Since X is a group variety, it's non-singular.

Therefore, Any weil divisor $Y \in \mathrm{Div}(X)$ define a cartier divisor
(integral codim 1 subscheme)

represented by $(U_i, \overset{D(Y)}{f(Y)_i})$, which gives a line bundle together with global section $(\mathcal{L}(D), f_i^{-1})$

$$\begin{array}{ccccc} \mathrm{Div}(X) & \longrightarrow & \mathrm{Cart}(X) = \Gamma(X, K^*/\mathcal{O}^*) = \{(U_i, f(Y)_i\} & & \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Cl}(X) & \longrightarrow & \mathrm{CaCl}(X) = \dfrac{\Gamma(X, K^*/\mathcal{O}^*)}{\Gamma(X, K^*)} & \longrightarrow & \mathrm{Pic}(X) \end{array}$$

$$\begin{array}{ccc} \mathrm{Cart}(X) & \longrightarrow & \text{line bundles + section} \\ \downarrow & & \downarrow \\ \mathrm{CaCl}(X) & \xrightarrow{\sim} & \mathrm{Pic}(X) \quad \text{always} \end{array}$$

* When X integral, seperated notherian schemes, where local ring is UFD, we have:

$$\begin{array}{ccccc} \mathrm{Div}(X) & \xrightarrow{\sim} & \Gamma(X, K^*/\mathcal{O}^*) = (1, s) & & \\ & & \downarrow & & \\ \mathrm{Cl}(X) & \xrightarrow{\sim} & \mathrm{CaCl}(X) & \xrightarrow{\sim} & \mathrm{Pic}(X). \end{array}$$

$(\mathcal{L})$

inparticular, X is non-singular variety, so we have $Cl(X) \xrightarrow{\sim} CaCl(X) \hookrightarrow Pic(X)$.

①: Theorem of the square:

If X, Y varieties s.t. X is complete. $\mathcal{L}, \mathcal{M}$ on $X \times Y$

If for all $y \in Y(k)$ we have $\mathcal{L}_y \cong \mathcal{M}_y$, then

$\exists N$ such that $\mathcal{L} \cong \mathcal{M} \otimes p^* N$, $p: X \times Y \to Y$

(X complete $\to$ X proper$/k$ $\to$ $p: X \times Y \to Y$ is proper.)

This is to say that If $\mathcal{L}_y \cong \mathcal{M}_y$ for all y,

Then $\mathcal{L} = \mathcal{M}$ as elements of $Pic(X/k)(Y)$.

we want to show that $(\mathcal{L} \otimes \mathcal{M}^{-1}) \cong p^* N$ for $N \in Pic(Y)$.

proof: $\mathcal{L}_y \cong \mathcal{M}_y \Rightarrow \mathcal{L}_y \otimes \mathcal{M}_y = (\mathcal{L} \otimes \mathcal{M})^{-1}|_{X \times \{y\}}$ is the trivial line bundle $\mathcal{O}_{X \times \{y\}}$.

and $X \times \{y\}$ is proper
$$\begin{array}{c} X \times \{y\} \\ \downarrow \pi \\ \{y\} \end{array}$$

· Thm 12.8: If $f: X \to Y$ flat and $\dim_{k(y)} H^0(X_y, \mathcal{O}_{X_y}) = r$

then $\pi_* \mathcal{O}_{X_y}$ is locally free of rank r.

since X complete, $H^0(X_y, O_{X_y}) = 1$ so $\pi_*(\mathcal{L}\otimes\mathcal{M}^{-1})$ is locally free of rank 1, say N.

Then $\pi^* N \to \mathcal{L}\otimes\mathcal{M}^{-1}$ is surjective at least on

$\pi: \operatorname{spec} A \longrightarrow \operatorname{spec} B$. So surjective + line bundle $\Rightarrow$ iso

see saw-principle: $X\times Y$ such that X is complete, Y is k-scheme

corollary: If we have $\mathcal{L}_y \cong \mathcal{M}_y$ for all y and $\mathcal{L}_x \cong \mathcal{M}_x$ for some x, then $\mathcal{L}\cong\mathcal{M}$.

