# INTEGRAL MODEL OF PEL TYPE

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## 1. Prelim on rational PEL data

Let $(B, *, V, H, h)$ be a $\mathbb{Q}$-PEL datum. That is, $B$ is semisimple finite dimensional $\mathbb{R}$ algebra, $*$ a positive involution on $B$, $V$ an $\mathbb{R}$ vector space and $B$ module, $H$ a non-degenerate skew Hermitian on $V$. That is, $H$ is non-degenerate $\mathbb{Q}$ valued alternating form on $V$ compatible with $B$ action. $h : \mathbb{C} \to \mathrm{End}_B(V)$ is an $\mathbb{Q}$ algebra homomorphism such that $h(\bar{z})$ is the adjoint of $h(z)$ under $H$. Then we can attach a triple $(C, *, h)$ such that:

**Definition 1.1.** *Let $(C, *, h)$ be a triple such that*

1. *$C$ is a semisimple finite dimensional $\mathbb{Q}$ algebra. Here, $C = \mathrm{End}_B(V)$.*
2. *$*$ is a involution on $C$. Here, $*$ is the adjoint map under $H$ on $C$.*
3. *$h : \mathbb{C} \to C_{\mathbb{R}}$ is $\mathbb{R}$ algebra homomorphism such that $h(z)^* = h(\bar{z})$. $h$ is the same as above*
4. *$\iota(x) = h(i)^{-1} x^* h(i)$ is a positive involution on $C$.*

*We can attach an algebraic group over $\mathbb{R}$, $(\mathcal{G}, \mathcal{G}_1)$ such that:*

$$\mathcal{G}(R) = \{x \in C \otimes_{\mathbb{Q}} R \mid xx^* \in R^{\times}\}$$

$$\mathcal{G}_1(R) = \{x \in C \otimes_{\mathbb{Q}} R \mid xx^* = 1\}$$

*Then $h$ restricted to $\mathbb{C}^{\times}$ can be regarded as a homomorphism of $\mathbb{R}$ algebraic group*

$$h : \mathbb{S} \to \mathcal{G}_{\mathbb{R}}$$

*Recall that $(B, *, V, H, h)$ is called:*

- *Type (A): If $(C, \iota)_{\mathbb{C}} \cong M_n(\mathbb{C}) \times M_n(\mathbb{C}), (x, y)^* = (y^t, x^t) \iff \mathcal{G}_1/\mathbb{C}$ has Dynkin diagram $A_{n-1}$;*
- *Type (C): If $(C, \iota)_{\mathbb{C}} \cong M_{2n}(\mathbb{C}), x^* = \bar{x}^t \iff \mathcal{G}_1/\mathbb{C}$ has Dynkin diagram $C_n$;*
- *Type(D): If $(C, \iota)_{\mathbb{C}} \cong M_{2n}(\mathbb{C}), x^* = J\bar{x}^t J^{-1}, \iff \mathcal{G}_1/\mathbb{C}$ has Dynkin diagram $D_n$; Here $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$*

**Lemma 1.2.** *Given $(C, *, h)$, the pair $(\mathcal{G}, h)$ is a Shimura datum. The corresponding Hermitian symmetric domain $X_\infty = \mathcal{G}(\mathbb{R}) \cdot h$.*

The $\mathbb{Q}$ PEL datum define an scheme over $E = E(G, X)$, which we denote as $\mathrm{Sh}(B)/E$.

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## 2. Integral model of good reduction of a prime

Let $\mathrm{Sh}(B)/E$ be the PEL type Shimura variety attched to the $\mathbb{Q}$-PEL data $(B, *, V, H, h)$. Fix a prime $p$, and let $v$ be a place in $E$ above $V$. We want extends $\mathrm{Sh}(B)/E$ an **integral model**, defined over $\mathcal{O}_{E,(p)}$. The idea is the fix a lattice in $B$ that gives an integral version for $(B, *, V, H, h)$.

