Seminar Presentation

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1 Setup

First we recall our Global PEL data. Let B be a simple $\mathbb{Q}$-algebra with center F and maximal $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ that is stable under a positive involution $*$ of B. Let V be a finitely generated left B-module equipped with a nondegenerate $*$ Hermitian form $(,)$. Then suppose that there is a $\mathcal{O}_B$-stable self-dual $\mathbb{Z}_{(p)}$-lattice $\Lambda \subset V$. Furthermore, set $F_0 = F^{*=1}$ so that F_0 is totally real. We assume that all places above p are ramified in F/F_0 and that the F-algebra B is split. Letting $C = \mathrm{End}_B(V)$ and $\mathcal{O}_C = \mathrm{End}_{\mathcal{O}_B}(\Lambda)$, both have an involution induced from $(,)$. Furthermore, from Stephanie's talk, we know that the pair $(C, *)$, where C is viewed as a $\bar{\mathbb{Q}}$-algebra is of one of the following types:

A) $M_n \times M_n^{\mathrm{opp}}$ with $(x, y)^* = (y, x)$
C) M_{2n} with x^* being the adjoint of x wrt a nondegenerate alternating form in $2n$ variables
D) M_{2n} with x^* being the adjoint of x wrt a nondegenerate symmetric form in $2n$ variables.

We will assume that either case A) or C) holds. Then we get a reductive group $G/\mathbb{Q}$ of B-linear similitudes of V, which can be extended to an algebraic group over $\mathbb{Z}_{(p)}$ representing the functor $G(R) = \{g \in (\mathcal{O}_C \otimes_{\mathbb{Z}_p} R)^\times | gg^* \in R^\times\}$. Then fix a homomorphism $h_0 : \mathbb{C} \to C \otimes R$ with $h_0(\bar{z}) = h_0(z)^*$ for all $z \in \mathbb{C}$ and such that the symmetric real form $(v, h_0(i)w)$ on $V \otimes \mathbb{R}$ is positive definite. Let $h : \mathbb{S} \to G \otimes \mathbb{R}$ defined on $\mathbb{R}$-valued points by $h(z) = h_0(z), z \in \mathbb{C}^\times$. One then gets a tower $\mathrm{Sh}_K, K \subset G(\mathbb{A}_f)$ of Shimura varieties corresponding to compact open subgroups of $G(\mathbb{A}_f)$, which are associated to the pair (G, h^{-1}). These Shimura varieties can be defined over a number field E, which is the reflex field, as defined in Yuxin's talk.

With this PEL data in mind, first we prove a useful lemma.

Lemma 1: Fix our PEL data and choose $(V', (,)')$ a $*$-Hermitian B-module such that $V' \cong V$ as B-modules. Thrn assuming there is a self-dual $\mathcal{O}_B$-stable lattice $\Lambda' \subset V'$, we actually have an isomorphism of $*$-Hermitian $\mathcal{O}_B$-modules $V \cong V'$ carrying Λ into Λ'.

Note that F_0 is a field and choose some uniformizer $\pi \in F_0$ be a uniformizer of F_0. Note that the group N of $\mathcal{O}_B$-linear hermitian isomorphisms of Λ is unramified over $\mathcal{O}_{F_0}$. Furthermore, by our type A or C assumption on the simple factors of the PEL data, we know that N is connected. Hence Lang's Theorem implies there is an isomorphism $\Lambda/\pi\Lambda \cong \Lambda'/\pi\Lambda'$ as Hermitian-$*$-$\mathcal{O}_B$-modules. Then by a lifting process, we conclude that $\Lambda \cong \Lambda'$ in the same way, as desired.

(This lemma is why we assume such a lattice Λ exists to begin with)

Let $\mathfrak{p}$ be a prime of E lying above p and let $\mathcal{O}_{E_\mathfrak{p}}$ be the ring of integers of the complete local field at $\mathfrak{p}$. Let us now set up our integral model.

