Winter 2024 Caltech Number Theory Learning Seminar: Introduction

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1 Introduction

The Andre Oort conjecture can be seen as a generalization of the Manin Mumford conjecture to other families of varieties. The general form of the conjecture is as follows.

Conjecture 1 (Andre-Oort Conjecture)**.** *Let S be a variety and $V \subset S$ be a subvariety. If V has a Zariski dense subset of special points, then V is special.*

In order to make this more precise, we look at what these special subvarieties are for different cases of S.

S	Special points	Special Subvarieties	Theorem
Abelian Variety	Torsion Points	Translations of Abelian Subvarieties by Torsion Points	Manin-Mumford (Raynaud '83) and Mordell-Lang (Hindry '83, Faltings '91, McQuillan '95) Conjecture
Torus = $(\mathbb{C}^\times)^n$	Roots of Unity	Varieties cut out by $\prod z_i^{e_i} = \omega \in \mu_\infty$	Lang Conjecture (Laurent '84)
Powers of the Modular Curve $= Y(1)^n$	Elliptic Curves with CM	Varieties cut out by modular polynomials or coordinate equal to CM point	Andre-Oort for $\mathbb{C}^n$ (Pila '11)
Moduli Space for Abelian Varieties A_g	Abelian varieties with CM	Varieties parametrizing AVs with non-trivial endomorphisms	Andre-Oort for A_g (Tsimerman '18)
Shimura Varieties	Shimura Subvarieties of Dimension 0	Shimura Subvarieties	Andre-Oort for Shimura Varieties (Pila-Shankar-Tsimerman '2?)

Pila used o-minimality to prove Andre-Oort for the modular curve, and also to give an alternative proof for the Manin-Mumford conjecture. This powerful technique was used by Tsimerman to prove the Andre-Oort conjecture for more generalized settings. Most of the elements of the proof are identical in all the settings. However, the key difference between each of these proofs is how to show that the Galois orbits of a special point grow polynomially with respect to their height. We will briefly sketch the proof for the A_g case. To prove the Andre-Oort conjecture for general Shimura varieties, we will use the same ingredients as the A_g case except we will need a different technique to deal with Galois orbits.

2 Andre-Oort for A_g

Theorem 2. *Let $V \subset \mathcal{A}_g$ be a subvariety that is not special. Then V does not contain a Zariski dense set of special points.*

We will prove this by contradiction. Suppose that V does contain a Zariski dense set of special points.

2.1 O-minimality

First recall that $\mathbb{H}_g = \{X + iY \in Sym_{2g}(\mathbb{C}) : Y \succ 0\}$ and setting $\Gamma_g := Sp_{2g}(\mathbb{Z})$, there is a covering map $\pi : \mathbb{H}_g \to \mathcal{A}_g = \Gamma_g \setminus \mathbb{H}_g$. Let $F_g \subset \mathbb{H}_g$ be a fundamental domain for the action of Γ_g so that π is a homeomorphism from F_g to a Zariski open subset of $\mathcal{A}_g$.

Theorem 3 (Klinger-Ullmo-Yafaev '13)**.** *The fundamental domain F_g is definable and $\pi|_{F_g} : F_g \to \mathcal{A}_g$ is a definable function.*

Algebraic varieties are defineable, which means that $\pi^{-1}(V) \cap F_g$ is a definable analytic variety.

Theorem 4 (Pila-Wilkie '09)**.** *Let $X \subset \mathbb{R}^n$ be a defineable set and let X^{alg} denote the union of all positive dimensional algebraic sets in X. Then the number of points of $X \setminus X^{alg}$ with height at most N grows sub-polynomially in N.*

We will take our defineable set $X = \pi^{-1}(V) \cap F_g$ and then show that the number of points of X with height at most N grows polynomially in N, proving that $X^{alg} \neq \varnothing$.

2.2 Polynomial Subbounds

Let $x \in V$ be a special point and consider its preimage $y \in \pi^{-1}(x) \cap F_g$. Suppose that the variety V is defined over a number field L. Then the Galois orbit $Gal(\bar{\mathbb{Q}}/L) \cdot x$ are all also in V and taking their preimage, we get $\#Gal(\bar{\mathbb{Q}}/L) \cdot x$ points of the same height all within X. If we can show that the number of points in the Galois orbit grows polynomially in the height of the point, then we will have achieved our objective.