Specifically, we suppose that $B$ is a finite dimensional simple $\mathbb{Q}$ algebra that is unramifed over $p$. Let $F$ be the center of $B$. That is, $B_{\mathbb{Q}_p}$ is a product of matrix algebra over unramified extension of $\mathbb{Q}_p$, and $F_p$ is a product of these unramified extensions. In addition to $(B, *, V, H, h)$, we choose the following **integral PEL data**:

- $\mathcal{O}_B$ is a $\mathbb{Z}_{(p)}$ order in $B$ such that $\mathcal{O}_{B,p}$ is a maximal order in $B_{\mathbb{Q}_p}$. In addition $\mathcal{O}_B$ should be preserved under $*$.
- Let $\Lambda_0$ be a lattice in $V_{\mathbb{Q}_p}$ that is self-dual under $\Psi$ and preserved by $\mathcal{O}_B$.
- Let $\mathcal{K}_p$ be a compact open of $\mathcal{G}(\mathbb{A}_f^p)$.

We use $\mathcal{D} = (\mathcal{O}_B, *, \Lambda_0, \Psi, h)$ to denote the integral PEL data.

**Lemma 2.1.** *As $B_{\mathbb{C}}$ module $V_{\mathbb{C}} \cong V_1 \oplus V_2$, where $V_1$ is the $B \otimes \mathbb{C}$ module where $1 \otimes z$ acts as $\cdot z$. Here the reflex field $E(G, h)$ is exatly the same as the field of definition of isomorphism class of complex representation $V_1$ of $B$. Choose a basis $u_1, \cdots, u_t$ of $\mathcal{O}_B$ over $\mathbb{Z}_{(p)}$. Define the degree $n$ homogenous polynomial:*

$$f(x_1, \cdots, x_n) := \det(x_1u_1 + \ldots x_nu_n; V_1)$$

*Then coefficient of $f$ lies in $\mathcal{O}_{E,(p)}$*

*Proof.* Since $E$ is the field of definition of $V_1$ as a $B$ representation, coefficient of $f$ is in $E$. But $u_1, \cdots, u_i$ preserves $\Lambda_0$, so have entries in $\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$. $\square$

**Definition 2.2.** *Given $\mathcal{D}$ and $\mathcal{K}_p$, we defined the moduli problem $S_{\mathcal{K}^p}$ over $\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$ as the contravariant functor (LNS means locally notherian scheme)*

$$M_{\mathcal{K}^p} : LNS/(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}) \to \text{ quadruples}$$

$$S \to (A, \lambda, i, \eta_{\mathcal{K}})$$

*Such that:*

- *$A \to S$ an abelian scheme;*
- *$\lambda : A \to A^\vee$ a polarization of degree in $\mathbb{Z}_{(p)}^\times$.*
- *$i : \mathcal{O}_B \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ a $*$ homomorphism, for $*$ the involution on $\mathcal{O}_B$ and the Rosati involution on $\mathrm{End}(A)$. That is, $\phi^* = \lambda^\vee \circ \phi^\vee \circ \lambda$.*
- *$\eta_{\mathcal{K}^p}$ be a $\mathcal{K}^p$ level structure. More precisely, we choose $s \in S$ a geometric point and consider $A_s$ the corresponding abelian variety. The Tate module of $A_s$, $T_{\mathbb{A}_f}(A_s) = \prod_{l \neq p} T_l(A_s) \cong H_1(A_s, \mathbb{A}_f^p)$. A level structure of type $\mathcal{K}^p$ is a $\mathcal{K}^p$ orbit $\bar{\eta}$, where $\eta : V_{\mathbb{A}_f^p} \to H_1(A_s, \mathbb{A}_f^p)$ is fixed by $\pi_1(S, s)$ and is a isomorphism of skew-Hermitian $B$-module.*

