# Shimura Varieties Learning Seminar
# Symplectic Modules over Involution Algebracs

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## 1 Overview

We primarily follow the first part of Chapter 8 of [1] up to the section PEL Data. Throughout this talk, we fix $k$ to be a field of characteristic 0. Unless otherwise stated, we will take $k$-algebras to be of finite dimension as vector spaces over $k$.

We begin by recalling the following definitions.

**Definition 1.1.** Let $B$ be a $k$-algebra. We say that $B$ is a *simple* $k$-algebra if 0 and $B$ are its only two-sided ideals. We say that $B$ is a *semisimple* $k$-algebra if every $B$-module is *semisimple*, i.e. decomposes as a direct sum of simple $B$-modules. In particular, simple $k$-algebras are also semisimple.

Let $B$ be a $k$-algebra and let $M$ be a $B$-module. The *trace map* of $M$ is the $k$-linear map given by mapping $b \in B$ to the trace $\mathrm{Tr}_k(b|M) \in k$ of the $k$-linear map $M \to M$ given by left multiplication by $b$. If $B$ is a semisimple $k$-algebra the trace map determines $B$-modules up to isomorphism (c.f. [1, Prop 8.1]).

## 2 Algebras with involution

**Definition 2.1** (Involution)**.** Let $B$ be a $k$-algebra. An *involution* of $B$ is a $k$-linear bijection $B \to B$, $b \mapsto b^*$ satisfying the following two properties

(i) for all $a, b \in B$, $(ab)^* = b^* a^*$

(ii) for all $b \in B$, $(b^*)^* = b$

From the first property, it follows that $1^* = 1$, hence by $k$-linearity, $c^* = c$ for all $c \in C$. Properties (i) and (ii) also tell us that any involution maps the center of $B$ into itself. We say that an involution is *of the first kind* if it fixes the elements of the center and *of the second kind* otherwise.

When $k$ is algebraically closed, we have the following classification of semisimple $k$-algebras with involution.

**Proposition 2.2.** *Let $k$ be an algebraically closed field of characteristic 0. Let $(B, *)$ be a semisimple $k$-algebra with involution. Then, $(B, *)$ is isomorphic to a product of pairs of the following types*

***(A)*** $M_n(k) \times M_n(k)$ $\quad (a, b)^* = (b^t, a^t)$

***(C)*** *orthogonal type:* $M_n(k)$ $\quad b^* = b^t$

***(BD)*** *symplectic type:* $M_n(k)$ $\quad b^* = J \cdot b^t \cdot J^{-1} \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$

Skipped proof during talk.

*Proof.* In any given decomposition of $B \cong B_1 \times \ldots \times B_m$ into a product of simple $k$-algebras, the $B_i$ are the minimal two-sided ideals of $B$, hence the decomposition is unique up to permutation of the simple factors. Applying the involution $*$ to $B$ gives the decomposition $B \cong B^* \cong B_1^* \times \ldots \times B_m^*$, hence $B_1^*, \ldots, B_m^*$ is just a permutation of $B_1, \ldots, B_m$. As $(B_i^*)^* = B_i$, this tells us that $B$ decomposes into a product of semisimple $k$-algebras with involution such that each factor is either a simple $k$-algebra or the product of two simple $k$-algebra that are interchanged by $*$. We are thus reduced to checking that any semisimple $k$-algebra with involution satisfying one of these two cases must be one of the three types described in the proposition statement.

Suppose first that $(B, *)$ is a simple $k$-algebra with involution. As $k$ is algebraically closed, Wedderburn's Theorem tells us that $B \cong M_n(k)$ for some $n$. By the Noether-Skolem theorem (see below, c.f. [2, Thm 2.10], as $b \mapsto b^*$ and $b \mapsto b^t$ are both $k$-algebra homomorphisms $M_n(k) \to M_n(k)$ (where the involution on $M_n(k)$ is induced by an isomorphism $B \to M_n(k)$) we have that there exists some fixed $u \in M_n(k)^\times$ such that $b^* = ub^tu^{-1}$ for all $b \in B$. Then,

