^{-x} + \frac{1}{2}x\sin x + x^3 - 6x + \frac{1}{5}e^x(2\sin x + \cos x)$

42. $(c_1 + c_2x)e^{2x} + \frac{1}{4}(x^2 + 2x + \frac{3}{2})e^x + \frac{1}{8}\cos 2x$

43. $c_1 + c_2e^x + c_3e^{-x} + c_4\cos x + c_5\sin x + 3xe^x + 2x\sin x + x^2$

44. $c_1\cos x + c_2\sin x + \frac{1}{2}x\sin x + \frac{1}{25}e^x(5x - 4) - \frac{1}{5}e^x(2\cos x - \sin x)$

45. $(c_1 + c_2x)e^{-x} + \frac{1}{4}e^x + x^2 - 4x + 6 - \frac{1}{2}\cos x$

46. $c_1e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x} + 8\cos 2x - 3\sin 2x + x + 4 + 4e^{-x/2}$

47. $c_1e^x + c_2e^{3x} - \frac{1}{10}e^{2x}\sin 3x$

48. $c_1e^{2x} + c_2e^{3x} + \frac{1}{108}(18x^2 + 30x + 19) + \frac{e^x}{4}(2x + 3)$

49. $(c_1\cos 2x + c_2\sin 2x) + (c_3\cos x + c_4\sin x) - \frac{1}{12}x\sin 2x - \frac{x}{6}\cos x$

50. $c_1e^{2x} + (c_2 + c_3x)e^{-x} - e^{-x}(20x^2 - 20x^3 - 15x^4 - 9x^5)$

51. $c_1\cos ax + c_2\sin ax + \frac{1}{4a^2}(ax^2\sin ax + x\cos ax)$

52. $c_1e^x + c_2e^{-x} + (c_3\cos x + c_4\sin x) + \frac{1}{3}(x^2\cos x - 3x\sin x)$

54. $x = e^{-(n\cos\alpha)t}\left[-\frac{a}{n^2\sin 2\alpha}\sin(n\sin\alpha)t\right] + \frac{a}{2n^2\cos\alpha}\sin nt$

55. $3e^{-x}\cos 3x + 6\cos 3x - \sin 3x$

56. $u = k(\theta\sin\theta - \cos\theta)$

61. $\frac{c_1 + c_2\log x}{x^2} + c_3 x$

62. $\sqrt{x}(c_1\cos\frac{\sqrt{3}}{2}\log x + c_2\sin\frac{\sqrt{3}}{2}\log x) + x^2$

63. $\frac{c_1 + c_2\log x}{x^2} + \frac{x^4}{36}$

64. $c_1x^2 + c_2x^3 + x^5$

65. $(c_1 + c_2\log x)x + c_3x^{-1} + \frac{1}{4x}\log x$

66. $(c_1\cos\log x + c_2\sin\log x)x + x\log x$

67. $c_1x + \frac{c_2}{x^2} + \frac{1}{4}x^2 + \frac{1}{3}x\log x - \frac{1}{2}$

68. $(c_1 + c_2\log x)x^2 + \frac{1}{4} + 2x + \frac{1}{2}x^2(\log x)^2$

69. $c_1x^4 + c_2x^{-1} + \frac{x^4}{5}\log x$

70. $2(\log x)^3 + c_1\log x + c_2$

71. $\frac{c_1}{x} + c_2x^4 - \frac{x^2}{6} - \frac{\log x}{2} + \frac{3}{8}$

72. $c_1x^{-1} + [c_2\cos(\log x) + c_3\sin(\log x)]\,x + 5x + \frac{10\log x}{x}$

73. $c_1x^3 + c_2x^{-4} + \frac{x^3}{98}\log x(7\log x - 2)$

74. $c_1x + c_2x^{-1} + \frac{x^m}{(m^2-1)}$

75. $c_1x^2 + x^{\frac{5}{2}}(c_2x^{\frac{\sqrt{21}}{2}} + c_3x^{-\frac{\sqrt{21}}{2}}) - \frac{x^3}{5}$

76. $x^2(c_1x^{\sqrt{3}} + c_2x^{\sqrt{-3}}) + \frac{1}{6x} + \frac{\log x}{61x}(5\sin\log x + 6\cos\log x) + \frac{2}{3721x}(27\sin\log x + 191\cos\log x)$

77. $(c_1 + c_2\log r) + \pi\rho r^2$

78. $(c_1 + c_2\log x)\cos(\log x) + (c_3 + c_4\log x)\sin(\log x) + (\log x)^2 + 2\log x - 3$

79. $[c_1\cos(\sqrt{3}\log x) + c_2\sin(\sqrt{3}\log x)]x + \frac{x}{2}\sin\log x$

80. $\frac{1}{3x}(5x^2 + c_1x + c_2)$

81. $c_1\cos\log(1+x) + c_2\sin\log(1+x) + 2\log(1+x)\sin\log(1+x)$

82. $c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\log(3x+2)]$

83. $c_1(2x+3)^{(3+\sqrt{57})/4} + c_2(2x+3)^{(3-\sqrt{57})/4} - \frac{3}{14}(2x+3) + \frac{3}{4}$

84. $c_1\cos\log(1+x) + c_2\sin\log(1+x) - \cos x\log(1+x)$

85. $(1+2x)^2[\{\log(1+2x)\}^2 + c_1\log(1+2x) + c_2]$

86. $c_1 + c_2\log(x+1) + \{\log(x+1)\}^2 + x^2 + 8x$

87. $c_1x^2 + c_2x^3 + \frac{1}{6}(\log x + \frac{5}{6})$

88. $c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\log(3x+2) + 1]$

89. $c_1e^{-2x} + c_2e^{-x} + 2e^x$

90. $(c_1 + c_2x)e^{-x} + \frac{x^4e^{-x}}{12}$

91. $c_1\cos x + c_2\sin x - x^2\cos x + x\sin x$

92. $(c_1 + c_2x)e^x + x^2e^x(\frac{1}{2}\log x - \frac{3}{4})$

93. $e^x \log(1+e^{-x}) - 1 - e^{-x}\log(1+e^x) + c_1e^x + c_2e^{-x}$
94. $e^x \log(1+e^{-x}) + e^{2x}\{\log(1+e^{-x}) - (1+e^{-x})\} + c_1e^x + c_2e^{2x}$
95. $(c_1 + c_2x)e^{-x} - e^{-x}\log x - e^{-x}$
96. $c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$
97. $c_1e^{2x} + c_2e^x - 9xe^x - 9e^x$
98. $c_1 \cos 3x + c_2 \sin 3x + \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x \log \cos 3x$
99. $c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log \sin x$
100. $(c_1 + c_2x)e^x + xe^x \log x$
101. $(c_1 \cos x + c_2 \sin x)e^x - e^x \cos x \log(\sec x + \tan x)$
102. $c_1e^{2x} + c_2e^{3x} + xe^{3x} + \frac{1}{10}(\sin x + \cos x)$
103. $c_1 + c_2e^{2x} - \frac{1}{2}e^x \sin x$
104. $c_1 \cos 2x + c_2 \sin 2x - 1 + \sin 2x \log \sec 2x + \tan 2x$
105. $c_1 \cos x + c_2 \sin x + \frac{1}{6}\tan^2 x \sin x$
106. $c_1x + c_2x\log x$
107. $\frac{c_1}{x} + c_2x + \frac{x}{2}\log x$

CHAPTER 5

Simultaneous Linear Differential Equations with Constant Coefficients

5.1 Introduction

In many engineering problems, it is observed that the number of dependent variables may be more than one, but the independent variable is one. In such cases the problem can be formulated as a system of differential equations.

To solve a system of differential equations, the simplest technique is construct an equation of higher order in one dependent variable by eliminating the other dependant variables. The elimination can be done in many different ways.

5.2 Solution of First Order Simultaneous Equations

EXAMPLE 5.2.1 Solve the equations: $\frac{dx}{dt} + y = e^t$, $\frac{dy}{dt} - x = e^{-t}$. (WBUT 2003)

Solution Given:

$$\frac{dx}{dt} + y = e^t \tag{i}$$

and

$$\frac{dy}{dt} - x = e^{-t} \tag{ii}$$

From (i)

$$y = -\frac{dx}{dt} + e^t \tag{iii}$$

Substituting y in (ii), we get $-\frac{d^2x}{dt^2} + e^t - x = e^{-t}$, or $\frac{d^2x}{dt^2} + x = e^t - e^{-t}$.

Let $x = ce^{mt}$ be a trial solution of $\frac{d^2x}{dt^2} + x = 0$.

$\therefore$ A.E. is $m^2 + 1 = 0$ or $m = \pm i$.

$\therefore$ C.F. is $c_1 \cos t + c_2 \sin t$.

$$\text{P.I.} = \frac{1}{D^2+1}(e^t - e^{-t}) = \frac{1}{D^2+1}e^t - \frac{1}{D^2+1}e^{-t} = \frac{1}{2}e^t - \frac{1}{2}e^{-t}$$

Thus

$$x = c_1 \cos t + c_2 \sin t + \frac{1}{2}(e^t - e^{-1})$$

Now, from (iii)

$$y = -\frac{d}{dt}[c_1 \cos t + c_2 \sin t + \frac{1}{2}(e^t - e^{-t})] + e^t$$

$$= c_1 \sin t - c_2 \cos t + \frac{1}{2}(e^t + e^{-t}) + e^t$$

$$= c_1 \sin t - c_2 \cos t + \frac{1}{2}(3e^t + e^{-t})$$

Hence the required solution is

$$x = c_1 \cos t + c_2 \sin t + \frac{1}{2}(e^t - e^{-t}) \quad \text{and} \quad y = c_1 \sin t - c_2 \cos t + \frac{1}{2}(3e^t + e^{-t})$$

where c_1 and c_2 are arbitrary constants.

EXAMPLE 5.2.2 Solve: $\dfrac{dx}{dt} - \dfrac{dy}{dt} + 2y = \cos 2t$ and $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x = \sin 2t$.

Solution The given equations can be written as

$$Dx - (D-2)y = \cos 2t \tag{i}$$

and

$$(D-2)x + Dy = \sin 2t \tag{ii}$$

Operating (i) and (ii) by D and $D-2$ respectively

$$D^2x - D(D-2)y = -2\sin 2t$$

and

$$(D-2)^2x + D(D-2)y = 2\cos 2t - 2\sin 2t$$

Adding these two equations, we get

$$(2D^2 - 4D + 4)x = 2\cos 2t - 4\sin 2t \qquad \text{or} \qquad (D^2 - 2D + 2)x = \cos 2t - 2\sin 2t$$

Let $x = ce^{mt}$ be a trial solution of $(D^2 - 2D + 2)x = 0$.

$\therefore$ A.E. is $m^2 - 2m + 2 = 0$ or $m = 1 \pm i$.

Hence C.F. is $e^t(c_1 \cos t + c_2 \sin t)$.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}(\cos 2t - 2\sin 2t) = \frac{1}{-2D-2}(\cos 2t - 2\sin 2t) \quad (\because D^2 = -4)$$

$$= -\frac{1}{2}\frac{D-1}{D^2-1}(\cos 2t - 2\sin 2t)$$

$$= -\frac{1}{2}\frac{D-1}{-4-1}(\cos 2t - 2\sin 2t)$$

$$= \frac{1}{10}(-2\sin 2t - 4\cos 2t - \cos 2t + 2\sin 2t)$$

$$= -\frac{1}{2}\cos 2t$$

$\therefore\ x = e^t(c_1 \cos t + c_2 \sin t) - \frac{1}{2}\cos 2t$. Adding given equations

$$2\frac{dx}{dt} - 2x + 2y = \cos 2t + \sin 2t$$

Therefore

$$\begin{aligned} y &= \frac{1}{2}(\cos 2t + \sin 2t) - \frac{dx}{dt} + x \\ &= \frac{1}{2}(\cos 2t + \sin 2t) - [e^t(c_1 \cos t + c_2 \sin t) + e^t(-c_1 \sin t + c_2 \cos t) + \sin 2t] \\ &\quad + e^t(c_1 \cos t + c_2 \sin t) - \frac{1}{2}\cos 2t \\ &= e^t(c_1 \sin t - c_2 \cos t) - \frac{1}{2}\sin 2t \end{aligned}$$

Hence the general solution is

$$x = e^t(c_1 \cos t + c_2 \sin t) - \frac{1}{2}\cos 2t \qquad \text{and} \qquad y = e^t(c_1 \sin t - c_2 \cos t) - \frac{1}{2}\sin 2t$$

EXAMPLE 5.2.3 Solve: $\dfrac{dx}{dt} + 4x + 3y = t,\ \dfrac{dy}{dt} + 2x + 5y = e^t$.

Solution The given equations are

$$(D+4)x + 3y = t \tag{i}$$

and

$$2x + (D+5)y = e^t \tag{ii}$$

Operating (i) by $(D+5)$, multiplying (ii) by 3 and subtracting, we get

$$(D^2 + 9D + 14)x = 1 + 5t - 3e^t$$

Let $x = ce^{mt}$ be a trial solution of $(D^2 + 9D + 14)x = 0$.

$\therefore$ A.E. is $m^2 + 9m + 14 = 0$ or $m = -2, -7$.

$\therefore$ C.F. is $c_1e^{-2t} + c_2e^{-7t}$.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+9D+14}(1+5t-3e^t) \\
&= \frac{1}{D^2+9D+14}(1+5t) - \frac{1}{D^2+9D+14}3e^t \\
&= \frac{1}{14}\left(1+\frac{D^2+9D}{14}\right)^{-1}(1+5t) - \frac{3}{1+9+14}e^t \\
&= \frac{1}{14}\left(1-\frac{D^2+9D}{14}-\cdots\right)(1+5t) - \frac{1}{8}e^t \\
&= \frac{1}{14}\left(1+5t-\frac{45}{14}\right) - \frac{1}{8}e^t \\
&= \frac{1}{14}\left(5t-\frac{31}{14}\right) - \frac{1}{8}e^t
\end{aligned}$$

Thus $x = c_1e^{-2t} + c_2e^{-7t} + \frac{1}{14}(5t - \frac{31}{14}) - \frac{1}{8}e^t$.

From the first equation

$$\begin{aligned}
y &= \frac{1}{3}\left(t-4x-\frac{dx}{dt}\right) \\
&= \frac{1}{3}\Big[t - 4c_1e^{-2t} - 4c_2e^{-7t} - \frac{2}{7}\left(5t-\frac{31}{14}\right) \\
&\quad + \frac{1}{2}e^t + 2c_1e^{-2t} + 7c_2e^{-7t} - \frac{5}{14} + \frac{1}{8}e^t\Big] \\
&= \frac{1}{3}\left(-2c_1e^{-2t} + 3c_2e^{-7t} + \frac{5}{8}e^t - \frac{3}{7}t + \frac{27}{98}\right)
\end{aligned}$$

Hence the general solution is

$$x = c_1e^{-2t} + c_2e^{-7t} + \frac{1}{14}\left(5t-\frac{31}{14}\right) - \frac{1}{8}e^t$$

and

$$y = \frac{1}{3}\left(-2c_1e^{-2t} + 3c_2e^{-7t} + \frac{5}{8}e^t - \frac{3}{7}t + \frac{27}{98}\right)$$

where c_1 and c_2 are arbitrary constraints.

EXAMPLE 5.2.4 Solve: $\dfrac{dx}{dt} = 2y, \dfrac{dy}{dt} = 2z, \dfrac{dz}{dt} = 2x.$

Solution We have $\dfrac{dx}{dt} = 2y.$

$$\therefore \quad \frac{d^2x}{dt^2} = 2\frac{dy}{dt} = 4z \text{ and } \frac{d^3x}{dt^3} = 4\frac{dz}{dt} = 8x$$

Let $x = ce^{mt}$ be a trial. solution of $\dfrac{d^3x}{dt^3} - 8x = 0$.

$\therefore$ A.E. is $m^3 - 8 = 0$, or $(m-2)(m^2+2m+4) = 0$, or $m = 2, -1 \pm i\sqrt{3}$.

$\therefore$ C.F. is $c_1e^{2t} + e^{-t}(c_2 \cos\sqrt{3}t + c_3 \sin\sqrt{3}t)$.

Hence $x = c_1e^{2t} + e^{-t}(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t)$.

Now

$$y = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}\left[2c_1e^{2t} - e^{-t}(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t) + e^{-t}(-\sqrt{3}c_2\sin\sqrt{3}t + \sqrt{3}c_3\cos\sqrt{3}t)\right]$$

$$= c_1e^{2t} + \frac{e^{-t}}{2}\left[(-c_2+\sqrt{3}c_3)\cos\sqrt{3}t + (-c_3 - \sqrt{3}c_2)\sin\sqrt{3}t\right]$$

$$z = \frac{1}{2}\frac{dy}{dt} = \frac{1}{2}\left[2c_1e^{2t} - \frac{e^{-t}}{2}\{(-c_2+\sqrt{3}c_3)\cos\sqrt{3}t - (c_3+\sqrt{3}c_2)\sin\sqrt{3}t\}\right.$$

$$\left. + \frac{e^{-t}}{2}\{-\sqrt{3}(-c_2+\sqrt{3}c_3)\sin\sqrt{3}t - \sqrt{3}(c_3+\sqrt{3}c_2)\cos\sqrt{3}t\}\right]$$

$$= c_1e^{2t} - \frac{e^{-t}}{2}\left[(\sqrt{3}c_3 + c_2)\cos\sqrt{3}t + (c_3 - \sqrt{3}c_2)\sin\sqrt{3}t\right]$$

Hence the general solution is

$$x = c_1e^{2t} + e^{-t}(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t)$$

$$y = c_1e^{2t} + \frac{e^{-t}}{2}\left[\left(-c_2+\sqrt{3}c_3\right)\cos\sqrt{3}t - \left(c_3+\sqrt{3}c_2\right)\sin\sqrt{3}t\right]$$

$$z = c_1e^{2t} - \frac{e^{-t}}{2}\left[\left(\sqrt{3}c_3+c_2\right)\cos\sqrt{3}t + \left(c_3-\sqrt{3}c_2\right)\sin\sqrt{3}t\right]$$

where c_1, c_2, c_3 are arbitrary constants.

EXAMPLE 5.2.5 Solve: $\dfrac{dx}{dt} + \dfrac{2}{t}(x-y) = 1$, $\dfrac{dy}{dt} + \dfrac{1}{t}(x+5y) = t$.

Solution We have

$$\frac{dx}{dt} + \frac{2}{t}(x-y) = 1 \qquad \text{(i)}$$

$$\frac{dy}{dt} + \frac{1}{t}(x+5y) = t \qquad \text{(ii)}$$

Differentiating (i) with respect to t, we get

$$\frac{d^2x}{dt^2} - \frac{2}{t^2}(x-y) + \frac{2}{t}\left(\frac{dx}{dt} - \frac{dy}{dt}\right) = 0$$

or

$$\frac{d^2x}{dt^2} - \frac{2}{t^2}(x-y) + \frac{2}{t}\left(\frac{dx}{dt} - t + \frac{1}{t}(x+5y)\right) = 0$$

or

$$\frac{d^2x}{dt^2}+\frac{2}{t}\frac{dx}{dt}-2+\frac{2}{t^2}(x+5y-x+y)=0$$

or

$$\frac{d^2x}{dt^2}+\frac{2}{t}\frac{dx}{dt}-2+\frac{12}{t^2}y=0$$

From (i), $x-y=\frac{t}{2}\left(1-\frac{dx}{dt}\right)$, or $y=x-\frac{t}{2}\left(1-\frac{dx}{dt}\right)$.

The above equation becomes

$$\frac{d^2x}{dt^2}+\frac{2}{t}\frac{dx}{dt}-2+\frac{12}{t^2}\left(x-\frac{t}{2}+\frac{t}{2}\frac{dx}{dt}\right)=0$$

or

$$\frac{d^2x}{dt^2}+\frac{8}{t}\frac{dx}{dt}+\frac{12x}{t^2}=2+\frac{6}{t}$$

or

$$t^2\frac{d^2x}{dt^2}+8t\frac{dx}{dt}+12x=t^2(2+6/t)=2t^2+6t$$

Putting $\log t=z$, or $t=e^z$, we have

$$t\frac{dx}{dt}=Dx \quad \text{and} \quad t^2\frac{d^2x}{dt^2}=D(D-1)x$$

where $D\equiv\frac{d}{dz}$

The above equation becomes

$$D(D-1)x+8Dx+12x=2e^{2z}+6e^z \quad \text{or} \quad (D^2+7D+12)x=2e^{2z}+6e^z$$

Let $x=ce^{mz}$ be a trial solution.

$\therefore$ A.E. is $m^2+7m+12=0$, or $m=-3,-4$.

Thus C.F. is $c_1e^{-3z}+c_2e^{-4z}$.

$$\text{P.I.}=\frac{1}{D^2+7D+12}(2e^{2z}+6e^z)$$

$$=\frac{1}{4+14+12}2e^{2z}+\frac{1}{1+7+12}6e^z=\frac{e^{2z}}{15}+\frac{3}{10}e^z.$$

The general solution is

$$x=c_1t^{-3}+c_2t^{-4}+\frac{t^2}{15}+\frac{3}{10}t \text{ and}$$

$$y=x-\frac{t}{2}(1-\frac{dx}{dt})$$

$$=x-\frac{t}{2}+\frac{t}{2}\left(-3c_1t^{-4}-4c_2t^{-5}+\frac{2t}{15}+\frac{3}{10}\right)$$

where c_1, c_2 are arbitrary constants.

5.3 Solution of Second Order Simultaneous Equations

EXAMPLE 5.3.1 Solve: $2\dfrac{d^2x}{dt^2} + 3\dfrac{dy}{dt} = 4,\ 2\dfrac{d^2y}{dt^2} - 3\dfrac{dx}{dt} = 0$, given $x(0) = y(0) = x'(0) = y'(0) = 0$.

Solution Differentiating first equation with respect to t, we get

$$2\frac{d^3x}{dt^3} + 3\frac{d^2y}{dt^2} = 0 \quad \text{or} \quad 2\frac{d^3x}{dt^3} + 3\cdot\frac{3}{2}\frac{dx}{dt} = 0 \qquad \text{(using second equation)}$$

or

$$(4D^3 + 9D)x = 0$$

Let $x = ce^{mt}$ be a trial solution.

$\therefore$ A.E. is $4m^3 + 9m = 0$ or $m(4m^2 + 9) = 0$ or $m = 0, \pm\frac{3}{2}i$.

$\therefore$ $x = c_1 + c_2\cos\dfrac{3}{2}t + c_3\sin\dfrac{3}{2}t.$

Integrating first equation with respect to t, we get $2\dfrac{dx}{dt} + 3y = 4t + c_4$.

That is

$$\begin{aligned} y &= \frac{1}{3}\left(4t + c_4 - 2\frac{dx}{dt}\right) \\ &= \frac{1}{3}\left[4t + c_4 - 2\left\{-\frac{3}{2}c_2\sin\frac{3}{2}t + \frac{3}{2}c_3\cos\frac{3}{2}t\right\}\right] \\ &= \frac{4}{3}t + c_2\sin\frac{3}{2}t - c_3\cos\frac{3}{2}t + \frac{1}{3}c_4 \end{aligned}$$

Now

$$x' = -\frac{3}{2}c_2\sin\frac{3}{2}t + \frac{3}{2}c_3\cos\frac{3}{2}t \qquad \text{and} \qquad y' = \frac{4}{3} + \frac{3}{2}c_2\cos\frac{3}{2}t + \frac{3}{2}c_3\sin\frac{3}{2}t$$

Using the initial condition $t = 0,\ x = 0,\ y = 0,\ x' = 0,\ y' = 0$, we obtain

$$0 = c_1 + c_2 \qquad 0 = -c_3 + \frac{1}{3}c_4 \qquad 0 = \frac{3}{2}c_3 \qquad 0 = \frac{4}{3} + \frac{3}{2}c_2$$

Solution of these equations is $c_3 = 0,\ c_4 = 0,\ c_1 = -c_2 = \dfrac{8}{9}$.

Hence the required solution is

$$x = \frac{8}{9}\left(1 - \cos\frac{3}{2}t\right) \qquad y = \frac{4}{3}t - \frac{8}{9}\sin\frac{3}{2}t$$

EXAMPLE 5.3.2 Solve: $\dfrac{d^2x}{dt^2} - \dfrac{dy}{dt} = 2x + 2t,\ \dfrac{dx}{dt} + 4\dfrac{dy}{dt} = 3y.$

Solution We have

$$\frac{d^2x}{dt^2} - \frac{dy}{dt} = 2x + 2t \tag{i}$$

and

$$\frac{dx}{dt} + 4\frac{dy}{dt} = 3y$$

Differentiating (ii) with respect to t, we get

$$\frac{d^2x}{dt^2} + 4\frac{d^2y}{dt^2} = 3\frac{dy}{dt}$$

Subtracting (i) from (iii), we get

$$4\frac{d^2y}{dt^2} + \frac{dy}{dt} = 3\frac{dy}{dt} - 2x - 2t \qquad \text{or} \qquad 2\frac{d^2y}{dt^2} - \frac{dy}{dt} = -x - t$$

Again differentiating it with respect to t

$$2\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} = -\frac{dx}{dt} - 1 = -\left(3y - 4\frac{dy}{dt}\right) - 1$$

or

$$2\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = -1$$

Let $y = ce^{mt}$ be a trial solution.

$\therefore$ A.E. is $2m^3 - m^2 - 4m + 3 = 0$, or $(m-1)^2(2m+3) = 0$, or $m = 1, 1, -3/2$.

$\therefore$ C.F. is $(c_1 + c_2t)e^t + c_3e^{-3t/2}$.

$$\text{P.I.} = \frac{1}{(D-1)^2(2D+3)}(-1) = \frac{1}{3}(1-D)^{-1}\left(1 + \frac{2}{3}D\right)^{-1}(-1) = -\frac{1}{3}$$

$\therefore \quad y = (c_1 + c_2t)e^t + c_3e^{-3t/2} - \frac{1}{3}$.

Now

$$\frac{dy}{dt} = c_2e^t + (c_1 + c_2t)e^t - \frac{3}{2}c_3e^{-3t/2}$$

and hence

$$\frac{d^2y}{dt^2} = 2c_2e^t + (c_1 + c_2t)e^t + \frac{9}{4}c_3e^{-3t/2}$$

Therefore, from (iv)

$$\begin{aligned} x = -2\frac{d^2y}{dt^2} + \frac{dy}{dt} - t &= -2\left[2c_2e^t + (c_1 + c_2t)e^t + \frac{9}{4}c_3e^{-3t/2}\right] \\ &+ \left[c_2e^t + (c_1 + c_2t)e^t - \frac{3}{2}c_3e^{-3t/2}\right] - t \\ &= -3c_2e^t - (c_1 + c_2t)e^t - 6c_3e^{-3t/2} - t \end{aligned}$$

Equations (v) and (vi) gives the solution, where c_1, c_2, and c_3 are constants.

5.4 Using Eigenvalues

In previous sections, a system of differential equations has been solved by converting it into a higher order differential equation. But it is well known that the solution of higher order equation is a very difficult task. Here an efficient method is used to solve a system of differential equations. Let us consider a system of differential equation of the form

$$\begin{aligned}
\dot{y}_1 &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\
\dot{y}_2 &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{y}_n &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n
\end{aligned}$$

where $y_1, y_2, y_3, \ldots, y_n$ are dependent variables. The solution of this system of equations can be done by finding eigenvalues and eigenvectors of the associated system.

The above equation can be written as

$$\dot{Y} = AY \tag{5.1}$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The solution of the above equation may be of the form

$$Y(t) = Xe^{\lambda t} \tag{5.2}$$

where X is a nonzero constant vector and λ is a constant scalar, and both are unknown.

Substituting (5.2) in (5.1), we get

$$\lambda X e^{\lambda t} = AXe^{\lambda t}$$

from which we have

$$AX = \lambda X \qquad \text{i.e.} \qquad (A - \lambda I)X = 0$$

Thus λ is the eigenvalue of A and X is the corresponding eigenvector. Hence we can determine $Y(t)$ using (5.2).

EXAMPLE 5.4.1 Solve: $\dfrac{dx}{dt} = x + 2y, \quad \dfrac{dy}{dt} = x.$

Solution The given equation can be written as

$$\dot{Y}(t) = AY(t)$$

where $Y(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

The eigenvalues λ of the matrix A are given by $\begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = 0$, or $\lambda = 2, -1$.

For $\lambda_1 = 2$ let the eigenvector be $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then $\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

That is $-x_1 + 2x_2 = 0$, or $x_1 - 2x_2 = 0$. This equation gives $x_1 = k_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -1$, let the eigenvector be $X_2 = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$.

Then $\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, or $x_1' + x_2' = 0$.

This gives $x_2 = k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Hence the required solution is

$$Y(t) = k_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \quad \text{or} \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2k_1 e^{2t} + k_2 e^{-t} \\ k_1 e^{2t} - k_2 e^{-t} \end{bmatrix}$$

Thus

$$x(t) = 2k_1 e^{2t} + k_2 e^{-t} \quad \text{and} \quad y(t) = k_1 e^{2t} - k_2 e^{-t}$$

EXAMPLE 5.4.2 Solve: $\dfrac{dx}{dt} - 7x + y = 0, \quad \dfrac{dy}{dt} - 2x - 5y = 0.$

(WBUT 2005, 2007)

Solution The given equations are

$$\frac{dx}{dt} = 7x - y \quad \text{and} \quad \frac{dy}{dt} = 2x + 5y$$

and these can be written as

$$\dot{Y} = AY$$

where $Y = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} 7 & -1 \\ 2 & 5 \end{bmatrix}$.

The eigenvalues λ of the matrix A are given by

$$\begin{vmatrix} 7-\lambda & -1 \\ 2 & 5-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 12\lambda + 37 = 0 \quad \text{or} \quad \lambda = 6 \pm i$$

Let $\lambda_1 = 6 + i$ and $\lambda_2 = 6 - i$.

For $\lambda_1 = 6 + i$, let the eigenvector be $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then
$$\begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or
$$(1-i)x_1 - x_2 = 0 \qquad \text{or} \qquad 2x_1 - (1+i)x_2 = 0$$

The solution of these equations is $x_1 = k_1$, $x_2 = (1-i)k_1$.

This gives $\quad X_1 = k_1 \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$.

The eigenvector corresponding to $\lambda_2 = 6 - i$ is obtained by replacing i by $-i$.

This gives $\quad X_2 = k_2 \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$.

Hence the required solution is

$$Y(t) = k_1 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(6+i)t} + k_2 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(6-i)t}$$

Thus

$$\begin{aligned} x &= k_1 e^{(6+i)t} + k_2 e^{(6-i)t} \\ &= e^{6t}(k_1 e^{it} + k_2 e^{-it}) \\ &= e^{6t}[k_1(\cos t + i \sin t) + k_2(\cos t - i \sin t)] \\ &= e^{6t}[(k_1 + k_2)\cos t + (k_1 - k_2) i \sin t] \\ &= e^{6t}(c_1 \cos t + c_2 \sin t) \end{aligned}$$

where

$$c_1 = k_1 + k_2 \qquad \text{and} \qquad c_2 = (k_1 - k_2)i$$

and

$$\begin{aligned} y &= k_1(1-i)e^{(6+i)t} + k_2(1+i)e^{(6-i)t} \\ &= e^{6t}[k_1(1-i)(\cos t + i \sin t) + k_2(1+i)(\cos t - i \sin t)] \\ &= e^{6t}[\{(k_1+k_2) - (k_1-k_2)i\}\cos t + \{(k_1+k_2) + (k_1-k_2)i\}\sin t] \\ &= e^{6t}[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t] \end{aligned}$$

where c_1, c_2 are arbitrary constants.

EXERCISES

Section A Multiple Choice Questions

1. The solution of the equations $y' = z,\ z' = -y$ is

   (a) $y = c_1 \cos x + c_2 \sin x,\ z = -c_1 \sin x + c_2 \cos x$
   (b) $y = c_1 e^x + c_2 e^{-x};\ z = c_1 e^x - c_2 e^{-x}$
   (c) $y = e^x(c_1 \cos x + c_2 \sin x),\ z = e^{-x}(c_1 \cos x + c_2 \sin x)$
   (d) none of these.

2. The solution of the equations $\frac{dy}{dx} = z + 2x,\ \frac{dz}{dx} = y + 2x$ is given by

   (a) $y = xe^{-z}$ (b) $y = z + ce^{-x}$ (c) $y + z = \log(xc)$ (d) $x + y + z = c$

3. The solution of $\dfrac{dx}{dt} = x,\ \dfrac{dy}{dt} + \dfrac{dx}{dt} = 0$ is

   (a) $x = xt + c_1,\ y = x + c_2$ (b) $x + y = c$ (c) $x = ce^x$
   (d) $x = c_1 e^t,\ y + c_1 e^t = c_2$.

4. $x = c_1 e^t + c_2 e^{-t},\ y = -c_1 e^t + c_2 e^{-t} + c_3$ is solution of the equations

   (a) $\frac{dy}{dt} = -x,\ \frac{d^2x}{dt^2} + \frac{dy}{dt} = 0$ (b) $\frac{dy}{dt} = -x,\ \frac{dx}{dt} = -y$
   (c) $\frac{dx}{dt} + \frac{dy}{dt} = e^{-t},\ \frac{dx}{dt} = t$ (d) none of these.