  *Recall: Let $(M, H_1)$ and $(N, H_2)$ be Hermitian $B$ modules over $R$. An isomorphism of Hermitian $B$ module is an $R$ homomorphism $\eta : M \to N$ such that $H_2(\eta x, \eta y) = rH_1(x, y)$, $r \in R^\times$. In our case, $R = \mathbb{A}_f^p$ and $(V_{\mathbb{A}_f^p}, H)$ is a Hermitian $B$ module, $(H_1(A_s, \mathbb{A}_f^p), e_\lambda)$ is a Hermitian $B$ module with $e_\lambda$ taking value in $\mathbb{A}_f^p(1)$. Hence $\eta : V_{\mathbb{A}_f^p} \to H_1(A_s, \mathbb{A}_f^p)$ is an isomorphism of $B$ module that sends $H$ to a $\mathbb{A}_f^p$ multiple of $e_\lambda$. The*

*$\mathcal{G}(\mathbb{A}_f^p)$ acts on $\eta$ by $\eta \to \eta \circ g$, and the action is simply transitive. This is partitioned into several $\mathcal{K}^p$ orbit, and the level structure is to choose such a orbit. In addition, $\pi_1(S,s)$ acts on $H_1(A_s, \mathbb{A}_f^p)$ hence acts on $\bar{\eta}$ on the left. We have $\bar{\eta}$ to be fixed by $\pi_1(S,s)$ so that $\eta_{\mathcal{K}^p}$ does not depend on the geometric point $s$.*

- *We require $(A, \lambda, i, \bar{\eta})$ satisfying the **determinant condition** on Lie$(A) = \pi_* \Omega^\vee_{A/S}$. Lie$(A)$ is locally free $\mathcal{O}_S$ module where $\mathcal{O}_B$ acts. and let $\alpha_1, \dots, \alpha_t$ generator of $\mathcal{O}_B$ over $\mathcal{O}_S$. Let $g(X_1, \dots, X_t) = \det(\alpha_1 X_1 + \cdots + \alpha_t X_t \mid \text{Lie}(A))$, coefficient in $\mathcal{O}_S$. Then the determinant condition says that*

$$g(X_1, \dots, X_t) = f(x_1, \dots, x_t)$$

*This is equivalent to Lie$(A) \cong V$ as $B$ modules.*

*Two quadriple $(A, \lambda, i, \bar{\eta})$ and $(A', \lambda', i', \bar{\eta}')$ are isomorphic if there exists an isogeny of degree in $\mathbb{Z}_{(p)}^\times$ that sends $A \to A'$, carries $\bar{\eta} \to \bar{\eta}', \lambda \to \mathbb{Z}_p^\times \lambda'$ and is compatible with action of $\mathcal{O}_B$.*

$S_{\mathcal{K}^p}$ can be understood as replacing the $\mathbb{Q}$ isogeny class (can invert any number in $\mathbb{Z}$) by $\mathbb{Z}_{(p)}$ isogeny class (can not inver $p$).

**Proposition 2.3** ($S_{\mathcal{K}^p}$ has good reduction). *For $\mathcal{K}^p$ sufficiently small, the moduli problem $S_{\mathcal{K}^p}$ is representable by a **smooth, quasi-projective** scheme over $\mathcal{O}_{E,(p)}$. This $S_{\mathcal{K}^p}$ is called the integral canonical model of* $\text{Sh}(B)$ *with good reduction at any prime $v$ above $p$.*

**Remark 2.4.** The smoothness is generally proven using the criteria for formal smoothness. A scheme $X$ is formally smooth over $S$ if $\text{Hom}_S(T, X) \to \text{Hom}_S(T', X)$ is surjective, where $T'$ is a first order thickening of $T$. That is, $T$ is a closed subscheme in $T'$ defined by $\mathcal{I}$ such that $\mathcal{I}^2 = 0$.

