P-ADIC HODGE THEORY

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1. Motivation

We have two cohomology theories that are mainly use for coefficient in $\mathbb{C}$: singular cohomology and De Rham cohomology. In the following, we let X be a complex manifold. Then we have the following result:

(1) comparison isomorphism:
$$H^i(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^i_{dR}(X)$$
In particular, if X is Kahler, then we have the following Hodge decomposition of the singular cohomology:
$$H^n(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \oplus_{j=0}^n H^{n-j}(X,\Omega_X^j)$$
This is the consequence of the Hodge-De Rham spectral sequence degenerates for X Kahler.
$$E_1^{i,j} = H^j(X,\Omega_X^i) \implies H^{i+j}_{dR}(X)$$
(Indeed, we always have this spectral sequence. When it degenerates, we know that the Hodge filtration $Fil^i(H^n_{dR}(X))$ gives $Fil^i/Fil^i = E_1^{i,n-i}$ and since the $E_1^{i,j}$ are $\mathbb{C}$ vector space the exact sequence split)

(2) $H^n(X,\mathbb{R})$ is a pure Hodge-structure of weight n. Equivalently, $H^n(X,\mathbb{R})\otimes_{\mathbb{R}} \mathbb{C} \cong H^{i,j} = H^j(X,\Omega^i)$ has a Galois action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ given by $\overline{H^{i,j}} = H^{j,i}$.

We observe the following pattern. For $H^n(X,\mathbb{R})$ the singular cohomology, there is a "period ring" $\mathbb{C}$, such that by tensoring $H^n(X,\mathbb{R})$ with $\mathbb{C}$, we get

- Comparison isomorphism with another cohomology theory (De Rham cohomology)
- a simpler structure ($\mathbb{C}$ vector space with a Galois action $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and filtration $Fil^n = \oplus_{i+j=n} H^{i,j}$).

These will be the ingredients for p-adic Hodge theory. Generally speaking, p-adic Hodge theory is a tool to study the p-adic representation of local fields given by the etale cohomology. So in the above results, we will replace $\mathbb{R}$ by $\mathbb{Q}_p$ and the singular cohomology by etale cohomology, X a Kahler manifold by X a proper variety over K. Then $H^n_{et}(X_{\bar{K}},\mathbb{Q}_p)$ is a representation of Gal_K on a finite dimensional $\mathbb{Q}_p$ vector space. It even has an integral model given by $H^n_{et}(X_{\bar{K}},\mathbb{Z}_p)$. On the other hand, the etale cohomology is generally not computable. The solution is to tensor $H^n_{et}(X_{\bar{K}},\mathbb{Q}_p)$ with some period ring $B_\cdot$ and compare $H^n_{et}(X_{\bar{K}},\mathbb{Q}_p)$ with $H^*_\cdot(X/K)$, where the other side have simpler structure, usually a Galois module with additional structure (Graded algebra, filtration, Frobenius). For the $\cdot$, we can

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fill in three cohomology theory where the coefficients are allowed to be in K: Hodge Tate, De Rham, Crystalline.

2. Review for p-adic field and cyclotomic character

Let's set up some notation. Let $k \subset \overline{\mathbb{F}}_p$ be either $\overline{\mathbb{F}}_p$ or some finite extension of $\mathbb{F}_p$. Let $K = W(k)[\frac{1}{p}]$ be the fraction field of ring of Witt vector over k, so $\mathcal{O}_K$ is complete DVR with p-adic valuation. This theory work if we replace K by some finite ramified extension of K).

Let $\mathbb{C}_K = \hat{\overline{K}} = \hat{\overline{\mathbb{Q}_p}}$ (The completion of algebraic closure of K).

2.1. cyclotomic character.

Let $\mu_m = \mu_m(\bar{K})$ denote the set of m^{th} root of unity. For $n \geq 1$, we fix a choice of a compatible sequence of primitive p^n-th roots of unity ϵ_n such that $\epsilon_0 = 1, \epsilon_n \in \mu_{p^n}(\overline{K})$, and $\epsilon_n^p = \epsilon_{n-1}$.

