The S-Unit Equation and Faltings' Theorem

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1 General Strategy

Let K be a number field, S a finite set of places including all places at infinity, $\mathcal{O}_S$ the ring of S-integers of K, $p \in \mathbb{Z}$ a rational prime not lying below any prime of S and v a place of K lying above p.

The setting is a sequence of morphisms

$$X \to Y' \to Y$$

where $\pi\colon Y' \to Y$ is finite étale, and $X \to Y'$ is a polarized abelian scheme. By expanding S as necessary, we construct their smooth integral models

$$\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$$

over $\mathcal{O}_S$. Moreover, by exanding S even more, we may assume that the relative de Rham cohomology sheaf $\mathscr{H}^1 := R^1 f_* \Omega^\bullet_{\mathcal{X}/\mathcal{Y}}$ is a locally free $\mathcal{O}_S$-sheaves. As we saw, the relative de Rham cohomology comes equipped with a Gauss–Manin connection on the generic fiber, which we can extend to $\mathcal{O}_S$, by expanding S as necessary to get

$$\nabla\colon \mathscr{H}^1 \to \mathscr{H}^1 \otimes \Omega^1_{\mathcal{Y}/\mathcal{O}_S}.$$

Given a point $y_0 \in \mathcal{Y}(\mathcal{O}_S)$, we will show that there are finitely many choices for y_0. Viewing it as a point of $Y(K)$, let X_y be the fiber of X above y. Then, we have a Galois representation

$$\rho_{y_0}\colon G_K \to \operatorname{End} H^1_{\acute{e}t}(X_{y,\overline{K}}, \mathbb{Q}_\ell).$$

In the case of abelian varieties, it was shown by Faltings that all ρ_{y_0} are semisimple. He also proved that there are only finitely many isomorphism classes of semisimple representations ρ_{y_0}. So, it suffices to show that there are a proper Zariski subset of $y \in \mathcal{Y}(\mathcal{O}_S)$ such that $y \equiv y_0 \pmod{v}$ and the local Galois representation $\rho_{y,v} = \rho_{y_0,v}$ agree.

Since $\mathcal{Y}$ is smooth, the representation $\rho_{y,v}$ is crystalline and thus the crystalline comparison theorem of Faltings gives a triple $(H^1_{dR}(X_y/K_v), \phi_v, \Phi_v(y))$, where ϕ_v is a Frobenius semi-linear automorphism of $H^1_{dR}(X_y/K_v)$, and $\Phi_v(y)$ a filtration on $H^1_{dR}(X_y/K_v)$ coming from the p-adic period morphism.

Then the first thing to show is the following:

Proposition 1. *The Gauss–Manin connection gives isomorphisms*

$$H^1_{dR}(X_{y_0}/\mathbb{C}) \cong H^1_{dR}(X_y/\mathbb{C})$$

for all $|y - y_0| < \varepsilon$ *and*

$$H^1_{dR}(X_{y_0}/K_v) \cong H^1_{dR}(X_y/K_v)$$

for all $y \equiv y_0 \pmod{v}$*. Moreover, in the* p*-adic case, this isomorphism is compatible with the action of* ϕ_v*.*

The Hodge filtration on $H^1_{dR}(X_y/\mathbb{C})$ gives a point in the flag variety $Gr(2g,g)_{\mathbb{C}}$ so we get a map from an open neighborhood of $Y(\mathbb{C}) \to Gr(2g,g)$. On the p-adic side, the filtration $\Phi_v(y)$ on $H^1_{dR}(X_y/K_v)$ gives a point on the flag variety $Gr(2g,g)$ now over K_v. Thus, we get a map from a p-adic open neighborhood of $Y(K_v) \to Gr(2g,g)$. The point is the Zariski closure of both images have the same dimension.

Proposition 2. *The Zariski closure of the complex and* p*-adic period morphisms have the same dimension.*

The idea now is to contrast the two. Namely for characteristic 0, we have the monodromy representation

$$\pi_1(Y(\mathbb{C}), y_0) \to \mathrm{GL}(H^1_{dR}(X_{y_0}/\mathbb{C}).$$

The goal is to show the Zariski closure is large using this.