## 3. Complex points of $S_{\mathcal{K}^p}(\mathbb{C})$

**Lemma 3.1.** *Isomorphism class of skew-Hermitian $B$ modules $V'$ such that $V' \cong V$ as a $\mathbb{Q}$ vector space is classified by $H^1(\mathbb{Q}, \mathcal{G})$,*

**Lemma 3.2.** *For each place $v$ of $\mathbb{Q}$, the skew-Hermitian $B_{\mathbb{Q}_v}$ modules $H_{\mathbb{Q}_v} = H_1(A, \mathbb{Q}_v)$ and $V_{\mathbb{Q}_v}$ are isomorphic.*

**Corollary 3.2.1.** *The isomorphism class of Hermitian $B$ module $H_1(A, \mathbb{Q})$ parameterized by $S_{\mathcal{K}^p}(\mathbb{C})$ is classified by* $\ker^1(\mathbb{Q}, \mathcal{G})$

*Proof.* The isomorphism class of Hermitian $B$ module $V' = H_1(A, \mathbb{Q})$ are exactly the ones classified by $H^1(\mathbb{Q}, \mathcal{G})$ whose image in $H^1(\mathbb{Q}_v, \mathcal{G})$ become trivial. Hence, it is classified by the kernel of

$$H^1(\mathbb{Q}, \mathcal{G}) \to \prod_{v \leq \infty} H^1(\mathbb{Q}_v, \mathcal{G})$$

□

**Proposition 3.3** (connected component of $S_{\mathcal{K}^p}(\mathbb{C})$ is indexed by $|\ker^1(\mathbb{Q}, \mathcal{G})|$). *Suppose $|\ker^1(\mathbb{Q}, \mathcal{G})| = m$. We let $V^1, \dots, V^m$ be the representatives. Fix the local isomorphism $\varphi_{i,v} : V^i_v \cong V_{\mathbb{Q}_v}$. Without lost of generality, let $(V^1, H^1)$ representing the identity element. That is, $(V^1, H^1) \cong (V, H)$ as Hermitian $B$ module. Let $G^i$ be* $\text{Aut}(V^i, H^i)$. *The local isomorphism $\varphi_{i,v}$ induces isomorphisms $G^i_{\mathbb{Q}_v} \to \mathcal{G}_{\mathbb{Q}_v}$ for* *all $v$. Then $S_{\mathcal{K}^p}(\mathbb{C}) = \bigsqcup_{i=1}^m S^i_{\mathcal{K}^p}(\mathbb{C})$, where $S^i_{\mathcal{K}^p}(\mathbb{C})$ classifies $(A, \lambda, i, \bar{\eta})$ such that $H = H_1(A, \mathbb{Q})$ is isomorphic to $V^i$ as a skew-Hermitian $B$ modules.*

**Proposition 3.4.** *Let $\mathcal{K}_p$ be the stabilizer of $\Lambda_0$ in $\mathcal{G}(\mathbb{Q}_p)$. Let $\mathcal{K} = \mathcal{K}_p\mathcal{K}^p \subseteq \mathcal{G}(\mathbb{A}_f)$. Then we have*

$$S^1_{\mathcal{K}^p}(\mathbb{C}) \cong \mathcal{G}(\mathbb{Q})\backslash((\mathcal{G}(\mathbb{A}_f)/\mathcal{K}) \times X_\infty)$$

*Proof.* Consider a point $(A, \lambda, i, \bar{\eta})$ of $S^1_{\mathcal{K}^p}(\mathbb{C})$. Fix an isomorphism $\varphi : H \to V$.

- $\bar{\eta}$ gives an element $\mathcal{G}(\mathbb{A}^p_f)/\mathcal{K}^p$, denoted as $g_{\bar{\eta}}$
- Under the isomorphism $\varphi_\mathbb{R}$, the complex structure on $H_\mathbb{R}$ becomes a complex structure on $V_\mathbb{R}$ and that induce a homomorphism $h' : \mathbb{C} \to C_\mathbb{R}$, $h'$ differs from $h$ by conjugation of $\mathcal{G}(\mathbb{R})$. Hence $h' \in X_\infty$.
- Using the fact that $\mathcal{G}(\mathbb{Q}_p)$ acts transitively on the set of self-dual lattice of $V_{\mathbb{Q}_p}$, we know that there exists $g \in \mathcal{G}(\mathbb{Q}_p)$ such that $g \cdot \varphi_{\mathbb{Q}_p}(\Lambda') = \Lambda_0$. $g$ is unique up to $\mathcal{K}_p$, so $g \in \mathcal{K}_p$.