$$b = (b^*)^* = u(ub^tu^{-1})^tu^{-1} = u(u^t)^{-1}bu^tu^{-1} = (u^tu^{-1})^{-1}b(u^tu^{-1}),$$

hence $c := u^tu^{-1} \in k \subseteq M_n(k)$. Then, $cu = u^t$, so $u = (u^t)^t = c^2u$, hence $c^2 = 1$. This tells us that $u^t = \pm u$, i.e. $u$ is either symmetric or skew-symmetric. By composing the isomorphism $B \to M_n(k)$ with the automorphism $M_n(k) \to M_n(k)$ given by $x \mapsto gxg^{-1}$ for an appropriate choice of $g \in GL_n(k)$, we may assume that $u = I, J$ in the symmetric and skew-symmetric cases respectively. Thus, $(B, *)$ is either of type (C) or type (BD).

**Theorem 2.3** (Noether-Skolem Theorem)**.** *Let $f, g : A \to B$ be $k$-algebra homomorphisms. If $A$ is simple and $B$ is central simple, then there exists $b \in B^\times$ such that $f(a) = b \cdot g(a) \cdot b^{-1}$ for all $a \in A$.*

Now suppose that $(B, *)$ is a semisimple $k$-algebra with involution such that $(B, *)$ is the product of two simple $k$-algebras which are interchanged by the involution. Writing $B = B_1 \times B_2$, we then see that $*$ gives an isomorphism of $B_2$ onto $B_1^{\mathrm{opp}}$, hence $(B, *)$ is isomorphic to $M_n(k) \times M_n(k)^{\mathrm{opp}}$ with involution $(a, b)^* = (b, a)$. Using the isomorphism $M_n(k)^{\mathrm{opp}} \to M_n(k)$, $a \mapsto a^t$ gives us that $(B, *)$ is of type (A). ☐

*Remark* 2.4. Observe that the involution is of the first kind for types (C) and (BD) while it is of the second kind for type (A).

**Definition 2.5** (Adjoint involution)**.** Let $W$ be a finite-dimensional vector space over $k$. Let $\phi : W \times W \to k$ be a non-degenerate bilinear form on $W$. For any $\alpha \in \mathrm{End}_k(W)$, define $\alpha^*$ to be the endomorphism $W \to W$ such that

$$\phi(\alpha^*(v), w) = \phi(v, \alpha(w)) \qquad \text{for any } v, w \in W.$$

Then, $\alpha \mapsto \alpha^*$ is an involution of the $k$-algebra $\mathrm{End}_k(W)$ if and only if $\phi$ is symmetric or skew-symmetric. When one of these two cases hold, $\alpha \mapsto \alpha^*$ is called the *adjoint involution of $\phi$*.

Proposition 2.2 then takes the following form.

**Proposition 2.6.** *Let $k$ be an algebraically closed field of characteristic 0 and let $(B, *)$ be a semisimple $k$-algebra with involution. If the only elements of the center of $B$ fixed by $*$ are those in $k$, then $(B, *)$ is isomorphic to one of the following:*

***(A)*** $(End_k(W) \times End_k(W^\vee), *)$ $(a, b)^* = (b^\vee, a^\vee)$

***(C)*** $(End_k(W), *)$ *$*$ is the adjoint involution of a symmetric bilinear form on $W$*

***(BD)*** $(End_k(W), *)$ *$*$ is the adjoint involution of an alternating bilinear form on $W$*

# 3 Symplectic modules

**Definition 3.1** (Symplectic module)**.** Let $(B, *)$ be a $k$-algebra with involution. Let $V$ be a $B$-module. A $k$-bilinear form $\psi : V \times V \to k$ is said to be *balanced* if it satisfies

$$\psi(b^*u, v) = \psi(u, bv) \quad \text{for all } b \in B, u, v \in V. \tag{1}$$

A *symplectic* $(B,*)$*-module* is a $B$-module $V$ equipped with a skew-symmetric balanced nondegenerate bilinear form $\psi$.