5. If $\frac{dy}{dt} = 2x + 3y,\ \frac{dx}{dt} = -x - 2y$. Then the solution of these equations are

   (a) $x + y = ce^t$ (b) $x - y = ce^{-t}$ (c) $x = c_1 e^t + c_2 e^{-t},\ y = -x$
   (d) none of these.

Section B Review Questions

Solve the following system of equations:

1. $\frac{dx}{dt} = -\omega y,\ \frac{dy}{dt} = \omega x$. Also show that the point (x, y) lies on a circle .
2. $4\frac{dx}{dt} - \frac{dy}{dt} + 3x = \sin t,\ \frac{dx}{dt} + y = \cos t$.
3. $\frac{dx}{dt} - y = t,\ \frac{dx}{dt} + x = t^2$.
4. $x' + y = \sin t,\ y' + x = \cos t,\ x(0) = 2$ *and* $y(0) = 0$.
5. $\frac{dx}{dt} + x - y = 0,\ \frac{dy}{dt} + 2x + 5y = 0$.
6. $\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0,\ \frac{dy}{dt} + 5x + 3y = 0$.
7. $\frac{dx}{dt} + 2x - 3y = t,\ \frac{dy}{dt} - 3x + 2y = e^{2t}$.
8. $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t,\ \frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$.
9. $\frac{dy}{dx} = 3z - y,\ \frac{dz}{dx} = 4z - 2y$.
10. $5\frac{dy}{dx} - 2\frac{dz}{dx} + 4y - z = e^{-x},\ \frac{dy}{dx} + 8y - 3z = 5e^{-x}$.
11. $\frac{dy}{dx} + y - z = e^x,\ \frac{dz}{dx} - y + z = e^x$.

12. $\frac{d^2x}{dt^2} - 3x - 4y = 0,\ \frac{d^2y}{dt^2} + x + y = 0.$

13. $\frac{d^2x}{dt^2} - 3x - y = e^t,\ \frac{dy}{dt} - 2x = 0.$

14. $2\frac{d^2x}{dt^2} - \frac{dy}{dt} - 4x = 2t,\ 2\frac{dx}{dt} + 4\frac{dy}{dt} - 3y = 0.$

15. $2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4,\ 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0,\ x(0) = y(0) = x'(0) = y'(0) = 0.$

16. $2x'' + 3y' = 4,\ 2y'' - 3x' = 0,\ \ x(0) = y(0) = x'(0) = y'(0) = 0.$

17. The coordinates x and y of the position of a particle at time t satisfy the differential equations $y' + 2x = \sin 2t$ and $x' - 2y = \cos 2t$. If $x(0) = 1$ and $y(0) = 0$, show that the path of the particle is given by $4x^2 + 4xy + 5y^2 = 4$.

Answers

Section A Multiple Choice Questions

1. (a) 2. (b) 3. (d) 4. (a) 5. (a)

Section B Review Questions

1. $x = -c_1 \sin wt + c_2 \cos \omega t,\ y = c_1 \cos \omega t + c_2 \sin \omega t$
2. $x = c_1 e^{-t} + c_2 e^{-3t},\ y = c_1 e^{-t} + c_2 e^{-3t} + \cos t$
3. $x = c_1 \cos t + c_2 \sin t + t^2 - 1,\ y = -c_1 \sin t + c_2 \cos t + t$
4. $x = e^t + e^{-t},\ y = \sin t - e^t + e^{-t}$
5. $y = c_1 e^{(-3+\sqrt{2})t} + c_2 e^{(-3-\sqrt{2})t},\ x = -(1 + \frac{1}{\sqrt{2}})c_1 e^{(-3+\sqrt{2})t} + (-1 + \frac{1}{\sqrt{2}})c_2 e^{(-3-\sqrt{2})t}$
6. $x = c_1 \cos t + c_2 \sin t,\ y = -\frac{1}{2}(c_1 + 3c_2)\sin t + \frac{1}{2}(c_2 - 3c_1)\cos t$
7. $x = c_1 e^t + c_2 e^{-5t} + \frac{3}{7}e^{2t} - \frac{2}{5}t - \frac{36}{25},\ y = c_1 e^t - c_2 e^{-5t} + \frac{4}{7}e^{2t} - \frac{3}{5}t - \frac{12}{25}$
8. $x = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + 3\cos t,\ y = (1 + \sqrt{2})c_1 e^{\sqrt{2}t} + (1 - \sqrt{2})c_2 e^{-\sqrt{2}t} - 5\sin t$
9. $y = -\frac{1}{2}c_1 e^{5x} + 3c_2 e^{-2x},\ z = c_1 e^{5x} - c_2 e^{-2x}$
10. $y = 2e^{-x} + c_1 e^x + c_2 e^{-2x},\ z = 3e^{-x} + c_3 e^x + c_4 e^{-2x}$
11. $y = e^x + c_1 + c_2 e^{-2x},\ z = e^x + c_1 - c_2 e^{-2x}$
12. $x = (c_1 + c_2 t)e^t + (c_3 + c_4 t)e^{-t},\ y = \frac{1}{2}e^t(c_2 - c_1 - c_2 t) - \frac{1}{2}e^{-t}(c_3 + c_4 + c_4 t)$
13. $y = (c_1 + c_2 t)e^{-t} + c_3 e^{2t} - \frac{1}{2}e^t,\ x = \frac{1}{2}[e^{-t}(c_2 - c_1 - c_2 t) + 2c_3 e^{2t} - \frac{1}{2}e^t]$
14. $x = (c_1 + c_2 t)e^t + c_3 e^{-\frac{3t}{2}} - \frac{1}{2}t,\ y = -2(c_1 + c_2 t - 3c_2)e^t - \frac{1}{3}c_3 e^{-\frac{3t}{2}} - \frac{1}{3}$
15. $x = \frac{8}{9}(1 - \cos\frac{3t}{2}),\ y = \frac{4}{3}t - \frac{8}{9}\sin\frac{3t}{2}$
16. $x = \frac{8}{9}(1 - \cos\frac{3t}{2}),\ y = \frac{4}{3} - \frac{8}{9}\sin\frac{3t}{2}$

CHAPTER 11

Laplace Transforms

11.1 Introduction

The Laplace transform is a very important tool for applied mathematics. This transformation provides an easy and effective means for the solution of many problems arising in engineering. The method of Laplace transform solves differential equations and corresponding initial and boundary value problems. This method reduces the problem of solving a differential equation to that of solving an algebraic equation. This process is made easier by using the table of functions and their transformations. Then, on solving this algebraic equation, one can solve the differential equation.

The partial differential equations can also be solved by using Laplace transforms.

11.2 Definition

Suppose $f(t)$ is a real-valued function defined over the interval $(-\infty, \infty)$ such that $f(t) = 0$ for all $t < 0$.

The Laplace transform of $f(t)$, denoted by $L\{f(t)\}$, is defined as

$$\overline{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \tag{11.1}$$

Here L is called the **Laplace transformation operator**. The parameter s is either a real or a complex number. In general, the parameter s is taken to be a positive number.

11.3 Sufficient Conditions for Existence of Laplace Transform

Definition 11.3.1 *A function is called* ***sectionally*** *or* ***piecewise continuous*** *in an interval if the interval can be subdivided into a finite number of intervals, in each of which the function is continuous and has finite left and right hand limits.*

Definition 11.3.2 *A function $f(t)$ is said to be of* ***exponential order of*** α *(or briefly as* ***exponential order****) if there exist constant α, M and N such that*

$$|f(t)| \leq Me^{\alpha t} \text{ for all } t \geq N$$

Theorem 11.1 *If $f(t)$ is sectionally continuous in every finite interval $0 \le t \le N$, and of exponential order γ for $t > N$, then its Laplace transform $\overline{f}(s)$ exists for all $s > \gamma$.*

Proof. We have

$$\int_0^\infty e^{-st} f(t)dt = \int_0^N e^{-st} f(t)dt + \int_N^\infty e^{-st} f(t)dt \tag{i}$$

Since $f(t)$ is sectionally continuous in every finite interval $0 \le t \le N$, $\int_0^\infty f(t)dt$ exists.

Again

$$\left|\int_N^\infty e^{-st} f(t)dt\right| \le \int_N^\infty |e^{-st} f(t)|dt = \int_N^\infty e^{-st}|f(t)|dt$$

$$\le \int_N^\infty e^{-st} M e^{\gamma t} dt \qquad (\because |f(t)| \le M e^{\gamma t}, \text{ for all } t > N)$$

$$\le M \int_0^\infty e^{-(s-\gamma)t} dt = M\left[\frac{e^{-(s-\gamma)t}}{-(s-\gamma)}\right]_0^\infty = \frac{M}{s-\gamma}, \qquad \text{if } s > \gamma$$

Thus, $\int_0^\infty e^{-st} f(t)dt$ exists if $s > \gamma$.

For example, let $f(t) = \sin \alpha t$. Then

$$\lim_{t\to\infty} |f(t)|e^{-\beta t} = \lim_{t\to\infty} |\sin \alpha t| e^{-\beta t} \le \lim_{t\to\infty} e^{-\beta t} = 0, \beta > 0 \tag{11.2}$$

Therefore, $\sin \alpha t$ is exponential order; also, $\sin \alpha t$ is continuous. Hence the Laplace transform of $\sin \alpha t$ exists.

Again, let $f(t) = e^{t^2}$. For any $\gamma > 0$, we have $e^{-\gamma t}|e^{t^2}| = e^{-\gamma t} e^{t^2} = e^{(t^2 - \gamma t)}$; this can be larger and larger for large t.

$\therefore$ $\lim_{t\to\infty} e^{-\gamma t}|e^{t^2}|$ does not exist, and also, the Laplace transform of e^{t^2} does not exist.

The conditions of this theorem are not necessary but sufficient, as $L\{t^{-1/2}\} = \sqrt{\pi/s}$, but $t^{-1/2}$ is not sectionally continuous, since it is finite at $t = 0$.

It may be noted that $L\{f(t)\}$ is a function of s and we denote it by $\overline{f}(s)$, i.e. $L\{f(t)\} = \overline{f}(s)$. This can be written as $f(t) = L^{-1}\{\overline{f}(s)\}$. Then $f(t)$ is called the **inverse Laplace transform** of $\overline{f}(s)$.

11.4 Transformations of Elementary Functions

A table of Laplace transforms of some functions is given below

(i) $L\{k\} = \frac{k}{s}$, k is a constant, $s > 0$ (11.3)

(ii) $L\{t^n\} = \frac{n!}{s^{n+1}}$, when $n = 0, 1, 2, \ldots$ (11.4)

(iii) $L\{e^{at}\} = \frac{1}{s-a}$, $s > a$ (11.5)

(iv) $L\{\sin at\} = \frac{a}{s^2+a^2}$, $s > 0$ (11.6)

(v) $L\{\cos at\} = \frac{s}{s^2+a^2}$, $s > 0$ (11.7)

(vi) $L\{\sinh at\} = \frac{a}{s^2-a^2}$, $s > |a|$ (11.8)

(vii) $L\{\cosh at\} = \frac{s}{s^2-a^2}$, $s > |a|$ (11.9)

Proof.

(i) $L\{k\} = \int_0^\infty ke^{-st}dt = \left[-k\frac{e^{-st}}{s}\right]_0^\infty = \frac{k}{s}$ if $s > 0$

(ii) $L\{t^n\} = \int_0^\infty t^n e^{-st}dt = \int_0^\infty e^{-z}(z/s)^n \frac{1}{s}dz$ where $z = st$

$= \frac{1}{s^{n+1}} \int_0^\infty z^n e^{-z}dz = \frac{\Gamma(n+1)}{s^{n+1}}$ if $n > -1$ and $s > 0$

where Γ is the Gamma function.

If n is a positive integer, then $\Gamma(n+1) = n!$ and therefore $L\{t^n\} = \frac{n!}{s^{n+1}}$.

(iii) $L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \left[\frac{e^{-(s-a)}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}$ if $s > a$

(iv) $L\{\sin at\} = \int_0^\infty \sin at\, e^{-st}dt = \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty = \frac{a}{s^2+a^2}$

The proof of (v) is similar to (iv).

(vi) $L\{\sinh at\} = \int_0^\infty e^{-st}\sinh at\, dt = \int_0^\infty e^{-st}\left[\frac{e^{at}-e^{-at}}{2}\right]dt$

$= m\frac{1}{2}\left(\int_0^\infty e^{-(s-a)t}dt - \int_0^\infty e^{-(s+a)t}dt\right) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$

$= \frac{a}{s^2-a^2}$ if $s > |a|$

EXAMPLE 11.4.1 Find the $L\{f(t)\}$, when

(i) $f(t) = \begin{cases} 1, & \text{if } 0 < t < 2 \\ 2, & \text{if } t > 2 \end{cases}$ (WBUT 2002)

(ii) $f(t) = \begin{cases} e^t, & \text{if } 0 < t \le 1 \\ 0, & \text{if } t > 1 \end{cases}$ (WBUT 2003)

(iii) $f(t) = \begin{cases} 0, & \text{if } 0 < t \le 1 \\ t, & \text{if } 1 < t \le 2 \\ 0, & \text{if } t > 2 \end{cases}$ (WBUT 2005)

(iv) $f(t) = \begin{cases} 1, & \text{if } t > \alpha \\ 0, & \text{if } t < \alpha \end{cases}$ (WBUT 2006)

(v) $f(t) = \begin{cases} \sin t, & \text{if } 0 < t < \pi \\ 0, & \text{if } t > \pi. \end{cases}$ (WBUT 2008)

Solution (i)

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^2 f(t)e^{-st}dt + \int_2^\infty f(t)e^{-st}dt$$

$$= \int_0^2 1 \cdot e^{-st}dt + \int_2^\infty 2 \cdot e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^2 + 2\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= \left[\frac{e^{-2s}}{-s} + \frac{1}{s}\right] + 2\left[0 + \frac{e^{-2s}}{s}\right] = \frac{1}{s}(1 + e^{-2s})$$

(ii)

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^1 e^t e^{-st}dt + \int_1^\infty 0 \cdot e^{-st}dt$$

here take $e^{(1-s)t}$ ✗

$$= \int_0^1 e^{-(s-1)t}dt = \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1 = \frac{1}{s-1}[1 - e^{-(s-1)}]$$

take $e^{-(s-1)t}$ ✓

(iii)

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^1 0 \cdot e^{-st}dt + \int_1^2 t \cdot e^{-st}dt + \int_2^\infty 0 \cdot e^{-st}dt$$

$$= \int_1^2 te^{-st}dt = \left[\frac{te^{-st}}{-s}\right]_1^2 - \int_1^2 1 \cdot \frac{e^{-st}}{-s}dt$$

$$= \frac{1}{s}(e^{-s} - 2e^{-2s}) + \frac{1}{s^2}(e^{-s} - 2e^{-2s})$$

(iv)

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^\alpha 1 \cdot e^{-st}dt + \int_\alpha^\infty 0 \cdot e^{-st}dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\alpha = \frac{1}{s}[1 - e^{-s\alpha}]$$

(v)

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^\pi \sin t \cdot e^{-st}dt + \int_\pi^\infty 0 \cdot e^{-st}dt$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^\pi = \frac{1}{s^2+1}(1 + e^{-s\pi})$$

11.5 Properties of Laplace Transforms

11.5.1 Linear Property

Suppose $\overline{f}_1(s)$ and $\overline{f}_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$$\begin{aligned} L\{af_1(t) + bf_2(t)\} &= aL\{f_1(t)\} + bL\{f_2(t)\} \\ &= a\overline{f}_1(s) + b\overline{f}_2(s) \end{aligned}$$

where a and b are any constants.

The proof of this result follows directly from the definition.

EXAMPLE 11.5.1 Find $L\{at+b\}$. (WBUT 2005)

Solution $L\{at+b\} = aL\{t\} + L\{b\} = \dfrac{a}{s^2} + \dfrac{b}{s}$ [using (11.3) and (11.4)]

11.5.2 Shifting Property

Theorem 11.2 (First shifting theorem). *If $L\{f(t)\} = \overline{f}(s)$, then $L\{e^{at}f(t)\} = \overline{f}(s-a)$.*

Proof. $L\{e^{at}f(t)\} = \displaystyle\int_0^\infty e^{at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s-a)t}dt = \int_0^\infty f(t)e^{-pt}dt$

where $p = s - a$

$$= \overline{f}(p) = \overline{f}(s-a)$$

Corollary 11.5.1 *If $L\{f(t)\} = \overline{f}(s)$, then $\overline{f}(s+a) = L\{e^{-at}f(t)\}$.*

Using this result, we have the following transformations:

(i) Since $L\{1\} = \frac{1}{s}$

$$L\{e^{at}\} = \frac{1}{s-a} \tag{11.10}$$

(ii) Since $L\{t^n\} = \frac{n!}{s^{n+1}}$

$$L\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}} \tag{11.11}$$

(iii) Since $L\{\sin bt\} = \frac{b}{s^2+b^2}$

$$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2} \tag{11.12}$$

(iv) Since $L\{\cos bt\} = \frac{s}{s^2+b^2}$

$$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \tag{11.13}$$

(v) Since $L\{\sinh bt\} = \frac{b}{s^2-b^2}$

$$L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2} \tag{11.14}$$

(vi) Since $L\{\cosh bt\} = \frac{s}{s^2-b^2}$

$$L\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2} \tag{11.15}$$

in every case $s > a$.

EXAMPLE 11.5.2 Find the Laplace transforms of

(i) te^{2t} (WBUT 2006) (ii) $e^t \sin t \cos t$ (WBUT 2006)

(iii) $e^{3t}\cos^2 t$ (iv) $e^{4t}\sin 2t \cos t$.

Solution (i) We know $L\{t\} = \frac{1}{s^2}$. Then by shifting property, $L\{te^{2t}\} = \frac{1}{(s-2)^2}$.

(ii) $L\{\sin t\cos t\} = L\{\sin 2t/2\} = \frac{1}{2}\frac{2}{s^2+4} = \frac{1}{s^2+4}$.

By shifting property, $L\{e^t \sin t\cos t\} = \frac{1}{(s-1)^2+4}$.

(iii) $L\{\cos^2 t\} = L\{(1+\cos 2t)/2\} = \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+4}\right]$.

Then $\{e^{3t}\cos^2 t\} = \frac{1}{2}\left[\frac{1}{s-3}+\frac{s-3}{(s-3)^2+4}\right]$.

(iv) $L\{\sin 2t\cos t\} = \frac{1}{2}L\{\sin 3t + \sin t\} = \frac{1}{2}\left[\frac{3}{s^2+3^2}+\frac{1}{s^2+1}\right]$.

By shifting property, $L\{e^{4t}\sin 2t\cos t\} = \frac{1}{2}\left[\frac{3}{(s-4)^2+9}+\frac{1}{(s-4)^2+1}\right]$.

Theorem 11.3 (Second shifting theorem). *If* $L\{f(t)\} = \overline{f}(s)$ *and*

$$g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

then $L\{g(t)\} = e^{-as}\overline{f}(s)$.

Proof.

$$\begin{aligned} L\{g(t)\} &= \int_0^\infty e^{-st}g(t)dt = \int_0^a e^{-st}g(t)dt + \int_a^\infty e^{-st}g(t)dt \\ &= \int_0^a e^{-st}\cdot 0dt + \int_a^\infty e^{-st}f(t-a)dt = \int_a^\infty e^{-st}f(t-a)dt \end{aligned}$$

Substituting, $t-a=p$. Then

$$L\{g(t)\} = \int_0^\infty e^{-s(p+a)}f(p)dp = e^{-sa}\int_0^\infty e^{-sp}f(p)dp = e^{-sa}\overline{f}(s)$$

11.5.3 Change of Scale Property

Theorem 11.4 *If* $L\{f(t)\} = \overline{f}(s)$, *then* $L\{f(at)\} = \frac{1}{a}\overline{f}\left(\frac{s}{a}\right)$

Proof.

$$\begin{aligned} L\{f(at)\} &= \int_0^\infty e^{-st} f(at)dt \\ &= \int_0^\infty e^{-sx/a} f(x)\frac{dx}{a} \qquad \text{where } at = x \\ &= \frac{1}{a}\int_0^\infty e^{-sx/a} f(x)dx = \frac{1}{a}\int_0^\infty e^{-pt} f(t)dt \qquad \text{where } p = \frac{s}{a} \\ &= \frac{1}{a}\overline{f}(p) = \frac{1}{a}\overline{f}\left(\frac{s}{a}\right) \end{aligned}$$

EXAMPLE 11.5.3 (i) Find $L\left\{\frac{\sin at}{t}\right\}$, given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$. (WBUT 2007)

(ii) If $L\{f(t)\} = \frac{s^2 - s + 1}{(2s+1)^2(s-1)}$, then show that $L\{f(2t)\} = \frac{s^2 - 2s + 4}{4(s+1)^2(s-2)}$

(WBUT 2002)

Solution (i) Let $f(t) = \frac{\sin t}{t}$. Then $L\{f(t)\} = L\left\{\frac{\sin t}{t}\right\} = \overline{f}(s)$. Then

$$L\left\{\frac{\sin at}{t}\right\} = aL\left\{\frac{\sin at}{at}\right\} = aL\{f(at)\} = a\cdot\frac{1}{a}\overline{f}\left(\frac{s}{a}\right) = \overline{f}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right).$$

(ii) Here $\overline{f}(s) = \frac{s^2 - s + 1}{(2s+1)^2(s-1)}$.

Therefore

$$L\{f(2t)\} = \frac{1}{2}\overline{f}\left(\frac{s}{2}\right) = \frac{1}{2}\frac{(s/2)^2 - (s/2) + 1}{\{2(s/2)+1\}^2(s/2-1)} = \frac{1}{4}\frac{s^2 - 2s + 4}{(s+1)^2(s-2)}$$

11.5.4 Multiplication by Powers of t

$L\{t^n f(t)\} = -\frac{d}{ds^n} L\{f(t)\}$

Theorem 11.5 *If* $L\{f(t)\} = \overline{f}(s)$, *then* $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\overline{f}(s)$, $n = 1, 2, 3, \ldots$.

(WBUT 2005)

Proof. Let $L\{f(t)\} = \overline{f}(s)$ and $\frac{d^n\overline{f}(s)}{ds^n} = \overline{f}^{(n)}(s)$.

Then $\overline{f}(s) = \int_0^\infty e^{-st} f(t)dt$. Therefore

$$\frac{d\overline{f}(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial}{\partial s}\{e^{-st} f(t)\}dt = -\int_0^\infty te^{-st} f(t)dt$$

or

$$(-1)\frac{d\overline{f}(s)}{ds} = \int_0^\infty e^{-st}\{tf(t)\}dt = L\{tf(t)\}$$

Thus, $(-1)\overline{f}'(s) = L\{tf(t)\}$.

This proves that the theorem is true for $n = 1$.

Let us suppose that the theorem is true for $n = m$, so that

$$(-1)^m\overline{f}^{(m)}(s) = L\{t^m f(t)\} = \int_0^\infty e^{-st}t^m f(t)dt$$

Now

$$(-1)^m\frac{d^{m+1}\overline{f}(s)}{ds^{m+1}} = \int_0^\infty \frac{\partial}{\partial s}\{e^{-st}t^m f(t)\}dt = -\int_0^\infty te^{-st}t^m f(t)dt$$

or

$$(-1)^{m+1}\overline{f}^{(m+1)}(s) = \int_0^\infty e^{-st}\{t^{m+1}f(t)\}dt = L\{t^{m+1}f(t)\}$$

This proves that the theorem is true for $n = m + 1$. Hence by mathematical induction, the theorem is true for $n = 1, 2, 3, \ldots$.

EXAMPLE 11.5.4 Find the Laplace transforms of (i) $t^2\cos at$, (ii) $te^{-t}\sin 2t$, (iii) $t^2(1-e^t)$.

Solution (i) $L\{\cos at\} = \dfrac{s}{s^2+a^2} = \overline{f}(s)$.

Thus,

$$L\{t^2\cos at\} = (-1)^2\frac{d^2}{ds^2}\overline{f}(s) = \frac{d^2}{ds^2}\left\{\frac{s}{s^2+a^2}\right\} = \frac{d}{ds}\left\{-\frac{s^2-a^2}{(s^2+a^2)^2}\right\} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

(ii) $L\{\sin 2t\} = \dfrac{2}{s^2+4} = \overline{f}(s)$.

Then

$$L\{t\sin 2t\} = (-1)\frac{d}{ds}\overline{f}(s) = -\frac{d}{ds}\left\{\frac{2}{s^2+4}\right\} = \frac{4s}{(s^2+4)^2}$$

Now, by shifting property

$$L\{e^{-t}t\sin 2t\} = \frac{4(s+1)}{[(s+1)^2+4]^2} = \frac{4(s+1)}{(s^2+2s+5)^2}$$

(iii) $L\{1-e^t\} = L\{1\} - L\{e^t\} = \dfrac{1}{s} - \dfrac{1}{s-1} = \overline{f}(s)$. Therefore

$$\begin{aligned} L\{t^2(1-e^t)\} &= (-1)^2\frac{d^2}{ds^2}\overline{f}(s) = \frac{d^2}{ds^2}\left\{\frac{1}{s} - \frac{1}{s-1}\right\} \\ &= \frac{d}{ds}\left\{-\frac{1}{s^2} + \frac{1}{(s-1)^2}\right\} = \frac{2}{s^3} - \frac{2}{(s-1)^3} \end{aligned}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s)\,ds$$

11.5.5 Division by t

Theorem 11.6 *If $L\{f(t)\} = \overline{f}(s)$, then $L\{f(t)/t\} = \int_s^\infty \overline{f}(s)ds$, provided the integral exists.*

Proof. Let $L\{f(t)\} = \overline{f}(s)$. Then $\overline{f}(s) = \int_0^\infty e^{-st} f(t)dt$.

Integrating this with respect to s between s and ∞

$$\int_s^\infty \overline{f}(s)ds = \int_s^\infty ds \int_0^\infty e^{-st} f(t)dt$$

Here s and t are independent variables and hence the order of integration in the repeated integral can be interchanged.

Thus

$$\begin{aligned}\int_s^\infty \overline{f}(s)ds &= \int_0^\infty f(t)dt \int_s^\infty e^{-st}ds = \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty ds \\ &= \int_0^\infty \frac{e^{-st}}{t} f(t)dt = \int_0^\infty e^{-st}\left\{\frac{f(t)}{t}\right\}dt = L\left\{\frac{f(t)}{t}\right\}\end{aligned}$$

EXAMPLE 11.5.5 Find the Laplace transforms of

(i) $\dfrac{1-e^t}{t}$ (WBUT 2004) (ii) $\dfrac{\sin at - \sin bt}{t}$ (iii) $\dfrac{\sin t}{t}$ (WBUT 2003)

Solution (i) $L\{1-e^t\} = L\{1\} - L\{e^t\} = \dfrac{1}{s} - \dfrac{1}{s-1} = \overline{f}(s)$ (say). Therefore

$$L\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty \left[\frac{1}{s} - \frac{1}{s-1}\right]ds \leftarrow \text{same time Integration}$$

$$\begin{aligned}&= \Big[\log s - \log(s-1)\Big]_s^\infty = \left[\log\frac{s}{s-1}\right]_s^\infty = \left[\log\frac{1}{1-1/s}\right]_s^\infty \\ &= \log 1 - \log\left(\frac{1}{1-1/s}\right) = \log\left(\frac{s-1}{s}\right)\end{aligned}$$

(ii) $L\{\sin at - \sin bt\} = L\{\sin at\} - L\{\sin bt\} = \dfrac{a}{s^2+a^2} - \dfrac{b}{s^2+b^2} = \overline{f}(s)$ (say).

Now

$$\begin{aligned}L\left\{\frac{\sin at - \sin bt}{t}\right\} &= \int_s^\infty \left[\frac{a}{s^2+a^2} - \frac{b}{s^2+b^2}\right]ds \\ &= \left[\tan^{-1}\frac{s}{a} - \tan^{-1}\frac{s}{b}\right]_s^\infty = \tan^{-1}\frac{s}{b} - \tan^{-1}\frac{s}{a}\end{aligned}$$

(iii) $L\{\sin t\} = \dfrac{1}{s^2+1}$.

Therefore $$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}ds = \Big[\tan^{-1} s\Big]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

11.6 Laplace Transform of Periodic Functions

Let a function $f(t)$ be periodic with period w, so that $f(t+nw) = f(t)$, for $n = 1, 2, 3, \ldots$. Then

$$L\{f(t)\} = \frac{\int_0^w e^{-st} f(t)dt}{1 - e^{-sw}} \tag{11.16}$$

Proof.

$$\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt = \int_0^w e^{-st} f(t)dt + \int_w^{2w} e^{-st} f(t)dt + \cdots \\
&= \sum_{n=0}^{\infty} \int_{nw}^{(n+1)w} e^{-st} f(t)dt = \sum_{n=0}^{\infty} \int_0^w e^{-s(x+nw)} f(x+nw)dx \qquad \text{where } t = x + nw \\
&= \sum_{n=0}^{\infty} \int_0^w e^{-sx} e^{-snw} f(x)dx \quad [\because f(x+nw) = f(x) \text{ for } n = 0, 1, 2, \ldots] \\
&= \sum_{n=0}^{\infty} e^{-snw} \int_0^w e^{-sx} f(x)dx = (1 + e^{-sw} + e^{-2sw} + \cdots) \int_0^w e^{-sx} f(x)dx \\
&= \frac{1}{1 - e^{-sw}} \int_0^w e^{-sx} f(x)dx \qquad \text{as } e^{-sw} < 1 \\
&= \frac{1}{1 - e^{-sw}} \int_0^w e^{-st} f(t)dt
\end{aligned}$$

EXAMPLE 11.6.1 Find the Laplace transform of the periodic function $f(t)$ given by

$$f(t) = \begin{cases} t, & \text{if } 0 < t < c \\ 2c - t, & \text{if } c < t < 2c \end{cases}$$

(WBUT 2003)

Solution Here $f(t)$ is a periodic function with period $2c$. Therefore

$$\begin{aligned}
L\{f(t)\} &= \frac{1}{1 - e^{-2cs}} \int_0^{2c} e^{-st} f(t)dt \\
&= \frac{1}{1 - e^{-2cs}} \left[\int_0^c e^{-st} f(t)dt + \int_c^{2c} e^{-st} f(t)dt \right] \\
&= \frac{1}{1 - e^{-2cs}} \left[\int_0^c te^{-st}dt + \int_c^{2c} (2c - t)e^{-st}dt \right]
\end{aligned}$$

$$= \frac{1}{1-e^{-2cs}}\left\{\left[\frac{-te^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_0^c+\left[(2c-t)\frac{e^{-st}}{-s}+\frac{e^{-st}}{s^2}\right]_c^{2c}\right\}$$

$$= \frac{1}{1-e^{-2cs}}\left[\frac{-ce^{-sc}}{s}-\frac{e^{-sc}}{s^2}+\frac{1}{s^2}+\frac{e^{-2cs}}{s^2}+\frac{ce^{-sc}}{s}-\frac{e^{-sc}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2cs}}\frac{(1-e^{-sc})^2}{s^2}=\frac{1}{s^2}\frac{1-e^{-sc}}{1+e^{-sc}}$$

EXAMPLE 11.6.2 Find the Laplace transform of the function

$$f(t) = \begin{cases} \sin wt, & \text{if } 0 < t < \pi/w \\ 0, & \text{if } \pi/w < t < 2\pi/w \end{cases}$$

Solution Here $f(t)$ is a periodic function with period $2\pi/w$. Therefore

$$\begin{aligned} L\{f(t)\} &= \frac{1}{1-e^{-2\pi s/w}}\int_0^{2\pi/w} e^{-st}f(t)dt \\ &= \frac{1}{1-e^{-2\pi s/w}}\left[\int_0^{\pi/w} e^{-st}\sin wt dt+\int_{\pi/w}^{2\pi/w} e^{-st}\cdot 0dt\right] \\ &= \frac{1}{1-e^{-2\pi s/w}}\left[\frac{e^{-st}(-s\sin wt-w\cos wt)}{s^2+w^2}\right]_0^{\pi/w} \\ &= \frac{we^{-\pi s/w}+w}{(1-e^{-2\pi s/w})(s^2+w^2)}=\frac{w}{(1-e^{-2\pi s/w})(s^2+w^2)} \end{aligned}$$

11.7 Laplace Transforms of Derivatives

Theorem 11.7 *If $L\{f(t)\}=\overline{f}(s)$ the $L\{f'(t)\}=s\overline{f}(s)-f(0)$, provided $f(t)$ is continuous for $0 \le t \le N$ and of exponential order γ for $t > N$, $s > \gamma$ where $f'(t)$ is sectionally continuous for $0 \le t \le N$.*

Proof.

$$\begin{aligned} L\{f'(t)\} &= \int_0^\infty e^{-st}f'(t)dt=\lim_{A\to\infty}\int_0^A e^{-st}f'(t)dt \\ &= \lim_{A\to\infty}\left\{\left[e^{-st}f(t)\right]_0^A+s\int_0^A e^{-st}f(t)dt\right\} \\ &= \lim_{A\to\infty} e^{-sA}f(A)-f(0)+s\overline{f}(s) \\ &= s\overline{f}(s)-f(0), \qquad \text{since } \lim_{A\to\infty} e^{-sA}f(A)=0 \end{aligned}$$

Corollary 11.7.1 *If $f(t)$ fails to be continuous at $t=0$ but $\lim_{t\to 0+} f(t)=f(0+)$ exists, then $L\{f'(t)\}=s\overline{f}(s)-f(0+)$.*

Theorem 11.8 *If $L\{f(t)\} = \bar{f}(s)$, then $L\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0)$ provided $f(t)$ and $f'(t)$ are continuous for $0 \le t \le N$ and of exponential order for $t > N$, where $f''(t)$ is sectionally continuous for $0 \le t \le N$.*

Proof. $L\{f''(t)\} = sL\{f'(t)\} - f'(0) = s\{s\bar{f}(s) - f(0)\} - f'(0) = s^2\bar{f}(s) - sf(0) - f'(0)$.