Hence from $(A, \lambda, i, \bar{\eta})$, we obtaine $([g_{\bar{\eta}} \times g] \times h') \in (\mathcal{G}(\mathbb{A}_f)/\mathcal{K}) \times X_\infty$. The group $\mathcal{G}(\mathbb{A}) = \mathcal{G}(\mathbb{A}_f) \times \mathcal{G}(\mathbb{R})$ acts on $(\mathcal{G}(\mathbb{A}_f)/\mathcal{K}) \times X_\infty$ and so is $\mathcal{G}(\mathbb{Q})$. Finally, notice that the choice of $\varphi$ is unique up to an element in $\mathcal{G}(\mathbb{Q})$. Hence the element is well-definied up to $\mathcal{G}(\mathbb{Q})$ orbit. So we obtain an element in $\mathcal{G}(\mathbb{Q})\backslash((\mathcal{G}(\mathbb{A}_f)/\mathcal{K}) \times X_\infty$. Hence we have the isomorphism:

$$S^1_{\mathcal{K}^p}(\mathbb{C}) \cong \mathcal{G}(\mathbb{Q})\backslash((\mathcal{G}(\mathbb{A}_f)/\mathcal{K}) \times X_\infty)$$

Similarly, $S^i_{\mathcal{K}^p}(\mathbb{C}) \cong \mathcal{G}^i(\mathbb{Q})\backslash((\mathcal{G}(\mathbb{A}_f)/\mathcal{K}) \times X_\infty)$ □

**Lemma 3.5.** *These components $S^i_{\mathcal{K}^p}(\mathbb{C})$ are all isomorphic to each other.*

*Proof.* (Skip the proof in the talk) We define an automorphism of $S_{\mathcal{K}^p}, \varphi$, by sending

$$\varphi : (A, \lambda, i, \bar{\eta}) \to (A, \lambda \circ i(\alpha), i, \bar{\beta\eta})$$

Here $\alpha$ represents the element $z_i \in \ker^1(\mathbb{Q}, Z)$, and $\beta = (\beta_v) \in \prod_v H^1(\mathbb{Q}, Z)$ whose component in $F_l$ is $\beta_l$ for every $l \nmid p$. This automorphism sends $S^i_{\mathcal{K}^p} \to S^1_{\mathcal{K}^p}$. Indeed, let $z_i \in \ker(\mathbb{Q}, Z)$ corresponding to the $i$ th component, or the element in $\ker(\mathbb{Q}, \mathcal{G})$ determined by the skew-Hermitian $B$ module $V^i$. The the above defined isomorphism sends $S^1_{\mathcal{K}_p}$ to $S^i_{\mathcal{K}_p}$. □

## 4. Examples of integral models with good reduction

**Example 4.1** (Integral model of Siegel modular variety)**.** The Siegel modular variety $\mathcal{A}_g$ has an integral model over $\mathbb{Z}_(p)$ that is smooth (and proper??). Here I treat $\mathcal{A}_g$ as a special case of PEL type Shimura variety. Indeed, we can take $B = \mathbb{Q}$ and $*$ to be the identity, $i : B \to \mathrm{End}(V)$ to be trivial. Let $V = \mathbb{Q}^{2g}$, $\Psi$ be the standard symplectic form, $h : \mathbb{C}^\times \to \mathcal{G}_\mathbb{R}$ that maps $z$ to $J$. Then $(B, *, i, V, h)$ gives rise to $(\mathcal{G}(\Psi), X(\Psi))$ in the same recipe.