Let $K_n = K(\epsilon(n))$ and $K_\infty = \cup K_n$. Notice that K_n are totally ramified extension of K with degree $\varphi(p^n)$. We have the inclusion $K \subset K_n \subset K_\infty \subset \overline{K}$

Let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group

Definition 2.1. *We define the cyclotomic character*

$$\chi : G_K \to \mathbb{Z}_p^*$$

where for $\sigma \in G_K, \chi(\sigma) \in \mathbb{Z}_p^*$ satisfies $\sigma(\epsilon_n) = \epsilon_n^{\chi(\sigma) \mod p^n}$ The kernel of χ is $H_K = Gal(\overline{K}/K_\infty)$, Hence, $\chi : \Gamma_K = Gal(K_\infty/K) = G_K/H_K \cong U$ where U is an open subgroup of $\mathbb{Z}_p^*$.

Definition 2.2 (p-adic representation)**.** *A p-adic representation V of G_K is a finite dimensional $\mathbb{Q}_p$-vector space with a continuous $\mathbb{Q}_p$ linear action of G_K. We write $V \in Rep_{\mathbb{Q}_p}(G_K)$.*

Definition 2.3 (Tate twist)**.** *For $r \in \mathbb{Z}$, we define the r^{th}* ***Tate twist*** *as the* 1 *dimensional representation $\mathbb{Q}_p(r)$, where $\mathbb{Q}_p(r) = \mathbb{Q}_p \cdot e_r$, and $\sigma(e_r) = \chi(\sigma)^r e_r$.*

More generally, if V is a G_K representation over $\mathbb{Q}_p$ (Not necessarily finite dimensional) , we can also define the r^{th} Tate twist of V, denoted as $V(r) = V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$. In particular, $\sigma(v) = \chi(\sigma)\rho(\sigma)v$, for $v = V(r)$ and $\rho : G_K \to V$ the representation.

One important instance of Tate-twist to keep in mind is $\mathbb{C}_K(r)$. Notice that $\mathbb{C}_K = \hat{\overline{K}}$ has G_K action via the continuous extension of G_K action on $\overline{K}$.

Theorem 2.4 (Tate Sen)**.** *We have that*

$$(\mathbb{C}_K(r))^{G_K} = \begin{cases} K & \text{if } r = 0 \\ 0 & \text{if } r \neq 0 \end{cases}$$

Also, $H^1_{cont}(G_K, \mathbb{C}_K(r)) = 0$ if $r \neq 0$ and $H^1_{cont}(G_K, \mathbb{C}_K(r))$ is 1-dimensional over K if $r = 0$.

Remark 2.5. This is true for K a p-adic field.

3. Comparison of Etale and Hodge-Tate

Analogous to the Hodge decomposition for the Kahler manifold, we have the following Hodge decomposition for etale:

Theorem 3.1 (Faltings)**.** *Let X be a smooth proper K-schemes X. Then there is a canonical isomorphism*

$$\mathbb{C}_K \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \cong \bigoplus_{p+q=n} (\mathbb{C}_K(-q) \otimes_K H^{n-q}(X, \Omega^q_{X/K}))$$

This is an isomorphism as G_K modules. The G_K action on the left is tensoring $\mathbb{C}_K$ with the G_K action on $\mathbb{Q}_p$, while on the right side is defined through the action on each Tate twist $\mathbb{C}_K(-q)$. In particular, non-canonically $\mathbb{C}_K \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \cong \oplus_q \mathbb{C}_K(-q)^{h_{n-q,q}}$, where $h_{p,q} = \dim_K H^p(X, \Omega^q_{X/K})$.

Using this isomorphism, we can recover each piece $H^{n-q}(X, \Omega^q)$ from etale. Indeed, suppose we know $\mathbb{C}_K \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)$, then $(\mathbb{C}_K(q) \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p))^{G_K} = H^{n-q}(X, \Omega^q)$.

We also call $\bigoplus_{p+q=n} H^{n-q}(X, \Omega^q_{X/K}) = H^n_{HT}(X/K)$ the Hodge-Tate cohomology. We can rewrite Falting's theorem in a way that provides comparison isomorphism between the Etale and Hodge Tate cohomology theory.

