- Definition, fibre product, kernel, elementary properties.
- when $S = \operatorname{spec} k$, when G/k is a variety (finite type, separated, geometrically integral)

. Rigidity lemma, abelian variety.

(1): Yoneda embedding : S scheme

Let Sch/S be the category of schemes/S.

Then $h: Sch/S \longrightarrow Fun(Sch/S^{op}, set)$ fully faithful.

$$\begin{matrix} X \\ \downarrow \\ S \end{matrix} \longrightarrow h_X : \begin{matrix} T \\ \downarrow \\ S \end{matrix} \longrightarrow \operatorname{Hom}_S(T, X).$$

Q: when is $h_X(T)$ a group ?

A: when X is a group object in the category of Sch/S.

Def: $X \in Sch/S$ is a group scheme over S if it has the following structure: $(m, i, e$

$m: X \times_S X \longrightarrow X$, $i: X \longrightarrow X$, $e: S \longrightarrow X$

satisfying associativity, inverse, identity.

ask them

. $m \circ (id_X \times m) = m \circ (m \times id) : X \times_S X \times_S X \longrightarrow X$

. $m \circ (i \times id) = e \circ \pi : X \longrightarrow X$

. $m \circ (e \times_S id) = m \circ (id \times_S e) = id : X \longrightarrow X$

- Given G_1, G_2 group scheme $/S$,

  $\varphi \in \mathrm{Hom}_S(G_1, G_2)$ is a homomorphism of group scheme

  if $\varphi \circ e_{G_1} = e_{G_2}$, $m_{G_2} \circ (\varphi \times \varphi) = \varphi \circ m_{G_1}$,

  $\varphi \circ i_{G_1} = i_{G_2} \circ \varphi$

Proposition (Group scheme can be determined by its functor of point).

- The following data are eq:

$\left(\begin{array}{l} S \text{ Group scheme } G \\ \text{with } m, e, i, \end{array}\right)$ & $\left(\begin{array}{l} m_T, e_T, i_T \text{ on } G(T) \\ \text{functorial in } T \end{array}\right)$

Furthermore, if our base is a field, it's determined by the value on affine k-scheme.

Little exercise: how to recover $m: G\times G \to G$ via Functor of point?

$(h_G(G\times G) \ni pr_1^* id_G \cdot pr_2^* id_G)$

- $(\text{homomorphism } f: G_1 \to G_2) \Leftrightarrow \left(\begin{array}{l} f(T): G_1(T) \to G_2(T) \\ \text{group hom functorial in } T \end{array}\right)$

Now, let $s: G\times G \to G\times G$, $(x, y) \to (y, x)$. Then G is <u>commutative</u> if $m \circ s = m$

(Note: Not the same as saying it's abelian scheme/variety!!)

- Base change group scheme is group scheme:

for $\begin{matrix} T \\ \downarrow \\ S \end{matrix} \in Sch/S$, $\quad$ $Grpsch/S \longrightarrow Grpsch/T$

$$G \longrightarrow G \times_S T$$

$$m: G \times_S G \to G \longrightarrow m_T: G \times_S G \times_S T \to G \times_S T$$

$$(m_G \times id_T)$$

Moreover, if $T' \to T \in Sch/T$,

$$G(T') \cong G_T(T')$$

- open and closed subgroup.

Let $\iota: H \hookrightarrow G$ be an open/closed immersion of group scheme. then H is an open/closed subgroup of G if H is a grp scheme and ι is a group scheme hom

- Fibre product and kernel:

we know Fibre product exists in the category of scheme.

it turns out that fibre product of group scheme is group scheme.

First, direct product is $G_1 \times_S G_2$

$$m_{G_1 \times_S G_2} : G_1 \times_S G_2 \times_S G_1 \times_S G_2 \to (G_1 \times_S G_1) \times_S (G_2 \times_S G_2) \xrightarrow{m_{G_1} \times m_{G_2}} G_1 \times_S G_2$$

$i_{G_1 \times_S G_2} = i_{G_1} \times i_{G_2}$, $e_{G_1 \times G_1} = (e_{G_1}, e_{G_2})$

and $G_1 \times_G G_2 \hookrightarrow G_1 \times_S G_2$ is an immersion

$$\begin{tikzcd} \arrow[d] & \arrow[d] \\ G \arrow[r, "\Delta"] & G \times_S G \end{tikzcd}$$ because it's base change of local immersion.