proof: we know $\mathcal{L} \cong \mathcal{M}\otimes p^* N$ $\qquad$ $p: X\times Y \to Y$, $X\times Y \to X$

since $\mathcal{L}_x \cong \mathcal{M}_x \otimes (p^* N)_x$ $\qquad$ so $(p^*N)_{\{x\}\times Y}$ is O_{Y_x}

so, $\{i^* p^* N\}$ is O_{Y_x}

$\{x\}\times Y \hookrightarrow X\times Y \longrightarrow Y$ $\qquad$ But $i^* p^*$ is isomorphism of variety, so N is trivial prop: X complete, Y k-scheme, $\mathcal{L}$ line bundle on $X\times Y$

Then $\exists\, Y_0 \hookrightarrow Y$ max subscheme s.t. $\mathcal{L}|_{X\times Y_0}$ trivial.

denot

Thm: X, Y complete, Z connected, locally Noetherian k-scheme

$\mathcal{L}$ is a line bundle on $X\times Y\times Z$ s.t

$\mathcal{L}|_{\{x\}\times Y\times Z}$, $\mathcal{L}|_{X\times\{y\}\times Z}$, $\mathcal{L}|_{X\times Y\times\{z\}}$ trivial

Then $\mathcal{L}$ is trivial on $X\times Y\times Z$

☆: Thm of the cube: X/k abelian variatey $\mathcal{L}$ a line bundle on X

$\Theta(\mathcal{L}) = p_{123}^*\mathcal{L} \otimes p_{12}^*\mathcal{L}^{-1} \otimes p_{13}^*\mathcal{L}^{-1} \otimes p_{23}^*\mathcal{L}^{-1} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L}$
is trivial on $X \times X \times X$

$p_{123}(x_1, x_2, x_3) = x_1 + x_2 + x_3$

$p_{12}(x_1, x_2, x_3) = x_1 + x_2$ ...

proof: we only need to show that are trivial on restriction: $\{e\} \times X \times X \xrightarrow{i} X \times X \times X \xrightarrow{p_{123}} X$ ($\uparrow s$)

$\Theta(\mathcal{L})|_{\{e\} \times X \times X}$ is $e_{X \times X}^*(\mathcal{L})$, so trivial. then a section is $s \circ (e_X \circ X)$ which is constant.

$p_{123} \circ i : (e, x, y) \to (x+y) = \cancel{m}$

$p_{12} \circ i : (e, x, y) \to x = \cancel{pr_1}$

$p_{13} \circ i : (e, x, y) \to y : \cancel{pr_2}$

$p_{23} \circ i : (e, (x, y)) \to x+y = \cancel{m}$

$p_1 : (e, x, y) \to e$

$p_2 : (e, x, y) \to x$ $\cancel{pr_1}$

$p_3 : (e, x, y) \to y$ $\cancel{pr_2}$

Corollary: Y scheme, X complete group variety. $\forall (f, g, h) : Y \to X$ and $\mathcal{L}$ on X,

$(f+g+h)^*\mathcal{L} \otimes (f+g)^*\mathcal{L}^{-1} \otimes (f+h)^*\mathcal{L}^{-1} \otimes (g+h)^*\mathcal{L}^{-1}$
$\otimes f^*\mathcal{L} \otimes g^*\mathcal{L} \otimes h^*\mathcal{L}$ is trivial, on Y

Since it's the pull back of $\Theta(\mathcal{L})$ on (f, g, h)

$$Y \xrightarrow{(f,g,h)} X\times X\times X \xrightarrow{p_{123},\ p_{12},\ p_{13},\ p_{23},\ p_1,\ p_2,\ p_3} X$$

Consequently:

Thm of the □: (makes $\lambda : A \to A^\vee$ a group hom)

X abelian variety (complete group variety)

$t_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong t_x^*\mathcal{L} \otimes t_y^*\mathcal{L}$ as line bundle on X.

$$X \to X\times X\times X$$

$$u \mapsto (u, x, y)$$

$$t_{x+y}^*\mathcal{L} \otimes t_x^*\mathcal{L}^{-1} \otimes t_y^*\mathcal{L}^{-1} \otimes \text{trivial} \otimes \mathcal{L} \text{ is trivial}$$

conseq: $\varphi_{\mathcal{L}} : X(k) \to \mathrm{Pic}(X) \ (= \underline{\mathrm{Pic}}(X/k)(k))$

$$a \mapsto t_a^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

is a group homomorphism.

more generally, for $T \in Sch/k$

$x, y \in X(T) = h_X(T) = \mathrm{Hom}(T, X)$ and $\mathcal{L}_T := pr_X^*\mathcal{L}$ on $X \times T$.

we have: $t_{x+y}^*\mathcal{L}_T \otimes \mathcal{L}_T \cong t_x^*\mathcal{L}_T \otimes t_y^*\mathcal{L}_T$ on X_T

so $\varphi_{\mathcal{L}}(T) : X(T) \to \mathrm{Pic}(X/k)(T)$ is a group homomorphism.