*Example* 1. Let $F$ be $k\times k$ or a field of degree 2 over $k$. In both cases, there is a unique nontrivial involution $*$ of $F$ that fixes the elements of $k$. In the first case, $k$ embeds in $F$ as the diagonal and $*$ swaps the factors, i.e. $(a,b)^* = (b,a)$. In the second case, writing $F = k(\sqrt{\alpha})$, $(a+b\sqrt{\alpha})^* = a - b\sqrt{\alpha}$. Let $B$ be a matrix algebra over $F$ with $(B,*)$ of type (A). Then, $B \cong \mathrm{End}_F(W)$ where $W$ is any simple $B$-module, and $*$ is the adjoint involution of a hermitian form $\phi : W\times W \to F$.

Any symplectic $(B,*)$-module $(V,\psi)$ is then of the following form. There is a free $F$-module of finite rank $V_0$ equipped with a skew-hermitian form $\psi_0 : V_0\times V_0 \to F$ such that $V \cong W\otimes_F F_0$ with the action of $B$ on $V$ corresponding to the action on the first factor and $\psi : V\times V \to k$ corresponds to

$$\psi(w\otimes v, w'\otimes v') = \mathrm{Tr}_{F/k}(\phi(w,w')\psi_0(v,v')).$$

See Example 8.5 in [1] for more details.

*Example* 2. Let $B$ be a $k$-algebra that is isomorphic to a matrix algebra over $k$ and $*$ be an involution on $B$ such that $(B,*)$ is of type (C). Similar to the previous example, we then have that $B\cong \mathrm{End}_k(W)$ for any simple $B$-module $W$, but in this case, $*$ is the adjoint involution of a symmetric bilinear form $\phi : W\times W \to k$.

Any symplectic $(B,*)$-module $(V,\psi)$ is then of the following form. There is a $k$-vector space $V_0$ equipped with a skew-symmetric form $\psi_0 : V_0\times V_0 \to k$ such that $V\cong W\otimes_k V_0$ and $\psi : V\times V\to k$ corresponds to

$$\psi(w\otimes v, w'\otimes v') = \phi(w,w')\psi_0(v,v').$$

See Example 8.6 in [1] for more details.

## 3.1 Reduced determinant

Let $B$ be a semisimple $k$-algebra and let $F$ be its center.

If $F$ is a field, then $B$ is a central simple $F$-algebra, hence (c.f. [3, Prop 5.2.2]) there exists a finite Galois extension $E/F$ such that $B\otimes_F E \cong M_n(E)$ for some $n\geq 1$. Let $f : B\otimes_F E \to M_n(E)$ be an isomorphism. Let $V$ be a $B$-module that is finite-dimensional as a $k$ vector space. Then, $V\otimes_F E$ is a $B\otimes_F E$ module, and there is a determinant map

$$\det_{B\otimes E} : \mathrm{End}_{B\otimes E}(V\otimes E) \to B\otimes E.$$

Composing with the isomorphism $f$ and the determinant map $\det : M_n(E)\to E$ gives us a reduced determinant map

$$\det_E : \mathrm{End}_{B\otimes E}(V\otimes E)\to E.$$

If $g\in \mathrm{End}_B(V)\subseteq \mathrm{End}_{B\otimes E}(V\otimes E)$, then $\det_E(g)\in E$ is fixed by $\mathrm{Gal}(E/F)$, hence $\det_E(g)\in F$. We thus have a reduced determinant map

$$\det : \mathrm{End}_B(V)\to F.$$

In general, $F$ will be a product of fields $F = \prod_i F_i$ with $B = \prod_i B_i$ where each $B_i = B\otimes_F F_i$ is a central simple algebra over $F_i$. If $B$ is free as a $F$-module, the degree $[B_i : F_i] = n^2$ is independent of $i$. Given a $B$-module $V$ that is finite-dimensional as a vector space over $k$, the above discussion gives a reduced determinant map

$$\det : \mathrm{End}_B(V)\to F$$

induced by the reduced determinant maps $\det : \mathrm{End}_{B_i}(V)\to F_i$.