In general

$$\begin{aligned} L\{f^{(n)}(t)\} &= s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots \\ &\quad - s^1 f^{(n-2)}(0) - f^{(n-1)}(0) \end{aligned}$$

provided $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous for $0 \le t \le N$ and of exponential order for $0 \le t \le N$.

11.8 Laplace Transform of Integral

Theorem 11.9 *If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)$.*

Proof. Let

$$L\{f(t)\} = \bar{f}(s) \qquad \text{and} \qquad g(t) = \int_0^t f(u)du \tag{i}$$

From (i), it is clear that $g(0) = 0$ and $g'(t) = \frac{d}{dt}\left(\int_0^t f(u)du\right) = f(t)$.

Now

$$L\{g'(t)\} = sL\{g(t)\} - g(0)$$

or

$$L\{f(t)\} = sL\{g(t)\}$$

i.e.

$$L\{g(t)\} = \frac{1}{s}\bar{f}(s)$$

Hence $L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)$.

EXAMPLE 11.8.1 Find the Laplace transforms of (i) $\int_0^t \frac{\sin t}{t}dt$, (ii) $\int_0^t e^{-t}\cos t\, dt$.

Solution (i) We know $L\{\sin t\} = \frac{1}{1+s^2}$. Therefore

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{1+s^2}ds = \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

Hence $L\left\{\int_0^t \frac{\sin t}{t}dt\right\} = \frac{1}{s}\cot^{-1} s$.

(ii) We know $L\{\cos t\} = \dfrac{s}{s^2+1}$.

By shifting property

$$L\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$$

Hence $L\left\{\displaystyle\int_0^t e^{-t}\cos t\,dt\right\} = \dfrac{1}{s}\dfrac{s+1}{s^2+2s+2}$.

11.9 Evaluation of Integrals by Laplace Transforms

EXAMPLE 11.9.1 Evaluate (i) $\displaystyle\int_0^\infty \frac{\sin at}{t}dt$, (ii) $\displaystyle\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt$, (iii) $\displaystyle\int_0^\infty \frac{e^{-t}\sin^2 t}{t}dt$.

Solution (i) $L\{\sin at\} = \dfrac{a}{s^2+a^2} = \overline{f}(s)$ (say).

Then

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \overline{f}(s)ds = \int_s^\infty \frac{a}{s^2+a^2}ds = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$$

Thus

$$\int_0^\infty e^{-st}\frac{\sin at}{t}dt = \frac{\pi}{2} - \tan^{-1}\frac{s}{a}$$

Now

$$\lim_{s\to 0}\tan^{-1}\frac{s}{a} = \begin{cases} 0, & \text{if } a>0 \\ \pi, & \text{if } a<0 \end{cases}$$

Thus, taking limit as $s \to 0$, we get

when $\tan^{-1} 0$, $\frac{\pi}{2} - 0 = \frac{\pi}{2}$

when $\tan^{-1}\pi$, $\frac{\pi}{2} - \pi = -\frac{\pi}{2}$

$$\int_0^\infty \frac{\sin at}{t}dt = \begin{cases} \dfrac{\pi}{2}, & \text{if } a>0 \\ -\dfrac{\pi}{2}, & \text{if } a<0 \end{cases}$$

(ii) We know $L\{e^{-at}-e^{-bt}\} = \dfrac{1}{s+a} - \dfrac{1}{s+b} = \overline{f}(s)$ say.

Then

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty \overline{f}(s)ds = \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right]ds$$

$$= \left[\log\frac{s+a}{s+b}\right]_s^\infty = \left[\log\frac{1+a/s}{1+b/s}\right]_s^\infty = \log 1 - \log\left(\frac{1+a/s}{1+b/s}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

That is, $\displaystyle\int_0^\infty e^{-st}\frac{e^{-at}-e^{-bt}}{t}dt = \log\left(\frac{s+b}{s+a}\right)$.

Taking limit as $s \to 0$, we get

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \log\left(\frac{b}{a}\right)$$

(iii) We know that $L\{\sin^2 t\} = \frac{1}{2}L\{1 - \cos 2t\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$.

Then

$$\begin{aligned}
L\left\{\frac{\sin^2 t}{t}\right\} &= \int_s^\infty \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds = \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty \\
&= \frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty = -\frac{1}{4}\log\left(\frac{s^2}{s^2+4}\right)
\end{aligned}$$

That is

$$\int_0^\infty e^{-st}\frac{\sin^2 t}{t} dt = \frac{1}{4}\left(\frac{s^2+4}{s^2}\right)$$

Now, taking limit as $s \to 1$, we get

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t} dt = \frac{\log 5}{4}$$

11.10 Laplace Transforms of Bessel's Functions

We know from the definition of Bessel function

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

Therefore

$$\begin{aligned}
L\{J_0(x)\} &= \frac{1}{s} - \frac{1}{2^2} \cdot \frac{2!}{s^3} + \frac{1}{2^2 \cdot 4^2} \cdot \frac{4!}{s^5} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{6!}{s^7} + \cdots \\
&= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{s^4}\right) + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{s^6}\right) + \cdots\right] \\
&= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{1+s^2}}
\end{aligned}$$

Again, we know that $J_0'(x) = -J_1(x)$.

Thus, $L\{J_1(x)\} = -L\{J_0'(x)\} = -[sL\{J_0(x)\} - J_0(0)] = 1 - \dfrac{s}{\sqrt{1+s^2}}$.

EXAMPLE 11.10.1 Find the Laplace transforms of the following functions

(i) $e^{-at}J_0(bt)$ (ii) $tJ_1(t)$ (iii) $\int_0^\infty te^{-3t}J_0(4t)dt$.

Solution (i) We know that $L\{J_0(t)\} = \dfrac{1}{\sqrt{1+s^2}}$.
Then by change of scale property

$$L\{J_0(bt)\} = \frac{1}{b}\frac{1}{\sqrt{1+(s/b)^2}} = \frac{1}{\sqrt{s^2+b^2}}$$

Now by shifting property, $L\{e^{-at}J_0(bt)\} = \dfrac{1}{\sqrt{(s-a)^2+b^2}}$.

(ii) We know that $L\{J_1(t)\} = 1 - \dfrac{s}{\sqrt{1+s^2}}$.
Therefore

$$L\{tJ_1(t)\} = (-1)\frac{d}{ds}\left\{1 - \frac{s}{\sqrt{1+s^2}}\right\} = \frac{1}{(1+s^2)^{3/2}}$$

(iii) Again, $L\{J_0(t)\} = \dfrac{1}{\sqrt{1+s^2}}$. Therefore

$$L\{J_0(4t)\} = \frac{1}{4}\frac{1}{\sqrt{1+(s/4)^2}} = \frac{1}{\sqrt{s^2+16}}$$

Also

$$L\{tJ_0(4t)\} = (-1)\frac{d}{ds}\left\{\frac{1}{\sqrt{s^2+16}}\right\} = \frac{s}{(s^2+16)^{3/2}}$$

That is

$$\int_0^\infty e^{-st}tJ_0(4t)dt = \frac{s}{(s^2+16)^{3/2}}$$

Now, taking limit as $s \to 3$, we get

$$\int_0^\infty e^{-3t}tJ_0(4t)dt = \frac{3}{(9+16)^{3/2}} = \frac{3}{125}$$

11.11 Unit Step Function and Its Laplace Transformation

The **unit step function** is generally denoted by u and is defined as

$$u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases} \tag{11.17}$$

This function has a unit jump at any point a. The unit step function is also called the **Heaviside function**. It is shown in Fig. 11.1.

The unit step function is a typical engineering function made to measure for engineering applications.

The unit step function can also be written as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

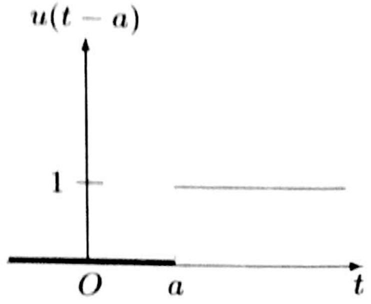

Figure 11.1: Unit step function $u(t-a)$.

The function

$$f(t) = \begin{cases} f_1(t), & t < a \\ f_2(t), & t \geq a \end{cases}$$

can be expressed by a unit step function as

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\}u(t-a)$$

Because, when $t \geq a$, then $u(t-a) = 1$. In this case, $f(t) = f_2(t)$, and when $t < a$, then $u(t-a) = 0$. Hence $f(t) = f_1(t)$.

Similarly, if

$$f(t) = \begin{cases} f_1(t), & t < a_1 \\ f_2(t), & a_1 < t < a_2 \\ f_3(t), & a_2 < t \end{cases}$$

then $f(t)$ can be expressed as

$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\}u(t-a_1) + \{f_3(t) - f_2(t)\}u(t-a_2)$$

Theorem 11.10 *If $u(t-a)$ is a unit step function then $L\{u(t-a)\} = \dfrac{e^{-as}}{s}$.*

Proof.

$$\begin{aligned} L\{u(t-a)\} &= \int_0^\infty e^{-st}\, u(t-a)dt = \int_0^a e^{-st} \cdot 0dt + \int_a^\infty e^{-st} \cdot 1dt \\ &= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-st}}{s} \end{aligned}$$

Theorem 11.11 *Let $L\{f(t)\} = \overline{f}(s)$ and $u(t-a)$ be a unit step function. Then $L\{f(t-a)u(t-a)\} = e^{-st}\overline{f}(s) = e^{-as}L\{f(t)\}$.*

Proof.

$$
\begin{aligned}
& L\{f(t-a)\,u(t-a)\} \\
&= \int_0^\infty e^{-st} f(t-a)\,u(t-a)dt \\
&= \int_0^a e^{-st} f(t-a) \times 0\,dt + \int_a^\infty e^{-st} f(t-a)\,dt \\
&= \int_a^\infty e^{-st} f(t-a)dt = \int_0^\infty e^{-s(x+a)} f(x)dx \qquad \text{where } x = t-a \\
&= e^{-sa} \int_0^\infty e^{-sx} f(x)dx = e^{-sa} L\{f(t)\} = e^{-sa}\bar{f}(s)
\end{aligned}
$$

EXAMPLE 11.11.1 Express the function

$$f(t) = \begin{cases} e^{-t}, & 0 < t < 2 \\ 0, & t \geq 2 \end{cases}$$

in terms of unit step function and hence find $L\{f(t)\}$. (WBUT 2004)

Solution The function $f(t)$ can be written in terms of unit step function as $f(t) = e^{-t} + (0 - e^{-t})u(t-3)$, where $u(t-3)$ is the unit step function.
Now

$$L\{f(t)\} = L\{e^{-t}\} - L\{e^{-t}u(t-3)\}$$

We know that

$$L\{e^{-t}\} = \frac{1}{s+1} \qquad \text{and} \qquad L\{u(t-3)\} = \frac{e^{-st}}{s}$$

Therefore

$$L\{e^{-t}u(t-3)\} = \frac{e^{-3(s+1)}}{s+1}$$

Hence

$$L\{f(t)\} = \frac{1}{s+1} - \frac{e^{-3(s+1)}}{s+1} = \frac{1}{s+1}\left[1 - e^{-3(s+1)}\right]$$

EXAMPLE 11.11.2 Express

$$f(t) = \begin{cases} 2, & \text{if } 0 < t < \pi \\ 0, & \text{if } \pi < t < 2\pi \\ \sin t, & \text{if } t > 2\pi \end{cases}$$

n terms of unit step function and hence evaluate $L\{f(t)\}$.

olution The function $f(t)$ in terms of step function is

$$
\begin{aligned}
f(t) &= 2 + (0-2)u(t-\pi) + (\sin t - 0)u(t-2\pi) \\
&= 2 - 2\,u(t-\pi) + \sin t\; u(t-2\pi)
\end{aligned}
$$

The function $\sin t\, u(t-2\pi)$ is equal to $\sin(t-2\pi)\, u(t-2\pi)$ because of periodicity. Thus

$$L\{\sin(t-2\pi)\cdot u(t-2\pi)\} = e^{-2\pi s}L\{\sin t\} = \frac{e^{-2\pi s}}{s^2+1}$$

Hence

$$\begin{aligned} L\{f(t)\} &= L\{2\} - L\{2u(t-\pi)\} + L\{\sin t\, u(t-2\pi)\} \\ &= \frac{2}{s} - \frac{2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1} \end{aligned}$$

11.12 Additional Worked-Out Examples

EXAMPLE 11.12.1 Find the Laplace transforms of

(i) $e^{-2t}(2\cos 3t - 3\sin 4t)$ (ii) $e^{5t}\sin 2t \cos t$
(iii) $e^{3t} + 5t^2 - 2\sin 4t + 3\cos 2t$ (iv) $\cos(at+b)$.

Solution (i) We know $L\{\cos 3t\} = \dfrac{s}{s^2+9}$ and $L\{\sin 4t\} = \dfrac{4}{s^2+16}$.

Therefore

$$L\{2\cos 3t - 3\sin 4t\} = \frac{2s}{s^2+9} - \frac{12}{s^2+16}$$

Hence by shifting property

$$\begin{aligned} L\{e^{-2t}(2\cos 3t - 3\sin 4t)\} &= \frac{2(s+2)}{(s+2)^2+9} - \frac{12}{(s+2)^2+16} \\ &= \frac{2(s+2)}{s^2+2s+13} - \frac{12}{s^2+4s+20} \end{aligned}$$

(ii) $L\{\sin 2t\ \cos t\} = \dfrac{1}{2}L\{\sin 3t + \sin t\} = \dfrac{1}{2}\left\{\dfrac{3}{s^2+9} + \dfrac{1}{s^2+1}\right\}$

By shifting property

$$L\{e^{5t}\sin 2t\ \cos t\} = \frac{1}{2}\left\{\frac{3}{(s-5)^2+9} + \frac{1}{(s-5)^2+1}\right\}$$

(iii) We have

$$\begin{aligned} & L\{e^{3t} + 5t^2 - 2\sin 4t + 3\cos 2t\} \\ =\ & L\{e^{3t}\} + 5L\{t^2\} - 2L\{\sin 4t\} + 3L\{\cos 2t\} \\ =\ & \frac{1}{s-3} + 5\frac{2}{s^3} - 2\frac{4}{s^2+16} + 3\frac{s}{s^2+4} \\ =\ & \frac{1}{s-3} + \frac{10}{s^3} - \frac{8}{s^2+16} + \frac{3s}{s^2+4} \end{aligned}$$

(iv) Since $\cos(at+b) = \cos at \cos b - \sin at \sin b$

$$L\{\cos(at+b)\} = \cos b\, L\{\cos at\} - \sin b\, L\{\sin at\}$$

$$= \cos b \frac{s}{s^2+a^2} - \sin b \frac{a}{s^2+a^2} = \frac{1}{s^2+a^2}(s\cos b - a\sin b)$$

EXAMPLE 11.12.2 If $L\{f(t)\} = \overline{f}(s)$, show that

$$L\{\sinh at\ f(t)\} = \frac{1}{2}[\overline{f}(s-a) - \overline{f}(s+a)]$$

and

$$L\{\cosh at\ f(t)\} = \frac{1}{2}[\overline{f}(s-a) + \overline{f}(s+a)]$$

Hence find the value of $L\{\sinh 2t\ \cos 4t\}$.

Solution We have

$$L\{\sinh at\ f(t)\} = L\{\frac{1}{2}(e^{at} - e^{-at})f(t)\}$$

$$= \frac{1}{2}[L\{e^{at}f(t)\} - L\{e^{-at}f(t)\}]$$

$$= \frac{1}{2}[\overline{f}(s-a) - \overline{f}(s+a)] \qquad \text{(by shifting property)}$$

Similarly

$$L\{\cosh at\ f(t)\} = L\left\{\frac{1}{2}(e^{at} + e^{-at})f(t)\right\} = \frac{1}{2}[\overline{f}(s-a) + \overline{f}(s+a)]$$

Also

$$L\{\sinh 2t\ \cos 4t\} = \frac{1}{2}[\overline{f}(s-2) - \overline{f}(s+2)] \qquad \text{and} \qquad L\{\cos 4t\} = \frac{s}{s^2+16} = \overline{f}(s)$$

Hence

$$L\{\sinh 2t\ \cos 4t\} = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+16} - \frac{s+2}{(s+2)^2+16}\right]$$

EXAMPLE 11.12.3 If $L\{f(t)\} = \dfrac{e^{-2s}}{s}$, then find the value of $L\{e^{-3t}f(4t)\}$.

Solution Since $L\{f(t)\} = \dfrac{e^{-2s}}{s}$, then by change of scale property

$$L\{f(4t)\} = \frac{1}{4}\frac{e^{-2(s/4)}}{(s/4)} = \frac{e^{-s/2}}{s}$$

Now, by shifting property

$$L\{e^{-3t}f(4t)\} = \frac{e^{-(s+3)/2}}{s+3}$$

EXAMPLE 11.12.4 Find the Laplace transform of $t^3 \sin t$ and hence find the value of $\int_0^\infty t^3 e^{-t} \sin t \, dt$.

Solution We know $L\{\sin t\} = \dfrac{1}{s^2+1}$.

Therefore

$$L\{t^3 \sin t\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right) = -\frac{d^2}{ds^2}\left(\frac{-2s}{(1+s^2)^2}\right)$$

$$= \frac{d}{ds}\left\{\frac{2-6s^2}{(1+s^2)^3}\right\} = \frac{24(s^2-1)s}{(1+s^2)^4}$$

Putting $s = 1$, we have $\int_0^\infty t^3 e^{-t} \sin t \, dt = 0.$

EXAMPLE 11.12.5 Evaluate $\int_0^\infty \int_0^t \dfrac{e^{-t} \sin u}{u} du \, dt.$

Solution We know $L\{\sin t\} = \dfrac{1}{s^2+1}$.

Therefore

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1} ds = \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

Again, $L\left\{\int_0^t \dfrac{\sin u}{u} du\right\} = \dfrac{1}{s}\cot^{-1} s.$

Now, by shifting property, $L\left\{e^{-t}\int_0^t \dfrac{\sin u}{u} du\right\} = \dfrac{1}{s+1}\cot^{-1}(s+1).$

That is

$$\int_0^\infty e^{-st}\left\{e^{-t}\int_0^t \frac{\sin u}{u} du\right\} dt = \frac{1}{s+1}\cot^{-1}(s+1)$$

or

$$\int_0^\infty e^{-st} \int_0^t \frac{e^{-t}\sin u}{u} du \, dt = \frac{1}{s+1}\cot^{-1}(s+1)$$

Putting $s = 0$, we get

$$\int_0^\infty \int_0^t \frac{e^{-t}\sin u}{u} du \, dt = \cot^{-1}(1) = \frac{\pi}{4}$$

EXAMPLE 11.12.6 Evaluate

(i) $L\left\{t\int_0^t \dfrac{e^{-t}\sin t}{t} dt\right\}$ (ii) $L\left\{\int_0^t \int_0^t \int_0^t (t\cos t) dt \, dt \, dt\right\}.$

Solution (i) We know $L\{\sin t\} = \dfrac{1}{s^2+1}$.
Thus,

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}ds = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$\therefore \quad L\left\{\int_0^t \frac{\sin t}{t}dt\right\} = \frac{1}{s}\cot^{-1} s.$$

Now, by shifting property

$$L\left\{e^{-t}\left(\int_0^t \frac{\sin t}{t}dt\right)\right\} = \frac{1}{s+1}\cot^{-1}(s+1)$$

(ii) $L\{\cos t\} = \dfrac{s}{s^2+1}$.
Therefore

$$L\{t\cos t\} = -\frac{d}{ds}\left\{\frac{s}{s^2+1}\right\} = \frac{s^2-1}{(s^2+1)^2}$$

Hence

$$L\left\{\int_0^t\int_0^t\int_0^t (t\cos t)dt\,dt\,dt\right\} = \frac{1}{s^3}L\{t\cos t\} = \frac{1}{s^3}\frac{s^2-1}{(s^2+1)^2}$$

EXAMPLE 11.12.7 Find the Laplace transform of $f(t) = |t-1| + |t+1|, t \geq 0$.

Solution This function is written as

$$f(t) = \begin{cases} -(t-1)+(t+1) = 2, & \text{when } 0 \leq t \leq 1 \\ (t-1)+(t+1) = 2t, & \text{when } t > 1 \end{cases}$$

Therefore

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st}f(t)dt = \int_0^1 e^{-st}\cdot 2dt + \int_1^\infty 2te^{-st}dt \\ &= \left[\frac{2e^{-st}}{-s}\right]_0^1 + \left[2t\frac{e^{-st}}{-s}\right]_1^\infty - \int_1^\infty 2\frac{e^{-st}}{-s}dt \\ &= \left[\frac{2e^{-s}}{-s} + \frac{2}{s}\right] + \frac{2e^{-s}}{s} - \left[\frac{2e^{-st}}{s^2}\right]_1^\infty \\ &= \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right) \end{aligned}$$

EXERCISES

Section A Multiple Choice Questions

1. The Laplace transform of 5 is
   (a) 5 (b) $5s$ (c) $5/s$ (d) s^5.

2. The Laplace transform of t^3 is

(a) $\frac{1}{3}$ (b) $\frac{3}{s}$ (c) $\frac{1}{s^4}$ (d) $\frac{6}{s^4}$ $\frac{6}{s^4}$

3. The Laplace transform of e^{-4t} is

(a) $\frac{1}{s+4}$ (b) $\frac{1}{s-4}$ (c) $(s-4)$ (d) e^{-4s}.

4. $L\{\sin 3t\}$ is

(a) $\frac{3}{s^2+9}$ (b) $\frac{s}{s^2+9}$ (c) $\frac{s}{s^2-9}$ (d) $\frac{3}{s^2-9}$.

5. If $L\{f(t)\} = 1/s^2$ then $L\{e^{2t}f(t)\}$ is

(a) $\frac{e^{2t}}{s^2}$ (b) $\frac{e^{2s}}{s^2}$ (c) $\frac{1}{(s-2)^2}$ (d) $\frac{1}{(s+2)^2}$.

6. $L\{te^{2t}\}$ is equal to

(a) $\frac{1}{s-2}$ (b) $2(s-2)^2$ (c) $\frac{1}{(s-2)^2}$ (d) $\frac{2!}{s^2}$. $\frac{2!}{(s-2)^2}$ (WBUT 2006)

7. The Laplace transform of the function $\cos at$ is

(a) $\frac{s}{s^2-a^2}$ (b) $\frac{a}{s^2+a^2}$ (c) $\frac{s}{s^2+a^2}$ (d) $\frac{1}{s^2-a^2}$. (WBUT 2007)

8. $L\{e^{2t}\sin 3t\}$ is equal to

(a) $\frac{3}{(s-2)^2+9}$ (b) $\frac{3}{(s+2)^2+9}$ (c) $\frac{s}{(s-2)^2+9}$ (d) $\frac{s}{(s-2)^2+1}$.

9. $L\{t\sin t\}$ is equal to

(a) $\frac{s}{(1+s^2)^2}$ (b) $\frac{1}{s}\frac{1}{(s^2+1)^2}$ (c) $-\frac{2s}{(1+s^2)^2}$ (d) $-\frac{2}{(1+s^2)^2}$.

10. If $L\{f(t)\} = \frac{s^2+1}{s^2-1}$, then $L\{f(2t)\}$ is

(a) $\frac{s^2+4}{s^2-4}$ (b) $\frac{1}{2}\frac{s^2+4}{s^2-4}$ (c) $\frac{1}{2}\frac{s^2+1}{s^2-1}$ (d) $\frac{1}{2}\frac{s^2+2}{s^2-2}$.

11. If $L\{f(t)\} = \frac{1}{s}$ and $f(0) = 1$, then $L\{f'(t)\}$ is

(a) 0 (b) s (c) $\frac{1}{s}$ (d) $s-1$.

12. If $L\{f(t)\} = \tan^{-1}(1/s)$ then $L\{tf(t)\}$ is

(a) $\tan^{-1}\left(\frac{1}{s}\right)$ (b) $\frac{1}{s^2+1}$ (c) $\frac{1}{s+1}$ (d) $\tan^{-1}\left(\frac{2}{\pi s}\right)$. (WBUT 2008)

13. If $f(t) = \begin{cases} 1, & 0 < t < 2 \\ 0, & t > 2 \end{cases}$, then $L\{f(t)\}$ is equal to

(a) $\frac{1}{s}(1-e^{-2s})$ (b) $\frac{1}{s}(1+e^{-2s})$ (c) $\frac{1}{s}$ (d) $\frac{e^{-2s}}{s}$.

14. If $L\{f(t)\} = \frac{1}{s^2}$, then $L\{f(t)/t\}$ is equal to

(a) $1/s^2$ (b) $1/s^3$ (c) $1/s$ (d) s.

15. $L\{\sin at + \cos at\}$ is

(a) $\frac{2a}{s^2+a^2}$ (b) $\frac{a+s}{s^2+a^2}$ (c) $\frac{2s}{s^2+a^2}$ (d) $\frac{a+s}{s^2-a^2}$.

16. $L\left\{\frac{t\sin t}{e^t}\right\}$ is

(a) $\frac{s+1}{s^2+2s+2}$ (b) $\frac{s+1}{(s^2+2s+2)^2}$ (c) $\frac{2(s+1)}{(s^2+2s+2)^2}$ (d) $\frac{s+2}{(s^2+2s+2)^2}$.

17. $\int_0^\infty e^{-t}\cos 2t\, dt$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{1}{5}$.

18. $\int_0^\infty t\sin 2t\, dt$ is

(a) 0 (b) 1 (c) -1 (d) 4.

19. If $f(t) = \begin{cases} 1, & t > 2 \\ 0, & t < 2 \end{cases}$, then $L\{f(t)\}$ is equal to

(a) $\frac{e^{-2s}}{s}$ (b) $\frac{e^{2s}}{s}$ (c) $\frac{e^s}{s}$ (d) $\frac{e^{-2s}}{5^s}$.

20. If $L\{f(t)\} = e^{-s}$, then $L\left\{\frac{f(t)}{t}\right\}$ is

(a) e^s (b) e^{-s} (c) $\frac{e^{-s}}{s}$ (d) $\frac{e^s}{s}$.

21. $\int_0^\infty t^2 e^{2t}\, dt$ is equal to

(a) $\frac{1}{4}$ (b) $\frac{-1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{8}$.

22. If $L\{f(t)\} = \cot^{-1} s$, then $L\left\{\int_0^t f(t)dt\right\}$ is equal to

(a) $\cot^{-1} s$ (b) $-\frac{1}{1+s^2}$ (c) $\frac{1}{s}\cot^{-1} s$ (d) $s\cot^{-1} s$.

23. $L\{1/t\}$ is equal to

(a) $\frac{1}{s}$ (b) s (c) 1 (d) none of these.

Section B Review Questions

1. Prove that the function e^{t^3} does not satisfy the sufficiency condition for existence of Laplace transform.

2. Find the Laplace transform of the following functions:
(a) $3t^2 + 4e^{2t} + 5$
(b) $a_0t^n + a_1t^{n-1} + a_2t^{n-2} + \cdots + a_{n-1}t + a_n$.

3. Find the value of $L\{4t^3 - 3\sin 4t + 3e^{-2t}\}$.

4. Evaluate $L\{(t^2-2)^2\}$.

5. Evaluate $L\{e^{-4t}\sin 2t\ \cos t\}$.

6. Evaluate $L\{\sin^3 2t\}$.

7. Evaluate $L\{3\sin 3t\ \sin 4t\}$.

8. Find the Laplace transform of $2\cosh 3t + 4\sinh 2t$.

9. Find the Laplace transform of $4e^{2t}\cos 3t$.

10. Find the values of $L\{t\sin at\}$ and $L\{t\cos at\}$.

11. Evaluate $L\{(\sin t - \cos t)^2\}$.

12. Evaluate $L\{\cosh at - \cos at\}$.

13. Evaluate $L\{\sin 2t\ \cos 3t\}$.

14. Evaluate $L\{\sin^5 t\}$.

15. Evaluate $L\{(1 + te^{-t})^3\}$.

16. Evaluate $L\{\sinh 3t\ \cos^2 t\}$.

17. Find the Laplace transform of $f(t) = |t-1| + |t-2|, t \geq 0$.

18. Find the Laplace transform of $f(t)$, where

$$f(t) = \begin{cases} \sin(t - \pi/3), & \text{if } t > \pi/3 \\ 0, & \text{if } t < \pi/3 \end{cases}$$ (WBUT 2004)

19. Find $L\{f(t)\}$, where

$$f(t) = \begin{cases} \sin t, & \text{if } 0 < t < \pi \\ 0, & \text{if } t > \pi \end{cases}$$

20. Find $L\{f(t)\}$, if

$$f(t) = \begin{cases} (t-1)^2, & t > 1 \\ 0, & 0 < t < 1 \end{cases}$$

21. Find $L\{f(t)\}$, where

$$f(t) = \begin{cases} 1, & \text{if } 0 < t < 2 \\ 2, & \text{if } t > 2 \end{cases}$$ (WBUT 2002)

22. Find the Laplace transform of $f(t)$, where

$$f(t) = \begin{cases} \cos(t - 2\pi/3), & \text{if } t > 2\pi/3 \\ 0, & \text{if } t < 2\pi/3 \end{cases}$$

23. Find $L\{f(t)\}$, where

$$f(t) = \begin{cases} e^t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

24. Find the Laplace transform of $f(t)$, where

$$f(t) = \begin{cases} 1, & 0 \le t \le a/2 \\ -1, & a/2 < t \le a \end{cases}$$

25. Find $L\{f(t)\}$, where

$$f(t) = \begin{cases} t^2, & 0 < t < 2 \\ t-1, & 2 < t < 3 \\ 7, & t > 3 \end{cases}$$

26. Find the Laplace transform of the triangular wave function of period $2c$ given by

$$f(t) = \begin{cases} t, & 0 \le t \le c \\ 2c - t, & c < t < 2c \end{cases}$$

27. Find the Laplace transform of the square wave function of period w defined by

$$f(t) = \begin{cases} 1, & 0 < t < w/2 \\ -1, & w/2 < t < w \end{cases}$$

28. Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$$

29. Find $L\{f(t)\}$ where $f(t)$ is a periodic function of period $2c$ defined by

$$f(t) = \begin{cases} t/c, & 0 < t < c \\ (2c - t)/c, & c < t < 2c \end{cases}$$

Draw the graph of the function.

30. Find $L\{f(t)\}$ where

$$f(t) = \begin{cases} 2\sin 3t, & 0 < t < \pi/3 \\ 0, & \pi/3 < t < 2\pi/3 \end{cases}$$

where $f(t) = f(t + 2\pi/3)$.

31. Find $L\{f(t)\}$, where

$$f(t) = \begin{cases} 5\sin 3(t - \pi/4), & t > \pi/4 \\ 0, & t < \pi/4 \end{cases}$$

32. Find (a) $L\{J_0(ax)\}$ and (b) $L\{e^{-at}J_0(bt)\}$.

33. If

$$f(t) = \begin{cases} 0, & 0 \le t \le 1 \\ t^2, & t > 1 \end{cases}$$

find $L\{f(t)\}$ and $L\{f'(t)\}$.

34. Find $L\{f(t)\}$ and $L\{f'(t)\}$, where

$$f(t) = \begin{cases} t, & 0 \le t \le 2 \\ 2t, & t > 2 \end{cases}$$

35. Find $L\{\sin 2t\}$ and hence find $L\left\{\frac{\sin 2t}{t}\right\}$.

36. Find the value of $L\left\{\frac{e^{at} - \cos bt}{t}\right\}$.

37. Evaluate $L\left\{\frac{e^{-2t} - e^{-4t}}{t}\right\}$.

38. Evaluate $L\{(\cos 4t - \cos 5t)/t\}$.

39. Evaluate $L\{(1 - \cos t)/t^2\}$.

40. Find the Laplace transform of $1 - \cos 3t$ and hence evaluate $L\left\{\frac{1 - \cos 3t}{t}\right\}$.

41. Evaluate $L\left\{2^t + \frac{\cos 2t - \cos 3t}{t}\right\}$.

42. Evaluate $L\left\{\frac{\cos at - \cos bt}{t}\right\}$.

43. Express

$$f(t) = \begin{cases} 0, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$$

as unit step function and hence evaluate $L\{f(t)\}$.

44. Find the Laplace transform of
(a) $(t-1)\,u(t-1)$ (b) $t^2\,u(t-1)$ (c) $4u(t-\pi)\,\cos t$.

45. Express the following function in terms of unit step function

$$f(t) = \begin{cases} t - 1, & 1 < t < 2 \\ 3 - t, & 2 < t < 3 \end{cases}$$

Also, find $L\{f(t)\}$. (WBUT 200

46. Express the function

$$f(t) = \begin{cases} e^t, & 0 < t < 2 \\ 0, & t > 2 \end{cases}$$

in terms of unit step function and hence find $L\{f(t)\}$. (WBUT 2004)

47. Express

$$f(t) = \begin{cases} 2t, & 0 < t < \pi \\ 1, & t > \pi \end{cases}$$

in terms of unit step function and hence find $L\{f(t)\}$.