Mumford bundle:

corollary 2: If X, Y variety, X complete for any $\mathcal{L}$ on $X \times Y$,

$S = \{y \in Y : \mathcal{L}_y = \mathcal{L}|_{X \times \{y\}}\}$ is closed

proof: if M is an line bundle on a complete variety,

Then M is trivial iff $H^0(M), H^0(M^{-1})$ both non-zero.

why? b/c. $s \in H^0(M)$, $\sigma \in H^0(M^{-1})$.

we have $s \otimes \sigma \in H^0(\mathcal{O}_X) = k$ (constant)

Since s non-zero, σ non zero, $\exists U_s, U_\sigma \subseteq X$

s.t. s, σ non-zero, and $U_s \cap U_\sigma$ non-empty.

Thus, $\exists U$ s.t. s, σ non-zero. So $s \otimes \sigma$ non-zero,

so $c \neq 0$. Thus $s \otimes \sigma = c$, so s vanishes

no where, so it's a trivialization.

Then $\{y \in Y : L_y \text{ is trivial}\} = \{y : h^0(\mathcal{L}_y) \geq 0\} \cap \{y : h^0(\mathcal{L}_y^{-1}) \geq 1\}$

which is semi-upper cts $= \{y : h^0(\mathcal{L}_y) \leq 1\}^c$

let $\mathcal{L}$ on X, on $X \times X$ have:

$\Lambda(\mathcal{L}) = m^*\mathcal{L} \otimes P_X^*\mathcal{L}^{-1} \otimes P_Y^*\mathcal{L}^{-1}$

Then by the previous prop, $\exists K(L) \subseteq X$ closed subscheme

such that $\Lambda(\mathcal{L})|_{X \times K(L)}$ is pull back of N on $K(L)$, and $K(L)$ is the

largest subscheme that does that, meaning $\forall Y \subseteq X$

such that $\Lambda(\mathcal{L})|_{X \times Y}$ is trivial, $i : Y \hookrightarrow X$ facto thru

such that $(id_X \times i)^*\mathcal{L}$ is pull back of linebundle on Y, i

φ

factor thru Z.

Lemma 2.17:

T k-scheme, $x: T \to X$

Then: (1): $x: T \to X$ factors through $K(L)$

if and only if $t_x^* \mathcal{L}_T \otimes \mathcal{L}_T^{-1} \cong Pr_T^* N$.

on $X \times T$

(2): If $t_x^* \mathcal{L}_T \otimes \mathcal{L}_T^{-1} \cong Pr_T^* M$ then $M \cong x^* \mathcal{L}$

(3): $\Lambda(\mathcal{L})|_{X \times K(L)} \cong \mathcal{O}_{X \times K(L)}$

Proof: notice that

(1) $t_x: X_T \cong X_T \times_T T \xrightarrow{id_{X_T} \times x_T} X_T \times_T X_T \xrightarrow{m} X_T$.

Thus $Pr_X \circ t_x: X_T \to X_T \to X = m \circ (id_X \times x)$

Thus: $t_x^* \mathcal{L}_T = (id_X \times x)^* m^* \mathcal{L}$ on $X \times T$

$t_x^* \mathcal{L}_T \otimes \mathcal{L}_T^{-1} \cong (id_X \times x)^* \Lambda(\mathcal{L}) \otimes (id_X \times x)^* p_2^* \mathcal{L}$

$= (id_X \times x)^* \Lambda(\mathcal{L}) \otimes (Pr_T^* x^* \mathcal{L})$

so $t_x^* \mathcal{L}_T \otimes \mathcal{L}_T^{-1} \cong Pr_T^* x^* \mathcal{L}$ iff $(id_X \times x)^* \Lambda(\mathcal{L})$ is trivial iff x factors through $K(L)$.

②: If $t_x^* \mathcal{L}_T \otimes \mathcal{L}_T^{-1} \cong Pr_T^* \mathcal{M}$ Notice they are trivial on $X \times \{t\}$. and on $\{0\} \times T$ we have LHS restricted to $x^* \mathcal{L}$. Thus $\mathcal{M} = x^* \mathcal{L}$.