## 3.2 The groups

Let $(B,*)$ be a semisimple $k$-algebra with involution. Let $F$ be the center of $B$ and $F_0\subsetneq F$ the subalgebra that is fixed pointwise by $*$. We say that $(B,*)$ *has type (A), (C), or (BD)* if $(B\otimes_{F_0,\rho}\overline{k},*)$ has that type for all $k$-homomorphisms $\rho : F_0\to \overline{k}$, where $\overline{k}$ denotes the algebraic closure of $k$. If $F_0$ is a field, $(B,*)$ will always have one of the three types. If further $B$ is free as an $F$-module, then $F$ will be free of rank 2 over $F_0$ in case (A) and equal to $F_0$ in cases (C), (BD). Let $n^2 = [B:F], g = [F_0 : k]$.

Let $(V,\psi)$ be a symplectic $(B,*)$-module such that $V$ is free over $F$. The reduced dimension of $V$ is given by

$$m = \frac{\dim_F(V)}{[B:F]^{\frac{1}{2}}} = \frac{\dim_F(V)}{n}.$$

We set $G_1, G$ to be the algebraic subgroups of $\mathrm{GL}_B(V)$ such that

$$G_1(k) = \{g \in GL_B : \ \psi(gx,gy) = \psi(\mu(g)x,y) \text{ for some } \mu(g) \in F_0^\times\} \tag{2}$$

$$G(k) = \{g \in GL_B : \ \psi(gx,gy) = \mu(g)\cdot\psi(x,y) \text{ for some } \mu(g) \in k^\times\} \tag{3}$$

We then have a homomorphism $\mu : G_1 \to (\mathbb{G}_m)_{F_0/k}$ such that $G = \mu^{-1}(\mathbb{G}_m)$. Set

$$G' = \ker(\mu)\cap\ker(\det) \qquad T_1 = G_1/G' \qquad T = G/G'$$

where $\det : GL_B(V) \to F^\times$ is the reduced determinant (which is a group homomorphism on the multiplicative group).

Diagram from [1] that summarizes this:

$$\begin{tikzcd}
B \arrow[r, dash] \arrow[d, dash, "n"'] & V \arrow[dl, dash, "mn"] \\
F \arrow[d, dash, "1 \text{ or } 2"'] & \\
F_0 \arrow[d, dash, "g"'] & \\
k &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G' \arrow[d] \arrow[dr] & \\
Z(G) \arrow[r] \arrow[dr] & G_1 \arrow[r, "\mathrm{ad}"'] \arrow[d, "v"'] & G_1^{\mathrm{ad}} \\
 & T_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G' \arrow[d] \arrow[dr] & \\
Z(G) \arrow[r] \arrow[dr] & G \arrow[r, "\mathrm{ad}"] \arrow[d, "v"'] & G^{\mathrm{ad}} \\
 & T &
\end{tikzcd}$$

Let $*$ be the adjoint involution on $\mathrm{End}_k(V)$ with respect to $\psi$. As $(V,\psi)$ is a symplectic $(B,*)$-module, the adjoint involution $*$ induces the involution $*$ on $B$ (consider the endomorphisms on $V$ by left multiplication by elements of $b$). Let $C = \mathrm{End}_B(V)$. Then $C$ is a semisimple algebra that is stable under $*$ on $\mathrm{End}_k(V)$, and the groups $G, G_1, G'$ admit the following description

$$G_1(k) = \{x \in C : \ x^*x \in F_0^\times\}$$

$$G(k) = \{x \in C : \ x^*x \in k^\times\}$$

$$G'(k) = \{x \in C : \ x^*x = 1, \mathrm{Nrd}(x) = 1\}$$

where Nrd denotes the reduced norm.

We consider the following examples from [4].