Fix a prime $p$. Let $\mathbb{Z}_{(p)}$ be the localization. We make a choice of the integral datum. Let $\Lambda_0 \subseteq \mathbb{Q}_p^{2g}$ be $\mathbb{Z}_p^{2g}$, then $\Psi$ is a perfect pairing on $\Lambda_0$. Let $\mathcal{K}_p \subseteq \mathcal{G}(\Psi)(\mathbb{Q}_p)$ be the stailizer of $\Lambda_0$, so it can be taken as $\mathcal{G}(\mathbb{Z}_p)$. For all $N$ relatively prime to $p$, we let $\mathcal{K}(N) \subseteq \mathcal{G}(\mathbb{A}^p_f)$ be the principle level $N$ subgroup. That is,

$$\mathcal{K}(N) = \{g \in \mathcal{G}(\hat{\mathbb{Z}}^p) : g \equiv 1 \mod N\mathcal{G}(\mathbb{Z}_l)\}$$

Given $(\Lambda, \Psi, h)$ and $\mathcal{K}(N)$ such that $N$ is sufficiently large, we define the moduli problem $S_{\mathcal{K}(N)} : LNS/\mathbb{Z}_{(p)} \to (A, \lambda, \eta)$ such that

- $A$ abelian scheme over $\mathbb{Z}_{(p)}$
- $\lambda : A \to \mathbb{A}^t$ is a polarization of degree $\mathbb{Z}_{(p)}^\times$
- $\bar{\eta} : V_{\mathbb{A}_f^p} \to H^1(A_s, \mathbb{A}_f^p)$ is an level $\mathcal{K}(N)$ structure.

  Alternatively, $\bar{\eta} : \Lambda_0/N\Lambda_0 \to A_s[N]$ is a principal level $N$ structure. In other words, it is a $\mathcal{K}(N)$ orbit of symplectic isomorphism

$$\eta : L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p \to T^p(A_s)$$

Then by the previous section, $S_{\mathcal{K}(N)}$ is smooth and quasi-projective, with generic fibre be the Siegel modular variety. Therefore, $S_{\mathcal{K}(N)}$ is an integral model.

**Example 4.2.** (integral model of unitary Shimura variety of signature $(2, 1)$) Let $E/\mathbb{Q}$ be a totally imaginary extension. Let $p$ be a prime unramified in $E$ and let $v$ be a place of $E$ above $p$. We fix an isomorphism $E \otimes_{\mathbb{Q}} (\mathbb{R})$. We consider the following PEL datum:

- $B = E$ with $*$ the unique non-trivial element in the Galois group. Let $\mathcal{O}_B = \mathcal{O}_E$ be the ring of integer
- $V = E^3, \Lambda = \mathcal{O}_E^3$. Defined the $E$-valued skew Hermitian form $H : \Lambda \times \Lambda \to 2\pi_i\mathbb{Z}$ defined by the matrix $M_H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Define the alternating form $\Psi : V \times V \to 2\pi i\mathbb{Z}$ be taking $\Psi(x, y) = \mathrm{tr}_{\mathcal{O}_E/\mathbb{Z}}(\alpha H(x, y))$, where $\alpha$ satisfies:
  - $\alpha \in \mathcal{O}_E$ and $p \nmid \alpha$.
  - $\alpha^* = -\alpha$

  and $i : E \to \mathrm{End}(V)$ the scalar matrix.
- $h : \mathbb{C}^\times \to M_3(\mathbb{C})$

$$h(z) = \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & \bar{z} \end{bmatrix}$$

Then $(\mathcal{O}, *, i, \Lambda_0, h)$ is an integral PEL datum. The associated moduli problem $S_{\mathcal{K}^p}$ for $\mathcal{K}^p \subseteq \mathcal{G}(\hat{\mathbb{Z}}^p)$ sufficiently small is representable by a smooth and quasi-projective scheme over $\mathcal{O}_{E,(p)}$.