Definition 3.2 (Hodge-Tate period ring)**.** *The Hodge–Tate ring of K is the $\mathbb{C}_K$-algebra $B_{HT} = \bigoplus_{q \in \mathbb{Z}} \mathbb{C}_K(q)$ in which multiplication is defined via the natural maps $\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} \mathbb{C}_K(q') \cong \mathbb{C}_K(q + q')$.*

From the definition, we see that B_{HT} is a graded $\mathbb{C}_K$-algebra with respect to which multiplication is additive in the degrees, and that the natural G_K-action respects the gradings and the ring structure. With our choice of $\epsilon = (\epsilon_n)_n$, $B_{HT} \cong \mathbb{C}_K[t, \frac{1}{t}]$ and G_K acts via $g(t^i) = \chi(g)^i t^i$.

Moreover, from Tate-Sen, we know that $B_{HT}^{G_K} = K$.

Definition 3.3. *Let B_{HT} be as stated above. Then, we can define the functor $D_{HT} : Rep_{\mathbb{Q}_p}(G_K) \to Gr_K$*

$$V \to (B_{HT} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

Here, the grading on $(B_{HT} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is induced from the grading on B_{HT}.

Proposition 3.4. *The natural multiplication map:*

$$\xi_V : B_{HT} \otimes_K D_{HT}(V) \to B_{HT} \otimes_{\mathbb{Q}_p} V$$

is always injective. When ξ_V is an isomorphism, we say that V is Hodge-Tate.

Example 3.5.

- The Tate twist $V = \mathbb{Q}_p(r)$ is Hodge-Tate. Indeed, $D_{HT}(V) = K$, and so the comparison morphism is $B_{HT} \to B_{HT}(r) = B_{HT}$.
- Falting theorem is equivalent to saying that $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)$ is Hodge-Tate. Indeed, we tensor the equation in the theorem by $C_K(r)$ and add them up, then we get: $B_{HT} \otimes_{\mathbb{Q}_p} H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{HT} \otimes_{\mathbb{Q}_p} H^n_{HT}(X/K)$.

B_{HT} is the period ring for Hodge Tate cohomology as it provides isomorphism from etale to Hodge Tate. Notice that Hodge-Tate cohomology only have the structure of Gr_K, hence B_{HT} has the structure of graded $\mathbb{C}_K$ algebra compatible with Galois action.

Now we want the comparison isomorphism from Etale to De Rham and Etale to Crystalline. Let's put the period ring B and the associated functor in a much more general frame work.

4. B-admissible representation

Definition 4.1 ($(\mathbb{Q}_p, G_K)$ regular)**.** *Let B be a $\mathbb{Q}_p$-algebra with an action of G_K. We denote by C the fraction field of B, endowed with a natural action of G_K which extends the action on B. We say that B is $(\mathbb{Q}_p, G_K)$-regular if it satisfies the following conditions:*

- $B^{G_K} = C^{G_K}$.
- *An element $b \in B$ is a unit if the set $\mathbb{Q}_p \cdot b := \{c \cdot b : c \in \mathbb{Q}_p\}$ is stable under the action of G_K.*

Definition 4.2 (The functor D)**.** *Let B be a $(\mathbb{Q}_p, G_K)$-regular ring. Let $E = B^{G_K}$, and denote by Vec_E the category of finite dimensional vector spaces over E.*

(1) We define the functor $D_B : Rep_{\mathbb{Q}_p}(G_K) \to Vec_E$ by

$$D_B(V) := (V \otimes_{\mathbb{Q}_p} B)^{G_K}$$

(2) For all V, we always have $\dim_E D_B(V) \leq \dim_{\mathbb{Q}_p}(V)$. More preciesly, we have the natural multiplication map

$$B \otimes_E D_B(V) \hookrightarrow B \otimes_{\mathbb{Q}_p} (V)$$

That is B-linear, G_K-equivariant, and injective.

We say that $V \in Rep_{\mathbb{Q}_p}(G_K)$ is B-admissible if it satisfies

$$\dim_E D_B(V) = \dim_{\mathbb{Q}_p} V$$

(3) We write $Rep^B_{\mathbb{Q}_p}(G_K)$ for the category of B-admissible p-adic G_K-representations. Equivalently, when α_V is an isomorphism.