③: $\Lambda(\mathcal{L})|_{X \times K(L)} = Pr_T^* \mathcal{M}$ for some $\mathcal{M}$ on $K(L)$

Take $T = K(\mathcal{L})$ and $x : K(\mathcal{L}) \to X$ the inclusion.

$$t_x^* \mathcal{L}_T \otimes \mathcal{L}_T^{-1} = \Lambda(\mathcal{L})|_{X \times K(\mathcal{L})} \otimes Pr_2^*(\mathcal{L})|_{X \times K(\mathcal{L})}$$

$$= P_2^* \mathcal{M} \otimes P_2^* (\mathcal{L}|_{K(\mathcal{L})})$$

$$= P_2^* (\mathcal{M} \otimes \mathcal{L}|_{K(\mathcal{L})}) \text{ then}$$

$$= P_2^* (\mathcal{M} \otimes x^* \mathcal{L})$$

Thus $\mathcal{M} \otimes x^* \mathcal{L} \cong x^* \mathcal{L}$

thus $\mathcal{M} \cong \mathcal{O}_{K(L)}$.

In fact, $\mathcal{L}(k)$ lands inside a subgroup $\mathrm{Pic}^0(X/k)$

they are the line bundles $\mathcal{L} \in \mathrm{Pic}^0(X/k)$ satisfying:

$t_b^* \mathcal{L} \otimes \mathcal{L}^{-1} \simeq \mathcal{O}_X$

when $\mathcal{L}$ is ample and symmetric, $\varphi_{\mathcal{L}} : X \to \mathrm{Pic}^0(X)$

is a "polarization".

Example of $\varphi_{\mathcal{L}} : X(k) \to \mathrm{Pic}(X/k)$ For $X = E$ elliptic curve

$$\begin{tikzcd} X(k) \arrow[r] \arrow[dr] & \mathrm{Pic}(X/k) \\ & \mathrm{Pic}^0(X/k) \arrow[u] \end{tikzcd}$$

we consider $\mathcal{L} = \mathcal{O}_E(O)$, where O is the rational point (point at ∞). $\mathcal{L}$ is ample since $\deg \mathcal{L} > 0$.

and $\mathrm{Pic}^0(E)(k)$ is the group of degree 0 divisors modulo linear equivalence.

Then what is $\varphi_{\mathcal{L}} : E(\bar{k}) \to \mathrm{Pic}^0(E)(k)$?

$$x \to t_x^* \mathcal{O}_E(O) \otimes \mathcal{O}_E(O)^{-1}$$

$t_x^* \mathcal{O}_E(O)$: as a Cartier divisor, represented by $\{U_i, s_i\}$: s_i has a pole of order at most 1 at O and regular otherwise. $s_i \cdot t_x(O - x)$

$t_x^* s_i = s_i \circ t_x = s_i'$ then s_i' has pole at $t_x(O)$

Therefore, $t_x^* \mathcal{O}_E(0) = \mathcal{O}_E(t_{-x}(0))$

$$t_x^* \mathcal{O}_E(0) \otimes \mathcal{O}_E(-0) = \mathcal{O}_E(t_{-x}(0) - 0)$$

$$= \mathcal{O}_E([0-x] - [0])$$

$[0-x] - [0] \sim [0] - [x]$ since

coefficient sums to 0 and

$[0 - x - 0 - 0 + x] = [-0] = [0]$ in group law

$$= \mathcal{O}_E([0] - [x])$$

$$x \to [0] - [x]$$

More generally, let $\mathcal{L}$ be degree m line bundle on E

Then $\mathcal{L} = \mathcal{O}_E(D)$. Then $D - m(0)$ has degree 0,

so $\exists!\, P \in E(k)$ s.t. $D - m(0) \sim (P) - (0)$

$$D \sim (P) + (m-1)(0)$$

$$\underbrace{t_a D - D}_{\text{degree } 0} \sim (-ma) - (0) \approx (0) - (ma)$$

Thus, $\varphi_{\mathcal{L}}$ only depend on degree of m.

Corollary: For line bundle $\mathcal{L}$ on X,

$$n^*\mathcal{L} \cong \mathcal{L}^{n(n+1)/2} \otimes (-1)^* \mathcal{L}^{n(n-1)/2}$$

proof: use $(f+g+h)^*\mathcal{L} \otimes \cdots$

$f = [n]$, $g = [1]$, $h = [-1]$

$$n^*\mathcal{L} \otimes [n+1]^*\mathcal{L}^{-1} \otimes [n-1]^*\mathcal{L}^{-1} \otimes [n]^*\mathcal{L} \otimes \mathcal{L} \otimes [-1]^*\mathcal{L}$$

- $\mathcal{L}$ symm iff $(-1)^*\mathcal{L} \cong \mathcal{L} \Rightarrow [n]^*\mathcal{L} \cong \mathcal{L}^{n^2}$

Thm: Abelian varieties are projective.