Proposition 3. *Let* Γ *be the Zariski closure of* $\pi_1(Y(\mathbb{C}), y_0) \subset \mathrm{GL}(H^1_{dR})$*. The Zariski closure of the image of the period map inside* $Gr(2g,g)$ *contains the Zariski closure of* Γh_0*.*

Now the reason we chose $Y' \to Y$ finite étale at the beginning was to force the Zariski closure of $\pi_1(Y(\mathbb{C}), y_0)$ to be large.

Then to show that it is small, we look at the p-adic picture. As stated before, the isomorphism given by the Gauss–Manin connection must be compatible with the action of ϕ_v, and hence must lie in $Z(\phi_v)$. However, one can show that

$$\dim_{\mathbb{Q}_p} Z(\phi_v) = \dim_{K_v} Z(\phi_v^{[K_v:\mathbb{Q}_p]}).$$

Thus, we get that the dimension of the Zariski closure of the p-adic period morphism is at most $\dim_{K_v} Z(\phi_v^{[K_v:\mathbb{Q}_p]})$. The last thing is to conclude.

Proposition 4. *If* $\dim_{K_v} Z(\phi_v^{[K_v:\mathbb{Q}_p]}) < \dim_C \Gamma \cdot h_0$*, then the set of* $y \equiv y_0 \pmod{v}$ *is contained in a proper* K_v*-analytic subvariety of the residue disk of* $Y(K_v)$ *at* y_0*.*

2 Proofs of Propositions

Fix a local basis $\{v_1, \ldots, v_r\}$ for $\mathscr{H}^1$, the relative de Rham cohomology of $\mathcal{X}/\mathcal{Y}$, in a neighborhood of $y_0 \in \mathcal{Y}(\mathcal{O}_S)$. Then, we can write $\nabla v_i = \sum_j A_{ij} v_j$, where $A_{ij} \in \Omega^1_{\mathcal{Y}/\mathcal{O}_S}$. So, a section $\sum f_i v_i \in \mathscr{H}^1$ is flat exactly when

$$df_i = -\sum_j A_{ji} f_j.$$

The A_{ij} can be written in terms of $\sum a_{ijk}dz_k$, where z_k generate the kernel of $\mathcal{O}_{\mathcal{Y},\overline{y_0}} \to \mathcal{O}_{(v)}$ and form a system of parameters of $\mathcal{O}_{\mathcal{Y},\overline{y_0}}$, so that $\mathcal{O}_{\mathcal{Y},\overline{y_0}} \subset \mathcal{O}_{(v)}[[z_1,\dots,z_m]]$. So, we can solve the equation for f_i to get that actually $f_i \in K[[z_1,\dots,z_m]]$ and they are power series that are v-adically absolutely convergent whenever $|z_i|_v < |p|_v^{1/(p-1)}$, which proves Proposition 1.

The fact that the flat sections of the Gauss–Manin connection are given by power series in K allows us to compare the image of monodromy in the p-adic and complex period maps. Applying the following lemma to the Gass–Manin connection, we get a proof of Proposition 2.

Lemma 5. *Suppose $B_0,\dots,B_N \in K[[z_1,\dots,z_m]]$ are absolutely convergent power series with no common zero in both v-adic and complex disks $U_v, U_{\mathbb{C}}$, and let $B_v, B_{\mathbb{C}}$ be the maps from $U_v \to \mathbb{P}^N_{K_v}$, $U_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$. Then, there exists a K-scheme $Z \subset \mathbb{P}^N_K$ whose base changes give the Zariski closures of $B_v(U_v)$ and $B_{\mathbb{C}}(U_{\mathbb{C}})$.*

To show Proposition 3, the preimage of the Zariski closure of the image of the period map must be a complex-analytic subvariety of $Y(\mathbb{C})$ which contains an open neighborhood of y_0, and thus must be all of $Y(\mathbb{C})$.

3 S-Unit Equation

3.1 Reductions

prove the following theorem.

Theorem 6. *The set*

$$U = \{t \in \mathcal{O}_S^\times : 1 - t \in \mathcal{O}_S^\times\}$$

is finite.

Set

$$U_1 = \{t \in U : t \notin (\mathcal{O}_S^\times)^2\},$$

and let m be the largest power of 2 so that $\zeta_m \in K$. Then $U \subset U_1 \cup U_1^2 \cup \cdots U_1^m$, so it suffices to show that U_1 is finite. Note that for each $t \in U_1$, the field $K(\sqrt[m]{t})$ is a cyclic degree m extension with bounded relative discriminant (e.g. by 2^{m^2}), and thus there are finitely many fields. Thus, it suffices to show that the set

$$U_{1,L} = \{t \in U_1 : K(\sqrt[m]{t}) \cong L\}$$

is finite.