48. Express

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \sin 2t, & \pi < t < 2\pi \\ \sin 3t, & t > 2\pi \end{cases}$$

in terms of unit step function and hence evaluate $L\{f(t)\}$.

49. Evaluate $L\left\{e^{-2t}\int_0^t \frac{\sin t}{t}\,dt\right\}$.

50. Evaluate $L\left\{e^{2t}\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$.

51. Evaluate $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$.

52. Find the value of $L\left\{\int_0^t e^{-2t}\cos 3t\,dt\right\}$.

53. Evaluate $L\left\{\cosh t\int_0^t u\cosh u\,du\right\}$.

54. Find the Laplace transform of $\int_0^t \frac{e^{2t}\sin t}{t}\,dt$.

55. Find the Laplace transform of $\frac{\sin at}{t}$. Hence show that $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$.

(WBUT 200

56. Evaluate (a) $\int_0^\infty \frac{e^{-2t} - e^{-t}}{t}\,dt$, (b) $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\,dt$.

57. Using the Laplace transform, prove that $\int_0^\infty te^{-3t}\sin t\,dt = \frac{3}{50}$.

58. Find the value of $\int_0^\infty \frac{\cos at - \cos bt}{t}\,dt$.

59. Show that $\displaystyle\int_0^\infty \frac{\sin^3 t}{t}\,dt = \frac{\pi}{4}$.

60. Prove that $\displaystyle\int_0^\infty te^{-2t}\sin 3t\,dt = \frac{12}{169}$.

61. Use the Laplace transform to prove that $\displaystyle\int_0^\infty \frac{e^{-\sqrt{2}t}\sinh t\,\sin t}{t}\,dt = \frac{\pi}{8}$.

62. Show that $\displaystyle\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt = \frac{1}{4}\log 5$.

63. Prove that $\displaystyle\int_0^\infty \frac{1-\cos t}{t^2}\,dt = \frac{\pi}{4}$.

64. Show that $\displaystyle\int_0^\infty te^{-3t}J_0(4t)\,dt = \frac{3}{125}$.

65. Prove that $\displaystyle L\left\{\int_0^t\int_0^t\int_0^t t\,\sin t\,dt\right\} = \frac{2}{s^2(s^2+1)^2}$.

Answers

Section A Multiple Choice Questions

1. (c) 2. (d) 3. (a) 4. (a) 5. (c) 6. (c) 7. (c) 8. (a) 9. (c)
10. (b) 11. (a) 12. (b) 13. (a) 14. (c) 15. (b) 16. (c) 17. (d) 18. (a)
19. (b) 20. (b) 21. (a) 22. (c) 23. (d)

Section B Review Questions

2. (a) $\dfrac{6}{s^3}+\dfrac{4}{s-2}+\dfrac{5}{s}$ (b) $a_0\dfrac{n!}{s^{n+1}}+a_1\dfrac{(n-1)!}{s^n}+\cdots+a_{n-1}\dfrac{1}{s^2}+a_n\dfrac{1}{s}$

3. $\dfrac{24}{s^4}-\dfrac{12}{s^2+16}+\dfrac{3}{s+2}$

4. $\dfrac{24}{s^5}-\dfrac{8}{s^3}+\dfrac{4}{s}$

5. $\dfrac{1}{2}\left\{\dfrac{3}{(s+4)^2+9}+\dfrac{1}{(s+4)^2+1}\right\}$

6. $\dfrac{48}{(s^2+4)(s^2+36)}$

7. $\dfrac{3s}{2}\left[\dfrac{1}{s^2+1}-\dfrac{1}{s^2+40}\right]$

8. $\dfrac{2s}{s^2-9}+\dfrac{8}{s^2-4}$

9. $\dfrac{4(s-2)}{s^2-2s+13}$

10. $\dfrac{2as}{(s^2+a^2)^2}$ and $\dfrac{s^2-a^2}{(s^2+a^2)^2}$

11. $\dfrac{s^2 - 2s + 4}{s(s^2 + 4)}$

12. $\dfrac{2a^2 s}{s^4 - a^4}$

13. $\dfrac{2(s^2 - 5)}{(s^2 + 1)(s^2 + 25)}$

14. $\dfrac{5}{4}\left[\dfrac{1}{s^2 + 1} - \dfrac{3/2}{s^2 + 9} + \dfrac{1/2}{s^2 + 25}\right]$

15. $\dfrac{1}{s} + \dfrac{3}{(s + 1)^2} + \dfrac{6}{(s + 2)^3} + \dfrac{6}{(s + 3)^4}$

16. $\dfrac{3}{2}\left[\dfrac{1}{s^2 - 9} + \dfrac{s^2 - 13}{s^4 - 10s^2 + 169}\right]$

17. $\dfrac{1}{s^2}(2e^{-2s} - 2e^{-s} + 3s + 2)$

18. $\dfrac{1}{1 + s^2}e^{-\pi s/3}$

19. $\dfrac{1 + e^{-\pi s}}{1 + s^2}$

20. $\dfrac{2e^{-s}}{s^3}$

21. $\dfrac{1}{s}(1 + e^{-2s})$

22. $e^{-2\pi s/3}\dfrac{s}{1 + s^2}$

23. $\dfrac{1}{1 - s}[e^{1-s} - 1]$

24. $\dfrac{1}{s}[1 + e^{-as} - 2e^{-as/2}]$

25. $\dfrac{2}{s^3} - \dfrac{e^{-2s}}{s^3}(2 + 3s + 3s^2) + \dfrac{e^{-3s}}{s^2}(5s - 1)$

26. $\dfrac{1}{s^2}\sinh\dfrac{cs}{2}$

27. $\dfrac{1}{s}\tanh\dfrac{sw}{4}$

28. $\dfrac{(1 - e^{-\pi s})(1 + \pi s)}{(1 + e^{-\pi s})s^2}$

29. $\dfrac{1}{cs^2}\tanh\dfrac{cs}{2}$

30. $\dfrac{12}{(s^2 + 9)(1 - e^{-2\pi s/3})}$

31. $\dfrac{15e^{-\pi s/4}}{s^2 + 9}$

32. (a) $\dfrac{1}{\sqrt{s^2+a^2}}$ (b) $\dfrac{1}{\sqrt{(s-a)^2+b^2}}$

33. $\dfrac{e^{-s}}{s^2}(s+2-2e^{s})$ and $\dfrac{e^{-s}}{s^2}(s+2-2e^{-s})$

34. $\dfrac{e^{-2s}}{s^2}(2s+1+e^{2s})$ and $\dfrac{e^{-2s}}{s}(2s+1+e^{2s})$

35. $\dfrac{2}{s^2+4}, \cot^{-1}\dfrac{s}{2}$

36. $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{(s-a)^2}\right)$

37. $\log\left(\dfrac{s+4}{s+2}\right)$

38. $\dfrac{1}{2}\log\left(\dfrac{s^2+25}{s^2+16}\right)$

39. $\cot^{-1}s-\dfrac{1}{2}s\log(1+\dfrac{1}{s^2})$

40. $\dfrac{1}{s}-\dfrac{s}{s^2+9}, \dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2}\right)$

41. $\dfrac{1}{s-\log 2}+\dfrac{1}{2}\log\left(\dfrac{s^2+9}{s^2+4}\right)$

42. $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$

43. $\left(\dfrac{1}{s}+\dfrac{1}{s^2}\right)e^{-s}-\left(\dfrac{1}{s^2}+\dfrac{2}{s}\right)e^{-2s}$

44. (a) $\frac{e^{-s}}{s^2}$, (b) $e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$, (c) $-4e^{\pi s}s/(1+s^2)$

45. $\dfrac{1}{s^2}(1-2e^{-2s})-\dfrac{1}{s}$

46. $\dfrac{1}{s+1}-\dfrac{e^{-2(s+1)}}{s+1}$

47. $\dfrac{2}{s^2}+\left(\dfrac{1-2\pi}{s}-\dfrac{2}{s^2}\right)e^{-as}$

48. $\dfrac{1}{s^2+1}-\dfrac{2e^{-\pi s}}{s^2+4}+\dfrac{e^{-\pi s}}{s^2+1}+\dfrac{3e^{-2\pi s}}{s^2+9}-\dfrac{2e^{-2\pi s}}{s^2+4}$

49. $\dfrac{1}{s+2}\cot^{-1}(s+2)$

50. $\dfrac{1}{s-2}\cot^{-1}(s-1)$

51. $\dfrac{s+(s^2+2s+2)\cot^{-1}(s+1)}{s^2(s^2+2s+2)}$

52. $\dfrac{1}{s}\dfrac{s+2}{s^2+2s+11}$

53. $\dfrac{1}{2}\left[\dfrac{s^2-2s+2}{(s-1)(s^2-2s)^2}+\dfrac{s^2+2s+2}{(s+1)(s^2+2s)^2}\right]$

54. $\frac{1}{s}\cot^{-1}(s-2)$

55. $\frac{\pi}{2}$ if $a > 0$, $\frac{-\pi}{2}$ if $a < 0$, 0 if $a = 0$

56. (a) $\log\frac{1}{2}$, (b) $\log\frac{b}{a}$

58. $\log\frac{b}{a}$.

Inverse Laplace Transforms

12.1 Introduction

It is seen that the Laplace transform converts ordinary derivatives to algebraic functions. To solve a differential equation, we convert it into an algebraic equation, and then after simplification, we apply the inverse Laplace transform to obtain the value of the dependent variable. Thus the inverse Laplace transform is very essential to solve a differential equation.

12.2 Definition and Some Standard Results

If $L\{f(t)\} = \overline{f}(s)$, then we say that $f(t)$ is the **inverse Laplace transform** of $\overline{f}(s)$ and symbolically

$$L^{-1}\{\overline{f}(s)\} = f(t)$$

For example, $L\{1\} = \frac{1}{s}$, thus $L^{-1}\left\{\frac{1}{s}\right\} = 1$.

Some Standard Results

(i) Since $L\{1\} = \frac{1}{s}$

$$L^{-1}\left\{\frac{1}{s}\right\} = 1 \tag{12.1}$$

(ii) Since $L\{e^{at}\} = \frac{1}{s-a}$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{12.2}$$

(iii) Since $L\{t^n\} = \frac{n!}{s^{n+1}}$

$$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!} \quad n = 1, 2, \ldots \tag{12.3}$$

(iv) Since $L\{\sin at\} = \frac{a}{s^2+a^2}$

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at \tag{12.4}$$

(v) Since $L\{\cos at\} = \frac{s}{s^2+a^2}$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at \tag{12.5}$$

(vi) Since $L\{\sinh at\} = \frac{a}{s^2-a^2}$

$$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at \tag{12.6}$$

(vii) Since $L\{\cosh at\} = \frac{s}{s^2-a^2}$

$$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at \tag{12.7}$$

(viii)

$$L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{e^{at}t^{n-1}}{(n-1)!} \tag{12.8}$$

(ix)

$$L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt \tag{12.9}$$

(x)

$$L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt \tag{12.10}$$

The inverse Laplace transform may not be unique. Lerch's theorem gives the condition for uniqueness of the inverse Laplace transform.

Theorem 12.1 (Lerch's theorem) *If the function $f(t)$ is sectionally continuous in $[0, N]$ for each positive integer N and if there exists a real constant $M > 0$ such that for all $t > N$, $|f(t)| < Me^{\alpha t}$ for some α, then $L^{-1}\{\overline{f}(s)\} = f(t)$ is unique.*

12.3 Properties of Inverse Laplace Transform

12.3.1 Linear Property

Theorem 12.2 *If $L\{f_1(t)\} = \overline{f_1}(s)$ and $L\{f_2(t)\} = \overline{f_2}(s)$, then $L^{-1}\{c_1\overline{f_1}(s) + c_2\overline{f_2}(s)\} = c_1L^{-1}\{\overline{f_1}(s)\} + c_2L^{-1}\{\overline{f_2}(s)\}$, where c_1 and c_2 are arbitrary constants.*

The proof of this theorem is trivial.

For example

$$\begin{aligned} L^{-1}\left\{\frac{1}{s^2}+\frac{4}{s-5}-\frac{s}{s^2+9}\right\} &= L^{-1}\left\{\frac{1}{s^2}\right\}+L^{-1}\left\{\frac{4}{s-5}\right\}-L^{-1}\left\{\frac{s}{s^2+9}\right\} \\ &= t+4e^{5t}-\cos 3t \end{aligned}$$

12.3.2 Shifting Property

Theorem 12.3 (First shifting theorem) *If* $L\{f(t)\} = \overline{f}(s)$, *then* $L^{-1}\{\overline{f}(s-a)\} = e^{at} f(t)$ $= e^{at} L^{-1}\{\overline{f}(s).\}$

This result follows from the relation $L\{e^{at} f(t)\} = \overline{f}(s-a)$.

For example, $L^{-1}\left\{\frac{1}{s^2}\right\} = t$, therefore $L^{-1}\left\{\frac{1}{(s-1)^2}\right\} = e^t t$. (WBUT 2005)

EXAMPLE 12.3.1 Find $L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$. (WBUT 2003)

Solution We have

$$\begin{aligned}
L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} &= L^{-1}\left\{\frac{s+1/2+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\} \\
&= L^{-1}\left\{\frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\} + L^{-1}\left\{\frac{1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\} \\
&= e^{-t/2} L^{-1}\left\{\frac{s}{s^2+(\sqrt{3}/2)^2}\right\} + \frac{1}{2}\frac{2}{\sqrt{3}} L^{-1}\left\{\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right\} \\
&= e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}\right)t + \frac{1}{\sqrt{3}} e^{-t/2} L^{-1}\left\{\frac{\sqrt{3}/2}{s^2+(\sqrt{3}/2)^2}\right\} \\
&= e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}\right)t + \frac{1}{\sqrt{3}} e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}\right)t
\end{aligned}$$

Theorem 12.4 (Second shifting theorem) *If* $L^{-1}\{\overline{f}(s)\} = f(t)$, *then*

$$L^{-1}\{e^{-as}\overline{f}(s)\} = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

Proof. Let us define a function $g(t)$ as

$$g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

Since $L^{-1}\{\overline{f}(s)\} = f(t)$, $L\{f(t)\} = \overline{f}(s)$.
Now by second shifting property of Laplace transform, $L\{g(t)\} = e^{-as}\overline{f}(s)$.
Therefore, $L^{-1}\{e^{-as}\overline{f}(s)\} = g(t)$, i.e.

$$L^{-1}\{e^{-as}\overline{f}(s)\} = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

EXAMPLE 12.3.2 Find (i) $L^{-1}\left\{\frac{2e^{-2s}}{s^2+9}\right\}$ (ii) $L^{-1}\left\{\frac{3se^{-\pi s}}{s^2+16}\right\}$.

Solution (i) We know $L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin 3t.$

Thus by second shifting property

$$L^{-1}\left\{\frac{2e^{-2s}}{s^2+9}\right\} = \begin{cases}\frac{2}{3}\sin 3(t-2), & t>2\\ 0, & t<2\end{cases}$$

(ii) Since $L^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4t.$

Thus
$$L^{-1}\left\{\frac{3se^{-\pi s}}{s^2+16}\right\} = \begin{cases}3\cos 4(t-\pi), & t>\pi\\ 0, & t<\pi\end{cases}$$

12.3.3 Change of Scale Property

Theorem 12.5 *If $L^{-1}\{\overline{f}(s)\} = f(t)$, then $L^{-1}\{\overline{f}(as)\} = \frac{1}{a}f(t/a)$, a is a constant.*

(WBUT 2002)

Proof. Let us consider

$$L\left\{\frac{1}{a}f(t/a)\right\} = \frac{1}{a}L\{f(t/a)\} = \frac{1}{a}\cdot\frac{1}{1/a}\overline{f}\left(\frac{s}{1/a}\right) = \overline{f}(as)$$

(By change of scale property of Laplace transform)

Thus, $L^{-1}\{\overline{f}(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right).$

For example, $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$. So $L^{-1}\left\{\frac{3s}{9s^2+a^2}\right\} = \frac{1}{3}\cos\frac{at}{3}.$

12.4 On Derivatives

Theorem 12.6 *Let $L\{f(t)\} = \overline{f}(s)$. Then $L^{-1}\left\{\overline{f}'(s)\right\} = -tL^{-1}\{\overline{f}(s)\} = -tf(t)$.*

Proof. $L\{tf(t)\} = -\frac{d}{ds}\overline{f}(s) = -\overline{f}'(s)$, that is, $-L^{-1}\left\{\overline{f}'(s)\right\} = tf(t)$, or $L^{-1}\left\{\overline{f}'(s)\right\} = -tf(t)$.

In general, since

$$L\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}\left\{\overline{f}(s)\right\}$$

$$L^{-1}\left\{\overline{f}^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

where $\overline{f}^{(n)}(s) = \frac{d^n}{ds^n}\left\{\overline{f}(s)\right\}$

EXAMPLE 12.4.1 Find (i) $L^{-1}\left\{\log\frac{s(s^2+9)}{s^2+16}\right\}$ (ii) $L^{-1}\{\cot^{-1}\frac{s}{2}\}$.

Solution Let $\overline{f}(s) = \log \dfrac{s(s^2+9)}{s^2+16} = \log s + \log(s^2+9) - \log(s^2+16)$,

$\overline{f}'(s) = \dfrac{1}{s} - \dfrac{2s}{s^2+9} - \dfrac{2s}{s^2+16}$.

Therefore

$$\begin{aligned} L^{-1}\{\overline{f}'(s)\} &= L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{2s}{s^2+9}\right\} - L^{-1}\left\{\frac{2s}{s^2+16}\right\} \\ &= 1 - 2\cos 3t - 2\cos 4t \end{aligned}$$

or

$$-tf(t) = 1 - 2\cos 3t - 2\cos 4t$$

or

$$f(t) = \frac{2\cos 3t + 2\cos 4t - 1}{t} = L^{-1}\left\{\frac{s(s^2+9)}{s^2+16}\right\}$$

(ii) Let $\overline{f}(s) = \cot^{-1}\left(\frac{s}{2}\right)$

Therefore,

$$\overline{f}'(s) = -\frac{1}{1+(s/2)^2}\cdot\frac{1}{2} = -\frac{2}{4+s^2}$$

Now

$$L^{-1}\{\overline{f}'(s)\} = -L^{-1}\left\{\frac{2}{s^2+2^2}\right\} = -\sin 2t$$

That is, $-tf(t) = -\sin 2t$, or $f(t) = \dfrac{\sin 2t}{t}$, or $L^{-1}\{\overline{f}(s)\} = \cot^{-1}\left(\dfrac{s}{2}\right) = \dfrac{\sin 2t}{t}$.

12.5 Multiplication by s^n

Theorem 12.7 *If* $L^{-1}\{\overline{f}(s)\} = f(t)$ *and* $f(0) = 0$, *then* $L^{-1}\{s\overline{f}(s)\} = f'(t)$.

Proof. From the relation $L\{f'(t)\} = s\overline{f}(s) - f(0)$, we have $L^{-1}\{s\overline{f}(s)\} = f'(t)$ as $f(0) = 0$.

In general, $L^{-1}\{s^n\overline{f}(s)\} = f^{(n)}(t)$, provided $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$.

EXAMPLE 12.5.1 Find (i) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$, (ii) $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)^2}\right\}$, (iii) $L^{-1}\left\{\dfrac{s^2}{(s+1)^5}\right\}$

(WBUT 2006)

Solution (i) Let $f(t) = \sin at$. Then $L\{f(t)\} = L\{\sin at\} = \dfrac{a}{s^2+a^2}$.

Now, $L\{t\sin at\} = -\dfrac{d}{ds}\left\{\dfrac{a}{s^2+a^2}\right\} = \dfrac{2as}{(s^2+a^2)^2}$, or $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\} = \dfrac{1}{2a}t\sin at$.

(ii) Let $\phi(t) = \dfrac{1}{2a}t\sin at$. Then $\phi(0) = 0$.

Now, from the formula $L^{-1}\{s\overline{f}(s)\} = f'(t)$

$$L^{-1}\left\{s\cdot\frac{s}{(s^2+a^2)^2}\right\} = \phi'(t) = \frac{1}{2a}(\sin at + at\cos at)$$

(iii) We know that $L^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{4!}$.

Therefore, $L^{-1}\left\{\frac{1}{(s+1)^5}\right\} = e^{-t}\frac{t^4}{4!} = f(t)$, say, by first shifting property.

Now by the formula, $L^{-1}\{s\overline{f}(s)\} = f'(t)$, as $f(0) = 0$, we have

$$L^{-1}\left\{\frac{s}{(s+1)^5}\right\} = \frac{d}{dt}\left\{e^{-t}\frac{t^4}{4!}\right\} = e^{-t}\frac{t^3}{4!}(4-t) = \phi(t) \qquad \text{(say)}$$

Also, $\phi(0) = 0$.

Again, using the same formula

$$L^{-1}\left\{\frac{s^2}{(s+1)^5}\right\} = \frac{d}{dt}\left\{e^{-t}\frac{t^3}{4!}(4-t)\right\} = e^{-t}\frac{t^2}{4!}(t^2 - 8t + 12)$$

12.6 Division by s

Theorem 12.8 *If $L^{-1}\{\overline{f}(s)\} = f(t)$, then $L^{-1}\left\{\frac{\overline{f}(s)}{s}\right\} = \int_0^t f(t)dt$.*

Proof. Let $F(t) = \int_0^t f(u)du$. Thus, $F'(t) = f(t)$ and $F(0) = 0$.

Then

$$L\{F'(t)\} = sL\{F(t)\} - F(0)$$

or

$$L\{F'(t)\} = sL\{F(t)\}$$

or

$$L\{F(t)\} = \frac{1}{s}L\{F'(t)\} = \frac{1}{s}L\{f(t)\} = \frac{\overline{f}(s)}{s}$$

That is, $L^{-1}\left\{\frac{\overline{f}(s)}{s}\right\} = F(t) = \int_0^t f(u)du$.

EXAMPLE 12.6.1 Find (i) $\left\{\frac{1}{(s^2+a^2)^2}\right\}$ (ii) $\left\{\frac{1}{s^2(s^2+1)}\right\}$.

Solution (i) From previous example, we have

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at$$

Therefore

$$\begin{aligned}
L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} &= L^{-1}\left\{\frac{1}{s}\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \frac{t\sin at}{2a}dt \\
&= \frac{1}{2a}\left\{\left[t\frac{-\cos at}{a}\right]_0^t - \int_0^t 1\cdot\frac{-\cos at}{a}dt\right\} \\
&= \frac{1}{2a}\left[\frac{-t\cos at}{a} + \frac{\sin at}{a^2}\right] = \frac{1}{2a^3}[\sin at - at\cos at]
\end{aligned}$$

(ii) We know $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$.

Therefore

$$L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u du = -\Big[\cos u\Big]_0^t = 1 - \cos t$$

Again, applying the same rule

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos u)du = \Big[u - \sin u\Big]_0^t = t - \sin t$$

12.7 Inverse Laplace Transform of Integrals

Theorem 12.9 *If $L^{-1}\{\overline{f}(s)\} = f(t)$, then $\left\{\int_s^\infty \overline{f}(u)du\right\} = \frac{f(t)}{t}$.*
This result follows from the Laplace transform of $f(t)/t$, as

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(u)du, \text{ i.e. } L^{-1}\left\{\int_s^\infty \overline{f}(u)du\right\} = \frac{f(t)}{t}$$

EXAMPLE 12.7.1 Find the inverse Laplace transform of $\frac{s}{(s^2+a^2)^2}$.

Solution Let $f(t) = L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$.

Now

$$\begin{aligned} L\left\{\frac{f(t)}{t}\right\} &= \int_s^\infty \frac{s}{(s^2+a^2)^2} ds = \frac{1}{2}\int_s^\infty \frac{2s}{(s^2+a^2)^2} ds \\ &= -\frac{1}{2}\left[\frac{1}{s^2+a^2}\right]_s^\infty = \frac{1}{2}\frac{1}{s^2+a^2} \end{aligned}$$

Thus, $\frac{f(t)}{t} = \frac{1}{2}L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{2a}\sin at$.

Hence, $f(t) = \frac{1}{2a} t \sin at$.

12.8 Convolution Theorem

Theorem 12.10 (Convolution theorem) *If $L^{-1}\{\overline{f}(s)\} = f(t)$ and $L^{-1}\{\overline{g}(s)\} = g(t)$, then $L^{-1}\{\overline{f}(s)\,\overline{g}(s)\} = \int_0^t f(u)\, g(t-u)du = F * G$. ($F * G$ is called the* **convolution** *of two functions F and G.)*

Proof. Let $\phi(t) = \int_0^t f(u)g(t-u)du$. Then $L\{\phi(t)\} = \int_0^\infty e^{-st}\int_0^t f(u)g(t-u)du\, dt$.
Now, we change the order of the integration. The domain is shown in Fig. 12.1.

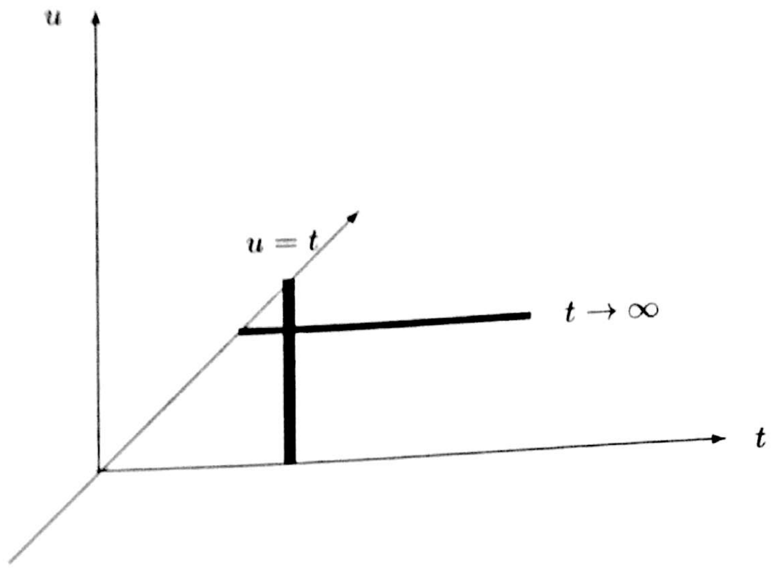

Figure 12.1: Change of order of integration.

Therefore

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)g(t-u)\,dt\,du$$

(Substituting $t - u = v$. Then $dt = dv$.)

$$\begin{aligned}
&= \int_0^\infty \int_0^\infty e^{-(u+v)s} f(u)g(v)\,dv\,du \\
&= \int_0^\infty e^{-su} f(u) \left\{ \int_0^\infty e^{-vs} g(v)\,dv \right\} du \\
&= \int_0^\infty e^{-su} f(u)\overline{g}(s)\,du = \overline{g}(s) \int_0^\infty e^{-su} f(u)\,du \\
&= \overline{g}(s)\overline{f}(s)
\end{aligned}$$

Therefore, $L^{-1}\{\overline{g}(s)\overline{f}(s)\} = \phi(t) = \displaystyle\int_0^t f(u)g(t-u)\,du.$

EXAMPLE 12.8.1 Use convolution theorem to evaluate

(i) $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$ (ii) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$.

Solution (i) Since $f(t) = L^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$ and $g(t) = L^{-1}\left\{\dfrac{s}{s^2+b^2}\right\} = \cos bt$.

Then by convolution theorem, we get

$$\begin{aligned}
L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\} &= \int_0^t \cos au \cos b(t-u)du \\
&= \frac{1}{2}\int_0^t [\cos\{(a-b)u+bt\}+\cos\{(a+b)u-bt\}]du \\
&= \frac{1}{2}\left[\frac{\sin\{(a-b)u+bt\}}{a-b}+\frac{\sin\{(a+b)u-bt\}}{a+b}\right]_0^t \\
&= \frac{1}{2}\left[\frac{\sin at-\sin bt}{a-b}+\frac{\sin at+\sin bt}{a+b}\right] \\
&= \frac{a\sin at-b\sin bt}{a^2-b^2}
\end{aligned}$$

(ii) Let $f(t)=L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at$ and $g(t)=L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at.$

Therefore, by convolution theorem

$$\begin{aligned}
L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} &= \int_0^t f(u)g(t-u)du \\
&= \int_0^t \frac{1}{a}\sin au\cos a(t-u)du = \frac{1}{2a}\int_0^t [\sin at-\sin(2au-at)]\,dt \\
&= \frac{1}{2a}\left[u\sin at+\frac{1}{2a}\cos(2au-at)\right]_0^t = \frac{1}{2a}t\sin at
\end{aligned}$$

Hence, $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{1}{2a}t\sin at.$

EXAMPLE 12.8.2 Use convolution theorem to find the values of

(i) $L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}$ (WBUT 2002)

(ii) $L^{-1}\left\{\frac{1}{(s-2)(s^2+1)}\right\}$ (WBUT 2004)

(iii) $L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\}.$ (WBUT 2008)

Solution (i) Let $f(t)=L^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t$ and $g(t)=L^{-1}\left\{\frac{1}{s^2+9}\right\}=\frac{1}{3}\sin 3t.$

Then

$$
\begin{aligned}
L^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+9}\right\} &= \int_0^t f(u)g(t-u)du \\
&= \int_0^t \sin u\ \frac{1}{3}\sin 3(t-u)du = \frac{1}{6}\int_0^t [\cos(4u-3t)-\cos(3t-2u)]du \\
&= \frac{1}{6}\left[\frac{\sin(4u-3t)}{4}-\frac{\sin(3t-2u)}{-2}\right]_0^t \\
&= \frac{1}{6}\left[\frac{1}{4}(\sin t-\sin 3t)+\frac{1}{2}(\sin t-\sin 3t)\right] \\
&= \frac{1}{8}(\sin t-\sin 3t)
\end{aligned}
$$

(ii) Let $f(t) = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $g(t) = L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$.
Therefore, by convolution theorem

$$
\begin{aligned}
L^{-1}\left\{\frac{1}{s-2}\cdot\frac{1}{s^2+1}\right\} &= \int_0^t \sin u\ e^{2(t-u)}du \\
&= e^{2t}\int_0^t \sin u\ e^{-2u}du = e^{2t}\frac{1}{5}\left[e^{-2u}\cos u - 2e^{-2u}\sin u\right]_0^t \\
&= \frac{1}{5}(e^{2t}-2\sin t-\cos t)
\end{aligned}
$$

(iii) Let

$$
\begin{aligned}
f(t) &= L^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} \\
&= e^{-t}L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{1}{2}e^{-t}\sin 2t
\end{aligned}
$$

By convolution theorem

$$
\begin{aligned}
L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\} &= L^{-1}\left\{\frac{1}{s^2+2s+5}\cdot\frac{1}{s^2+2s+5}\right\} \\
&= \int_0^t f(u)\ f(t-u)du \\
&= \int_0^t \frac{e^{-2u}\sin 2u}{2}\cdot\frac{e^{-(t-u)}\sin 2(t-u)}{2}du \\
&= \frac{1}{4}\int_0^t e^{-t}\sin 2u\ \sin(2t-2u)du
\end{aligned}
$$

$$= \frac{1}{8}\int_0^t e^{-t}[\cos(4u-2t)-\cos 2t]du = \frac{e^{-t}}{8}\left[\frac{\sin(4u-2t)}{4} - \cos 2t\cdot u\right]_0^t$$

$$= \frac{e^{-t}}{8}\left[\frac{\sin 2t+\sin 2t}{4} - t\cos 2t\right] = \frac{e^{-t}}{8}\left[\frac{1}{2}\sin 2t - t\cos 2t\right]$$

EXAMPLE 12.8.3 Use convolution theorem to prove that

$$\int_0^t \sin u\ \cos(t-u)du = \frac{t}{2}\sin t$$

(WBUT 2007)

Solution By convolution theorem

$$\int_0^t f(u)g(t-u)du = L^{-1}\left\{\overline{f}(s)\overline{g}(s)\right\}$$

Comparing, we get $f(t) = \sin t$ and $g(t) = \cos t$.

Therefore, $\overline{f}(s) = L\{\sin t\} = \dfrac{1}{s^2+1}$ and $\overline{g}(s) = L\{\cos t\} = \dfrac{s}{s^2+1}$.

Thus, $\displaystyle\int_0^t \sin u\ \cos(t-u)du = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$.

Again, $\overline{f}'(s) = \dfrac{-2s}{(s^2+1)^2}$.

We know that $L^{-1}\left\{\overline{f}'(s)\right\} = -tf(t)$, that is $L^{-1}\left\{\dfrac{-2s}{(s^2+1)^2}\right\} = -t\sin t$.

or $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{t\sin t}{2}$.

Hence, $\displaystyle\int_0^t \sin u\cos(t-u)du = \frac{t\sin t}{2}$.

12.9 Method of Partial Fractions

In this method the transformed expression is divided in such a way that each term has some known standard Laplace transformation.

EXAMPLE 12.9.1 Find the Laplace inverse transforms of

(i) $\dfrac{4s+5}{(s-4)^2(s+3)}$ (WBUT 2003)

(ii) $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$ (WBUT 2006)

(iii) $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$

(iv) $\dfrac{s}{s^4+4a^4}$.