Let X be AV. $\quad \mathfrak{d} \subseteq |D|$

proof: we want to find a base point free linear system $\mathfrak{d}$ that separates points and tangent vector.

- separates points iff $\forall$ $p \neq q$ in A, $\exists$ $D \in \mathfrak{d}$ s.t. $P \in \operatorname{supp}(D)$ and $Q \notin \operatorname{supp}(D)$

  i.e., If D is locally cut out by (f) and $P, Q \in U$, then $P \in Z(f)$ But $Q \notin Z(f)$

- separates tangent vector: if $\forall P \in X$, and $v \in T_p(X)$, $\exists D \in \partial$ s.t. $P \in \mathrm{supp}(D)$ but $t \notin T_p(D)$. so let U_p be a open of P on which $D = Z(f)$ for $f \in \mathcal{O}_X(U_p)$. then

$$T_p(X) = \mathrm{Hom}_{k(p)}(m_p/m_p^2, k(p))$$

$$\begin{tikzcd} m_p/m_p^2 \arrow[d] \arrow[dr, "t"] & \\ m_{p,D}/m_{p,D}^2 \arrow[r] & k(p) \end{tikzcd}$$

$m_{p,D} = (m_p)$

$m_{p,D}^2 = (m_p^2, f)$

$t \in T_p(D)$ if t vanishes on (f).

i.e., $\ker(t) \supseteq (f)$.

So we want to choose a linear system $\sigma \subseteq |D|$

Chose $D_1, D_2, \dots, D_n$ such that $\bigcap_{i=1}^n D_i = \{0\}$

Let $D = \sum_{i=1}^n D_i$, we claim that $3D$ is very ample i.e., satisfies the above.

First show that $3D$ separate points.

given $P \neq Q$, find $\Delta \in |3D|$ (effective, linearly eq) s.t. $P \in \mathrm{supp} D$ and $Q \notin \mathrm{supp} D$.

take $\Delta = \sum_{i=1}^{n} t_{a_i}^* D_i + t_{b_i}^* D_i + t_{-a_i-b_i}^* D_i$

notice that $t_{-a_i-b_i}^* D_i + D_i = t_{-a_i}^* D_i + t_{-b_i}^* D_i$

and $t_{-a}^* D_i + t_a^* D_i = 2D_i$

since $P \neq Q$, $\exists D_i$ s.t. $P-Q \notin \operatorname{Supp} D_i$

$O \in \operatorname{Supp} D_i$ $\quad t_{-Q}^*(P-Q) \notin t_{-Q}^*(\operatorname{Supp} D_i)$

$P \notin t_{-Q}^* D_i$, $\quad Q \in t_{-Q}^* D_i$

choose a_i, b_i, a_i s.t. $\sum_{i=1}^{n} t_{a_i}^* D_i + t_{b_i}^* D_i + t_{-a_i-b_i}^* D_i$

also does not have P as support.

Next show that $|3D|$ separates tangent vector.

so given $p \in X$ and $\tau \in T_{X,p}$

notice that $T_{D_1 \cap D_2, 0} = T_{D_1, 0} \cap T_{D_2, 0}$

So $\exists D_1$ s.t. $0 \in D_1$ and $t_{-p}^* \tau \notin T_{D_1, 0}$.

then $t_p^* D_1$ is s.t. $p \in D_1$ and $\tau \notin T_{D, p}$

(This is because X is a group variety, so $t_p^* T_{D,0} = T_{D,p}$)

choose b, a_i, b_i s.t. $p \notin \operatorname{Supp}(D_i)$

Remark: How to find D_i s.t. $\bigcap D_i = \{0\}$?

First: For 0 and P, can find affine that contains both.