We will do this in two steps. The first is to give a lower bound in terms of the data of this special point.

Definition 5. A **CM field extension** is an extension E/F of number fields so that $F/\mathbb{Q}$ is a totally real field and E/F is a quadratic totally imaginary extension. We say E is a **CM field** and F is its **totally real subfield**. Let $g = [F : \mathbb{Q}]$.

A CM-type is a subset $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ such that $\Phi \sqcup \bar{\Phi} = \mathrm{Hom}(E, \mathbb{C})$. One way to think of this is above every real place of F, we choose an embedding of E into $\mathbb{C}$. Yet another way to define a CM-type is a choice of isomorphism $\Phi : E \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{C}^g$.

We say that an abelian variety A has complex multiplication of type (E, Φ) if there is an embedding $E \to End^0(A)$ and there is an isomorphism $\Phi : E \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{C}^g \cong Lie(A)$ that respects E action.

Remark. If A has complex multiplication of type (E, Φ), then complex analytically we can write $A(\mathbb{C}) = \mathbb{C}^g/\Lambda = (E \otimes_{\mathbb{Q}} \mathbb{R})/\mathcal{O}$ where $\mathcal{O} \subset \mathcal{O}_E$ is an order of E.

For a special point $x \in V$, let A_x denote the abelian variety it parametrizes and let $R_x = Z(End(A_x))$ be the center of the endomorphism ring, which will be an order in a CM field.

Theorem 6 (Tsimerman '18). *There are constants $C, n > 0$ such that*

$$|Disc(R_x)| \leq C|Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \cdot x|^n.$$

This theorem changes the problem from thinking about Galois orbits to now thinking about discriminants of CM fields, and how they correspond to heights. Colmez proved that if two abelian varieties both have complex multiplication of the same type (E, Φ), then they have the same height. Thus, we want to now get a bound of $h(x) = h(A_x) = h(E, \Phi)$ in terms of $Disc(E)$.

Theorem 7 (Bost '96). *There exists a constant c_g depending on g alone so that if A is an abelian variety of dimension g, then $h(A) \geq c_g$.*

Theorem 8 (Yuan-Zhang, Andreatta-Goren-Howard-Madapusi-Pera '15).

$$\frac{1}{2^g}\sum_{\Phi} h(E, \Phi) = \frac{-1}{2}\frac{L'(0, \chi)}{L(0, \chi)} - \frac{1}{4}\log\frac{Disc(E)}{Disc(F)},$$

where $\chi : \mathbb{A}_F^\times \to \{\pm 1\}$ is the character associated to the extension E/F.

Corollary 9. *There are constants $C, n > 0$ so that $h(x) \leq C|Disc(E)|^n$.*

Proof. The theorem of Bost tells us that

$$h(E, \Phi) \leq -(2^g - 1)c_g + \sum_{\Phi'} h(E, \Phi').$$

Then the averaged Colmez conjecture bounds $\sum_{\Phi'} h(E, \Phi')$ in terms of $L(0, \chi)$ which we can bound using Brauer-Siegel in terms of $|Disc(E)|$. □

Corollary 10. *There are constants $C, \varepsilon > 0$ so that the number of points of X of height less than N grows as $C \cdot N^\varepsilon$.*

Proof. Combining our results, we have that $h(x) \leq C'|Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \cdot x|^n$. □

2.3 Finishing Up

Now we know that $X = \pi^{-1}(V) \cap F_g$ has an algebraic component. But π is a transcendental function and so this should be unlikely. In fact, we have a theorem that says precisely that.

Theorem 11 (hyperbolic Ax Lindemann Pila-Tsimerman '14). *Let $V \subset \mathcal{A}_g$ be an irreducible algebraic variety and suppose that $W \subset \pi^{-1}(V)$ is an irreducible algebraic subset of $\mathbb{H}_g$. Then there exists a weakly special subvariety $S \subset V$ such that $W \subset \pi^{-1}(S)$.*

Thus, we get that all but finitely many special points of V lie within X^{alg}, which correspond to weakly special subvarieties of V. Since the special points are Zariski dense, one of these weakly special subvarieties must be V, showing that V is special.