and we only need $m_{G_1 \times_S G_2} : (G_1 \times_G G_2) \times_S (G_1 \times_G G_2) \to G_1 \times_S G_2$

$$\begin{tikzcd} \overset{x \quad y}{(G_1 \times_G G_2)} \times_S \overset{z \quad w}{(G_1 \times_G G_2)} \arrow[rr, "m_{G_1 \times G_2} \circ (\iota \times \iota)"] \arrow[d, "\varphi_1 \circ m_{G_1} \circ (pr_{G_1} \times pr_{G_1})"'] \arrow[dr] & & G_1 \times_S G_2 \quad (xz, yw) \arrow[d, "(\varphi_1 \times \varphi_2)"] \\ G \arrow[rr, "\Delta"] & G_1 \times_G G_2 \arrow[ur] \arrow[l] & G \times_S G \quad (\varphi_1 \circ m(x,z), \varphi_2 \circ m(y,w)) \end{tikzcd}$$

$\varphi_1(x) \cdot \varphi_1(z)$ $\qquad$ $\varphi_1(x) \cdot \varphi_1(z)$

$\varphi_2(y) \cdot \varphi_2(w) = (m_G(\varphi_1(x), \varphi_1(z)), m_G(\varphi_2(y), \varphi_2(w)))$

$$(\varphi_1 \times \varphi_2) \circ m_{G_1 \times G_2} = m_{G \times_S G} \circ (\varphi_1 \times \varphi_2 \times \varphi_1 \times \varphi_2)$$

$$m_{G \times_S G} : (G \times_S G) \times (G \times_S G) \to G \times_S G$$

OR: can check in functor of points:

$$G_1 \times_G G_2(T) = G_1(T) \times_{G(T)} G_2(T)$$

$$= (a,b) \in (G_1(T) \times G_2(T)) \underset{\text{subgroup of}}{\leq} G_1(T) \times G_2(T).$$

$$\varphi_1(a) = \varphi_2(b) \text{ in } G(T)$$

corollary: $\forall\, \varphi : G_1 \to G_2$, $\exists\, \mathrm{Ker}(\varphi) \in \mathrm{grpsch}/S$

with an immersion $i : \mathrm{Ker}(\varphi) \hookrightarrow G_1$

with universal property that $\forall\, \psi : G \to G_1$,

with $\varphi \circ \psi = e_{G_2} \circ \pi$, then $\exists !$ factorization $\tilde{\psi} : G$

$\to \mathrm{Ker}(\varphi)$, the functor of point is

$\mathrm{Ker}(\varphi)(T) = \mathrm{Ker}\left(G_1(T) \xrightarrow{\varphi(T)} G_2(T) \right)$

· proof: $\mathrm{Ker}(\varphi) := G_1 \times_{\varphi \to G_2 \leftarrow e_{G_2}} S \hookrightarrow G_1 \times_S S$

$$\begin{tikzcd}
G_1 \times_{\varphi \to G_2 \leftarrow e_{G_2}} S \arrow[r, hook] \arrow[d] & G_1 \times_S S \arrow[d] \\
G_2 \arrow[r] & G_2 \times_S G_2
\end{tikzcd}$$

· Seperated Group schemes:

$G \in \mathrm{grpsch}/S$ is separated if $\Delta : G \to G \times_S G$ is closed immersion.

Noticed that this is equivalent to $e : S \to G$ is a closed immersion.

Indeed, $G \xrightarrow{\Delta} G \times_S G$

$$\begin{tikzcd} {} \arrow[d] & {} \arrow[d, "(\mathrm{id}_G, \pi)"] \\ S \arrow[r, "e"] & G \end{tikzcd}$$

So e closed im $\Rightarrow$ Δ closed imm.

$$\begin{tikzcd} S \times_G G \arrow[r, "e"] \arrow[d] & S \times_S G \arrow[d] \\ G \arrow[r, "\Delta"] & G \times_S G \end{tikzcd}$$

So Δ closed im $\Rightarrow$ e closed imm

Corollary: If G/S separated, then $\ker(\varphi): G_1 \xrightarrow{\varphi} G$ is closed subgroup of G_1.