Let $K^p \subset G(\mathcal{A}_f^p)$ be a sufficiently small compact open subgroup. Let $\mathfrak{M}_{K^p}$ be the contravariant set-valued functor defined on the category of locally Noetherian schemes over $\mathcal{O}_{E_\mathfrak{p}}$ associating to each scheme S the set of isomorphism classes of quadruples $(A, \iota, \lambda, \bar{\eta})$, where A is an abelian scheme up to prime to p isogeny over S, $\iota : \mathcal{O}_B \to \mathrm{End}(A)$, $\lambda : A \to A^\vee$ is a prime to p isogeny which is also a polarization, and $\bar{\eta}$ is a level structure of type K^p. Isomorphism classes are defined by prime to p isogenies $\alpha : A \to A'$ carrying ι into ι', $\bar{\eta}$ into $\bar{\eta}'$, and λ into a $\mathbb{Z}_{(p)}^\times$ multiple of λ' locally on S.

Recall that $\mathfrak{M}_{K_p}$ is represented by a quasiprojective scheme $\mathcal{M}_{K_p}$ over $\mathcal{O}_{E_\mathfrak{p}}$ and that we have an isomorphism $\mathcal{M}_{K_p} \otimes E_\mathfrak{p} \cong \coprod_{\ker^1(\mathbb{Q},G)} \mathrm{Sh}_{G(\mathbb{Z}_p)K^p} \otimes_E E_\mathfrak{p}$, where $\ker^1(\mathbb{Q}, G) = \ker(H^1(\mathbb{Q}, G) \to \prod_v H^1(\mathbb{Q}_v, G))$, which we know is a finite set.

2 Main Content

Now let $K_p \subset G(\mathbb{Z}_p)$ be a compact open subgroup and let $\mathcal{M}_{K_p,K^p}$ be the cover of the set $\mathcal{M}_{K_p}$ that parametrizes K_p-orbits of isomorphisms between $\Lambda \otimes \mathbb{Z}_p$ and the p-adic Tate module T_pA of A, compatible with the $\mathcal{O}_B$ action and the Hermitian forms up to a scalar. Then we have the following proposition:

Proposition 1: The covering map $\pi_{K_p,K^p} : \mathcal{M}_{K_p,K^p} \to M_{K^p} \otimes E_\mathfrak{p}$ is finite etale and whenever K_p is normal in $G(\mathbb{Z}_p)$, Galois with Galois group $G(\mathbb{Z}_p)/K_p$. Furthermore, we have that $\mathcal{M}_{K_p,K^p} \cong \coprod_{\ker^1(\mathbb{Q},G)} \mathrm{Sh}_{K_pK^p} \otimes_E E_\mathfrak{p}$.

Firstly, over the geometric points, the proposition follows immediately from Lemma 1. Then note that $V_\ell A \cong V \otimes \mathbb{Q}_\ell$ by the existence of K^p-level structures, and so since the characters of V_pA and $V_\ell A$ agree as B-representations, we conclude that $V_pA \cong V \otimes \mathbb{Q}_p$ as well. Now the proof proceeds in the same way as the previous result.

Let $\ell \neq p$ be a prime. Consider a continuous ℓ-adic representation ξ of $G(L)$, where L is some number field. This construction induces smooth ℓ-adic sheaves $\mathcal{F}_{\xi,K^p}$ on $\mathcal{M}_{K_p}$ by restricting the ℓ-adic representation to K^p and the tower over $\mathcal{M}_{K_p}$ with Galois group K^p to reconstruct $\mathcal{F}_{\xi,K^p}$). We call this an ℓ-adic local system on $\mathcal{M}_{K^p}$. We can similarly define such a local system on $\mathcal{M}_{K_p,K^p}$, which we denote $\mathcal{F}_{\xi,K_p,K^p}$.