Solution (i) Let

$$\frac{4s+5}{(s-4)^2(s+3)} = \frac{A}{s-4} + \frac{B}{(s-4)^2} + \frac{C}{(s+3)}$$

or

$$4s+5 = A(s-4)(s+3) + B(s+3) + C(s-4)^2$$

Substituting $s = 4,\ -3$ and 0 respectively

$$21 = 7B \qquad \text{or} \qquad B = 3$$

$$-7 = 49C \qquad \text{or} \qquad C = -\frac{1}{7}$$

$$5 = -12A + 3B + 16C$$

This gives $A = \frac{1}{7}$. Therefore

$$\frac{4s+5}{(s-4)^2(s+3)} = \frac{1}{7}\frac{1}{s-4} + \frac{3}{(s-4)^2} - \frac{1}{7}\frac{1}{(s+3)}$$

Hence

$$\begin{aligned} L^{-1}\left\{\frac{4s+5}{(s-4)^2(s+3)}\right\} &= \frac{1}{7}L^{-1}\left\{\frac{1}{s-4}\right\} + 3L^{-1}\left\{\frac{1}{(s-4)^2}\right\} - \frac{1}{7}L^{-1}\left\{\frac{1}{(s+3)}\right\} \\ &= \frac{1}{7}e^{4t} + 3te^{4t} - \frac{1}{7}e^{-3t} \end{aligned}$$

(ii) $\dfrac{1}{(s^2+a^2)(s^2+b^2)} = \dfrac{1}{b^2-a^2}\left[\dfrac{1}{s^2+a^2} - \dfrac{1}{s^2+b^2}\right].$

Therefore

$$\begin{aligned} L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} &= \frac{1}{b^2-a^2}\left[L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - L^{-1}\left\{\frac{1}{s^2+b^2}\right\}\right] \\ &= \frac{1}{b^2-a^2}\left[\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right] \end{aligned}$$

(iii) Denominator $s^3 - 6s^2 + 11s - 6 = (s-1)(s-2)(s-3)$.

Let

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

or

$$2s^2 - 6s + 5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

Substituting $s = 1, 2, 3$, we get $A = 1/2$, $B = -1$, $C = 5/2$.

Therefore

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{s-2} + \frac{5}{2}\frac{1}{s-3}$$

Hence

$$L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s-1}\right\}-L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{5}{2}L^{-1}\left\{\frac{1}{s-3}\right\}$$
$$= \frac{1}{2}e^t-e^{2t}+\frac{5}{2}e^{3t}$$

(iv) The denominator $s^4+4a^4=(s^2+2as+2a^2)(s^2-2as+2a^2)$.

Let

$$\frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}=\frac{As+B}{s^2+2as+2a^2}+\frac{Cs+D}{s^2-2as+2a^2}$$

or

$$s=(As+B)(s^2-2as+2a^2)+(Cs+D)(s^2+2as+2a^2)$$

Equating coefficients of s^3, s^2, s, we get respectively

$$0=A+C$$
$$0=-2aA+B+2aC+D$$
$$1=2a^2A-2aB+2a^2C+2aD$$

Substituting $s=0$, we obtain

$$0=2a^2B+2a^2D$$

The solution of these equations is

$$A=C=0,\ B=-\frac{1}{4a},\ D=\frac{1}{4a}$$

Therefore

$$L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = -\frac{1}{4a}L^{-1}\left\{\frac{1}{s^2+2as+2a^2}\right\}+L^{-1}\left\{\frac{1}{s^2-2as+2a^2}\right\}$$
$$= -\frac{1}{4a}L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}+\frac{1}{4a}L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}$$
$$= -\frac{1}{4a}e^{-at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\}+\frac{1}{4a}e^{at}L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$
$$= -\frac{1}{4a}e^{-at}\frac{1}{a}\sin at+\frac{1}{4a}e^{at}\frac{1}{a}\sin at$$
$$= \frac{1}{2a^2}\sin at\frac{e^{at}-e^{-at}}{2}$$
$$= \frac{1}{2a^2}\sin at\sinh at$$

12.10 Additional Worked-Out Examples

EXAMPLE 12.10.1 Evaluate $L^{-1}\left\{\dfrac{se^{-4s}}{s^2-2s+5}\right\}$.

Solution We have

$$
\begin{aligned}
L^{-1}\left\{\frac{s}{s^2-2s+5}\right\} &= L^{-1}\left\{\frac{s-1+1}{(s-1)^2+4}\right\} = L^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} + L^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} \\
&= e^{t}L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + e^{t}L^{-1}\left\{\frac{1}{s^2+2^2}\right\} \\
&= e^{t}\cos 2t + e^{t}\frac{1}{2}\sin 2t = e^{t}(\cos 2t + \frac{1}{2}\sin 2t)
\end{aligned}
$$

Hence by second shifting property

$$
L^{-1}\left\{\frac{se^{-4s}}{s^2-2s+5}\right\} = \begin{cases} e^{(t-4)}[\cos 2(t-4) + \frac{1}{2}\sin 2(t-4)], & t > 4 \\ 0, & t < 4 \end{cases}
$$

EXAMPLE 12.10.2 Evaluate $L^{-1}\left\{\tan^{-1}(s+3)\right\}$.

Solution Let $\overline{f}(s) = \tan^{-1}(s+3)$.

Therefore, $\overline{f}'(s) = \dfrac{1}{1+(s+3)^2}$, or

$$
L^{-1}\left\{\overline{f}'(s)\right\} = L^{-1}\left\{\frac{1}{1+(s+3)^2}\right\} = e^{-3t}L^{-1}\left\{\frac{1}{1+s^2}\right\} = e^{-3t}\sin t
$$

That is, $-tL^{-1}\left\{\overline{f}(s)\right\} = e^{-3t}\sin t$, or

$$
L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\tan^{-1}(s+3)\right\} = -\frac{e^{-3t}\sin t}{t}
$$

EXAMPLE 12.10.3 Find the inverse Laplace transform of $s\log\dfrac{s-1}{s+1}$.

Solution Let $\overline{f}(s) = \log\dfrac{s-1}{s+1} = \log(s-1) - \log(s+1)$.

Therefore, $\overline{f}'(s) = \dfrac{1}{s-1} - \dfrac{1}{s+1}$.

Then

$$
L^{-1}\left\{\overline{f}'(s)\right\} = L^{-1}\left\{\frac{1}{s-1}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}
$$

or

$$
-tf(t) = e^{t} - e^{-t} \quad \text{or} \quad f(t) = \frac{e^{-t}-e^{t}}{t}
$$

Hence

$$L^{-1}\left\{s\log\frac{s-1}{s+1}\right\} = \frac{d}{dt}\{f(t)\} = -\frac{(e^{t}+e^{-t})t+(e^{-t}-e^{t})}{t^2}$$

$$= \frac{2(\sinh t - t\cosh t)}{t^2}$$

EXAMPLE 12.10.4 Evaluate $L^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\}$.

Solution Let $\overline{f}(s) = \log\left(1+\frac{1}{s^2}\right)$.

Therefore, $\overline{f}'(s) = -\dfrac{2}{s(s^2+1)}$.

$$L^{-1}\left\{\overline{f}'(s)\right\} = -2L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = -2L^{-1}\left\{\frac{1}{s}-\frac{s}{s^2+1}\right\}$$

or

$$-tL^{-1}\left\{\overline{f}(s)\right\} = -2(1-\cos t)$$

or

$$L^{-1}\left\{\overline{f}(s)\right\} = \frac{2(1-\cos t)}{t}$$

Hence

$$L^{-1}\left\{\frac{1}{s}\overline{f}(s)\right\} = L^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\} = \int_0^t \frac{2(1-\cos t)}{t}\,dt$$

EXAMPLE 12.10.5 Use convolution theorem to evaluate $L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$.

(WBUT 2007)

Solution Let $\overline{f}(s) = \dfrac{1}{s^2+9}$ and $\overline{g}(s) = \dfrac{s}{s^2+9}$.

Then $f(t) = L^{-1}\left\{\dfrac{1}{s^2+9}\right\} = \dfrac{1}{3}\sin 3t$ and $g(t) = L^{-1}\left\{\dfrac{s}{s^2+9}\right\} = \cos 3t$.

Now by convolution theorem

$$L^{-1}\left\{\overline{f}(s)\overline{g}(s)\right\} = \int_0^t f(u)g(t-u)du$$

That is

$$L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = \int_0^t \frac{1}{3}\sin 3u \cdot \cos 3(t-u)du$$
$$= \frac{1}{3}\int_0^t [\sin 3t - \sin(6u-3t)]\, du$$
$$= \frac{1}{3}\left[u\sin 3t + \frac{1}{6}\cos(6u-3t)\right]_0^t$$
$$= \frac{1}{6}t\sin 3t$$

EXAMPLE 12.10.6 Find $L^{-1}\left\{\frac{1}{s}e^{1/s}\right\}$.

Solution We know that $e^{1/s} = 1 + \frac{1}{s} + \frac{1}{2!}\frac{1}{s^2} + \frac{1}{3!}\frac{1}{s^3} + \cdots$.

Therefore

$$\frac{1}{s}e^{1/s} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{2!}\frac{1}{s^3} + \frac{1}{3!}\frac{1}{s^4} + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{s^{n+1}}$$

Hence

$$L^{-1}\left\{\frac{1}{s}e^{1/s}\right\} = \sum_{n=0}^{\infty}\frac{1}{n!}L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{t^n}{(n!)^2}$$

EXAMPLE 12.10.7 Use convolution theorem to determine

$$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right\}$$

Solution Let $f(t) = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ and $g(t) = L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{\sin 3t}{3}$.

Then by convolution theorem

$$L^{-1}\left\{\frac{1}{s^2+1}\frac{1}{s^2+9}\right\} = \int_0^t \sin u\,\frac{1}{3}\sin 3(t-u)du$$
$$= \frac{1}{6}\int_0^t [\cos(4u-3t) - \cos(3t-2u)]du$$
$$= \frac{1}{6}\left[\frac{\sin(4u-3t)}{4} - \frac{\sin(3t-2u)}{-2}\right]_0^t$$
$$= \frac{1}{6}\left[\frac{1}{4}(\sin t - \sin 3t) + \frac{1}{2}(\sin t - \sin 3t)\right] = \frac{1}{8}[\sin t - \sin 3t]$$

Now we consider

$$f_1(t) = L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\} = \frac{1}{8}(\sin t - \sin 3t)$$

and

$$f_2(t) = L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

Again, by convolution theorem

$$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right\} = L^{-1}\left\{\overline{f_1}(s)\overline{f_2}(s)\right\}$$

$$= \int_0^t f_1(u)f_2(t-u)du$$

$$= \int_0^t \frac{1}{8}(\sin u - \sin 3u)\cos 2(t-u)du$$

$$= \frac{1}{8}\int_0^t [\sin u \cos 2(t-u) - \sin 3u \cos 2(t-u)]\, du$$

$$= \frac{1}{8}\int_0^t \left[\frac{1}{2}\{\sin(2t-u) - \sin(3u-2t)\} - \frac{1}{2}\{\sin(u+2t) - \sin(5u-2t)\}\right] du$$

$$= \frac{1}{16}\left\{\left[\cos(2t-u) + \frac{1}{3}\cos(3u-2t)\right]_0^t - \left[-\cos(u+2t) + \frac{1}{5}\cos(5u-2t)\right]_0^t\right\}$$

$$= \frac{1}{16}\left\{\frac{4}{3}(\cos t - \cos 2t) + \frac{4}{5}(\cos 3t - \cos 2t)\right\}$$

$$= \frac{1}{12}\cos t - \frac{2}{15}\cos 2t + \frac{1}{20}\cos 3t$$

EXAMPLE 12.10.8 Find $L^{-1}\left\{\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\}$.

Solution Let $\overline{f}(s) = \log\left(\frac{s^2+b^2}{s^2+a^2}\right) = \log(s^2+b^2) - \log(s^2+a^2)$.

Therefore,

$$\overline{f}'(s) = \frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}$$

Thus, $L^{-1}\left\{\overline{f}'(s)\right\} = 2\cos bt - 2\cos at$, or $-tf(t) = 2\cos bt - 2\cos at$, or $f(t) = 2\left(\frac{\cos at - \cos bt}{t}\right)$.

Hence

$$L^{-1}\left\{\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\} = \frac{\cos at - \cos bt}{t}$$

EXERCISES

Section A Multiple Choice Questions

1. The inverse Laplace transform of $1/s^2$ is

   (a) 1 (b) t (c) $\frac{t^2}{2!}$ (d) $2t^2$.

2. The value of $L^{-1}\left\{\dfrac{2}{s-3}\right\}$ is

(a) $2e^{3t}$ (b) $3e^{2t}$ (c) $2e^{-3t}$ (d) e^{-3t}.

3. The value of $L^{-1}\left\{\dfrac{1}{s}+\dfrac{4}{s-3}\right\}$ is

(a) $\dfrac{1}{t}+\dfrac{4}{t-3}$ (b) $1+4e^{-3t}$ (c) $1+4e^{3t}$ (d) $1+e^{3t}$.

4. $L^{-1}\left\{\dfrac{1}{(s^2+1)(s^2+4)}\right\}$ is equal to

(a) $\dfrac{1}{3}(\sin t-\sin 2t)$ (b) $\dfrac{1}{3}(\cos t-\cos 2t)$ (c) $\cos t-\cos 2t$
(d) $\sin t-\sin 2t$.

5. The inverse Laplace transform of $\dfrac{6}{(s-1)^4}$ is

(a) $\dfrac{5}{t^4}$ (b) $5e^{-t}$ (c) $e^t t^3$ (d) $6t^3e^{3t}$.

6. If $L^{-1}\{\overline{f}(s)\}=f(t)$ then $L^{-1}\left\{\overline{f}'(s)\right\}$ is equal to

(a) $tf(t)$ (b) $-tf(t)$ (c) $f'(t)$ (d) $-t^2f'(t)$.

7. If $f(t)=\dfrac{t+2}{t^2+2t+5}$ then $L^{-1}\{\overline{f}(2s)\}$ is equal to

(a) $\dfrac{t+4}{t^2+4t+20}$ (b) $\dfrac{t}{t^2+2t+1}$ (c) $\dfrac{1}{4}\dfrac{t+4}{t^2+4t+20}$ (d) $\dfrac{2(t+2)}{t^2+2t+5}$.

8. $L^{-1}\left\{\dfrac{s}{s^2-a^2}\right\}$ is

(a) $\sinh at$ (b) $\dfrac{1}{a}\sinh at$ (c) $\cosh at$ (d) $\dfrac{1}{a}\cosh at$.

9. $L^{-1}\left\{\dfrac{7}{(s-2)(s+5)}\right\}$ equal to

(a) $e^{-2t}-e^{5t}$ (b) $e^{2t}-e^{-5t}$ (c) $7(e^{2t}-e^{-5t})$ (d) $\dfrac{1}{7}(e^t-e^{-5t})$.

10. The value of $L^{-1}\left\{\dfrac{4}{s^2-16}+\dfrac{3s}{s^2-9}\right\}$ is

(a) $4\sin 4t+3\cos 3t$ (b) $\sinh 4t+3\cosh 3t$ (c) $4\sinh 4t+\cosh 3t$
(d) $\sin 4t+3\cos 3t$.

11. $L^{-1}\left\{\dfrac{s}{s^2+a^2}\right\}$ is equal to (WBUT 2006)

(a) $\dfrac{1}{2a}t\sin at$ (b) $\dfrac{1}{a}t\sin at$ (c) $\dfrac{1}{a^2}\sin at$ (d) $\cos at$

12. $L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$ is equal to

(a) $\frac{1}{t^2}$ (b) e^{2t} (c) te^{2t} (d) $\frac{e^{2t}}{t}$.

13. $L^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$ is equal to

(a) $\frac{1}{t^n}$ (b) t^n (c) t^{n+1} (d) $\frac{t^n}{n!}$.

14. $L^{-1}\left\{\frac{1}{(s-2)^2+9}\right\}$ is

(a) $e^{2t}\sin 3t$ (b) $\frac{1}{3}e^{2t}\sin 3t$ (c) $e^{2t}\cos 3t$ (d) $\frac{1}{3}e^{2t}\cos 3t$.

15. The inverse Laplace transform of $\frac{s-2}{(s-2)^2+16}$ is

(a) $e^{2t}\cos 4t$ (b) $e^{4t}\cos 4t$ (c) $e^{2t}\sin 4t$ (d) $\sin 4t$.

16. $L^{-1}\left\{\frac{1}{(s-2)(s+2)}\right\}$ is equal to

(a) $e^{2t}+e^{-2t}$ (b) $\cosh 2t$ (c) $2\cosh 2t$ (d) $2\sinh 2t$.

17. $L^{-1}\left\{\frac{5s^2-3s+4}{s^3}\right\}$ is equal to

(a) $1-3t+2t^2$ (b) $5-3t+2t^2$ (c) $5t^2-3t+4$ (d) $5-3t^2+4t^3$.

18. $L^{-1}\{\log(5+s)\}$ is

(a) e^{-5t} (b) $-\frac{e^{-5t}}{t}$ (c) $-e^{-5t}$ (d) $-te^{-5t}$.

19. $L^{-1}\left\{\int_s^{\infty}\overline{f}(u)du\right\}$ is equal to

(a) $\frac{1}{t}L^{-1}\{\overline{f}(s)\}$ (b) $\frac{1}{t^2}L^{-1}\{\overline{f}(s)\}$ (c) $L^{-1}\{\overline{f}(s)\}$ (d) $tL^{-1}\{\overline{f}(s)\}$.

20. If $L^{-1}\{\overline{f}(s)\}=f(t)$, then $L^{-1}\left\{\frac{1}{s^2}\overline{f}(s)\right\}$ is equal to

(a) $-\frac{d^2}{ds^2}\overline{f}(s)$ (b) $\int_0^t\int_0^p f(u)\,du\,dp$ (c) $\int_0^t\int_0^p \overline{f}(s)\,ds\,dp$
(d) $\int_0^t\int_0^p \frac{1}{t^2}f(t)\,dt\,dp$.

21. $L^{-1}\left\{\frac{1}{(s+a)^n}\right\}$ is

(a) $\frac{1}{t^n}$ (b) $\frac{a^n}{t^n}$ (c) $e^{-at}\frac{t^{n-1}}{(n-1)!}$ (d) $e^{-at}\frac{t^n}{n!}$.

Section B Review Questions

1. Prove that $L^{-1}\left\{\frac{d}{ds}\bar{f}(s)\right\} = -tL^{-1}\{\bar{f}(s)\}$.
2. Prove that $L^{-1}\{\bar{f}(s-a)\} = e^{at}L^{-1}\{\bar{f}(s)\}$.
3. Show that $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)dt$ where $L^{-1}\{\bar{f}(s)\} = f(t)$.

Find the inverse Laplace transforms of the following

4. $\dfrac{6s-4}{s^2-4s+20}$
5. $\dfrac{4s+12}{s^2+8s+16}$
6. $\dfrac{s^2+2s+8}{s^3}$
7. $\dfrac{s+2}{s^2-4s+13}$
8. $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$
9. $\dfrac{4s+5}{(s-1)^2(s+2)}$
10. $\dfrac{6e^{-2s}}{s^2+9}$
11. $\dfrac{(s+1)e^{-2\pi s}}{s^2+s+1}$
12. $\dfrac{s}{(s^2+9)^2}$
13. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$
14. $\dfrac{s}{s^4+64}$
15. $\dfrac{3s+2}{s^2-s-2}$
16. $\dfrac{1-7s}{(s-3)(s-1)(s+2)}$
17. $\dfrac{s}{(s^2-1)^2}$
18. $\dfrac{s^3}{s^4-a^4}$
19. $\dfrac{s}{s^4+s^2+1}$
20. $\dfrac{3s+12}{4s^2+12s+9}$
21. $\dfrac{1}{s^2-6s+10}$

22. $\dfrac{3s+1}{(s+1)^4}$

23. $\dfrac{e^{-4s}}{s^2}$

24. $\dfrac{s+1}{(s^2+2s+2)^2}$

25. $s\log\dfrac{s}{\sqrt{1+s^2}}+\cot^{-1}s$

26. $\log\dfrac{s+a}{s+b}$

27. $\tan^{-1}\dfrac{2}{s^2}$

28. $\tan^{-1}\left(\dfrac{s-2}{3}\right)$

29. $\log(1-\dfrac{1}{s^2})$

30. $\log\sqrt{\dfrac{s-1}{s+1}}$

31. $\log\dfrac{s^2+1}{s(s+1)}$

32. $\dfrac{1}{s(s+2)^3}$

33. $\dfrac{1}{s}\log\dfrac{s+2}{s+1}$

34. $\dfrac{s+1}{s^2+s+1}$

35. $\dfrac{1}{(s+1)(s^2+1)}$ (WBUT 20

36. $\dfrac{4s+5}{(s-4)^2(s+3)}$

37. $\dfrac{1}{(s^2+1)^3}$

38. $\dfrac{s}{(s^2+4)^2}$

39. $\dfrac{s^2}{(s^2+4)(s^2+9)}$

40. $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

41. $\log\dfrac{1+s}{s}$

42. $\log\sqrt{\dfrac{s^2+b^2}{s^2+a^2}}$

43. $\log \dfrac{s^2+1}{(s-1)^2}$

44. $3\log \dfrac{s+1}{(s+2)(s+4)}$

45. $\dfrac{6}{(s+2)(s-4)}$

46. $\dfrac{s^2+9s-9}{s^3-9s}$

47. $\dfrac{2s^3}{s^4-81}$

48. $\dfrac{s^4+3(s+1)^3}{s^4(s+1)^3}$

49. $\dfrac{s^3+6s^2+14s}{(s+2)^4}$

50. $\dfrac{s+1}{s(s-2)(s+3)}$

51. $\dfrac{s}{(s^2+\pi^2)^2}$

52. $\dfrac{w}{s^2(s^2+w^2)}$

53. $\dfrac{s}{s^4+4a^4}$

54. $\dfrac{s^2}{s^4+4a^4}$

55. $\dfrac{s^3}{s^4+4a^4}$

56. Show that

(a) $L^{-1}\left\{\dfrac{1}{s}\sin\dfrac{1}{s}\right\} = t - \dfrac{t^3}{(3!)^2} + \dfrac{t^5}{(5!)^2} - \dfrac{t^7}{(7!)^2} + \cdots$

(b) $L^{-1}\left\{\dfrac{1}{s}\cos\dfrac{1}{s}\right\} = 1 - \dfrac{t^2}{(2!)^2} + \dfrac{t^4}{(4!)^2} - \dfrac{t^6}{(6!)^2} + \cdots$

Use convolution theorem, evaluate

57. $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$

58. $L^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}$

59. $L^{-1}\left\{\dfrac{1}{(s^2+a^2)^2}\right\}$

60. $L^{-1}\left\{\dfrac{1}{(s+1)(s+9)^2}\right\}$

61. $L^{-1}\left\{\dfrac{s}{(s^2+9)^2}\right\}$ (WI

62. $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$ (WBUT 2005, 2006)

63. $L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\right\}$ (WBUT 2002)

64. $L^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\}$

65. $L^{-1}\left\{\frac{s}{(s^2+a^2)(s^2+b^2)}\right\}$

66. $L^{-1}\left\{\frac{1}{(s^2+2s+5)^2}\right\}$ (WBUT 200

67. $L^{-1}\left\{\frac{1}{(s-2)(s^2+1)}\right\}$ (WBUT 200

68. $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$

69. $L^{-1}\left\{\frac{1}{s^2(s^2+1)^2}\right\}$

70. Show that

$$L^{-1}\left\{\frac{1}{s}\log\frac{s^2+a^2}{s^2+b^2}\right\} = 2\int_0^t \frac{\cos bu - \cos au}{u}du$$

71. Prove that

$$\int_0^t\int_0^t\int_0^t f(t)\,dt\,dt\,dt = \int_0^t \frac{(t-u)^3}{2}f(u)du$$

72. Using convolution theorem prove that

$$\int_0^t e^u e^{2(t-u)}du = e^{2t} - e^t$$

73. Using convolution theorem, verify that

$$\int_0^t \sin u\cos(t-u)du = \frac{1}{2}t\sin t$$

Answers

Section A Multiple Choice Questions

1. (b)	2. (a)	3. (c)	4. (a)	5. (c)	6. (b)	7. (a)	8. (c)	9. (b)
10. (b)	11. (d)	12. (c)	13. (b)	14. (b)	15. (a)	16. (c)	17. (b)	18. (b)
19. (a)	20. (b)	21. (c)						

Section B Review Questions

4 $2e^{2t}(3\cos 4t + \sin 4t)$

5 $4(1-t)e^{-4t}$

6 $1 + 2t + 8t^2$

7. $e^{2t}(\cos 3t + \frac{4}{3}\sin 3t)$

8. $\frac{1}{2}e^{t} - e^{2t} + \frac{5}{2}e^{3t}$

9. $\frac{1}{3}e^{t} + 3te^{t} - \frac{1}{3}e^{-2t}$

10. $f(t) = 2\sin 3(t-2), t > 0, f(t) = 0, t < 3$

11. $e^{-(t-2\pi)/2}\{\cos\frac{\sqrt{3}}{2}(t-2\pi) + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-2\pi)\}, t > 2\pi$

12. $\frac{1}{6}t\sin 3t$

13. $e^{t} - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$

14. $\frac{1}{8}\sin 2t\ \sinh 2t$

15. $\frac{1}{3}8e^{2t} + \frac{1}{3}e^{-t}$

16. $e^{t} + e^{-2t} - 2e^{3t}$

17. $\frac{1}{2}t\sinh t$

18. $\frac{1}{2}[\cos at + \cosh at]$

19. $\frac{2}{\sqrt{3}}\sinh(\frac{t}{2})\ \sin(\sqrt{3}\frac{t}{2})$

20. $\frac{1}{8}e^{-3t/2}(6-5t)$

21. $e^{3t}\sin t$

22. $e^{-t}(\frac{3t^2}{2} - \frac{t^3}{3})$

23. $t-4, t > 4$

24. $\frac{1}{2}te^{-t}\sin t$

25. $\frac{1-\cos t}{t^2}$

26. $-e^{-at} + e^{bt}$

27. $2\sinh t\ \sin t$

28. $-\frac{1}{t}e^{2t}\sin 3t$

29. $\frac{2}{t}(1-\cosh t)$

30. $-\frac{1}{t}\sinh t$

31. $\frac{1}{t}(1+e^{-t}-2\cos t)$

32. $1-e^{-t}(\frac{t^2}{2}+t+1)$

33. $\int_0^t \frac{e^{-u}-e^{-2u}}{u}\,du$

34. $e^{-t/2}(\cos\frac{\sqrt{3}t}{2}+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}t}{2})$

35. $\frac{1}{2}(\sin t-\cos t+e^{-t})$

36. $-\frac{1}{7}e^{-3t}+\frac{1}{7}e^{4t}+3te^{4t}$

37. $\frac{1}{8}\{(3-t^2)\sin t-2t\sin t\}$

38. $\frac{1}{4}t\sin 2t$

39. $\frac{1}{5}(-2\sin 2t+3\sin 3t)$

40. $\frac{a\sin at-b\sin bt}{a^2-b^2}$

41. $\frac{1-e^t}{t}$

42. $\frac{1}{t}(\cos at-\cos bt)$

43. $\frac{2}{t}(e^t-\cos t)$

44. $3(e^{-t}-e^{-2t}-e^{-4t})$

45. $e^{4t}-e^{-2t}$

46. $1+3\sinh 3t$

47. $\cos 3t+\cosh 3t$

48. $\frac{1}{2}(t^2e^{-t}+t^3)$

49. $e^{-2t}(1+t^2-2t^3)$

50. $-\frac{1}{6}+\frac{3}{10}e^{2t}-\frac{2}{15}e^{-3t}$

51. $\frac{t\sin\pi t}{2\pi}$

52. $\frac{\cos t-\sin wt}{w^2}$

53. $\frac{1}{2a^2}\sinh at\ \sin at$

54. $\frac{1}{2a}(\cosh at\ \sin at+\sinh at\ \cos at)$

55. $\cosh at\ \cos at$

57. $\frac{e^{-bt} - e^{-at}}{a-b}$

58. $\frac{1}{4}(1 - \cos 2t)$

59. $\frac{1}{2a^3}(\sin at - at \cos at)$

60. $\frac{e^{-t}}{64}[1 - e^{-8t}(1 + 8t)]$

61. $\frac{1}{6}t \sin t$

62. $\frac{1}{2}t \sin t$

63. $\frac{1}{24}(3 \sin t - \sin 3t)$

64. $\frac{1}{b^2 - a^2}(\frac{\sin at}{a} - \frac{\sin bt}{b})$

65. $\frac{1}{b^2 - a^2}(\cos at - \cos bt)$

66. $\frac{1}{16}e^{-t} \sin 2t - \frac{1}{8}te^{-t} \cos 2t$

67. $\frac{1}{5}(e^{2t} - 2 \sin t - \cos t)$

68. $\frac{t^2}{2} + \cos t - 1$

69. $t(e^{-t} + 1) + 2(e^{-t} - 1)$

Chapter 14

Improper Integral, Gamma and Beta Functions

In the definite integral $\int_a^b f(x)\,dx$ it is assumed that

(i) the limits a and b both are finite, and

(ii) the integrand $f(x)$ is bounded within $a \le x \le b$.

If a definite integral satisfies these two conditions, then the integral is called **proper integral**.

But, if a or b or both are infinite or $f(x)$ is not finite in $a \le x \le b$, then the integral is called **improper integral** or **infinite integral** or **generalised integral**.

If the integral is proper and integrable, then it has a finite value. But the value of improper integral may be finite or infinite. If the value of the integral is finite, then it is said to be **convergent**, otherwise the integral is said to be **divergent**.

The improper integrals are of two types, viz., first type or first kind and second type or second kind.

Gamma and beta functions are two improper integrals and they are used to solve a large number of problems involved in integration.

14.1 First Type Improper Integrals

The integrals $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$ and $\int_{-\infty}^\infty f(x)\,dx$ are called **first type** improper integrals. The values of these integrals are evaluated as follows.

(i) Let $f(x)$ be bounded and integrable in $a \le x \le B$ for every $B > a$. Then $\int_a^\infty f(x)\,dx$ is said to be **converge** or **exist** if $\lim\limits_{B\to\infty} \int_a^B f(x)\,dx$ exists. Thus

$$\int_a^\infty f(x)\,dx = \lim_{B\to\infty} \int_a^B f(x)\,dx \tag{14.1}$$

If $\lim\limits_{B\to\infty} \int_a^B f(x)\,dx$ does not exist, then the integral $\int_a^\infty f(x)\,dx$ is said to be a diverge provided $\lim_{B\to\infty}\int_a^B f(x)\,dx$ tends to be an infinity with a fixed sign. If the improper integral $\int_a^\infty f(x)\,dx$ is neither a converge nor diverge, then it is called **oscillatory**.

The improper integral $\int_{-\infty}^b f(x)\,dx$ can be evaluated as

$$\int_{-\infty}^b f(x)\,dx = \lim_{A\to-\infty}\int_A^b f(x)\,dx$$

provided the limit exists and $f(x)$ is bounded and integrable in $A \le x \le b$.

The improper integral $\int_{-\infty}^\infty f(x)\,dx$ is broken up into two integrals of the previous forms,

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx \tag{14.2}$$

where a is any point.

If $f(x)$ be bounded and integrable in $A \le x \le a$ for every $A < a$, and in $a \le x \le B$ for every $B > a$ and $\lim_{A\to-\infty}\int_A^a f(x)\,dx$ and $\lim_{B\to\infty}\int_a^B f(x)\,dx$ for $A < a < B$ exist finitely, then the integral $\int_{-\infty}^\infty f(x)\,dx$ is said to be convergent. In this case, we write

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{A\to-\infty}\int_A^a f(x)\,dx + \lim_{B\to\infty}\int_a^B f(x)\,dx \tag{14.3}$$

EXAMPLE 14.1.1 Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$.

Solution Here the upper limit is infinite and the integrand $1/x^2$ is bounded in $1 \le x \le B$, for every $B \ge 1$. Now

$$\lim_{B\to\infty}\int_1^B \frac{1}{x^2}\,dx = \lim_{B\to\infty}\left[-\frac{1}{x}\right]_1^B = \lim_{B\to\infty}\left[-\frac{1}{B}+1\right] = 1$$

Hence the integral $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent and $\int_1^\infty \frac{1}{x^2}\,dx = 1$.

EXAMPLE 14.1.2 Evaluate $\int_1^\infty \frac{1}{x}\,dx$.

Solution This is first type improper integral since the upper limit is ∞. Here also the integrand $1/x$ is bounded in every $1 \le B$.

Now,

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{B\to\infty}\int_1^B \frac{1}{x}\,dx = \lim_{B\to\infty}\Big[\log x\Big]_1^B$$

$$= \lim_{B\to\infty}[\log B - 0] = \infty$$

Since $\lim\limits_{B\to\infty}\int_1^B \frac{1}{x}\,dx$ does not exist, therefore $\int_1^\infty \frac{1}{x}\,dx$ also does not exist.

EXAMPLE 14.1.3 Find the value of $\int_a^\infty \cos x\,dx$, if it exists.

Solution Here $\int_a^\infty \cos x\,dx = \lim\limits_{B\to\infty}\int_a^B \cos x\,dx = \lim\limits_{B\to\infty}\Big[\sin x\Big]_a^B = \lim\limits_{B\to\infty}(\sin B - \sin a)$.

Since the value of $\lim\limits_{B\to\infty}\sin B$ is not fixed (its value lies between -1 and 1), $\lim\limits_{B\to\infty}(\sin B - \sin a)$ oscillates finitely.