Since $\ker(\varphi) :=$

$$\begin{tikzcd} G_1 \times_G S \arrow[r, hook] \arrow[d] & G_1 \times_S S \arrow[d] \\ G \arrow[r, "\Delta"] & G \times_S G \end{tikzcd}$$

- Examples of group scheme.
  - Constant group scheme: $\forall$ M a group,
    $\underline{M}_S$ the constant group scheme attached to M over S
    is defined as follows:

as a scheme $\underline{M}_S = \coprod_{m \in M} S_m$

$m: \underline{M}_S \times \underline{M}_S \to \underline{M}_S$ $\qquad\qquad$ $e: S \hookrightarrow S_e \hookrightarrow \coprod S_m.$

$S_m \times S_n \xrightarrow{\pi_1} S \xrightarrow{\iota} S_{\mu(m,n)}.$

$i: \underline{M}_S \to \underline{M}_S$

$S_m \to S_{i(m)}$

Functor of points:

$\underline{M}_S(T) = \mathrm{Hom}_S(T, \coprod_{m \in M} S_m)$

given $f: T \to \coprod_{m \in M} S_m$, $f^{-1}(S_m)$ is open

and $\bigcup f^{-1}(S_m)$ disjoint open cover T.

so each connected component only goes inside one S_m.

$$\mathrm{Hom}_S(\coprod T_i, \coprod S_m)$$

$$= \prod_i \mathrm{Hom}_S(T_i, \coprod S_m)$$

$$= \prod_i (m_i)$$

$$= \prod_{\text{conn comp of } T} M$$

Now since M is a group, we can defined the group structure as $(f_{m_i})_i \cdot (f_{n_i})_i = (f_{m_i n_i})_i$

Exercise: If $S = \mathrm{Spec} R$:

show that: $\underline{M}_S = \mathrm{spec}(R^M)$

where $R^M = R[e_m] = \prod_{m \in M} R$

e_m are idempotent in R^M indexed by $m \in M$.

$$c: R^M \to R^M \otimes R^M$$
$$e_m \to \sum_{\sigma\tau = m} e_\sigma \otimes e_\tau$$

$$e: R^M \to R$$
$$e_m \to 0, \quad e_e \to 1$$

verify this is compatible with the group structure defined as above. i.e.,

given any R-algebra A, if A has no idempotents,

$\mathrm{Hom}_R(R^M, A) = \underline{M}_R(A) \cong M \;:\; m \in M \longrightarrow f_m : \begin{array}{l} e_m \to 1 \\ \text{others} \to 0 \end{array}$

Then the group structure on the functor of point is
$$\phi \to \phi \circ c$$

$$\underline{M}_R(A) \times \underline{M}_R(A) \longrightarrow \underline{M}_R(A)$$

$$\Vert$$

$$\underline{M}_R(A \otimes A)$$

suppose $\phi = f_\gamma \otimes f_{\gamma'}$ show: $\phi \circ C = f_{m(\gamma, \gamma')}$

- Affine group sch/k (automatically separated since $e: \operatorname{spec} k \to A$ is a closed immersion).

G_m, G_a,

$$G_m[n] = \ker(G_m \xrightarrow{[n]} G_m)$$

①: $G_m \xrightarrow{[n]} G_m$ $\qquad \ker = k[T, \tfrac{1}{T}] \otimes_{k[T, \frac{1}{T}]} k$

$T \to T^n$ $\qquad$ (with $T \to T^n$ on the left factor and $T = 1$ on the right)

$$= k[X, \tfrac{1}{X}] \otimes_{k[Y, \frac{1}{Y}]} k[Y, \tfrac{1}{Y}]/(Y-1)$$

$$Y \to X^n$$

$$= k[X, \tfrac{1}{X}]/(X^n - 1)$$

$$k[X, \tfrac{1}{X}]/(X-1)^p = k[X]/(X-1)^p$$

②: $\operatorname{char}(k) = p$, then G_a.

α_{p^n}: $\operatorname{spec} k[X]/X^{p^n}$ with additive structure

③: $\underline{\mathbb{Z}/p\mathbb{Z}}_k = \operatorname{Spec} \prod_{i \in \mathbb{Z}/p\mathbb{Z}} k_i = \operatorname{spec} k[e_i]$

show: if $p \neq char(k)$ then $\exists\, k'/k$ such that

$$k'[x]/_{(x^p-1)} \cong \prod k_i$$

- G^0, G_{red}, every group scheme locally of finite type over $char(k)=0$ is reduced.