When $B = B_{HT}$, we see that B is $(\mathbb{Q}_p, G_K)$ regular. Hence, $D_B(V)$ is the functor we defined earlier, and V is Hodge-Tate iff V is $B_{HT}-$admissible.

5. B_{dR} and De Rham-representation

Similarly, we have a period ring B_{dR} for De Rham representation, where B_{dR} is $(\mathbb{Q}_p, G_K)$ regular, and $D_B(V) = H^n_{dR}(X/K)$ for $V = H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)$. The construction of B_{dR} is very complicated, so instead we record some properties of B_{dR} and some intuition for the property.

Recall that from the algebraic De Rham cohomology, $H^n_{dR}(X/K)$ has a filter $Fil^\bullet H^n(X)$, such that $Fil^q/Fil^{q+1} \cong H^{n-q}(X, \Omega^q)$. Hence, we expected the B_{dR} to be a filtered K-vector space, such that $gr^\bullet B_{dR} \cong B_{HT}$. Also, it should have a G_K action compatible with its filtration. The following are useful properties of B_{dR}:

(1) B_{dR} is a filtered K-algebra endowed with a G_K-action respecting the filtration
(2) B_{dR} is $(\mathbb{Q}_p, G_K)$-regular, with $B_{dR}^{G_K} = K$
(3) B_{dR} is the fraction field of a complete DVR B^+_{dR}, and there is a choice of uniformizer $t \in B^+_{dR}$, such that for $\sigma \in G_K$, $\sigma t = \chi(\sigma)t$. Moreover, the filter on B_{dR} is given by $Fil^n = t^n B^+_{dR}$.

(4) $Fil^0/Fil^1 \cong C_K$. More generally, there is a canonical G_K-equivariant isomorphism $gr^\bullet(B_{dR}) \cong B_{HT}$ as graded $\mathbb{C}_K$ algebras.

Since B_{dR} is $(\mathbb{Q}_p, G_K)$ regular, we have the associated functor

$$D_{dR}(V) = (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

$$Rep_{\mathbb{Q}_p}(G_K) \to Fil_K$$

Indeed, there is a natural filtration

$$Fil^i D_{dR}(V) = (t^i B_{dR}^+ \otimes \mathbb{Q}_p V)^{G_K}$$

We record the following properties of the functor D_{dR}:

(1) We have $\dim_K D_{dR}(V) = \dim_K gr^\bullet(D_{dR}(V)) \leq \dim_K D_{HT}(V) \leq \dim_{\mathbb{Q}_p} V$. So, if V is B_{dR} admissible (De Rham) then V is Hodge Tate, and

$$gr^\bullet(D_{dR}(V)) \cong D_{HT}(V)$$

And the isomorphism respect the grading.

(2) The functor D_{dR} with values in Fil_K is faithful , exact and tensor-exact on B_{dR}-admissible representations.

(3) The comparison isomorphism respect the filtration. That is, if V is De-Rham,

$$D_{dR}(V) \otimes_K B_{dR} \cong V \otimes_{\mathbb{Q}_p} B_{dR}$$

As a filtered K vector space with G_K-action.

Example 5.1. (1) The Tate twist $\mathbb{Q}_p(n)$ is De Rham;

$$\dim_K D_{dR}(\mathbb{Q}_p(n)) \leq \dim_{\mathbb{Q}_p} \mathbb{Q}_p(n) = 1$$

is an equality, as $D_{dR}(\mathbb{Q}_p(n)) = (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{dR})^{G_K}$ contains a nonzero element $1 \otimes t^{-n}$. Consequently, $D_{dR}(\mathbb{Q}_p(n)) = K(-n)$.

What's the filter on $\mathbb{Q}_p(n)$? Since $\mathbb{Q}_p(n)$ is De Rham, it is Hodge Tate. We say that an integer $n \in \mathbb{Z}$ is a **Hodge-Tate** weight of V with multiplicity m if $\dim_K(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{G_K} = m > 0$. In particular, the Hodge-Tate weight of $\mathbb{Q}_p(n)$ is $-n$ with multiplicity 1, and we know from $gr(D_{dR}(V)) = D_{HT}(V)$ that $gr^n(D_{dR}(V)) \neq 0$ iff n is a Hodge-Tate weight of V, and the dimension of grading is the multiplicity. Hence, we have that $Fil^i(D_{dR}(V)) = K(-n)$ if $i \leq -n$, and is 0 if $i > -n$.