*Example* 3 (Unitary). Let $(B,*) = (E,*)$ where $E$ is an imaginary quadratic extension of $\mathbb{Q}$ and $*$ is given by complex conjugation. Then, $(E,*)$ is of type (A). Fix an isomorphism $E\otimes_{\mathbb{Q}}\mathbb{R} \to \mathbb{C}$. Let $a \geq b$ be two nonnegative integers. Take $V = E^{a+b}$ and $\psi$ to be the bilinear form with matrix

$$J_{a,b} = \begin{pmatrix} 0 & 0 & I_b \\ 0 & \epsilon I_{a-b} & 0 \\ -I_b & 0 & 0 \end{pmatrix} \in M_{a+b}(E)$$

where $\epsilon \in \mathcal{O}_E$ is such that $-i\epsilon \in \mathbb{R}_{\geq 0}$.

Observe that for any $g = (g_{i'j'}) \in GL_{a+b}(V)$, and $1 \leq i, j \leq a+b$

$$ge_i = \sum_k g_{ki}e_k \qquad ge_j = \sum_{k'} g_{k'j}e_{k'}$$

$$\psi(ge_i, ge_j) = \sum_{k,k'} g_{ki}g_{k'j}\psi(e_k, e_k') = \sum_{k,k'} g_{ki}g_{k'j}(J_{a,b})_{k,k'} = \sum_k g_{ki}(J_{a,b}g)_{k,j} = (\overline{g}^t J_{a,b}g)_{i,j}$$

so the condition $\psi(gx, gy) = \psi(\mu(g)x, y) = \mu(g)\psi(x, y)$ (noting that $\psi$ is $E$-bilinear),

$$\overline{g}^t J_{a,b} g = \mu(g) J_{a,b}.$$

Then,

$$G_1(\mathbb{Q}) = \{g \in GL_{a+b}(E) : \ \overline{g}^t J_{a,b} g = \mu(g) J_{a,b} \text{ for some } \mu(g) \in E^\times\}$$
$$G(\mathbb{Q}) = \{g \in GL_{a+b}(E) : \ \overline{g}^t J_{a,b} g = \mu(g) J_{a,b} \text{ for some } \mu(g) \in \mathbb{Q}^\times\}$$

We shall see next week that $GU_{a,b} := G$ is part of what is called the *unitary Shimura datum.*

*Example* 4 (Siegel Shimura Datum). Let $k = \mathbb{Q}$ and $(B, *) = (\mathbb{Q}, \mathrm{id})$. Then, $(B, *)$ is of type (C). Let $(V, \psi)$ be a symplectic $(B, *)$-module. Using notation as above, $F_0 = \mathbb{Q}$ in this case, so the subgroups $G_1, G$ are the same. From last week's talk, we see that

$$G(\mathbb{Q}) = \{g \in GL_{\mathbb{Q}}(\mathbb{Q}^{2n}) : \psi(gx, gy) = \mu(g)\psi(x, y) \text{ for some } \mu(g) \in \mathbb{Q}^\times\} = \mathrm{GSp}(\psi)(\mathbb{Q}),$$

so $G = \mathrm{GSp}(\psi)$, the group of symplectic similitudes with respect to $\psi$.

Now, let $V = \mathbb{Q}^{2n}$ and $J_n \in M_{2n}(\mathbb{Q})$ be

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Let $\psi : \mathbb{Q}^{2n} \times \mathbb{Q}^{2n} \to \mathbb{R}$ be the bilinear form corresponding to $J_n$ with respect to the standard basis of $\mathbb{Q}^{2n}$. Observe that $\psi$ is nondegenerate, skew-symmetric, bilinear form, so $(V, \psi)$ is a $(\mathbb{Q}^{2n}, \mathrm{id})$-symplectic module. Note that this is a generalization of the example Brian gave in his talk last week and

$$G_1 = G = \mathrm{GSp}(\psi) = \mathrm{GSp}_{2n}.$$

We saw last week that we have a corresponding Hodge structure

$$h_J : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \to GL(\mathbb{Q}^{2n}) \qquad (a + bi) \mapsto aI_{2n} + bJ_n$$

which gives rise to the Siegel Shimura datum $(\mathrm{GSp}_{2n}, \mathcal{H}^\pm)$.