- G^0: let G be a group scheme over k. localy of finite type then $e: \mathrm{Spec}\, k \to G$ is automatically an closed immersion, so G is separated. and locally Noetherian. so locally connected $\Rightarrow$ connected compunets of G are open.

  Then G^0 is the connected component of G containing the identity section, the identity component)

- G^0 is an open subgroup:

$$\begin{tikzcd} m: G^0 \times G^0 \arrow[r] \arrow[dr] & G \times G \arrow[r, "m"] & G \\ & G^0 \arrow[ur] & \end{tikzcd}$$

since imge of connected set is connected.

$m(G^0 \times G^0)$ is in the connected component containing e.

~~• If G is also Noetherian topolgoical space (for example, when G is a group variety)~~

~~Then G^0 is also the irre component containing e.~~

~~indeed, G is Noet $\Rightarrow$ finitely many irr component $\Rightarrow$ irr component are open $\Rightarrow$ G^0 can not intersect more than 1 irr comp $\Rightarrow$ G^0 = irr comp at e.~~

Example: $G = \mathbb{G}_m$: $G = G^0$

$G = \underline{\mathbb{Z}/p\mathbb{Z}}_k$: $G^0 = \operatorname{spec} k e = \{*\}$.

$G = \mu_{p_{\mathbb{Q}}} = \operatorname{spec} \mathbb{Q}[x]/x^p-1 = \mathbb{Q}[x]/(x-1) \otimes \mathbb{Q}[y]/\Phi_p(y)$

Then $\{\mu_p\}^0$ is $\mathbb{Q}[x]/(x-1)$ $\quad \begin{array}{c}\downarrow \pi \uparrow e \\ \operatorname{spec}\mathbb{Q}\end{array}$

prop: G/k l.f.t.

• G^0 is open and closed, and irreducible, and geo. irr and finite type $/k$, in particular, $(G^0)_K = (G_K)^0$

• TFAE: a) G_K is reduced for some $K \supseteq k$ perfect field a2): $\mathcal{O}_{G,e} \otimes_k K$ is reduced for some $K \geq k$

b1): G is smooth over k

b) 2 : G^0 is smooth over k

b3): G^0 is smooth over k at e.

(Smooth = flat + regular fibre)

So, smooth and reduced can be seen from the G^0

Corollary (Cartier): If G/k l.f.t and $\mathrm{char}(k)=0$, then G is reduced, hence smooth over k

· When G is not reduced, we denote the closed subscheme $G_{red} \overset{i}{\hookrightarrow} G$. it's not a group, since $G_{red} \times G_{red}$ might not be in G_{red}.

But if k is perfect, then $G_{red} \times G_{red}$ is reduced, the $m: G_{red} \times G_{red} \longrightarrow G$ factors thr G_{red}, so it's a closed subgroup of G.

e.g. $k[x]/(x^p-1) \rightsquigarrow G_{red} = k[x]/(x-1) = (*)$.

$G^0 = k[x]/(x-1)^p$

when k is perfect, $0 \to G^0 \to G \to G_{red} \to 0$

$G_{red} = G^{\acute{e}t}$ and we have section $G_{red} \hookrightarrow G \twoheadrightarrow G^{\acute{e}t}$
$$\begin{array}{c} G_{red} \\ \text{\rotatebox{90}{$\cong$}} \\ G^{\acute{e}t} \end{array}$$

Now, let's focus more specifically on grp variety

Variety: seperated, finite type, geometrically integral.

- Group variety are non-singular.

  (Non-sing locus is always dense, so $\forall x$, can find U_x smooth, and the translation cover G.