Hence $\int_a^\infty \cos x\,dx$ is oscillatory.

EXAMPLE 14.1.4 Examine for convergence of the following integrals:

(i) $\int_{-\infty}^\infty \frac{dx}{1+x^2}$

(ii) $\int_0^\infty xe^{-x^2}\,dx$

(iii) $\int_{-\infty}^\infty \frac{dx}{(1+x^2)^2}$

(iv) $\int_2^\infty \frac{dx}{x\log x}$.

Solution

(i) Here both the limits are infinite. We choose a point 0 within the interval of integration. Thus,

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$$

$$= \lim_{A\to-\infty}\int_A^0 \frac{dx}{1+x^2} + \lim_{B\to\infty}\int_0^B \frac{dx}{1+x^2}$$

$$= \lim_{A\to-\infty}\left[\tan^{-1}0 - \tan^{-1}A\right] + \lim_{B\to\infty}\left[\tan^{-1}B - \tan^{-1}0\right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Thus the integral converges.

(ii) In this problem,

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{B\to\infty}\int_0^B xe^{-x^2}\,dx$$

$$= \lim_{B\to\infty}\left[\frac{1}{2}\int_0^{B^2} e^{-z}\,dz\right]$$

[where $x^2 = z,\ 2x\,dx = dz$]

$$= \frac{1}{2}\lim_{B\to\infty}\left[-e^{-z}\right]_0^{B^2} = \frac{1}{2}\lim_{B\to\infty}\left[1 - e^{-B^2}\right] = \frac{1}{2}$$

Hence the integral converges.

(iii) We divide the interval at the point $x = 0$. That is,

$$I = \int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2} = \int_{-\infty}^{0}\frac{dx}{(1+x^2)^2} + \int_0^\infty \frac{dx}{(1+x^2)^2}$$

$$= 2\int_0^\infty \frac{dx}{(1+x^2)^2} = 2\lim_{B\to\infty}\int_0^B \frac{dx}{(1+x^2)^2}$$

Substituting $x = \tan\theta$. Then $dx = \sec^2\theta\,d\theta$.

When $x \to 0, \theta \to 0$ and when $x \to B$ then $\theta \to \tan^{-1}B$. Therefore,

$$I = 2\lim_{B\to\infty}\int_0^{\tan^{-1}B}\frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^2} = 2\lim_{B\to\infty}\int_0^{\tan^{-1}B}\cos^2\theta\,d\theta$$

$$= \lim_{B\to\infty}\int_0^{\tan^{-1}B}(1+\cos 2\theta)\,d\theta = \lim_{B\to\infty}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\tan^{-1}B}$$

$$= \lim_{B\to\infty}\left[\tan^{-1}B + \frac{B}{1+B^2}\right] = \frac{\pi}{2}$$

This integral also converges.

(iv) Here

$$\int_2^\infty \frac{dx}{x\,\log x} = \lim_{B\to\infty}\int_2^B \frac{dx}{x\log x}$$

[Putting $\log x = z, \frac{1}{x}\,dx = dz$.

When $x \to 2, z \to \log 2$ and when $x \to B, z \to \log B$]

$$= \lim_{B\to\infty}\int_{\log 2}^{\log B}\frac{dz}{z} = \lim_{B\to\infty}\left[\log z\right]_{\log 2}^{\log B}$$

$$= \lim_{B\to\infty}\left[\log\log B - \log\log 2\right]$$

But $\lim_{B\to\infty} \log B \to \infty$ and hence $\lim_{B\to\infty} \log\log B \to \infty$.

Therefore, $\lim_{B\to\infty} [\log\log B - \log\log 2]$ does not exist.

Hence $\displaystyle\int_2^{\infty} \frac{dx}{x\,\log x}$ diverges.

14.2 Second Type Improper Integrals

In this type, both the lower and upper limits of the integral are finite, but the integrand $f(x)$ is infinite for some point within $a \le x \le b$. $f(x)$ may be infinite

(i) at the lower limit a,

(ii) at the upper limit b, and

(iii) at a point c in $a < c < b$.

(i) If $f(x)$ has an infinite discontinuity only at the lower limit a, then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b f(x)\,dx, \qquad 0 < \varepsilon < b-a$$

If this limit exists, then we say that $\displaystyle\int_a^b f(x)\,dx$ converges, otherwise it diverges.

(ii) If $f(x)$ has an infinite discontinuity at the upper limit only, then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+} \int_a^{b-\varepsilon} f(x)\,dx, \qquad 0 < \varepsilon < b-a$$

(iii) If $f(x)$ has an infinite discontinuity at the point $x = c$ where $a < c < b$, then $\displaystyle\int_a^b f(x)\,dx$ can be evaluated as

$$\lim_{\varepsilon_1\to 0+} \int_a^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2\to 0+} \int_{c+\varepsilon_2}^b f(x)\,dx$$

provided all these limits exist. If either of the limits does not exist, we say that the integral does not exist.

If we take $\varepsilon_1 = \varepsilon_2 = \varepsilon$ then the integral

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0+} \left\{ \int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^b f(x)\,dx \right\}$$

This is called the **Cauchy principal value** of the integral. Sometimes it may happen that the Cauchy principal of the integral exists when the integral as per general definition does not exist.

EXAMPLE 14.2.1 Evaluate $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$, if it exists.

Solution The integrand $1/x^2$ has the an infinite discontinuity at $x = 0$. So, it is an improper integral of second type.

Now,

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{\varepsilon\to 0+}\int_{0+\varepsilon}^1 \frac{1}{x^2}\,dx = \lim_{\varepsilon\to 0+}\left[-\frac{1}{x}\right]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon\to 0+}\left[-1+\frac{1}{\varepsilon}\right] = \infty$$

Hence $\lim\limits_{\varepsilon\to 0+}\int_{0+\varepsilon}^1 \frac{1}{x^2}\,dx$ does not exist and consequently $\int_0^1 \frac{1}{x^2}\,dx$ does not exist.

EXAMPLE 14.2.2 Evaluate $\int_{-1}^1 \frac{1}{x^3}\,dx$.

Solution $1/x^3$ has the infinite discontinuity at $x = 0$, so it is an improper integral of second type.

We break up the interval at the point $x = 0$ as

$$\int_{-1}^0 \frac{1}{x^3}\,dx + \int_0^1 \frac{1}{x^3}\,dx$$

Now

$$\int_{-1}^1 \frac{1}{x^3}\,dx = \lim_{\varepsilon_1\to 0+}\int_{-1}^{0-\varepsilon_1} \frac{1}{x^3}\,dx + \lim_{\varepsilon_2\to 0+}\int_{0+\varepsilon_2}^{1} \frac{1}{x^3}\,dx$$

$$= \lim_{\varepsilon_1\to 0+}\left[-\frac{1}{2x^2}\right]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2\to 0+}\left[-\frac{1}{2x^2}\right]_{\varepsilon_2}^{1}$$

$$= \lim_{\varepsilon_1\to 0+}\left[\frac{1}{2}-\frac{1}{2\varepsilon_1^2}\right] + \lim_{\varepsilon_2\to 0+}\left[-\frac{1}{2}+\frac{1}{2\varepsilon_2^2}\right]$$

Since $\lim\limits_{\varepsilon_1\to 0+}\frac{1}{2\varepsilon_1^2}$ and $\lim\limits_{\varepsilon_2\to 0+}\frac{1}{2\varepsilon_2^2}$ do not exist, therefore $\int_{-1}^1 \frac{1}{x^3}\,dx$ does not exist.

But, according to Cauchy principal value the integral exists, as shown below

$$\lim_{\varepsilon\to 0+}\left[\int_{-1}^{0-\varepsilon} \frac{1}{x^3}\,dx + \int_{0+\varepsilon}^1 \frac{1}{x^3}\,dx\right]$$

$$= \lim_{\varepsilon\to 0+}\left[\left(\frac{1}{2}-\frac{1}{2\varepsilon^2}\right)+\left(-\frac{1}{2}+\frac{1}{2\varepsilon^2}\right)\right]$$

$$= \lim_{\varepsilon\to 0+}(0) = 0$$

Thus for Cauchy principal value sense $\int_{-1}^1 \frac{1}{x^2}\,dx = -2$, and for general sense it is divergent.

EXAMPLE 14.2.3 Evaluate $\int_0^1 \frac{dx}{(1-x)^{1/3}}$.

Solution Here $x = 1$ is the point of infinite discontinuity, so it is also a second type improper integral.

Now,

$$\int_0^1 \frac{dx}{(1-x)^{1/3}} = \lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{dx}{(1-x)^{1/3}} = \lim_{\varepsilon \to 0+} \left[\frac{(1-x)^{2/3}}{2/3}\right]_0^{1-\varepsilon}$$

$$= \lim_{\varepsilon \to 0+} \frac{3}{2}\left[\varepsilon^{2/3} - 1\right] = -\frac{3}{2}$$

Thus $\lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{dx}{(1-x)^{1/3}}$ exists and $\int_0^1 \frac{dx}{(1-x)^{1/3}} = -\frac{3}{2}$.

EXAMPLE 14.2.4 Test for convergence.

(i) $\int_{1/2}^1 \frac{dx}{x \log x}$ (ii) $\int_1^4 \frac{dx}{x-1}$ (iii) $\int_0^\pi \frac{dx}{\sin x}$.

Solution

(i) Here $x = 1$ is the point of infinite discontinuity. Therefore,

$$\int_{1/2}^1 \frac{dx}{x \log x} = \lim_{\varepsilon \to 0+} \int_{1/2}^{1-\varepsilon} \frac{dx}{x \log x}$$

[Putting $\log x = z$. $\frac{1}{x}\, dx = dz$.

When $x \to 1/2, z \to \log(1/2)$ and when $x \to 1-\varepsilon, z \to \log(1-\varepsilon)$.]

$$= \lim_{\varepsilon \to 0+} \int_{\log(1/2)}^{\log(1-\varepsilon)} \frac{dz}{z} = \lim_{\varepsilon \to 0+} \left[\log z\right]_{\log(1/2)}^{\log(1-\varepsilon)}$$

$$= \lim_{\varepsilon \to 0+} \left[\log\log(1-\varepsilon) - \log\log(1/2)\right].$$

Here $\lim_{\varepsilon \to 0+} \log\log(1-\varepsilon) = \log 0 \to -\infty$ and $\log\log(1/2) = \log(-\log 2)$, does not exist.

Hence the limit does not exist and consequently the integral diverges.

(ii) Here also $x = 1$ is the point of infinite discontinuity.

Therefore,

$$\int_1^4 \frac{dx}{x-1} = \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^4 \frac{dx}{x-1}$$

$$= \lim_{\varepsilon \to 0+} \left[\log(x-1)\right]_{1+\varepsilon}^4$$

$$= \lim_{\varepsilon \to 0+} [\log 3 - \log \varepsilon] \to \infty$$

Hence the integral does not converge.

(iii) Here 0 and π both are points of infinite discontinuity. Thus we write

$$\begin{aligned}
\int_0^{\infty} \frac{dx}{\sin x} &= \int_0^{\pi/2} \frac{dx}{\sin x} + \int_{\pi/2}^{\pi} \frac{dx}{\sin x} \\
&= \lim_{\varepsilon_1 \to 0+} \int_{\varepsilon_1}^{\pi/2} \frac{dx}{\sin x} + \lim_{\varepsilon_2 \to 0+} \int_{\pi/2}^{\pi-\varepsilon_2} \frac{dx}{\sin x} \\
&= \lim_{\varepsilon_1 \to 0+} \Big[\log|\tan(x/2)|\Big]_{\varepsilon_1}^{\pi/2} + \lim_{\varepsilon_2 \to 0+} \Big[\log|\tan(x/2)|\Big]_{\pi/2}^{\pi-\varepsilon_2} \\
&= \lim_{\varepsilon_1 \to 0+} \Big[\log|\tan(\pi/4)| - \log|\tan(\varepsilon_1/2)|\Big] \\
&\quad + \lim_{\varepsilon_2 \to 0+} \Big[\log|\tan\{(\pi-\varepsilon_2)/2\}| - \log|\tan(\pi/4)|\Big] \\
&= \lim_{\varepsilon_1 \to 0+} \Big[0 - \log|\tan(\varepsilon_1/2)|\Big] + \lim_{\varepsilon_2 \to 0+} \Big[\log|\tan\{(\pi-\varepsilon_2)/2\}| - 0\Big] \\
&= \infty
\end{aligned}$$

Hence the integral does not converge.

In the previous problems, whether an integral converges or diverges, is tested by computing its value. But, without computing the actual value of the integration one can determine the convergence of the integral. Several methods are available for testing the convergence of an integral, which are stated below.

14.3 Test for First Type Improper Integrals

It is mentioned earlier that the integral $\int_a^{\infty} f(x)\,dx$ evaluated as

$$\int_a^{\infty} f(x)\,dx = \lim_{B \to \infty} \int_a^B f(x)\,dx$$

where $f(x)$ is bounded and integrable in $[a, B]$ for every $B \geq a$.

Theorem 14.1 *A necessary and sufficient condition for the convergence of* $\int_a^{\infty} f(x)\,dx$, *where* $f(x)$ *is positive in* $[a, B]$ *is that there exists a positive number* M, *independent of* B, *such that*

$$\int_a^B f(x)\,dx < M$$

for every $B \geq a$, *i.e.* $\int_a^B f(x)\,dx$ *is bounded above.*

The integral $\int_a^\infty f(x)\,dx$ *is said to be convergent if* $\lim_{B\to\infty}\int_a^B f(x)\,dx$ *tends to a finite limit.*

If $\int_a^B f(x)\,dx$ *is not bounded above then* $\lim_{B\to\infty}\int_a^B f(x)\,dx$ *tends to* ∞, *i.e. the integral diverges.*

For example, let us consider the integral $\int_1^\infty \frac{1}{x^2}\,dx$. The integral $1/x^2$ is positive in $[1, B]$ for any $B \geq 1$. Also,

$$\int_1^B \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^B = 1 - \frac{1}{B} < 1 \text{ for any } B \geq 1$$

Thus the integral $\int_1^\infty \frac{1}{x^2}\,dx$ satisfies the necessary and sufficient conditions of the above theorem. Hence the integral $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

14.3.1 A Standard Integral

The improper integral

$$\int_a^\infty \frac{dx}{x^n},\ a > 0 \tag{14.4}$$

converges if $n > 1$ and diverges if $n \leq 1$.

Proof. We have

$$\begin{aligned}
\int_a^\infty \frac{dx}{x^n} &= \lim_{B\to\infty}\int_a^B \frac{dx}{x^n} \\
&= \lim_{B\to\infty}\left[\frac{1}{1-n}\frac{1}{x^{n-1}}\right]_a^B = \lim_{B\to\infty}\left[\frac{1}{1-n}\left\{\frac{1}{B^{n-1}} - \frac{1}{a^{n-1}}\right\}\right], n \neq 1 \\
&= \begin{cases} \dfrac{1}{n-1}\dfrac{1}{a^{n-1}}, & \text{if } n-1 > 0 \\ \infty, & \text{if } n-1 < 0. \end{cases}
\end{aligned}$$

Again, if $n = 1$,

$$\begin{aligned}
\lim_{B\to\infty}\int_a^B \frac{dx}{x} &= \lim_{B\to\infty}\left[\log B - \log a\right], n = 1 \\
&= \infty
\end{aligned}$$

That is,

$$\int_a^\infty \frac{dx}{x^n} = \begin{cases} \frac{1}{n-1}\frac{1}{a^{n-1}}, & \text{if } n > 1 \\ \infty, & \text{if } n \leq 1 \end{cases}$$

Hence $\int_a^\infty \frac{dx}{x^n}$ converges if $n > 1$ and diverges if $n \leq 1$.

Comparison test I (comparison of two integrals)

Theorem 14.2 *If $f(x)$ and $g(x)$ are two positive integrable functions and $f(x) \leq g(x)$ on $[a, B]$ for any $B \geq a$, then*

(i) $\int_a^\infty f(x)\,dx$ *converges, if* $\int_a^\infty g(x)\,dx$ *converges, and*

(ii) $\int_a^\infty g(x)\,dx$ *diverges, if* $\int_a^\infty f(x)\,dx$ *diverges.*

EXAMPLE 14.3.1 Examine the convergence of $\int_1^\infty \frac{\cos^2 x}{x^2}\,dx$.

Solution It is easy to verify that

$$0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2} \text{ for } 1 \leq x.$$

Also, $\int_1^\infty \frac{dx}{x^2}$ is convergent $(n = 2 > 1)$. Hence by comparison test $\int_1^\infty \frac{dx}{x^2}$ converges.

Comparison test II (limit form)

Theorem 14.3 *Let $f(x)$ and $g(x)$ be two positive integrable functions when $x \geq a$ and $g(x)$ be positive. Let*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$$

(i) If $l \neq 0$, then the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge and diverge together.

(ii) If $l = 0$ and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

(iii) If $l = \pm\infty$ and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

EXAMPLE 14.3.2 Test for convergence.

(i) $\int_1^\infty \frac{x^2}{x^4+1}\,dx$ (ii) $\int_1^\infty \frac{\log x}{x^2}\,dx$ (iii) $\int_1^\infty \frac{x^{3/2}}{x^2+2}\,dx$.

Solution

(i) Let $f(x) = \frac{x^2}{x^4+1}$ and $g(x) = \frac{1}{x^2}$.

Now, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^4}{x^4+1} = \lim_{x\to\infty} \frac{1}{1+1/x^4} = 1 \neq 0$

Also, $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^2}\,dx$ converges (since $n = 2 > 1$).

Therefore, by comparison test $\int_1^\infty \frac{x^2}{x^4+1}\,dx$ converges.

(ii) Let $f(x) = \dfrac{\log x}{x^2}$ and $g(x) = \dfrac{1}{x^{3/2}}$.

Now,

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{\log x}{x^2} \cdot x^{3/2} \\
&= \lim_{x\to\infty} \frac{\log x}{x^{1/2}} \quad \left(\frac{\infty}{\infty} \text{form}\right) \\
&= \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0
\end{aligned}$$

Also, $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\, dx$ converges (since $n = 3/2 > 1$), hence by comparison test $\displaystyle\int_1^\infty \frac{\log x}{x^2}\, dx$ converges.

(iii) Let $f(x) = \dfrac{x^{3/2}}{x^2+2}$ and $g(x) = \dfrac{1}{x}$

Then, $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{5/2}}{x^2+2} = \lim_{x\to\infty} \frac{\sqrt{x}}{1+2/x^2} = \infty$

Also, $\displaystyle\int_1^\infty g(x)\, dx = \int_1^\infty \frac{1}{x}\, dx$ diverges (since $n = 1$)

Hence, by comparison test $\displaystyle\int_1^\infty \frac{x^{3/2}}{x^2+2}\, dx$ diverges.

Alternate.

Let $g(x) = \dfrac{1}{\sqrt{x}}$. Then

$\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x^2+2} = 1 \neq 0$ and $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\, dx$ diverges (since $n = 1/2 < 1$). Hence $\displaystyle\int_1^\infty f(x)\, dx$ ~~con~~diverges.

EXAMPLE 14.3.3 Examine the convergence of the following integrals.

(i) $\displaystyle\int_1^\infty e^{-x} x^n\, dx$ (ii) $\displaystyle\int_{e^2}^\infty \frac{dx}{x \log\log x}$

Solution

(i) Let $f(x) = e^{-x}x^n$ and $g(x) = \dfrac{1}{x^2}$.

Now,

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{e^{-x}\cdot x^n}{1/x^2} = \lim_{x\to\infty} \frac{x^{n+1}}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right) \\
&= \lim_{x\to\infty} \frac{(n+1)x^n}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)
\end{aligned}$$

and so on, finally $\to 0$ for all n.

Also, $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent.

Hence, by comparison test $\int_1^\infty e^{-x}x^n\,dx$ is convergent for all n.

Substituting $\log x = z$. Then $\frac{1}{x}\,dx = dz$. When $x \to e^2$ then $z \to 2$ and when $x \to \infty, z \to \infty$. Therefore,

$$\int_{e^2}^\infty \frac{1}{x\,\log\log x}\,dx = \int_2^\infty \frac{dz}{\log z}$$

Now, let $f(z) = \frac{1}{\log z}$ and $g(z) = \frac{1}{z}$.

Then

$$\lim_{z\to\infty} \frac{f(z)}{g(z)} = \lim_{z\to\infty} \frac{z}{\log z}\ \left(\frac{\infty}{\infty}\text{ form}\right) = \lim_{z\to\infty} \frac{1}{1/z} \to \infty$$

Also, $\int_{e^2}^\infty \frac{1}{z}\,dz$ is divergent and hence $\int_2^\infty \frac{dz}{\log z}$, i.e. $\int_{e^2}^\infty \frac{1}{x\,\log\log x}\,dx$ is divergent.

14.3.2 Absolute and Conditional Convergent

The improper integral $\int_a^\infty f(x)\,dx$ is said to be **absolute convergent** if $\int_a^\infty |f(x)|\,dx$ is convergent.

It can easily be verified that, if $\int_a^\infty |f(x)|\,dx$ exists, then obviously $\int_a^\infty f(x)\,dx$ exists. But, the converse is not true. This leads to another type of convergence, called conditional convergent

If the integral $\int_a^\infty f(x)\,dx$ is convergent, but $\int_a^\infty |f(x)|\,dx$ does not converge, then the integral $\int_a^\infty f(x)\,dx$ is called **conditionally convergent**.

That is, if the integral $\int_a^\infty f(x)\,dx$ absolutely convergent, then both the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty |f(x)|\,dx$ convergent, but for conditionally convergent, only the integral $\int_a^\infty f(x)\,dx$ is convergent and $\int_a^\infty |f(x)|\,dx$ is dievergent.

EXAMPLE 14.3.4 Show that $\int_1^\infty \frac{\cos x}{x^p}\,dx$ converges absolutely if $p > 1$.

Solution Here $\left|\frac{\cos x}{x^p}\right| = \frac{|\cos x|}{x^p} \le \frac{1}{x^p}$ for all $x \ge 1$.

Again, $\int_1^\infty \frac{dx}{x^p}$ converges iff $p > 1$.

Therefore, $\int_1^\infty \left|\frac{\cos x}{x^p}\right| dx$ converges if $p > 1$.

Hence $\int_1^\infty \frac{\cos x}{x^p}\, dx$ converges absolutely if $p > 1$.

EXAMPLE 14.3.5 If $\phi(x)$ be bounded when $0 < a \leq x$ and $p > 1$, show that $\int_a^\infty \frac{|\phi(x)|}{x^p}\, dx$ converges.

Solution Since $\phi(x)$ is bounded for $0 < a \leq x$, then there exists a positive number M such that $|\phi(x)| \leq M$, for $0 < a \leq x < \infty$.

Thus
$$\frac{|\phi(x)|}{x^p} \leq \frac{M}{x^p}$$

Now,
$$\int_a^\infty \frac{|\phi(x)|}{x^p}\, dx \leq \int_a^\infty \frac{M}{x^p}\, dx = M \int_a^\infty \frac{1}{x^p}\, dx$$

Since $\int_a^\infty \frac{1}{x^p}\, dx$ exists for $p > 1$, therefore $\int_a^\infty \frac{|\phi(x)|}{x^p}\, dx$ exists for $p > 1$, i.e. the given integral is convergent for $p > 1$.

14.4 Test for Second Type Improper Integrals

14.4.1 A Standard Integral

The integral
$$\int_a^b \frac{dx}{(x-a)^p} \tag{14.5}$$

exists, if $p < 1$ and does not exist, if $p \geq 1$.

Proof. Here a is the point of infinite discontinuity. Let $p \neq 1$. Then

$$\begin{aligned}
\lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^p} &= \lim_{\varepsilon \to 0+} \left[\frac{1}{1-p}\,(x-a)^{1-p}\right]_{a+\varepsilon}^b \\
&= \lim_{\varepsilon \to 0+} \left[\frac{1}{1-p}\,(b-a)^{1-p} - \varepsilon^{1-p}\right] \\
&= \begin{cases} \frac{1}{1-p}(b-a)^{1-p}, & \text{if } 1-p > 0 \\ -\infty, & \text{if } 1-p < 0 \end{cases}
\end{aligned}$$

when $p = 1$ then

$$\begin{aligned}
\lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b \frac{dx}{x-a} &= \lim_{\varepsilon \to 0+} \left[\log(x-a)\right]_{a+\varepsilon}^b \\
&= \lim_{\varepsilon \to 0+} [\log(b-a) - \log\varepsilon] = -\infty
\end{aligned}$$

Thus $\int_a^b \frac{dx}{(x-a)^p}$ exists if $p < 1$ and does not exist if $p \geq 1$.

Similarly, the integral $\int_a^b \frac{dx}{(b-x)^p}$ exists if $p < 1$ and does not exist if $p \geq 1$.

Comparison test III (comparison of two integrals)

Theorem 14.4 *Let a be the only point of infinite discontinuity. If $f(x)$ and $g(x)$ be integrable functions in $a < x \leq b$ such that $0 \leq f(x) \leq g(x)$, then*

(i) $\int_a^b f(x)\, dx$ *converges, if* $\int_a^b g(x)\, dx$ *converges, and*

(ii) $\int_a^b g(x)\, dx$ *diverges, if* $\int_a^b f(x)\, dx$ *diverges.*

EXAMPLE 14.4.1 Show that $\int_0^1 \frac{\sin x}{x^{3/2}}\, dx$ convergence.

Solution Here $x = 0$ is the only point of infinite discontinuity.

Now,

$$\frac{\sin x}{x^{3/2}} = \frac{1}{x^{1/2}} \frac{\sin x}{x}.$$

The function $\frac{\sin x}{x}$ is bounded and $\frac{\sin x}{x} \leq 1$. Therefore, $\frac{\sin x}{x^{3/2}} \leq \frac{1}{x^{1/2}}$.

Since $\int_0^1 \frac{dx}{x^{1/2}}$ is convergent, then $\int_0^1 \frac{\sin x}{x^{3/2}}\, dx \left(\leq \int_0^1 \frac{1}{x^{1/2}}\, dx \right)$ is also convergent.

Comparison test IV (limit form)

Theorem 14.5 *Let a be the only point of infinite discontinuity. If $f(x)$ and $g(x)$ are two positive integrable functions in $[a, b]$ such that*

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = l$$

(i) If $l \neq 0$ and finite, then the integrals $\int_a^b f(x)\, dx$ *and* $\int_a^b g(x)\, dx$ *both converge and diverge together at a.*

(ii) If $l = 0$ and $\int_a^b g(x)\, dx$ *converges, then* $\int_a^b f(x)\, dx$ *converges at a.*

(iii) If $l = \pm\infty$ and $\int_a^b g(x)\, dx$ *diverges, then* $\int_a^b f(x)\, dx$ *diverges at a.*

EXAMPLE 14.4.2 Test the convergence.

(i) $\int_0^1 e^{-x} x^{n-1}\, dx$ (ii) $\int_0^{\pi/2} \frac{x^p}{\sin x}\, dx$ (iii) $\int_0^1 \frac{x^n}{1+x}\, dx$

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Solution

(i) Here $x = 0$ is the point of infinite discontinuity when $n - 1 < 0$.

Let $f(x) = e^{-x}x^{n-1}$ and $g(x) = \dfrac{1}{x^{1-n}}$.

Now
$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{x^{n-1}e^{-x}}{1/x^{1-n}} = \lim_{x\to 0+} e^{-x} = 1$$

Also, $\displaystyle\int_0^1 \frac{1}{x^{1-n}}\,dx$ converges if $1 - n < 1$, or $n > 0$

Therefore, by comparison test $\displaystyle\int_0^1 e^{-x}x^{n-1}\,dx$ converges for $n > 0$.

(ii) Here $x = 0$ is the only point of infinite discontinuity.

Let $f(x) = \dfrac{x^p}{\sin x}$ and $g(x) = \dfrac{1}{x^{1-p}}$.

Now
$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{x^p}{\sin x}\cdot x^{1-p} = \lim_{x\to 0+} \frac{x}{\sin x} = 1$$

Again, $\displaystyle\int_0^{\pi/2} \frac{1}{x^{1-p}}\,dx$ converges only if $1 - p < 1$, or $p > 0$

Hence by comparison test $\displaystyle\int_0^{\pi/2} \frac{x^p}{\sin x}\,dx$ is convergent only when $p > 0$.

(iii) Here also $x = 0$ is the point of infinite discontinuity.

Let $f(x) = \dfrac{x^n}{1+x}$ and $g(x) = \dfrac{1}{x^{1-n}}$.

Now
$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{x^n}{1+x}\cdot x^{1-n} = \lim_{x\to 0+} \frac{x}{1+x} = 0$$

Also, $\displaystyle\int_0^1 \frac{1}{x^{1-n}}\,dx$ is convergent when $1 - n < 1$ or $n > 0$

Hence by comparison test $\displaystyle\int_0^1 \frac{x^n}{1+x}\,dx$ converges only when $n > 0$.

EXAMPLE 14.4.3 Show that $I = \displaystyle\int_0^1 \frac{\log x}{\sqrt{x}}\,dx$ converges whereas $J = \displaystyle\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$ diverges.

Solution *For the integral I.*
In this case $x = 0$ is the point of discontinuity.

Let $f(x) = \dfrac{\log x}{\sqrt{x}}$ and $g(x) = \dfrac{1}{x^{3/4}}$

Now, $\displaystyle\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{\log x}{\sqrt{x}}\cdot x^{3/4} = \lim_{x\to 0+} \frac{\log x}{x^{-1/4}} = \lim_{x\to 0+} (-4x^{1/4}) = 0$

But, $\int_0^1 \frac{1}{x^{3/4}}\,dx$ is convergent ($p = 3/4 < 1$).

Hence by comparison test $I = \int_0^1 \frac{\log x}{\sqrt{x}}\,dx$ converges.

For the integral J.

Here $x = 1$ is the only point of infinite discontinuity.

Let $f(x) = \frac{\sqrt{x}}{\log x}$ and $g(x) = \frac{1}{x-1}$.

Now

$$\begin{aligned}
\lim_{x\to 1+} \frac{f(x)}{g(x)} &= \lim_{x\to 1+} \frac{\sqrt{x}}{\log x}\cdot(x-1) \\
&= \lim_{x\to 1+} \frac{x^{3/2} - x^{1/2}}{\log x} \quad \left(\tfrac{0}{0} \text{ form}\right) \\
&= \lim_{x\to 1+} \frac{\frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2}}{1/x} = \lim_{x\to 1+}\left(\frac{3}{2}x^{3/2} - \frac{1}{2}x^{1/2}\right) \\
&= 1
\end{aligned}$$

But, $\int_1^2 \frac{1}{x-1}\,dx$ is divergent ($p = 1$).

Hence by comparison test $I = \int_1^2 \frac{\sqrt{x}}{\log x}\,dx$ diverges.

14.4.2 Absolute Convergent

The improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b |f(x)|\,dx$ is convergent.

EXAMPLE 14.4.4 Show that $\int_0^1 \frac{\sin x}{x^p}\,dx$, $p > 0$ converges absolutely for $p < 1$.

Solution Since $p > 0$, $x = 0$ is the only point of infinite discontinuity.

Now, $\left|\frac{\sin x}{x^p}\right| = \frac{|\sin x|}{x^p} < \frac{1}{x^p}$, in $0 < x \le 1$

Also, $\int_0^1 \frac{1}{x^p}\,dx$ converges only if $p < 1$

Hence by comparison test, $\int_0^1 \left|\frac{\sin x}{x^p}\right| dx$ converges and hence $\int_0^1 \frac{\sin x}{x^p}\,dx$ converges absolutely only if $p < 1$.

14.5 Gamma Function

Definition 14.5.1 *The gamma function is denoted by $\Gamma(n)$ and is defined by*

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx, \qquad n > 0 \tag{14.6}$$

This is a first type improper integral if $n - 1 \geq 0$ and if $n - 1 < 0$, then it is also a second type improper integral, because in this case, $x = 0$ is a point of discontinuity. If $n \leq 0$, then the integral does not converge, i.e. the integral does not give any finite value.

The gamma function satisfies several important properties, they are studied here.

Property 14.5.1 $\Gamma(n+1) = n\Gamma(n), n > 0$.

Proof. We have

$$\begin{aligned}
\Gamma(n+1) &= \int_0^\infty e^{-x}x^n\,dx, n+1>0 \\
&= \lim_{\substack{B\to\infty \\ \epsilon\to 0+}} \left\{ \left[x^n \frac{e^{-x}}{-1}\right]_\epsilon^B - \int_\epsilon^B nx^{n-1}\frac{e^{-x}}{-1}\,dx \right\} \\
&= 0 + n\lim_{\substack{B\to\infty \\ \epsilon\to 0+}} \int_\epsilon^B x^{n-1}e^{-x}\,dx \\
&= n\Gamma(n)
\end{aligned}$$

$\therefore \Gamma(n+1) = n\,\Gamma(n)$.

Property 14.5.2 *If* $n > 0$ *is an integer, then* $\Gamma(n+1) = n!$.

Proof. If n being an integer, then

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) \\
&= n(n-1)\Gamma(n-1) \\
&= n(n-1)(n-2)\Gamma(n-2) \\
&\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots \\
&= n(n-1)(n-2)\ldots 1\cdot\Gamma(1)
\end{aligned}$$

Again, $\Gamma(1) = \int_0^\infty e^{-x}x^{1-1}dx = \int_0^\infty e^{-x}dx = \left[-e^{-x}\right]_0^\infty = 1$
Hence $\Gamma(n+1) = n!$.