*Example* 5. Let $B$ be a quaternion algebra over $\mathbb{Q}$ with $B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$, i.e. $B$ has a basis of the form $\{1, i, j, k\}$ over $\mathbb{Q}$ where the multiplication of the basis elements agrees with the multiplication in the quaternion group $Q_8$. There is then an involution $*$ on $B$ given by $1^* = 1, i^* = -i, j^* = -j, k^* = -k$. The pair $(B, *)$ is of type (BD). Let $V = B^{2n}$ and $\psi$ be the bilinear form with matrix $J_n$ (defined as before). In this case, the group $G$ will not be connected, hence we do not get a Shimura datum from it.

The three preceding examples gave examples where $(B, *)$ was of type (A), (C), and (BD) respectively. Of the three, (A) and (C) both gave rise to Shimura datum but (BD) did not. In general, types (A) and (C) are better behaved than type (BD). In particular, we have the following proposition.

**Proposition 3.2.** *Let $(B, *)$ be a semisimple $k$-algebra with involution. Let $(V, \psi)$ be a symplectic $(B, *)$-module. Let $F$ be the center of $B$ and $F_0$ the subalgebra of elements fixed by $*$. Assume that both $V$ and $B$ are free over $F$. Set*

$$n^2 = [B : F] \qquad g = [F_0 : k] \qquad mn = dim_F(V)$$

*When $(B, *)$ is of type (A):*

*The groups $G, G_1$ are connected and reductive (i.e. all representations are semisimple) and $G'$ is semisimple and simply connected. $F$ is a quadratic extension of $F_0$ and*

$$det(g) \cdot det(g)^* = \mu(g)^m \quad \text{for all } g \in G_1(k)$$

*If $k$ is algebraically closed, $G' \cong (Sl_m(k))^{[F_0:k]}$.*

*If $m = 2\ell$, then $(det^{-1} \cdot \mu^\ell, \mu)$ defines isomorphisms*

$$T_1 \cong \ker((\mathbb{G}_m)_{F/k} \xrightarrow{Nm} (\mathbb{G}_m)_{F_0/k}) \times (\mathbb{G}_m)_{F_0/k}$$

$$T \cong \ker((\mathbb{G}_m)_{F/k} \xrightarrow{Nm} (\mathbb{G}_m)_{F_0/k}) \times \mathbb{G}_m$$

*If $m = 2\ell - 1$, then $\kappa = det^{-1} \cdot \mu^{\ell}$ defines an isomorphism*

$$T_1 \cong (\mathbb{G}_m)_{F/k}$$

*and $\mu = \kappa \cdot \kappa^*$.*

*When $(B, *)$ is of type (C):*

*The groups $G, G_1$ are connected and reductive and $G'$ is semisimple and simply connected. The integer $m$ is even. Writing $m = 2\ell$,*

$$det(g) = \mu(g)^{\ell} \qquad \text{for all } g \in G_1(k)$$

*The map $\mu$ identifies $T_1$ with $(\mathbb{G}_m)_{F/k}$ and $T$ with $\mathbb{G}_m$.*

*If $k$ is algebraically closed, $G' \cong (Sp_m(k))^{[F:k]}$.*

See [1, Prop 8.7] for the proof.

The case (BD) is split into the case (B) where $m$ is odd and the case (D) where $m$ is odd. In the case (B), the groups are not part of a Shimura datum, so this case is not of interest to us. In case (D), the groups $G, G_1$ have $2^{[F:k]}$ connected components and their identity components are reductive. The group $G'$ is semisimple but not simply connected. If $k$ is algebraically closed, $G' \cong (\mathrm{SO}_m(k))^{[F:k]}$.

# 4 Algebras with positive involution

Let $(C, *)$ be a semisimple $\mathbb{R}$-algebra with involution. Let $V$ be a $C$-module that is finite-dimensional as a $\mathbb{R}$ vector space. We use the following terminology in this part of the talk.

**Definition 4.1.** A *hermitian form* on $V$ is a $C$-balanced symmetric $\mathbb{R}$-bilinear form $\psi : V \times V \to \mathbb{R}$. A hermitian form is said to be *positive-definite* if $\psi(v, v) > 0$ for all $v \in V \setminus \{0\}$.