Now, pick v unramified in K and inert in L. Then, we have that $K_v(\sqrt[m]{t}) \cong L \otimes K_v$ is also a field. And, it suffices to show that

$$\{t \in U_{1,L} : t \equiv t_0 \pmod{v}\}$$

is finite.

Our family $\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$ in this case is the Legendre family where $\mathcal{Y} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $\mathcal{Y}' = \mathbb{P}^1 \setminus \{0, \mu_m, \infty\}$, where $\mathcal{Y}' \to \mathcal{Y}$ is the mth-power map. Then $\mathcal{X}$ is the Legendre family over $\mathcal{Y}'$, so that over $\mathcal{Y}$, it is the disjoint union of m curves given by $y^2 = x(x-1)(x - \zeta_m \sqrt[m]{t})$.

For each $t \in U_{1,L}$ equivalent to t_0, we get a representation ρ_t on the Tate module $T_\ell(E_t) = H^1_{\acute{e}t}(E_{t,\overline{L}}, \mathbb{Q}_\ell)$. The strategy from before tells us that it suffices to show that there are finitely many t whose pairs $(K_v(\sqrt[m]{t}), \rho_t|_{G_{K_v(\sqrt[m]{t})}})$ are isomorphic. Under p-adic Hodge theory, we get that this representation of ρ_t gives a triple

$$(H^1_{dR}(X_{t,K_v}/K_v), \phi_v, \Phi_v).$$

The upshot of using $\mathcal{Y}' \to \mathcal{Y}$ is that $H^1_{dR}(X_{t,K_v}/K_v)$ has the action of $K_v(\sqrt[m]{t})$, and it is a 2-dimensional vector space over $K_v(\sqrt[m]{t})$. The Gauss–Manin connection gives

$$H^1_{dR}(X_{t,K_v}/K_v) \cong H^1_{dR}(X_{t_0,K_v}/K_v)$$

as well as

$$K_v(\sqrt[m]{t}) \cong H^0_{dR}(X_{t,K_v}/K_v) \cong H^0_{dR}(X_{t_0,K_v}/K_v) \cong K_v(\sqrt[m]{t_0}).$$

Therefore, we can view the one-step Hodge filtration Φ_v as giving a $K_v(\sqrt[m]{t_0})$ line inside of $H^1_{dR}(X_{t_0,K_v}/K_v)$, which is the K_v-analytic period mapping.

$$\Phi\colon \{t \in K_v : t \equiv t_0 \pmod{v}\} \to \mathbb{P}^1_{K_v(\sqrt[m]{t_0})} \to Gr(2m, m)(K_v).$$

From the discussion before Proposition 4, we get that the dimension of the Zariski closure of Φ inside of $Gr(2m, m)(K_v)$ is at most the dimension of Z, the centralizer of $Frob_v^{[K_v:\mathbb{Q}_p]}$ in the $K_v(\sqrt[m]{t_0}]$-linear automorphisms of $H^1_{dR}(X_{t_0,K_v}/K_v)$. Some linear algebra shows that this dimension is at most

$$\dim_{K_v} Z \leq \left(\dim_{K_v(\sqrt[m]{t_0})} H^1_{dR}\right)^2 = 4.$$

To get a larger lower bound, we turn to the complex period mapping. As we saw before, we have that $X_{t_0,\mathbb{C}}$ is the disjoint union of m curves and so we get

$$H^1_{dR}(X_{t_0}/\mathbb{C}) = \oplus V_i,$$

where each $V_i = H^1_{dR}(X_{\sqrt[m]{t_0}}/\mathbb{C})$ is a 2-dimensional vector space.

Let Γ be the Zariski closure of $\pi_1(\mathbb{C} \setminus \{0, 1\}, t_0) \to \mathrm{GL}(\oplus V_i)$. Then by taking a loop around 0, we see that Γ includes the element that sends $V_i \mapsto V_{i+1}$. Moreover, $\Gamma \cap \prod \mathrm{SL}(V_i)$ is surjective onto each component because the statement is true for the usual Legendre family. Finally, by taking a small loop around 1, we see that Γ contains an element of the form $(u, 1, \ldots, 1)$ where u is unipotent. By combining all these facts, we can show that $\prod \mathrm{SL}(V_i) \subset \Gamma$. Thus, we see that the image of the period map contains $\prod \mathbb{P}(V_i)$ which has dimension m. Now we can choose $m > 4$ in the beginning to get finiteness.