Some initial values of gamma function are:

$$\Gamma(1) = 0! = 1, \quad \Gamma(2) = 1! = 1, \quad \Gamma(3) = 2! = 2,$$

$$\Gamma(4) = 3! = 6, \quad \Gamma(5) = 4! = 24, \quad \Gamma(6) = 5! = 120.$$

Note that $\Gamma(1)$ and $\Gamma(2)$ have same value. It may be remembered that $\Gamma(0)$ does not exist. Also, $\Gamma(-n), n > 0$ does not exist.

Property 14.5.3 *For any $a > 0$,*

$$\int_0^\infty e^{-ax}x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}, n > 0$$

Proof. To prove this result substituting $ax = y$. Then $a\,dx = dy$. Also, when $x \to 0$ then $y \to 0$ and when $x \to \infty$ then $y \to \infty$.

Thus

$$\begin{aligned}\int_0^\infty e^{-ax}x^{n-1}\,dx &= \int_0^\infty e^{-y}\frac{y^{n-1}}{a^{n-1}}\frac{dy}{a}\\ &= \frac{1}{a^n}\int_0^\infty e^{-y}y^{n-1}\,dy = \frac{\Gamma(n)}{a^n}\end{aligned}$$

EXAMPLE 14.5.1 Find the value of $\int_0^\infty e^{-3x}x^6\,dx$.

Solution By the Property 14.5.3,

$$\int_0^\infty e^{-3x}x^6\,dx = \int_0^\infty e^{-3x}x^{7-1}\,dx = \frac{\Gamma(7)}{3^7} = \frac{6!}{3^7} = \frac{80}{243}$$

14.6 Beta Function

Definition 14.6.1 *The beta function is denoted by $B(m,n)$ and is defined by*

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \qquad m,n > 0. \tag{14.7}$$

It is a second type improper integral. If $m - 1 < 0$, then $x = 0$ is the point of infinite discontinuity and when $n - 1 < 0$, then $x = 1$ is the point of infinite discontinuity. It can be shown that the integral has a finite value if $m > 0$ and $n > 0$.

The beta function satisfies many interesting results which are discussed below.

Property 14.6.1 $B(m,n) = B(n,m)$, *i.e. beta function is commutative.*

Proof. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m,n > 0.$

Putting $x = 1-y$, $dx = -dy$ and when $x \to 1$, $y \to 0$ and when $x \to 0$, $y \to 1$.

$$\therefore B(m,n) = \int_1^0 (1-y)^{m-1}y^{n-1}(-dy) = \int_0^1 y^{n-1}(1-y)^{m-1}dx = B(n,m).$$

Property 14.6.2 $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$

Proof. Substituting $x = \frac{1}{1+y}$, $dx = -\frac{1}{(1+y)^2}dy$. Again, $x + xy = 1$, $y = \frac{1-x}{x}$. When $x \to 1$, $y \to 0$ and when $x \to 0$, $y \to \infty$.

Therefore

$$\begin{aligned} B(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}dx \\ &= \int_0^\infty \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy. \end{aligned}$$

Thus $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx.$

Since $B(m,n) = B(n,m)$,

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{n+m}}\,dx \text{ [interchanging } m \text{ and } n]$$

Therefore, $$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{n+m}}\,dx$$

Property 14.6.3

$$B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

Proof. Let $B(m,n) = \int_0^1 z^{m-1}(1-z)^{n-1}\,dz$. Putting $z = \frac{1}{1+y}$, $dz = -\frac{1}{(1+y)^2}\,dy$. Again, $z + zy = 1$ or, $y = \frac{1-z}{z}$, when $z \to 1$, $y \to 0$, and when $z \to 0$, $y \to \infty$.

Now,

$$\begin{aligned}
B(m,n) &= \int_0^1 z^{m-1}(1-z)^{n-1}dz \\
&= \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\frac{-1}{(1+y)^2}dy \\
&= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy \\
&= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} + \int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy
\end{aligned}$$

[Again substituting $y = \frac{1}{x}$ in second integral.

$dy = -\frac{1}{x^2}\,dx$ and when $y \to 1, x \to 1;\ y \to \infty, x \to 0.$]

$$\begin{aligned}
&= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx + \int_1^0 \frac{\left(\frac{1}{x}\right)^{n-1}}{\left(1+\frac{1}{x}\right)^{m+n}}\left(-\frac{1}{x^2}\right)dx \\
&= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{m+n}}{x^{n-1}(1+x)^{m+n}}\frac{1}{x^2}dx \\
&= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}dx + \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx \\
&= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx.
\end{aligned}$$

Property 14.6.4

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta, \qquad m,n>0$$

Proof. We have $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$. Substituting $x = \sin^2\theta$, $dx = \sin 2\theta\,d\theta$. When $x \to 1$, $\theta \to \frac{\pi}{2}$ and when $x \to 0$, $\theta \to 0$.

Now,

$$\begin{aligned}
B(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^{\frac{\pi}{2}} \sin^{2(m-1)}\theta(1-\sin^2\theta)^{n-1}\sin 2\theta\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \sin^{2m-2+1}\theta\cos^{2n-2+1}\theta\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta.
\end{aligned}$$

Property 14.6.5 *For any $p > -1, q > -1$,*

$$\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}B\Big(\frac{p+1}{2}, \frac{q+1}{2}\Big).$$

Proof. Substituting $\sin^2\theta = x$, $\sin 2\theta\, d\theta = dx$, when $\theta \to \frac{\pi}{2}$, $x \to 1$ and when $\theta \to 0$, $x \to 0$.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta &= \int_0^{\frac{\pi}{2}} (\sin\theta)^{p-1}(\cos\theta)^{q-1}\sin\theta\cos\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{\frac{p-1}{2}}(\cos^2\theta)^{\frac{q-1}{2}}\sin\theta\cos\theta\, d\theta \\
&= \int_0^1 x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}}\frac{1}{2}dx \\
&= \frac{1}{2}\int_0^1 x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}}dx \\
&= \frac{1}{2}B\Big(\frac{p-1}{2}+1, \frac{q-1}{2}+1\Big) \\
&= \frac{1}{2}B\Big(\frac{p+1}{2}, \frac{q+1}{2}\Big).
\end{aligned}$$

Property 14.6.6 $B(\frac{1}{2}, \frac{1}{2}) = \pi$.

Proof. Substituting $p = q = 0$ to the above property.

Then $\displaystyle\int_0^{\pi/2} \cos^0\theta\ \sin^0\theta\, d\theta = \frac{1}{2}B\Big(\frac{0+1}{2}, \frac{0+1}{2}\Big)$

This gives $\displaystyle\frac{\pi}{2} = \frac{1}{2}B\Big(\frac{1}{2}, \frac{1}{2}\Big)$ or, $\displaystyle B\Big(\frac{1}{2}, \frac{1}{2}\Big) = \pi$.

Property 14.6.7 (Relation between beta and gamma functions)

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Proof. From Property 14.6.4 and the definition of the gamma function

$$B(m,n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta\sin^{2n-1}\theta\, d\theta \tag{i}$$

$$\text{and } \Gamma(m) = \int_0^\infty t^{m-1}e^{-t}\, dt = 2\int_0^\infty r^{2m-1}e^{-r^2}\, dr \text{ where } t = r^2. \tag{ii}$$

Let us consider three regions (Fig. 14.1).

E_1 :	first quadrant of the circle $x^2 + y^2 = R^2$	: OABCO
E :	the square $0 \le x \le R$ and $0 \le y \le R$	: OADCO
E_2 :	first quadrant of the circle $x^2 + y^2 = 2R^2$	: OFGDHO

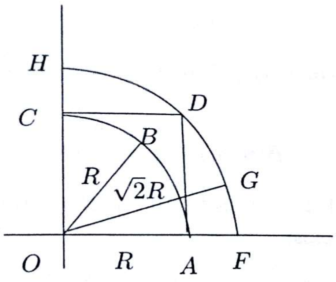

Figure 14.1: The regions E_1, E, E_2.

From (ii) it is observed that the integral

$$4\iint_E x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}\,dx\,dy$$

tends to $\Gamma(m)\Gamma(n)$ as $R \to \infty$.

The positive quadrant (E_1) of the circle $x^2+y^2 = R^2$ is a part of the square E which, again is a part of the positive quadrant (E_2) of the $x^2+y^2 = 2R^2$. Thus $E_1 \subseteq E \subseteq E_2$.

The integrand being positive, we have

$$4\iint_{E_1} x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy \le 4\iint_E x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy$$

$$\le 4\iint_{E_2} x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy.$$

Substituting $x = r\cos\theta, y = r\sin\theta$ to the first integral. Therefore,

$$4\iint_{E_1} x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy$$

$$= 4\int_0^{\pi/2} \cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta \times \int_0^R e^{-r^2}r^{2m+2n-1}\,dr$$

$$= 2B(m,n)\int_0^R e^{-r^2}r^{2m+2n-1}\,dr.$$

Similarly, $4\iint_{E_2} x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy = 2B(m,n)\int_0^{\sqrt{2}R} e^{-r^2}r^{2m+2n-1}\,dr.$

Therefore,

$$2B(m,n)\int_0^R e^{-r^2}r^{2m+2n-1}\,dr \le 4\iint_E x^{2m-1}y^{2n-1}e^{-(x^2+y^2)}dx\,dy$$

$$\le 2B(m,n)\int_0^{\sqrt{2}R} e^{-r^2}r^{2m+2n-1}\,dr.$$

Taking $R \to \infty$, we get

$$B(m,n)\Gamma(m+n) \le \Gamma(m)\Gamma(n) \le B(m,n)\Gamma(m+n)$$

or

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

It is easy to observe that $B(1,1) = 1$ and $B(0,n)$, $B(m,0)$ do not exist, for any m, n. Also, $B(-m,-n)$; $m, n > 0$ does not exist.

Property 14.6.8 $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof. Since $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$ and by Property 14.6.7,

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(1)} = \pi.$$

But, $\Gamma(1) = 1$, therefore $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Thus, $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$

Combining Properties 14.6.5 and 14.6.7, we obtain the following result.

Property 14.6.9 *Since* $\int_0^{\pi/2} \sin^p\theta\, \cos^q\theta\, d\theta = \int_0^{\pi/2} \cos^p\theta\, \sin^q\theta\, d\theta$, *thus*

$$\int_0^{\pi/2} \cos^p\theta\, \sin^q\theta\, d\theta = \int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\,\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q+2}{2})}$$

for any $p > -1, q > -1$

EXAMPLE 14.6.1 Find the value of the following integrals.

(i) $\int_0^{\pi/2} \sin^5 x\, dx$ (ii) $\int_0^{\pi/2} \cos^6 x\, dx$

(iii) $\int_0^{\pi/2} \sin^4 x\, \cos^6 x\, dx$ (iv) $\int_0^{\pi/2} \sin^4 x\, \cos^5 x\, dx$.

Solution

(i)
$$\int_0^{\pi/2} \sin^5 x\, dx = \int_0^{\pi/2} \sin^5 x\, \cos^0 x\, dx$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{5+0+2}{2}\right)}$$

[by Property 14.6.9]

$$= \frac{1}{2}\,\frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \frac{1}{2}\,\frac{2! \times \Gamma\left(\frac{1}{2}\right)}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)} = \frac{8}{15}$$

(ii)

$$
\begin{aligned}
\int_0^{\pi/2} \cos^6 x\, dx &= \int_0^{\pi/2} \cos^6 x\, \sin^0 x\, dx \\
&= \frac{1}{2}\frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{6+0+2}{2}\right)} \\
&= \frac{1}{2}\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)} = \frac{1}{2}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\times\Gamma\left(\frac{1}{2}\right)}{3!} = \frac{15}{16}\cdot\frac{\pi}{6} = \frac{5\pi}{32}.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\int_0^{\pi/2} \sin^4 x\, \cos^6 x\, dx &= \frac{1}{2}\frac{\Gamma\left(5/2\right)\Gamma\left(7/2\right)}{\Gamma\left(12/2\right)} \\
&= \frac{1}{2}\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2)\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(6)} = \frac{45\pi}{64\times 5!} = \frac{3\pi}{512}.
\end{aligned}
$$

(iv)

$$
\begin{aligned}
\int_0^{\pi/2} \sin^4 x\, \cos^5 x\, dx &= \frac{1}{2}\frac{\Gamma\left(5/2\right)\Gamma\left(6/2\right)}{\Gamma\left(11/2\right)} \\
&= \frac{1}{2}\frac{\Gamma(5/2)\cdot\Gamma(3)}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\Gamma(5/2)} = \frac{8}{315}
\end{aligned}
$$

Property 14.6.10 (Duplicating formula) *For any $m > 0$,*

$$2^{2m-1}\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2m)$$

Proof. By the property of beta function

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta. \qquad \text{(i)}$$

Substituting $n = m$. Therefore,

$$
\begin{aligned}
B(m,m) &= \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta\, d\theta \\
&= \frac{2}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1}(2\theta)\, d\theta \\
&= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\, d\phi \qquad [\text{where } 2\theta = \phi]
\end{aligned}
$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi \tag{ii}$$

Again, substituting $n = 1/2$ in (i) we get

$$\frac{\Gamma(m)\Gamma(1/2)}{\Gamma(m+1/2)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta$$

or
$$\frac{1}{2^{2m-1}} \frac{\Gamma(m)\Gamma(1/2)}{\Gamma(m+1/2)} = \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta \tag{iii}$$

From (ii) and (iii) for $m > 0$,

$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \frac{\Gamma(m)\Gamma(1/2)}{\Gamma(m+1/2)} = \frac{\sqrt{\pi}\Gamma(m)}{2^{2m-1}\Gamma(m+1/2)}$$

or
$$2^{2m-1}\Gamma(m)\Gamma(m+1/2) = \sqrt{\pi}\Gamma(2m), \qquad m > 0$$

From this relation, one can determine the value of $\Gamma(1/2)$ by putting $m = 1/2$.

Property 14.6.11

$$\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}, \qquad 0 < m < 1$$

Proof. From the property of beta function

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx.$$

Putting $n = 1 - m, 0 < m < 1$. Therefore,

$$B(m, 1-m) = \frac{\Gamma(m)\Gamma(1-m)}{\Gamma(m+1-m)} = \Gamma(m)\Gamma(1-m) = \int_0^\infty \frac{x^{m-1}}{1+x}\, dx$$

$$= \int_0^1 \frac{x^{m-1}}{1+x}\, dx + \int_1^\infty \frac{x^{m-1}}{1+x}\, dx$$

[It can be proved that both the integrals convergent for $0 < m < 1$]

Putting $x = 1/y$ in the second integral

$$= \int_0^1 \frac{x^{m-1}}{1+x}\, dx + \int_0^1 \frac{y^{-m}}{1+y}\, dy = \int_0^1 \frac{x^{m-1}}{1+x}\, dx + \int_0^1 \frac{x^{-m}}{1+x}\, dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{-m}}{1+x}\, dx$$

$$= \int_0^1 (x^{m-1} + x^{-m})\left(1 - \frac{x}{1+x}\right) dx$$

$$= \int_0^1 (x^{m-1} + x^{-m})\, dx - \int_0^1 \frac{x^m + x^{1-m}}{1+x}\, dx$$

Now
$$\int_0^1 (x^{m-1} + x^{-m})\,dx = \left[\frac{x^m}{m} + \frac{x^{1-m}}{1-m}\right]_0^1 = \frac{1}{m} + \frac{1}{1-m}$$

and

$$\int_0^1 \frac{x^m + x^{1-m}}{1+x}\,dx$$
$$= \int_0^1 (x^m + x^{1-m})(1 - x + x^2 - x^3 + x^4 + \cdots)\,dx$$
$$= \left[\frac{x^{m+1}}{m+1} + \frac{x^{2-m}}{2-m} - \frac{x^{m+2}}{m+2} - \frac{x^{3-m}}{3-m} + \cdots\right]_0^1$$

Thus

$$\Gamma(m)\Gamma(1-m) = \frac{1}{m} + \frac{1}{1-m} - \frac{1}{m+1} - \frac{1}{2-m} + \frac{1}{m+2} + \frac{1}{3-m} + \cdots$$
$$= \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{k+m} + \frac{1}{k+1-m}\right) = \pi\text{cosec}\, m\pi = \frac{\pi}{\sin m\pi}.$$

14.7 Additional Worked-Out Examples

EXAMPLE 14.7.1 Show that $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\,dx = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sqrt{2}}.$

Solution

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\,dx = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x \cos^{-\frac{1}{2}} x\,dx$$
$$= \frac{1}{2}\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$
$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$
$$= \frac{1}{2}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{2}\frac{\pi}{\sin 45^o} = \frac{1}{\sqrt{2}}\pi$$

EXAMPLE 14.7.2 Find the value of $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx.$

Solution Here the integral is even function. Thus, $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx.$

Now, substituting $x^2 = z$. Then $2x\,dx = dz$. When $x \to 0, z \to 0$ and when $x \to \infty$, $z \to \infty$.

Therefore,

$$\int_0^\infty e^{-x^2}\,dx = \int_0^\infty e^{-z}\cdot\frac{1}{2\sqrt{z}}\,dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z}z^{1/2-1}\,dz = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

Thus, $\int_{-\infty}^\infty e^{-x^2}\,dx = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$.

EXAMPLE 14.7.3 Evaluate $\int_0^\infty a^{-x^2}\,dx, a > 1$.

Solution We know $a^{-x^2} = e^{\log a^{-x^2}} = e^{-x^2\log a} = e^{-bx^2}$, where $b = \log a$.

Substituting $bx^2 = z$. Then $2bx\,dx = dz$ or $dx = \frac{1}{2bx}\,dz = \frac{1}{2b}\sqrt{\frac{b}{z}}\,dz = \frac{1}{2\sqrt{bz}}\,dz$.

Hence

$$\int_0^\infty a^{-x^2}\,dx = \int_0^\infty \frac{1}{2\sqrt{bz}}e^{-z}\,dz$$

$$= \frac{1}{2\sqrt{b}}\int_0^\infty e^{-z}z^{1/2-1}\,dz$$

$$= \frac{1}{2\sqrt{b}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{b}} = \frac{\sqrt{\pi}}{2\sqrt{\log a}}$$

EXAMPLE 14.7.4 Show that $\int_0^\infty e^{-x^4}dx \times \int_0^\infty e^{-x^4}.x^2dx = \frac{\pi}{8\sqrt{2}}$.

Solution Let $x^4 = z$, $4x^3dx = dz$, or $dx = \frac{dz}{4x^3}$, $x = z^{\frac{1}{4}}$.

Let

$$I_1 = \int_0^\infty e^{-x^4}dx$$

$$= \int_0^\infty e^{-z}\frac{dz}{4z^{\frac{3}{4}}}dz$$

$$= \int_0^\infty e^{-z}z^{\frac{1}{4}-1}dz = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

and

$$I_2 = \int_0^\infty e^{-x^4}\cdot x^2dx$$

$$= \frac{1}{4}\int_0^\infty e^{-z}z^{-\frac{1}{4}}dz$$

$$= \frac{1}{4}\int_0^\infty e^{-z}z^{\frac{3}{4}-1}dz = \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$$\therefore I = I_1 \times I_2 = \frac{1}{16}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{16}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{16}\pi\sqrt{2} = \frac{\pi}{8\sqrt{2}}.$$

EXAMPLE 14.7.5 Show that $\int_0^\infty e^{-x^2}x^{\alpha^2}dx = \frac{1}{2}\Gamma(\frac{\alpha^2+1}{2})$, $\alpha > 1$.

Solution Putting $x^2 = z$, $2x\,dx = dz$.

$$\begin{aligned}\int_0^\infty e^{-x^2}x^{\alpha^2}dx &= \frac{1}{2}\int_0^\infty e^{-z}(\sqrt{z})^{\alpha^2}\frac{dz}{2\sqrt{z}} = \frac{1}{2}\int_0^\infty e^{-z}z^{\frac{\alpha^2-1}{2}}dz \\ &= \frac{1}{2}\int_0^\infty e^{-z}z^{\frac{\alpha^2+1}{2}-1}dz = \frac{1}{2}\Gamma\left(\frac{\alpha^2+1}{2}\right),\ \alpha > 1\end{aligned}$$

EXAMPLE 14.7.6 Prove that $\displaystyle\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}}dx = \frac{3\pi}{128}.$

Solution

$$\begin{aligned}\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{3}{2}}dx &= \int_0^1 x^{\frac{5}{2}-1}(1-x)^{\frac{5}{2}-1}dx \\ &= \frac{\Gamma(\frac{5}{2})\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}+\frac{5}{2})} = \frac{\{\Gamma(\frac{5}{2})\}^2}{\Gamma(5)} = \frac{\{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\}^2}{4\cdot 3\cdot 2} = \frac{3\pi}{128}\end{aligned}$$

Since $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$ and $\Gamma(5) = 4!$

EXAMPLE 14.7.7 Prove that $\displaystyle\int_a^b (x-a)^3(b-x)^2dx = \frac{(b-a)^6}{60}.$

Solution Substituting $x = a+(b-a)t$, $dx = (b-a)dt$, $b-x = b-a-(b-a)t = (b-a)(1-t)$. When $x \to a$, $t \to 0$ and when $x \to b$, $t \to 1$.

$$\begin{aligned}\int_a^b (x-a)^3(b-x)^2dxdx &= \int_0^1 \{t(b-a)\}^3\{(b-a)(1-t)\}^2(b-a)dt \\ &= \int_0^1 (b-a)^6t^3(1-t)^2dt \\ &= (b-a)^6\int_0^1 t^{4-1}(1-t)^{3-1}dt \\ &= (b-a)^6B(4,3) \\ &= (b-a)^6\frac{\Gamma(4)\Gamma(3)}{\Gamma(7)} = (b-a)^6\frac{3\cdot 2\cdot\Gamma(3)}{6\cdot 5\cdot 4\cdot 3\Gamma(3)} \\ &= \frac{(b-a)^6}{60}\end{aligned}$$

EXAMPLE 14.7.8 Show that $\int_0^1 \frac{dx}{\sqrt{1-x^4}}dx = \frac{1}{4}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}$.

Solution Substituting $1-x^4 = y$, $x^4 = 1-y$, $4x^3dx = -dy$, $dx = -\frac{1}{4}\frac{dy}{(1-y)^{\frac{3}{4}}}$.

When $x \to 0$, $y \to 1$; when $x \to 1$, $y \to 0$.

$$\begin{aligned}
\int_0^1 \frac{dx}{\sqrt{1-x^4}}dx &= \int_1^0 \frac{1}{\sqrt{y}}\frac{1}{4}\frac{-dy}{(1-y)^{\frac{3}{4}}} \\
&= \int_0^1 \frac{1}{4}y^{-\frac{1}{2}}(1-y)^{-\frac{3}{4}}\,dy \\
&= \frac{1}{4}\int_0^1 y^{\frac{1}{2}-1}(1-y)^{\frac{1}{4}-1}dy \\
&= \frac{1}{4}B\Big(\frac{1}{2},\frac{1}{4}\Big) = \frac{1}{4}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}
\end{aligned}$$

EXAMPLE 14.7.9 Show that $\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}}dx = \frac{\pi}{3}$.

Solution Let $x^6 = z$, $6x^5dx = dz$

$$\begin{aligned}
\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}}dx &= \int_0^1 \frac{1}{6}\frac{z^{-\frac{5}{6}}}{(1-z)^{\frac{1}{6}}}dz = \frac{1}{6}\int_0^1 z^{\frac{1}{6}-1}(1-z)^{\frac{5}{6}-1}dz \\
&= \frac{1}{6}B\Big(\frac{1}{6},\frac{5}{6}\Big) = \frac{1}{6}\frac{\Gamma\big(\frac{1}{6}\big)\Gamma\big(\frac{5}{6}\big)}{\Gamma(1)} = \frac{1}{6}\Gamma\Big(\frac{1}{6}\Big)\Gamma\Big(1-\frac{1}{6}\Big) \\
&= \frac{1}{6}\frac{\pi}{\sin\frac{\pi}{6}} = \frac{\pi}{3}
\end{aligned}$$

EXAMPLE 14.7.10 Prove that $\int_0^{\pi/2} \sin^p x\,dx \times \int_0^{\pi/2} \sin^{p+1} x\,dx = \frac{\pi}{2(p+1)}$.

Solution

$$\begin{aligned}
\int_0^{\pi/2} \sin^p x\,dx \times \int_0^{\pi/2} \sin^{p+1} x\,dx &= \frac{1}{2}\frac{\Gamma\big(\frac{p+1}{2}\big)\Gamma\big(\frac{1}{2}\big)}{\Gamma\big(\frac{p+2}{2}\big)} \times \frac{1}{2}\frac{\Gamma\big(\frac{p+2}{2}\big)\Gamma\big(\frac{1}{2}\big)}{\Gamma\big(\frac{p+3}{2}\big)} \\
&= \frac{1}{4}\frac{\pi\Gamma\big(\frac{p+1}{2}\big)}{\frac{p+1}{2}\Gamma\big(\frac{p+1}{2}\big)} = \frac{\pi}{2(p+1)}.
\end{aligned}$$

EXAMPLE 14.7.11 Find the value of $\int_{-1}^{1}(1+x)^p(1-x)^q\,dx$.

Solution Putting $1+x=2y$, or $x=2y-1$. Therefore, $dx=2dy$. When $x\to -1, y\to 0$ and when $x\to 1$ then $y\to 1$.

$$\begin{aligned}
\int_{-1}^{1}(1+x)^p(1-x)^q\,dx &= \int_0^1 (2y)^p 2^q(1-y)^q 2\,dy \\
&= 2^{p+q+1}\int_0^1 y^{(p+1)-1}(1-y)^{q+1-1}\,dy \\
&= 2^{p+q+1}B(p+1,q+1) \\
&= 2^{p+q+1}\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}
\end{aligned}$$

provided $p>-1, q>-1$.

EXAMPLE 14.7.12 Prove that $\int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{1\cdot 2\cdot 3\ldots(m-1)}{n(n+1)\ldots(n+m-1)}$, where m is a positive integer and n is any positive quantity.

Solution We have

$$\begin{aligned}
B(m,n) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{\Gamma(m)\Gamma(n)}{(m+n-1)\Gamma(m+n-1)} \\
&= \frac{\Gamma(m)\Gamma(n)}{(m+n-1)(m+n-2)\Gamma(m+n-2)} \\
&= \frac{\Gamma(m)\Gamma(n)}{(m+n-1)(m+n-2)(m+n-3)\Gamma(m+n-3)} \\
&= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \frac{\Gamma(m)\Gamma(n)}{(m+n-1)(m+n-2)\ldots n\cdot\Gamma(n)} \\
&= \frac{1\cdot 2\cdot 3\ldots(m-1)}{n(n+1)\ldots(n+m-1)} \quad [\because\ m \text{ is a positive integer}]
\end{aligned}$$

EXAMPLE 14.7.13 Prove that

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx = (b-a)^{m+n-1}B(m,n), \quad m,n>0$$

Solution Putting $x = a + (b - a)t$, $dx = (b - a)dt$, $b - x = b - a - (b - a)t = (b - a)(1 - t)$.
When $x \to a$, $t \to 0$ and when $x \to b$, $t \to 1$.

$$\begin{aligned}
\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx &= \int_0^1 (b-a)^{m-1}t^{m-1}(b-a)^{n-1}(1-t)^{n-1}(b-a)dt \\
&= \int_0^1 (b-a)^{m+n-1}t^{m-1}(1-t)^{n-1}dt \\
&= (b-a)^{m+n-1}B(m,n)
\end{aligned}$$

EXAMPLE 14.7.14 Show that $\displaystyle\int_0^1 x^{m-1}(1-x^2)^{n-1}\,dx = \frac{1}{2}B\left(\frac{m}{2}, n\right)$, $m, n > 0$.

Solution Putting $x^2 = z$, $2x\,dx = dz$

$$\begin{aligned}
\int_0^1 x^{m-1}(1-x^2)^{n-1}dx &= \frac{1}{2}\int_0^1 z^{\frac{m-1}{2}-\frac{1}{2}}(1-z)^{n-1}dz \\
&= \frac{1}{2}\int_0^1 z^{\frac{m}{2}-1}(1-z)^{n-1}dz \\
&= \frac{1}{2}\left(\frac{m}{2}, n\right),\quad m, n > 0
\end{aligned}$$

EXAMPLE 14.7.15 Prove that $\displaystyle\int_0^1 x^p(1-x^q)^n\,dx = \frac{1}{q}B\left(\frac{p+1}{q}, n+1\right)$.

Solution Substituting $x^q = z$. Then $qx^{q-1}\,dx = dz$.

$$\begin{aligned}
\int_0^1 x^p(1-x^q)^n\,dx &= \int_0^1 z^{\frac{p}{q}-\frac{q-1}{q}}(1-z)^{n+1-1}\,dz = \frac{1}{q}\int_0^1 z^{(p-q+1)/q}(1-z)^n\,dz \\
&= \frac{1}{q}\int_0^1 z^{(p+1)/q-1}(1-z)^{n+1-1}\,dz \\
&= \frac{1}{q}B\left(\frac{p+1}{q}, n+1\right)
\end{aligned}$$

EXAMPLE 14.7.16 Prove that $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\cdots\Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$.

Solution

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\cdots\Gamma\left(\frac{8}{9}\right)$$

$$= \left\{\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{8}{9}\right)\right\}\left\{\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{7}{9}\right)\right\}\left\{\Gamma\left(\frac{3}{9}\right)\Gamma\left(\frac{6}{9}\right)\right\}\left\{\Gamma\left(\frac{4}{9}\right)\Gamma\left(\frac{5}{9}\right)\right\}$$

$$= \left\{\Gamma\left(\frac{1}{9}\right)\Gamma\left(1-\frac{1}{9}\right)\right\}\left\{\Gamma\left(\frac{2}{9}\right)\Gamma\left(1-\frac{2}{9}\right)\right\}\left\{\Gamma\left(\frac{3}{9}\right)\Gamma\left(1-\frac{3}{9}\right)\right\}\left\{\Gamma\left(\frac{4}{9}\right)\Gamma\left(1-\frac{4}{9}\right)\right\}$$

$$= \frac{\pi}{\sin\frac{\pi}{9}} \times \frac{\pi}{\sin\frac{2\pi}{9}} \times \frac{\pi}{\sin\frac{3\pi}{9}} \times \frac{\pi}{\sin\frac{4\pi}{9}}$$

$$= \frac{2}{\sqrt{3}}\pi^4\frac{1}{\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{4\pi}{9}} = \frac{2^3}{\sqrt{3}}\pi^4\frac{1}{2\sin\frac{\pi}{9}\left(\cos\frac{2\pi}{9}-\cos\frac{2\pi}{3}\right)}\ .$$

$$= \frac{2^3\pi^4}{\sqrt{3}}\frac{1}{2\sin\frac{\pi}{9}\left(\cos\frac{2\pi}{9}+\frac{1}{2}\right)}$$

$$= \frac{2^3\pi^4}{\sqrt{3}}\frac{1}{2\sin\frac{\pi}{9}\cos\frac{2\pi}{9}+\sin\frac{\pi}{9}}$$

$$= \frac{2^3\pi^4}{\sqrt{3}}\frac{1}{\sin\frac{3\pi}{9}-\sin\frac{\pi}{9}+\sin\frac{\pi}{9}}$$

$$= \frac{2^3\pi^4}{\sqrt{3}}\frac{2}{\sqrt{3}} = \frac{16}{3}\pi^4$$

EXAMPLE 14.7.17 Show that $\int_0^p x^m(p^q-x^q)^n\,dx = \frac{p^{nq+m+1}}{q}B\left(n+1,\frac{m+1}{q}\right)$, if $p > 0, q > 0, m+1 > 0, n+1 > 0$.

Solution Putting $p^q - x^q = zp^q$. Therefore, $-\frac{qx^{q-1}}{p^q}\,dx = dz$, i.e. $x = (1-z)^{1/q}\cdot p$.

$$\begin{aligned}\int_0^p x^m p^{nq}\left(1-\frac{x^q}{p^q}\right)^n dx &= -\int_1^0 (1-z)^{(m+1-q)/q}p^{nq+q}z^n\,dz\\ &= \int_0^1 \frac{p^{nq+m+1}}{q}z^n(1-z)^{(m+1)/q-1}\,dz\\ &= \frac{p^{nq+m+1}}{q}\int_0^1 z^{n+1-1}(1-z)^{(m+1)/q-1}\,dz\\ &= \frac{p^{nq+m+1}}{q}B\left(n+1,\frac{m+1}{q}\right),\end{aligned}$$

if $(m+1)/q > 0, n+1 > 0$. That is, if $n+1 > 0, m+1 > 0, q > 0$. Since p is the upper limit of the integral, so it must be positive as lower limit is 0.