Due to certain compatibility relations, this allows us to define the cohomology of the Shimura variety with coefficients in the local system, namely by setting $H^*_\xi = \varinjlim_{K_p,K^p} H^*(\mathcal{M}_{K_p,K^p} \otimes \overline{\mathbb{Q}}_p, \mathcal{F}_{\xi,K_p,K^p})$, which is equipped with actions by the groups $G_{E_\mathfrak{p}}$ and $G(\mathbb{Z}_p) \times G(\mathbb{A}^p_f)$. In fact, from Proposition 1, it follows that $H^*_\xi = \bigoplus_{\ker^1(\mathbb{Q},G)} H^*_{\mathrm{Sh},\xi}$, where $H^*_{\mathrm{Sh},\xi} = H^*(\mathrm{Sh}_{K_pK^p} \otimes_E E_\mathfrak{p}, \mathcal{F}_{\xi,K_p,K^p})$.

Set $K^p_{g^p} = K^p \cap g^{-p}K^pg^p$. Next we define the notion of a Kottwitz triple.

Definition: Let j be a positive integer and set $r = j[k_{E_\mathfrak{p}} : \mathbb{F}_p]$. Then a degree j Kottwitz triple consists of $(\gamma_0, \gamma, \delta)$, where γ_0 is a semisimple stable conjugacy class of $G(\mathbb{Q})$, γ is a conjugacy class of $G(\mathbb{A}^p_f)$ that is stably conjugate to γ_0, and δ is a σ-conjugacy class of $G(\mathbb{Q}_{p^r})$ such that $N\delta$ is stably conjugate to γ_0 such that γ_0 is elliptic in $G(\mathbb{R})$ and $\kappa_{G\otimes\mathbb{Q}_p}(p\delta) = \mu^\sharp$.

We will seek to count fixed points in our moduli problem. A fixed point in the correspondence is given by a point $(\bar{A}, \iota, \gamma, \bar{\eta}) \in \mathcal{M}_{K^p_{g^p}}(\overline{\mathbb{F}_p})$ such that $(\bar{A}, \iota, \gamma, \bar{\eta})$ and $\sigma^r(\bar{A}, \iota, \gamma, \eta\bar{g}^p)$ give the same point of $\mathcal{M}_{K^p}(\overline{\mathbb{F}_p})$, where σ is the pth power map on $\bar{\mathbb{F}}_p$ and the action of σ^r is obtained by extension of scalars. This is equivalent to saying that there is a prime to p isogeny $u : \sigma^r(\bar{A}) \to \bar{A}$ compatible with ι mapping $\sigma^r(\eta\bar{g}^p)$ into $\bar{\eta}$ and such that the induced action of u^* on λ satisfies $u^*\lambda = c_0\sigma^r(\delta)$.

Set $I/\mathbb{Q}$ to be the group of self-quasiisogenies of (A, ι, λ), which gives an algebraic group over $\mathbb{Q}$. Set L_r to be the unramified extension of $E_\mathfrak{p}$ with residue field of degree r over $k_{E_\mathfrak{p}}$ and then. Then let K_r be the stabilizer of $\Lambda_0 \subset V \otimes_\mathbb{Q} L_r$, which we define to be the extension of scalars of our original lattice Λ. Now recall the map $h : \mathbb{S} \to G \otimes \mathbb{R}$, which induces a map $\mu_h : G_m \to G_\mathbb{C}$. Then the $G(\mathbb{C})$-conjugacy class of homomorphisms of μ_h gives a $G(\mathbb{Q}_p)$-class of homomorphisms $\mu : G_m \to G_{\bar{\mathbb{Q}}_p}$ fixed by G_{L_r}. Furthermore, let $a = \mu^{-1}(p^{-1})$ and set $N' = K_raK_r$ to be a double coset. The key innovation of Kottwitz is a count of the fixed points in terms of orbital integrals, namely $\mathrm{Vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f))O_\gamma(f^p)\mathrm{TO}_{\sigma^{-1}(\delta)}(\phi_r)$, where ϕ_r is the characteristic function of K_raK_r, $O_\gamma(f^p) = \int_{I(\mathbb{A}^p_f)\backslash G(\mathbb{A}^p_f)} f^p(y^{-1}\gamma y)$ and $\mathrm{TO}_\delta(\phi_r) = \int_{I(\mathbb{Q}_p)\backslash G(L_r)} \phi_r(y^{-1}\delta\sigma(y))$