**Example 4.3** (integral model that has bad reduction)**.** Let $(r, s)$ be a pair such that $|r - s| > 1$. Say we can take $(r, s) = (3, 1)$. Define the same thing as example 4.2, exact that this time we choose a prime $p$ ramified in $E$ and let $v$ be the unique place above $p$. Then the corresponding moduli problem $S_{\mathcal{K}^p}$ is no longer smooth over $\mathcal{O}_{E,(p)}$. The generic fibre is still $\mathrm{Sh}(\mu_5, \underline{f})(B)/E$, and it has dimension $rs$ since this is the dimension of corresponding Hermitian symmetric domain. Let $S_{\mathcal{K}^p,\mathbb{F}_p}$ be the special fibre of $S_{\mathcal{K}^p}$ over $\mathbb{F}_p$

**Lemma 4.4.** *$S_{\mathcal{K}^p,\mathbb{F}_p}$ has dimension $> \frac{n^2-1}{4}$.*

*Proof.* Let $F$ be the base field of $V$ and $E$ be the reflex field, so $E \subseteq F$, $E_v \subseteq F_v$. Let $S'_{\mathcal{K}^p}$ be the moduli functor over $\mathcal{O}_{F_v}$ whose $S$ valued point has the same definition of $S_{\mathcal{K}^p}$ except that we remove the determinantal condition. We have a closed immersion

$$S_{\mathcal{K}^p} \times_{\mathcal{O}_{E,v}} \mathcal{O}_{F,v} \to S'_{\mathcal{K}^p}$$

Moreover, if $S$ is over $\text{Spec}_{\mathbb{F}_p} = \text{Spec}_{\mathcal{O}_{F,v}/m_v}$, Then

$$i : \det(T_0 \cdot 1 + T_1 \cdot u \mid H_1(A, \mathbb{Q}))$$

Now, notice that $1, \pi$ are generator of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \mathcal{O}_F/\pi^2$. Its generator over $\mathbb{Z}_{(p)}$ can be taken as $1, \pi$, where $\pi$ is the uniformizer of $v$. Then the coefficient of $\det(T_0 \cdot 1 + T_1 \cdot \pi \mid H_1(A, \mathbb{Q}))$ is in $m_v \subseteq \mathcal{O}_{F_v}$. Therefore, the determinant condition becomes redundant over $\text{Spec}(\mathbb{F}_p)$ and so $i$ is an isomorphism.

But then, for any other pairs of indices $(r', s')$ such that $(r' + s') = n$, we can again define $S^{r',s'}_{\mathcal{K}^p}$, and we have the relation

$$S_{\mathcal{K}^p}/\mathbb{F}_p \cong S'_{\mathcal{K}^p}/\mathbb{F}_p \cong S^{r',s'}_{\mathcal{K}^p}/\mathbb{F}_p$$

Also, since $S^{r,s}_{\mathcal{K}^p}$ has a $\text{Spec}(\mathcal{O}_N)$ valued point for $N$ some extension of $K_v$, the dimension of special fibre is $\geq$ dimension of generic fibre. Since $S^{r',s'}_{\mathcal{K}^p}$ has generic fibre of dimension $r's'$, we can conclude that the dimension of special fibre of $S_{\mathcal{K}^p}$ is at least $\max\{r's' \mid r' + s' = n\}$, which is $\frac{n^2-1}{4}$. In particular, the moduli problem is not smooth (Does not have good reduction) if we remove the assumption that $p$ is unramified in $K$. $\square$

**Remark 4.5.** Notice that if $p$ is unramified in $\mathcal{O}_K$, we are not going to run into the same issue. Indeed, suppose $p$ splits into $v_1, v_2$ in $\mathcal{O}_K$, then $\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \mathcal{O}_{K,v_1} \oplus \mathcal{O}_{K,v_2}$, and one can just take $u_1, u_2$ such that $u_i$ is not zero in the residue field $\cong \mathbb{F}_p$. Same as $u_2$. Then the polynomial will remain separable after reduction mod $p$. If $p$ inert, the basis can be taken as $1, u$, where the residue class of $u$ generates $\mathbb{F}_{p^2}$ over $\mathbb{F}_p$.

## References