$U_0 \ni 0$, and $p \in t_p^* U_0 = U_p$

take $u \in U \cap t_p^* U$.

$$u \in U \Rightarrow u + p \in U + p \Rightarrow p \in U + p - u$$

$$u \in t_p^* U \Rightarrow u \in U + p \Rightarrow 0 \in U + p - u.$$

then U' contain 0 and p.

take H contains 0 but not p

$U' \cap H \subseteq U'$ is a locally closed subscheme that does not contain u, let $H_i' = \overline{U' \cap H}$

Then $\bigcap\limits_{\substack{p \in X \\ p \neq 0}} H_p' = \{0\} \Rightarrow$ only finitely many does

So $\bigcap \{Z_i\} = 0$. set theoretically. Now choose U affine of P Now suppose $t \neq 0$, $t \in T_0, z_i$

choose H that does not contain t, then add $\overline{H \cap A}$ to $\{Z_i\}$.

Thm: dim g AV can not be embedded in $\mathbb{P}^{2g-1}$

If $g \geq 3$, No dim g AV can be embedded in $\mathbb{P}^{2g}$.

proof (Intersection theorey)

suppose $i: X \hookrightarrow \mathbb{P}^n$

get $0 \to T_X \hookrightarrow i^* T_{\mathbb{P}^n} \to N_{\mathbb{P}^n/X} \to 0$

These are vector bundle on $CH(X)$

$$\begin{array}{ccc} i^*: CH^*(\mathbb{P}^n) & \longrightarrow & CH^*(X) \\ \cong & & \\ \mathbb{Z}[H]/H^{n+1} & & \end{array}$$

$\therefore\ C(T_X)\, C(N) = i^* C(T_{\mathbb{P}^n})$

$$1 + \sum_{i=1}^{n-g} C_i(N) = (1+H)^{n+1}$$

so $H^{n-g+1}, H^{n-g+2}, \dots, H^n$ are 0 in $CH(X)$

on the other hand, $\deg(i^* H^g)$

i^*H is the very ample line bundle on X that gives the embedding, so $i^*H^g = i^{-1}(X \cap (\underbrace{H \cup \cdots \cup H}_{g\ \text{times}})^\vee)$

$= \deg(X) \neq 0$, therefore, $n-g+1 > g$ $\quad n > 2g-1$..

if $n = 2g$

Also, $C_g = \binom{2g+1}{g}\ i^*H^g$

$d^2 = \deg(C_g) = \binom{2g+1}{g} d \qquad d = \binom{2g+1}{g} = i H^g.$

on there other hand, $\frac{C_1(i^* H)^g}{g!} = \chi(L) \in \mathbb{Z}$

Thus $g! \mid \binom{2g+1}{g}$.

On the other hand, $g! \mid c_1(\mathcal{L})^g = \deg(h^g) = d$.

So $g! \mid \binom{2g+1}{g} \Rightarrow g < 3$.

Remark: let $i: X \hookrightarrow \mathbb{P}^m = \mathbb{P}(H^0(X, \mathcal{O}_X(mA)))$ an embedding into projective space, what's the degree of the image of the embedding?

. suppose the embedding is given by the very ample line bundle $\mathcal{L} = \mathcal{O}_X(A)$, $\mathcal{L}^{\otimes m} = \mathcal{O}_X(mA)$

Then $\deg(X) = \boxed{(\dim X)! \cdot \text{l.c. } P_{\mathcal{O}_X(mA)}} = A^{\dim X}$

where $P_{\mathcal{O}_X(mA)}(n) = \chi(X, \mathcal{O}_X(nA))$

$$= A^{\dim X} = (i^* H)^{\dim X}$$

(This is the self intersection number, can also think as $i^* h^g$ in $CH^g(X)$
the element.

Now, when $m = 2g$, $i^* h^g = \deg(X) = d$.

$c_g = \binom{2g+1}{g} i^* h^g$

on the other hand, $\deg(N) = c_g = X \cdot X = d^2$.

on the other hand,

$\chi(\mathcal{L}) = \sum ch_{n-i}(\mathcal{L}) \, td_i(X)$

$$= \frac{c_1(\mathcal{L})^g}{g!}$$

since $c_1(\mathcal{L}) = i^* h^g$. we have, $d\, g! \mid \binom{2g+1}{g}$.

How to think about it:

Let $Y = i(X)$ in $\mathbb{P}^N$. Then $\#(Y \cap H^n) = \deg(Y)$

with dim n

But $i^*(Y \cap H^n) = \quad D^n$ when $\mathcal{L}(D) = mA$

$= i^*[Y]^\vee \cdot i^*[H]^{\vee \cdot n}$

$= i^*[H]^{\vee \cdot n} \quad = D^n$ thinking D as a closed subscheme.

$D^n = (m \cdot A)^n$ thinking A as the divisor determine it.