(2) The etale-Cohomology $V = H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)$ is De Rham, and $D_{dR}(V) = H^n_{dR}(X/K)$. Let's check that the grading with D_{dR} is compatible with the Hodge Filtration. Indeed, the Hodge Tate weight of V is $0 \leq q \leq n$, with multiplicity $\dim_K H^{n-q}(X, \Omega^q)$, hence $gr^q = h^{n-q,q}$. But the Hodge Filtration gives $Fil^q/Fil^{q+1} = H^{n-q}(X, \Omega^q)$, so the two coincide.

(3) Let $\eta : G_K \to \mathbb{Z}_p^\times$ be a continuous character with finite image. Then there exists a finite extension L of K such that $\mathbb{Q}_p(\eta)$ is trivial as a representation of G_L. Then, for $D_{K,dR}(\mathbb{Q}_p(\eta)) = (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{dR})^G_K$, we have $D_{K,dR}(\mathbb{Q}_p(\eta)) \otimes_K L = D_{L,dR}(\mathbb{Q}_p(\eta)) = L$. Hence, $D_{K,dR}(\mathbb{Q}_p(\eta)) = K = D_{K,dR}(\mathbb{Q}_p)$.

6. B_{cris} and Crystalline representation

Analogus to B_{dR}, we want a period ring B_{cris} that gives comparison morphism between etale and Crystalline cohomology. From the property of $H^n(X/W)$, it turns out that B_{cris} have the following properties:

(1) B_{cris} with the action of G_K is $(G_K, \mathbb{Q}_p)$-regular. Morever, $B_{cris}^{G_K} = K_0$ ($K_0 = K$ in our case).
(2) There is a Frobenius-semilinear endomorphism φ on B_{cris}. The endomorphism φ in is referred to as the Frobenius action on B_{cris}. φ is G_K equivarient, with $\varphi(t) = pt$.
(3) There is a natural G_K equivariant injection $B_{cris} \otimes_{K_0} K \to B_{dR}$ which induces a filtration on B_{cris} from the filtration on B_{dR}. Moreover, there is a G_K equivariant isomorphism of graded algebra.
$$gr(B_{cris} \otimes_{K_0} K) \cong gr(B_{dR}) \cong B_{HT}$$

Then, we can use B_{cris} to define the functor D_{cris}. But this time, the value of the functor will be a "filtered isocrystal". That is, a finite dimensional vector space over N K_0 with $\varphi : N \to N$ a σ-linear automorphism, and a filter Fil^i on N_K. This is compatible with our intuition: $H^n_{cris}(X/W)[\frac{1}{p}]$ is also a filtered isocrystal, since it has filtration from Hodge cohomology and has Frobenius.

Example 6.1. Let X over K be a smooth proper scheme with with a proper smooth integral model $\mathcal{X}$ over $\mathcal{O}_K$. Let $\mathcal{X}_k$ be the special fibre over k. Then, the etale cohomology $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)$. The image of the functor D_{cris} is
$$D_{cris}(H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{cris}(\mathcal{X}_k/W(k))[1/p]$$

Hence we have the comparison isomorphism:
$$H^n_e t(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cris} \cong H^n_{cris}(\mathcal{X}_k/W(k))[1/p] \otimes_{K_0} B_{cris}$$
compatible with G_K-actions, filtrations, and Frobenius actions.

What is the filter, Frobenius on $D_{cris}(V)$? The crystalline cohomology $H^n_{cris}(\mathcal{X}_k/W(k))[\frac{1}{p}]$ is naturally a filtered isocrystal over K with the Frobenius automorphism $\varphi = \varphi^*_{\mathcal{X}_k}$ induced by the relative Frobenius of $\mathcal{X}_k$ and the filtration on $H^n_{cris}(\mathcal{X}_k/K_0) \otimes_{K_0} K$ is given by the Hodge filtration via the canonical comparison isomorphism
$$H^n_{cris}(\mathcal{X}_k/K_0) \otimes_{K_0} K \cong H^n_{dR}(X/K)$$