Note that we can bypass Faltings' proof that all the representations are semisimple using p-adic methods as well. We will show that there are only finitely many t such that $t, 1 - t \in \mathcal{O}_v$ but $H^1_{\acute{e}t}(X_t, \mathbb{Q}_p)$ is not semisimple (and hence simple since 2 dimensional).

If the Tate module is reducible, then there is a one-dimensional subrepresentation W_t and thus by p-adic Hodge theory we get a filtered K_v vector space W_t^{dR}. Some linear algebra shows that $F^1(W_t^{dR}) = W_t^{dR}$ and so the Newton and Hodge polygons have the same shape of a slope of 0 and a slope of 1 meaning the Frobenius $Frob_v^{[K_v:\mathbb{Q}_p]}$ has distinct eigenvalues, meaning that W_t^{dR} is uniquely determined by the eigenspace for Frobenius. But for $t \equiv t_0 \pmod{v}$, we have that the position of the Hodge line $F^1 H^1_{dR}(X_t/K_v)$ varies in $H^1_{dR}(X_{t_0}/K_v)$ and hence there are only finitely many t that are reducible.

4 Faltings' Theorem

For Faltings' Theorem, we again consider a family $\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$ where $\mathcal{X}$ is given by the Kodaira–Parshin family. As we saw in the S-unit equation, the purpose of $\mathcal{Y}' \to \mathcal{Y}$ is to constrain the dimension of the centralizer as well as boost the dimension of monodromy action. In the S-unit equation, this was a finite étale map where all fibers had m preimages, but in general, that may no longer be the case. The notion of how much this fails is given by the size function. The fibers above a K point $y_0 \in \mathcal{Y}$ form a G_K-set and let S be this set. We define

$$size_v(S) := \frac{|\{s \in S : |Frob_v(s)| < 8\}|}{|S|}.$$

Then, they prove using the methods from before the following.

Proposition 7. *Let Y/K be a curve of genus $g \geq 2$ and let $X \to Y' \to Y$ be the Kodaira–Parshin family. Let d be the relative dimension of $X \to Y'$. Then*

$$\{y \in Y(K) : size_v(\pi^{-1}(y)) < \frac{1}{d+1}\}$$

is finite.

Now all that is left is to construct the appropriate Kodaira–Parshin family $X \to Y' \to Y$ that satisfies this condition. They choose $Y'_\ell \to Y$ the Hurwitz space for $Aff(\ell)$ and X_ℓ is the Prym of the universal curve. For each $y_0 \in Y(K)$, there is a G_K map from $\pi^{-1}(y_0)$ with conjugacy classes of surjections $\pi_1(Y \setminus \{y_0\}, \cdot) \to Aff(\ell)$, which gives a map

$$\pi^{-1}(y) \to M := H^1_{\acute{e}t}(Y_{\overline{K}}, \mathbb{Z}/(q-1)\mathbb{Z}).$$

Then note that if $E \to E'$ is a morphism of G_K-sets such that all fibers have the same cardinality, then $size_v(E) \leq size_v(E')$. Applying it to the above morphism, it suffices to show that $size_v(I)$, the image if $\pi^{-1}(y)$ satisfies the given bound. This is done by using the perfect Weil pairing on M

$$\langle \cdot, \cdot \rangle \colon M \times M \to \mu_{\ell-1}^\vee := \operatorname{Hom}(\mu_{\ell-1}, \mathbb{Z}/(\ell-1)\mathbb{Z}).$$

The Frobenius acts on the Weil pairing by giving $T \colon M \to M$ satisfying

$$\langle Tv_1, Tv_2 \rangle = q_v^{-1} \langle v_1, v_2 \rangle.$$

And, T-orbits of size less than 8 correspond to $\ker(T^i - 1)$. Choosing ℓ appororiately gives that $2\langle m_1, m_2 \rangle = 0$ for all $m_1, m_2 \in \ker(T^i - 1)$, which gives a bound on the size of $\ker(T^i - 1)$ as needed.