**Proposition 4.2.** *Let $C$ be a semisimple algebra over $\mathbb{R}$. The following conditions on an involution $*$ of $C$ are equivalent:*

*(a) Some faithful $C$-module admits a positive-definite hermitian form*

*(b) Every $C$-module admits a positive-definite hermitian form*

*(c) $Tr_{C/\mathbb{R}}(c^*c) > 0$ for all $c \in C \setminus \{0\}$.*

See [1, Prop 8.10] for proof.

**Definition 4.3.** An involution $*$ on $C$ satisfying any of the equivalent conditions of Proposition 4.2 is said to be *positive.*

**Proposition 4.4.** *Let $(B, *)$ be a semisimple $\mathbb{R}$-algebra with positive involution and let $(V, \psi)$ be a symplectic $(B, *)$-module. Suppose $(B, *)$ is of type (A) or (C) and let $C$ be the centralizer of $B$ in $End_{\mathbb{R}}(V)$, There then exists a homomorphism of $\mathbb{R}$-algebras $h : \mathbb{C} \to C$ such that*

*(i) $h(\overline{z}) = h(z)^*$*

*(ii) The map $V \times V \to \mathbb{R}$ given by $(u, v) \mapsto \psi(u, h(i)v)$ is positive-definite and symmetric*

*Proof.* We give a very rough sketch of the proof. The proof is centered around two main ideas. The first is that any homomorphism of $\mathbb{R}$-algebras $h : \mathbb{C} \to C$ is completely determined by $h(i)$. To show that $h$ satisfying the conditions of the proposition exists then amounts to finding a element $J \in C$ such that

(i) $J^2 = -1$

(ii) $\psi(Ju, Jv) = \psi(u, v)$ and $\psi(v, Jv) > 0$ for $v \neq 0$

Note that these conditions coincide with the period relations for a Riemann form that Brian discussed in his talk last week when discussing complex abelian varieties.

To find such a $J$, we decompose $(B, *, V, \psi)$ into a product of systems as in Example 1 if $(B, *)$ is of type (A) and Example 2 if $(B, *)$ is of type (C). This allows us to reduce to the case where $(B, *, V, \psi)$ is of the form given in the two examples, from which $J$ can be explicitly defined. See [1, Prop 8.12] for the full proof. $\square$

Let $(B, *)$ be a semisimple $\mathbb{R}$-algebra with positive involution. Let $(V, \psi)$ be a symplectic $(B, *)$-module. Let $h : \mathbb{S} \to G_{\mathbb{R}}$ be a homomorphism such that $(V, h)$ is of type $\{(-1, 0), (0, -1)\}$ and $(u, v) \mapsto \psi(u, h(i)v)$ is symmetric and positive definite. Let $J = h(i)$. There are isomorphisms of complex vector spaces

$$(V, J) \cong V(\mathbb{C})/F_h^0 V(\mathbb{C}) \cong V^{-1,0}$$

that are compatible with the actions of $B$. Define $t(b)$ to be the trace of $b \in B$ on any one of the spaces above. Then,

$$t(b) = \mathrm{Tr}_{\mathbb{C}}(b|(V, J)) \quad b \in B, J = h(i)$$

**Proposition 4.5** (Deligne 1971)**.** *With notation as above,*

*(a) The $G'(\mathbb{R})$ conjugacy class of $h$ is uniquely determined by the map $t : B \to \mathbb{C}$*

*(b) In case (A), the isomorphism class of $(V, \psi)$ is determined by $t$. In cases (C) and (D), it is determined by $\dim_k(V)$.*

*(c) The centralizer of $h$ in $G(\mathbb{R})$ and $G_1(\mathbb{R})$ is connected.*

In the next talk, we will see how we can combine Propositions 4.4 and 4.5 to get a Shimura datum, called the simple PEL data of type (A) (resp. type (C)), out of $(B, *, V, \psi)$ when $(B, *)$ is of type (A) (resp. type (C)).

# References


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