EXAMPLE 14.7.18 Show that

$$B(m,n)B(m+n,l) = B(n,l)B(n+l,m) = B(l,m)B(l+m,n).$$

Solution We have

$$\begin{aligned}
B(m,n)B(m+n,l) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \cdot \frac{\Gamma(m+n)\Gamma(l)}{\Gamma(m+n+l)} \\
&= \frac{\Gamma(m)\Gamma(n)\Gamma(l)}{\Gamma(m+n+l)} \\
&= \frac{\Gamma(n)\Gamma(l)}{\Gamma(n+l)} \cdot \frac{\Gamma(n+l)\Gamma(m)}{\Gamma(n+l+m)} \\
&= B(n,l)B(n+l,m)
\end{aligned}$$

Again

$$\begin{aligned}
B(n,l)B(n+l,m) &= \frac{\Gamma(n)\Gamma(l)}{\Gamma(n+l)} \cdot \frac{\Gamma(n+l)\Gamma(m)}{\Gamma(n+l+m)} \\
&= \frac{\Gamma(m)\Gamma(n)\Gamma(l)}{\Gamma(m+n+l)} \\
&= \frac{\Gamma(l)\Gamma(m)}{\Gamma(m+l)} \cdot \frac{\Gamma(m+l)\Gamma(n)}{\Gamma(m+l+n)} \\
&= B(l,m)B(m+l,n)
\end{aligned}$$

EXAMPLE 14.7.19 Show that $B(m,n) = B(m+1,n) + B(m,n+1)$.

Solution We have

$$\begin{aligned}
B(m,n) &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \\
&= \frac{\Gamma(m)\Gamma(n)\cdot(m+n)}{\Gamma(m+n)\cdot(m+n)} \\
&= \frac{m\Gamma(m)\Gamma(n) + n\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} \\
&= \frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} + \frac{\Gamma(m)\cdot n\Gamma(n)}{(m+n)\Gamma(m+n)} \\
&= \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)} \\
&= B(m+1,n) + B(m,n+1)
\end{aligned}$$

EXAMPLE 14.7.20 Show that $\frac{B(p,q+1)}{q} = \frac{B(p+1,q)}{p} = \frac{B(p,q)}{p+q}$.

Solution

$$\begin{aligned}\frac{B(p,q+1)}{q} &= \frac{\Gamma(p)\Gamma(q+1)}{q\cdot\Gamma(p+q+1)}\\ &= \frac{\Gamma(p)\cdot q\Gamma(q)}{q\cdot\Gamma(p+q+1)}\\ &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)}\\ &= \frac{p\Gamma(p)\Gamma(q)}{p\Gamma(p+q+1)} = \frac{B(p+1,q)}{p}\end{aligned}$$

Again

$$\begin{aligned}\frac{B(p+1,q)}{p} &= \frac{\Gamma(p+1)\Gamma(q)}{p\cdot\Gamma(p+q+1)}\\ &= \frac{p\Gamma(p)\Gamma(q)}{p\Gamma(p+q+1)}\\ &= \frac{\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)} = \frac{B(p,q)}{p+q}\end{aligned}$$

EXAMPLE 14.7.21 Show that $\int_0^\infty \frac{x^{m-1}dx}{(a+bx)^{m+n}} = \frac{1}{a^n\cdot b^m}B(m,n)$.

Solution Substituting $bx = at$, $bdx = adt$. When $x\to\infty$, $t\to\infty$; when $x\to 0$, $t\to 0$.

$$\begin{aligned}\int_0^\infty \frac{x^{m-1}dx}{(a+bx)^{m+n}}dx &= \int_0^\infty \frac{\left(\frac{a}{b}t\right)^{m-1}}{(a+at)^{m+n}}\times\frac{a}{b}dt = \int_0^\infty \frac{\left(\frac{a}{b}\right)^{m-1+1}t^{m-1}}{a^{m+n}(1+t)^{m+n}}dt\\ &= \int_0^\infty \frac{a^m}{b^m a^m a^n}\frac{t^{m-1}}{(1+t)^{m+n}}dt = \frac{1}{a^n b^m}\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt\\ &= \frac{1}{a^n\cdot b^m}B(m,n),\ m,n>0\end{aligned}$$

EXAMPLE 14.7.22 Show that $\int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{a^m b^n\Gamma(m+n)}$.

Solution

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta}{\cos^{2m+2n}\theta(b + a\tan^2\theta)^{m+n}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^{2m-1}\theta\sec^2\theta\, d\theta}{(b + a\tan^2\theta)^{m+n}} = \int_0^{\frac{\pi}{2}} \frac{\tan^{2m-2}\theta\sec^2\theta\tan\theta\, d\theta}{(b + a\tan^2\theta)^{m+n}}$$

[Putting $a\tan^2\theta = bt$. $\therefore$ $2a\tan\theta.\sec^2\theta\, d\theta = b\, dt$.

When $\theta \to 0, t \to 0$ and when $\theta \to \pi/2, t \to \infty$.]

$$= \int_0^{\infty} \frac{\left(\frac{bt}{a}\right)^{m-1}\frac{b\,dt}{2a}}{(b+bt)^{m+n}}dt = \frac{1}{2}\int_0^{\infty} \frac{b^{m-1+1}t^{m-1}}{a^{m-1+1}b^{m+n}(1+t)^{m+n}}\, dt$$

$$= \frac{1}{2a^m b^n}\int_0^{\infty}\frac{t^{m-1}}{(1+t)^{m+n}}\, dt$$

$$= \frac{1}{2a^m b^n}B(m,n) = \frac{1}{2a^m b^n}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad \left(\because\ B(m,n) = \int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m+n}}\, dx\right)$$

EXAMPLE 14.7.23 Show that $\displaystyle\int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{(b+cx)^{l+m}}\, dx = \frac{B(l,m)}{(b+c)^l b^m}$.

Solution Substituting $x = \dfrac{1}{1+y}, dx = -\dfrac{1}{(1+y)^2}\, dy$. When $x \to 1, y \to 0$ and when $x \to 0, y \to \infty$.

$$\int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{(b+cx)^{l+m}}\, dx = \int_{\infty}^{0} \frac{\left(\frac{1}{1+y}\right)^{l-1}\left(\frac{y}{1+y}\right)^{m-1}}{\left(b + c\frac{1}{1+y}\right)^{l+m}} \cdot \frac{-1}{(1+y)^2}\, dy$$

$$= \int_0^{\infty}\frac{y^{m-1}(1+y)^{l+m}}{(1+y)^{l-1+m-1+2}(b+by+c)^{l+m}}\, dy$$

$$= \int_0^{\infty}\frac{y^{m-1}\, dy}{(b+c)^{l+m}\left(1 + \frac{b}{b+c}y\right)^{l+m}}$$

$$= \int_0^{\infty}\frac{\left(\frac{b+c}{c}\right)^{m-1} z^{m-1}\frac{b+c}{b}\, dz}{(b+c)^{l+m}(1+z)^{l+m}}$$

$$= \frac{(b+c)^m}{b^m(b+c)^{l+m}}\int_0^{\infty}\frac{z^{m-1}}{(1+z)^{l+m}}\, dz$$

$$= \frac{1}{(b+c)^l b^m}B(m,l)$$

EXAMPLE 14.7.24 Prove that $\displaystyle\int_0^{\pi} \frac{\sin^{n-1} x\, dx}{(a+b\cos x)^n} = \frac{2^{n-1}\cdot B(n/2, n/2)}{(a^2-b^2)^{n/2}}$

Solution From trigonometry

$$\begin{aligned}
a+b\cos x &= a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)\\
&= (a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}\\
&= \left\{(a+b)+(a-b)\tan^2\frac{x}{2}\right\}\cos^2\frac{x}{2}\\
&= \left(1+\frac{a-b}{a+b}\tan^2\frac{x}{2}\right)(a+b)\cos^2\frac{x}{2}
\end{aligned}$$

Let

$$\begin{aligned}
I=\int_0^{\pi}\frac{\sin^{n-1}x\,dx}{(a+b\cos x)^n} &= \int_0^{\pi}\frac{(2\sin\frac{x}{2}\cos\frac{x}{2})^{n-1}\,dx}{\left(1+\frac{a-b}{a+b}\tan^2\frac{x}{2}\right)^n(a+b)^n\cos^{2n}\frac{x}{2}}\\
&= \int_0^{\pi}\frac{2^{n-1}(\sin\frac{x}{2}\cos\frac{x}{2})^{n-1}\sec^{2n}\frac{x}{2}\,dx}{\left(1+\frac{a-b}{a+b}\tan^2\frac{x}{2}\right)^n(a+b)^n}\\
&= \int_0^{\pi}\frac{2^{n-1}\tan^{n-1}\frac{x}{2}\sec^2\frac{x}{2}\,dx}{\left(1+\frac{a-b}{a+b}\tan^2\frac{x}{2}\right)^n(a+b)^n}
\end{aligned}$$

Putting $\dfrac{a-b}{a+b}\tan^2\dfrac{x}{2}=z$, $\quad\therefore \dfrac{a-b}{a+b}2\cdot\dfrac{1}{2}\tan\dfrac{x}{2}\sec^2\dfrac{x}{2}\,dx=dz$

When $x\to 0$ then $z\to 0$ and when $x\to\pi$ then $z\to\infty$ also $\tan\dfrac{x}{2}=\sqrt{\dfrac{a+b}{a-b}}\sqrt{z}$.

Therefore

$$\begin{aligned}
I &= \int_0^{\infty}\frac{2^{n-1}\left(\frac{a+b}{a-b}\right)^{\frac{n-2}{2}+1}z^{(n-2)/2}\,dz}{(a+b)^n(1+z)^n}\\
&= 2^{n-1}\left(\frac{a+b}{a-b}\right)^{n/2}\frac{1}{(a+b)^n}\int_0^{\infty}\frac{z^{n/2-1}\,dz}{(1+z)^n}\\
&= 2^{n-1}\frac{1}{\{(a-b)(a+b)\}^{n/2}}\int_0^{\infty}\frac{z^{n/2-1}\,dz}{(1+z)^{n/2+n/2}}\\
&= \frac{2^{n-1}}{(a^2-b^2)^{n/2}}B\left(\frac{n}{2},\frac{n}{2}\right)
\end{aligned}$$

EXAMPLE 14.7.25 Show that $\int_0^1 \log\Gamma(x)\,dx = \frac{1}{2}\log(2\pi)$ where $\int_0^{\pi/2} \log\sin x\,dx = \frac{\pi}{2}\log\frac{1}{2}$.

Solution Let $I = \int_0^1 \log\Gamma(x)\,dx$.

Substituting $x = 1 - y$. Therefore, $dx = -dy$.

Therefore

$$\begin{aligned}
I &= \int_0^1 \log\Gamma(1-y)\,dy = \int_0^1 \log\Gamma(1-x)\,dx \\
\therefore\ 2I &= \int_0^1 \log\Gamma(x)\,dx + \int_0^1 \log\Gamma(1-x)\,dx \\
&= \int_0^1 \log\{\Gamma(x)\Gamma(1-x)\}\,dx \\
&= \int_0^1 \log\frac{\pi}{\sin\pi x}\,dx \\
&= \int_0^1 \log\pi\,dx - \int_0^1 \log\sin\pi x\,dx \\
&= \log\pi - \frac{1}{\pi}\int_0^{\pi} \log\sin z\,dz = \log\pi - \frac{2}{\pi}\times\int_0^{\pi/2}\log\sin z\,dz \quad [\text{ where } \pi x = z] \\
&= \log\pi - \frac{2}{\pi}\cdot\frac{\pi}{2}\log\frac{1}{2} = \log(2\pi)
\end{aligned}$$

Hence $I = \frac{1}{2}\log(2\pi)$.

EXAMPLE 14.7.26 Show that

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx = \frac{1}{9^{1/3}}B(2/3,1/3) = \frac{1}{9^{1/3}}\frac{2\pi}{\sqrt{3}}$$

Solution Putting $\frac{x}{1-x} = \frac{at}{1-t}$, where a is a suitable constant, to be chosen latter.

That is, $x = \frac{at}{1-t+at}$, $dx = \frac{a}{(1-t+at)^2}\,dt$.

When $x \to 0, t \to 0$ and when $x \to 1, t \to 1$.

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx$$

$$= \int_0^1 \left\{\frac{at}{(1-t+at)}\right\}^{-1/3}\left\{\frac{1-t}{1-t+at}\right\}^{-2/3}\left\{\frac{1-t+3at}{1-t+at)}\right\}^{-1}\cdot\frac{a}{(1-t+at)^2}\,dt$$

$$= \int_0^1 a^{2/3}t^{-1/3}(1-t)^{-2/3}(1-t+3at)^{-1}\,dt$$

If we choose $a = 1/3$, the above integral becomes beta function. Therefore, the given integral is equal to

$$\int_0^1 (1/3)^{2/3}t^{-1/3}(1-t)^{-2/3}\,dt$$

$$= \int_0^1 \frac{1}{9^{1/3}}t^{2/3-1}(1-t)^{1/3-1}\,dt = \frac{1}{9^{1/3}}B(2/3,1/3)$$

$$= \frac{1}{9^{1/3}}\frac{\Gamma(2/3)\Gamma(1/3)}{\Gamma(1)} = \frac{1}{9^{1/3}}\cdot\frac{\pi}{\sin\pi/3}$$

$$= \frac{1}{9^{1/3}}\cdot\frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}\cdot 9^{1/3}}$$

EXAMPLE 14.7.27 Find the values of

$$\int_0^\infty e^{-ax}x^{m-1}\cos bx\,dx \text{ and } \int_0^\infty e^{-ax}x^{m-1}\sin bx\,dx,$$

$m > 0$ in terms of gamma function. Hence or otherwise show that

$$\int_0^\infty x^{m-1}\cos bx\,dx = \frac{\Gamma(m)\cos\left(\frac{m\pi}{2}\right)}{b^m}$$

$$\text{and} \int_0^\infty x^{m-1}\cos bx\,dx = \frac{\Gamma(m)\sin\left(\frac{m\pi}{2}\right)}{b^m}$$

Solution Let $I_1 = \int_0^\infty e^{-ax}x^{m-1}\cos bx\,dx$ and $I_2 = \int_0^\infty e^{-ax}x^{m-1}\sin bx\,dx$.

Now, let

$$I = I_1 + iI_2 = \int_0^\infty e^{-ax}x^{m-1}(\cos bx + i\sin bx)\,dx, \text{ where } i = \sqrt{-1},$$

$$= \int_0^\infty e^{-ax}x^{m-1}e^{ibx}\,dx = \int_0^\infty e^{-(a-ib)x}x^{m-1}\,dx$$

Putting $(a-ib)x = z$. Then $(a-ib)dx = dz$. When $x \to 0, z \to 0$ and when $x \to \infty, z \to \infty$. Therefore,

$$\begin{aligned} I &= \int_0^\infty e^{-z} \frac{z^{m-1}}{(a-ib)^{m-1}} \cdot \frac{1}{a-ib}\, dz \\ &= \frac{1}{(a-ib)^m} \int_0^\infty e^{-z} z^{m-1}\, dz = \frac{\Gamma(m)}{(a-ib)^m} \\ &= \frac{\Gamma(m)\,(a+ib)^m}{(a^2+b^2)^m} \end{aligned}$$

To separate real and imaginary parts of $(a+ib)^m$, let $a = r\cos\theta, b = r\sin\theta$, where $a^2+b^2 = r^2$ and $\theta = \tan^{-1}\frac{b}{a}$. Therefore, $(a+ib)^m = (r\cos\theta + ir\sin\theta)^m = r^m(\cos m\theta + i\sin m\theta)$.

Thus,

$$\begin{aligned} I_1 + iI_2 &= \frac{\Gamma(m)(a^2+b^2)^{m/2}(\cos m\theta + i\sin m\theta)}{(a^2+b^2)^m} \\ &= \frac{\Gamma(m)(\cos m\theta + i\sin m\theta)}{(a^2+b^2)^{m/2}} \end{aligned}$$

Separating real and imaginary parts, we get

$$\begin{aligned} I_1 &= \int_0^\infty e^{-ax} x^{m-1} \cos bx\, dx = \frac{\cos m\theta\, \Gamma(m)}{(a^2+b^2)^{m/2}} \\ \text{and } I_2 &= \int_0^\infty e^{-ax} x^{m-1} \sin bx\, dx = \frac{\sin m\theta\, \Gamma(m)}{(a^2+b^2)^{m/2}} \end{aligned}$$

where $\theta = \tan^{-1}\dfrac{b}{a}$.

Second part: Substituting $a = 0$. Then $\theta = \pi/2$.

Therefore, from above integrals

$$\int_0^\infty x^{m-1} \cos bx\, dx = \frac{\Gamma(m)\cos\left(\frac{m\pi}{2}\right)}{b^m}$$

$$\text{and } \int_0^\infty x^{m-1} \cos bx\, dx = \frac{\Gamma(m)\sin\left(\frac{m\pi}{2}\right)}{b^m}$$

EXERCISES

Section A Multiple Choice Questions

1. If $\int_a^{\infty} f(x)\, dx$ converges then its value is

 (a) finite (b) infinite (c) oscillates (d) none of these

2. If $\int_a^{\infty} f(x)\, dx$ diverges then its value is

 (a) finite (b) infinite (c) oscillates (d) none of these

3. $\int_1^{\infty} \frac{dx}{x^p}$ converges only if

 (a) $p > 1$ (b) $p \leq 1$ (c) $p = 1$ (d) for all p

4. $\int_1^{3} \frac{dx}{(x-1)^p}$ converges only if

 (a) $p < 1$ (b) $p \geq 1$ (c) $p = 1$ (d) for all p

5. The Cauchy principal value of $\int_{-1}^{1} \frac{dx}{x^2}$ is

 (a) 0 (b) 2 (c) -2 (d) 3

6. The integral $\int_{-1}^{1} \frac{dx}{x^2}$

 (a) converges (b) diverges

7. Let $f(x)$ be a function such that $f(x) \leq \frac{1}{x^2}$ for all $x \geq 1$. Then the integral $\int_1^{\infty} f(x)\, dx$

 (a) converges (b) diverges

8. If $|\phi(x)| < M$ for all x in $1 \leq x < \infty$, then $\int_1^{\infty} \frac{|\phi(x)|}{x^2}\, dx$ is

 (a) converges (b) diverges

9. If $f(x) \geq \frac{1}{\sqrt{x}}$ for all $x \geq a, a > 0$. Then $\int_a^{\infty} f(x)\, dx$

 (a) converges (b) diverges

10. If $\lim\limits_{x\to\infty} \frac{f(x)}{1/x^2}$ is a non-zero finite number, then $\int_a^{\infty} f(x)\, dx, a > 0$

 (a) converges (b) diverges

11. If $0 \leq f(x) \leq \frac{1}{\sqrt{x}}$ for all x in $[0,1]$. Then the integral $\int_0^{1} f(x)\, dx$

 (a) converges (b) diverges

12. If $\lim\limits_{x\to 1+} \dfrac{f(x)}{1/(x-1)} = 1$, then $\displaystyle\int_1^2 f(x)\,dx$ is

    (a) convergent (b) divergent

13. If $\lim\limits_{x\to 0+} \dfrac{f(x)}{1/x^{3/4}} = 2$, then $\displaystyle\int_0^4 f(x)\,dx$ is

    (a) convergent (b) divergent

14. If $\lim\limits_{x\to 0+} \dfrac{1/x^2}{g(x)} = 1$, then $\displaystyle\int_0^1 g(x)\,dx$ is

    (a) convergent (b) divergent

15. $\displaystyle\int_0^\infty \cos x\,dx$ is

    (a) convergent (b) divergent (c) oscillates (d) none of these

16. $\displaystyle\int_0^1 e^{-x}x^{n-1}\,dx$ converges if

    (a) $n < 0$ (b) $n > 0$ (c) $n \geq 0$ (d) $n \leq 0$

17. $\Gamma(1)$ is equal to

    (a) 0 (b) 1 (c) 2 (d) ∞

18. The value of $\Gamma(0)$ is

    (a) 0 (b) 1 (c) 2 (d) ∞

19. The value of $\Gamma(-5)$ is

    (a) 0 (b) 1 (c) 4! (d) ∞

20. $\Gamma(7)$ is

    (a) $6\Gamma(6)$ (b) $7\Gamma(7)$ (c) 7! (d) none of these

21. The value of $\Gamma(1/2)$ is

    (a) 1/2 (b) π (c) $\sqrt{\pi}$ (d) none of these

22. The value of $B(1/2, 1/2)$ is

    (a) 1 (b) 2 (c) π (d) $\sqrt{\pi}$

23. The value of $B(1, 1)$ is

    (a) 1 (b) 2 (c) 0 (d) none of these

24. $B(1/2, 1)$ is equal to

    (a) 1/2 (b) 2 (c) 1 (d) none of these

25. $B(0, 1)$ is equal to

    (a) 0 (b) 1 (c) 2 (d) ∞

26. $\displaystyle\int_0^{\infty} \frac{1}{(1+x)^2}\, dx$ is

(a) $B(1,1)$ (b) $\Gamma(1)$ (c) $B(1/2,1/2)$ (d) ∞

27. $\displaystyle 2\int_0^{\pi/2} \sin^3\theta\, \cos^4\theta\, d\theta$ is equal to

(a) $B(2,5/2)$ (b) $B(2,2)$ (c) $B(3,4)$ (d) $B(3/2,5/2)$

28. $\displaystyle 2\int_0^{\pi/2} \sin^6\theta\, d\theta$ is

(a) $\frac{\pi}{16}$ (b) $\frac{5\pi}{16}$ (c) 5π (d) 6

29. The beta function $B(x,y)$ is defined as

(a) $\int_0^1 t^{x-1}(1-t)^{y-1}\, dt$ (b) $\int_0^{\infty} t^{x-1}(1-t)^{y-1}\, dt$ (c) $\int_0^1 t^x(1-t)^y\, dt$

(d) $\int_0^{\infty} t^x(1-t)^y\, dt$

30. $\displaystyle\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx$ is equal to

(a) $B(m,n)$ (b) $2B(m,n)$ (c) $\Gamma(m)$ (d) $\Gamma(m+n)$

31. $\displaystyle\int_0^{\infty} \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx$ is equal to

(a) $B(m,n)$ (b) $2B(m,n)$ (c) $\Gamma(m)$ (d) $\Gamma(m+n)$

32. $\displaystyle\int_0^{\pi/2} \sin^5 x\, \cos^7 x\, dx$ is equal to

(a) $B(5,7)$ (b) $2B(3,4)$ (c) $\frac{1}{2}B(3,4)$ (d) $\frac{1}{2}B(2,3)$

33. $\displaystyle 2\int_0^{\pi/2} \sin^3 x\, \cos^5 x\, dx$ is equal to

(a) $B(3,5)$ (b) $\frac{1}{2}B(3,5)$ (c) $\frac{1}{2}B(2,3)$ (d) $B(2,3)$

34. $\displaystyle 2\int_0^{\pi/2} \sin^6 x\, dx$ is equal to

(a) $B(7/2,1/2)$ (b) $B(7/2,0)$ (c) $\frac{1}{2}B(7/2,1/2)$ (d) $B(6,0)$

35. The value of $\Gamma(3/4)\Gamma(1/4)$ is

(a) $\sqrt{2}\,\pi$ (b) π (c) $\sqrt{2}$ (d) $\pi/\sqrt{2}$

36. $\displaystyle\int_0^{\pi/2} \sqrt{\cot x}\, dx$ is equal to

(a) π (b) $\sqrt{2}$ (c) $\pi/\sqrt{2}$ (d) $\pi\sqrt{2}$

37. $\displaystyle\int_0^1 x^{5/2}(1-x)^{3/2}\, dx$ is

(a) 3π (b) $\pi/256$ (c) $3\pi/2$ (d) $3\pi/256$

38. $\int_0^\infty e^{-x^4}\,dx$ is equal to

(a) $\Gamma(1/4)$ (b) $\frac{1}{4}\Gamma(1/4)$ (c) $\Gamma(3/4)$ (d) $B(1/2, 3/4)$

39. $\Gamma(7/4)\Gamma(1/4)$ is equal to

(a) $\pi/\sqrt{2}$ (b) $\sqrt{2}\,\pi$ (c) $3\pi/2\sqrt{2}$ (d) none of these

40. $\Gamma(m)\Gamma(1-m), 0 < m < 1$ is equal to

(a) $\pi \sin m\pi$ (b) $\pi \cos m\pi$ (c) $\pi \operatorname{cosec} m\pi$ (d) $\pi \tan m\pi$

41. $B(2,1)\Gamma(3)$ is equal to

(a) 2 (b) 3 (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

42. $\int_0^1 x^2(1-x^3)\,dx$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{1}{6}$ (d) 6

Section B Review Questions

1. Show that the following integrals converge.

(a) $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$ (b) $\displaystyle\int_{-\infty}^\infty e^{-(x-1/x)^2}\,dx$

(c) $\displaystyle\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ (d) $\displaystyle\int_0^\infty x^3 e^{-x^2}\,dx$

(e) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ (f) $\displaystyle\int_0^\infty \frac{\cos x}{\sqrt{1+x^3}}\,dx$

(g) $\displaystyle\int_0^\infty \sqrt{x}e^{-x}\,dx$ (h) $\displaystyle\int_e^\infty \frac{dx}{x(\log x)^{3/2}}$

(i) $\displaystyle\int_0^\infty \frac{x\log x}{(1+x^2)^2}\,dx$ (j) $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$

(k) $\displaystyle\int_1^\infty \frac{x^2\,dx}{(1+x^2)^2}$ (l) $\displaystyle\int_1^\infty \frac{x\,dx}{(1+x)^3}$

(m) $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x(1-x)}}$ (n) $\displaystyle\int_0^\infty \frac{\sqrt{x}\,dx}{(1+x)^2}$

(o) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$ (p) $\displaystyle\int_1^\infty \frac{\log x\,dx}{x^2}$

(q) $\displaystyle\int_1^\infty \frac{dx}{x^2(1+x)}$ (r) $\displaystyle\int_1^\infty \frac{\tan^{-1} x}{x^2}\,dx.$

2. Show that the following integrals diverge.

(a) $\int_0^\infty \frac{\log x}{x^2}\,dx$ (b) $\int_0^\infty \frac{x\,dx}{x^2+9}$

(c) $\int_2^\infty \frac{x^3\,dx}{\sqrt{x^7+1}}$ (d) $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$

(e) $\int_1^\infty \frac{dx}{x\log x}$ (f) $\int_1^\infty \frac{dx}{\sqrt{x^2+x}}$

(g) $\int_0^\infty \frac{x^{3/2}}{4x^2+9}\,dx$ (h) $\int_1^\infty \frac{x^3+1}{x^4}\,dx$

(i) $\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx$ (j) $\int_0^\infty x\sin x\,dx.$

3. Show that the following integrals converge.

(a) $\int_0^1 \log x\,dx$ (b) $\int_0^2 \frac{dx}{(2-x)^{3/4}}$

(c) $\int_1^2 \frac{x\,dx}{\sqrt{x-1}}$ (d) $\int_0^1 \frac{dx}{x+\sqrt{x}}$

(e) $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ (f) $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$

(g) $\int_0^{1/e} \frac{dx}{x(\log x)^2}$ (h) $\int_0^1 \frac{dx}{x\log x}$

(i) $\int_{-1}^1 \frac{x-1}{x^{5/3}}\,dx$ (j) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

(k) $\int_1^\pi \frac{\sqrt{x}}{\sin x}\,dx$ (l) $\int_0^1 \frac{\log x\,dx}{\sqrt{1-x^2}}$

(m) $\int_0^1 \frac{\sin^{-1}x}{x}\,dx.$

4. Show that the following integrals diverge.

(a) $\int_0^\pi \frac{\sqrt{x}}{\sin x}\,dx$ (b) $\int_0^1 \frac{dx}{x(1+x)}$

(c) $\int_2^3 \frac{x^2+1}{x^2-4}\,dx$ (d) $\int_{-1}^1 \frac{x-1}{x^{5/3}}\,dx$

(e) $\int_1^2 \frac{dx}{x\log x}.$

5. Show that $\int_0^{\pi/2} \frac{x^m}{\sin^n x}\,dx$ exists iff $n < m+1$.

6. Show that $\int_0^1 \frac{x^n \log x}{(1+x)^2}\, dx$ is convergent if $n > -1$.

7. Prove that the following integrals are convergent under specified conditions.

(a) $\int_0^\infty \frac{x^{n-1}}{1+x}\, dx, \quad 0 < n < 1$

(b) $\int_0^\infty x^{2n+1} e^{-x^2}\, dx$, for all n

(c) $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\, dx, \quad m, n > 0$ and $n - m > 1/2$.

8. Show that $\int_0^{\pi/2} x^p \sin^q x\, dx$ converges if $p + q > -1$.

9. Show that $\int_{-\infty}^{\infty} x\, dx$ is not convergent, but the Cauchy principal value is equal to 0.

10. Show that $\int_0^1 \frac{\sin 1/x}{x^p}\, dx, \quad p > 0$, converges absolutely for $p < 1$.

11. Show that $\int_0^1 e^{-mx} x^n\, dx$ is not convergent for $n > -1$, irrespective of the values of m.

12. Prove the following results.

(a) $\Gamma(7/2) = \frac{15}{8}\sqrt{\pi}$

(b) $\Gamma(7) = 6!$

(c) $\Gamma(1/2) = \sqrt{\pi}$

(d) $B(1/2, 1/2) = \pi$

(e) $B(x, 1) = \frac{1}{x}$

(f) $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

(g) $\Gamma(1/3)\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}$

(h) $\Gamma(1/4)\Gamma(7/4) = \frac{3\pi}{2\sqrt{2}}$

(i) $\int_0^1 \sqrt{1-x^4}\, dx = \frac{1}{12}\sqrt{\frac{2}{\pi}}\left[\Gamma(1/4)\right]^2$.

13. Show that

(a) $$\int_0^{\pi/2} \cos^m x\, dx = \int_0^{\pi/2} \sin^m x\, dx = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}, m > -1$$

(b) $$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4}\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)}$$

(c) $$\int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \frac{\pi}{3}$$

(d) $$\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\, dx = \frac{B(7/4, 1/4)}{2^{9/4}}$$

(e) $$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\, dx = \frac{1}{9^{1/3}} B(2/3, 1/3).$$

14. If $m > 0$, prove that

$$B(m, m)B(m + 1/2, m + 1/2) = \frac{\pi}{m.2^{4m-1}}.$$

15. If n is a positive integer, prove that

$$\frac{\Gamma(n+1/2)}{\Gamma(1/2)} = \frac{(2n)!}{2^{2n} \cdot n!}.$$

16. Prove that

$$\int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}\, dx = 2^{p+q-1}B(p,q), \text{ where } p>0, q>0.$$

17. If $n \geq 1$ is an integer and $m > 0$ is any number, prove that
$$\Gamma(m+n) = (m+n-1)\cdots(m+1)m \cdot \Gamma(m).$$

18. Prove that

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \quad \text{and} \quad 2B(m,n) = \int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx.$$

19. Show that

$$B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx.$$

20. Prove that

(a) $\Gamma(n+1/2) = (2n)!\sqrt{\pi}/4^n \cdot n!,\ n = 1,2,3,\ldots.$

(b) $B(m,m) = 2^{1-2m}B(m,1/2)$

(c) $B(m,m)B(m+1/2, m+1/2) = \dfrac{\pi}{m}2^{1-4m}.$

21. Show that

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\, dt = n^x B(x, n+1), \text{ where } x>0.$$

22. Prove that

$$\int_0^p x^m(p^q - x^q)^n\, dx = \frac{p^{qn+m+1}}{q} B(n+1, (m+1)/q)$$

if $p>0, q>0, m+1>0, n+1>0$.

23. Prove that

$$\int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{(b+cx)^{p+q}}\, dx = \frac{B(p,q)}{(b+c)^p b^q}$$

when $p>0, q>0$ and $(b+c)>0, b>0$.

24. If $p>0, q>0$ then prove that

$$\int_{-1}^{1} \frac{(1+x)^{2p-1}(1-x)^{2q-1}}{(1+x^2)^{p+q}}\, dx = 2^{p+q-2}B(p,q).$$

25. Show that $\displaystyle\int_0^1 \frac{x^{n/2-1}(1-x)^{n/2-1}}{(a+b-2bx)^n}\, dx = \frac{B(n/2, n/2)}{(a^2-b^2)^{n/2}}.$

26. Prove that

$$\int_0^\infty \frac{x^a}{a^x}\,dx = \frac{\Gamma(a+1)}{(\log a)^{n+1}},$$

provided $a > 1$.

27. Show that

$$\int_0^\infty \frac{\sin^{m-1} x}{(a+b\cos x)^m}\,dx = \frac{2^{m-1}}{(a^2-b^2)^{m/2}} B(m/2, m/2),$$

if $m > 0$ and $|a| > |b|$.

28. Evaluate $I = \displaystyle\int_0^t x^{\alpha+k-1}(t-x)^{\beta+k-1}\,dx$ and find its value when $\alpha = \beta = 1/2$.

29. Prove that

$$\int_0^{\pi/2} \frac{\sin^{2m-1}\theta\,\cos^{2n-1}\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}}\,d\theta = \frac{1}{2}\,\frac{\Gamma(m)\Gamma(n)}{a^m b^n \Gamma(n+m)}, \quad m, n > 0.$$

Answers

Section A Multiple Choice Questions

1. (a)	2. (b)	3. (a)	4. (a)	5. (c)	6. (b)	7. (a)	8. (a)	9. (b)
10. (a)	11. (a)	12. (b)	13. (a)	14. (b)	15. (c)	16. (b)	17. (b)	18. (d)
19. (d)	20. (a)	21. (c)	22. (c)	23. (a)	24. (b)	25. (d)	26. (a)	27. (b)
28. (b)	29. (a)	30. (a)	31. (b)	32. (c)	33. (d)	34. (a)	35. (a)	36. (c)
37. (d)	38. (b)	39. (c)	40. (c)	41. (d)	42. (c)			

Section B Review Questions

28 Hint. Put $x = ty$. $I = t^{\alpha+\beta+2k-1}\dfrac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\alpha+\beta+2k)}$.