- G/k has trivial canonical bundle

  $\Omega^{\vee}_{X/k} \cong T_{X,e} \otimes_k \mathcal{O}_X$. It's trivial.

  idea: $\forall \tau \in T_{X,e}$, can associate $\xi(\tau) \in H^0(X, T_{X/k})$

  such that $\xi(\tau)_x \bmod m_x = t^{\#}_{x,e}(\tau)$

  in particular, $\{\tau_1, \dots, \tau_n\}$ linearly indep in $T_{X,e}$,

  we have a global trivialization $\{\xi(\tau_i)\}$

  Concretely, $\alpha : T_{X,e} \otimes k \mathcal{O}_X \longrightarrow \Gamma(X, \Omega^{\vee}_{X/k})$

  $\tau \longrightarrow \xi(\tau)$

  $\alpha_x \bmod m_x$ is surjection as $k(x)$ v.s.

$\alpha_x : T_{X,e} \otimes_k \mathcal{O}_{X,x} \longrightarrow T_{X/k,x}$ surjection

of $\mathcal{O}_{X,x}$ module by Nakayama

free + same rank $\Rightarrow$ iso.

- Elliptic curve E/k: the global section of $\Omega_{X/k}$ is the $\frac{dx}{y}$ invariant differential.

Abelian variety:

Defn: AV is a complete group variety.

- Rigidity lemma: if X is complete. Y, Z variety. suppose for some $y \in Y(k)$, $X \times \{y\} \xrightarrow{\varphi|_{X\times\{y\}}} Z$ is constant, then $X \times Y \to Y \to Z$ f factors thru projection.. conseq, if $f: X \times Y \to Z$ has $X \times \{y\}$ constant and $\{x\} \times Y$ constant, then f is constant

The meat is:

- $\underset{\text{complete}}{\varphi : X} \longrightarrow \underset{\text{affine}}{U}$ is a constant

- $\varphi : X \times Y \longrightarrow Y$ is closed morphism.

- X is variety, so morphism agree on non-empty open agree on the whole thing.

proof: (might skip). we show that $f: X\times\{p\} \to Z$ is a constant, for any $p \in Y(k)$. That is, $f = g \circ Pr_Y$ for $g(y) = f(x_0, y)$

Since Y is algebraic variety, suffices to show that the p that satisfies this property is an non-empty open set in Y, then $f = g \circ Pr_Y$ on $X \times Q$
Q

which is a dense open, so $f = g \circ Pr_Y$ everywhere.

Now, say $f: X\times\{y\} \to Z$ has image z and pick an affine open U_z containing z.

then $f^{-1}(Z - U_z) \subseteq X \times Y$ is closed, so

$Pr_Y : X \times Y \to Y$ is proper, $Pr_Y(f^{-1}(Z-U_z)) \subseteq Y$.

is closed, say it's V. Let $Q = Y \setminus V$.

Then for $p \in Q(k)$, $\forall x \in X(k)$, $f(x,p) \notin Z - U_z$

so $f(x,p) \in U_z$. But $f: X\times\{p\} \to Z$ factors thru U_z, since X complete, U_z affine, $f(X\times\{p\}) = \{x\}$.

therefore, $\forall p \in Q(k)$, $f = g \circ Pr_Y$ on $X \times \{p\}$. So so $=$ on a open dense $\Rightarrow$ $=$ everywhere.

• Corollary: Every $f: X \to Y$ of abelian variety

is $f = t_{f(e_X)} \circ h$ where h is grp hom.

to check grp hom, suffice if $f(e_X) = e_Y$.

$\operatorname{spec} k \xrightarrow{a} X$.

$t_a : X \times_k k \xrightarrow{(id_X, a)} X \times_k X \xrightarrow{m} X$.

pf: we consider $h = t_{-f(e_X)} \circ f$

show: $m_Y \circ (h \times h) = h \circ m_X \quad X \times X \to Y$

same as show:

$$X \times X \xrightarrow{(m_Y \circ (h \times h),\, h \circ m_X)} Y \times Y \longrightarrow Y \times Y \longrightarrow Y$$

$m_Y \circ (h \times h),\ i \circ h \circ m_X$

$\varphi = m_Y \circ (m_Y \circ (h \times h),\ i \circ h \circ m_X)$

show φ is constant with value e_Y

$\Leftrightarrow$ $\varphi|_{X \times \{e_X\}}$ and $\varphi|_{\{e_X\} \times X}$ is.

- Commutative :

  $m : X \times X \longrightarrow X$ agree with $s \circ m$

Check

$\Rightarrow m_X(i \circ m, s \circ m)$ is constant on $X \times \{e_X\}$ & $\{e_X\} \times X$.