Winter 2024 Caltech Number Theory Learning Seminar: André–Oort Conjecture

(1) **1/4 Introduction and Organization** Give an introduction to the André–Oort conjecture and how it appears in different settings as other theorems (Raynaud, Manin–Mumford, etc.). List the three main ingredients that go into the Pila–Zannier strategy and give a sketch of how these three ideas get put together to prove the theorem.

(2) **1/11 Introduction to Heights** Begin with the definition of heights on projective space and the Northcott property. Then introduce heights with respect to line bundles and Weil's height machinery. Give the definition of metrized line bundles and the height of a point with respect to a line bundle. Show the compatibility of this height with the original height on projective space. Finally, give the definition of the Faltings height.

(3) **1/18 Introduction to o-minimality** Introduce structures over $\mathbb{R}$, definable sets and maps, and basic properties of o-minimal structures. Introduce $\mathbb{R}_{alg}, \mathbb{R}_{exp}, \mathbb{R}_{an}, \mathbb{R}_{an,exp}$. State and explain the Pila–Wilkie counting theorem (no proof). Cover Section 3, and $4.1 - 4.6$ of [Sca17]. Cover Section 5 of [KUY18].

(4) **1/25 Ax–Schanuel for the exponential** Give the motivation of Ax–Schanuel and Ax–Lindemann–Weierstrass theorems and bi-algebraic geometry from Section 4 of [KUY18] (ignore the Shimura variety setting for now). Then give a proof of the Ax–Schanuel Theorem and Ax–Lindemann–Weierstrass theorem for the exponential function following [Tsi15].

(5) **2/1 No Meeting** AIM Conference

(6) **2/8 Manin–Mumford Conjecture for Powers of an Elliptic Curve** Give the proof of the Manin–Mumford Conjecture following the original

paper of Pila and Zannier [PZ08]. Emphasize the main ingredients of having definability, functional transcendence, and lower bounds for sizes of Galois orbits.

(7) **2/15 André–Oort Conjecture for Powers of the Modular Curve** Give a proof of the André–Oort Conjecture for products of the modular curve following Sections $10, 11, 12$ of [Pil11]. Review the Pila–Wilkie theorem in this setting. State (no need to prove) the necessary Ax–Schanuel results from Section 6.

(8) **2/22 Introduction to Shimura Varieties** Give an introduction to Shimura varieties to non-experts. Avoid too many technicalities and instead give many examples and definitions. Explain how the modular curve $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ can be viewed as a double quotient $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})/SO(2)$ and show how the definition of Shimura varieties follows from this. Give the example of Siegel modular space. Cover sections $5, 6$ of [Mil05]. Another source is Section 3.1 of [Lan17].

(9) **2/29 Canonical Models of Shimura Varieties** Continue the introduction to Shimura varieties by defining their canonical models and proving their existence. This should cover Sections $12 - 14$ of [Mil05].

(10) **3/7 Ax–Lindemann for Shimura Varieties** Give an idea of the proof of Ax–Lindemann–Weierstrass for Shimura varieties following [KUY16]. This can follow the summary given in Section 7 of [KUY18]. Use this to prove Theorem 8.1 of [KUY18].

(11) **3/14 André–Oort Conjecture for $\mathcal{A}_g$** State Conjecture 2 of [BSY23] and then define Faltings heights and prove Conjecture 2 in the setting of $\mathcal{A}_g$ following [Tsi18] using the averaged Colmez conjecture. Prove how the André–Oort conjecture follows from this. This is also the content of Theorem 8.2 and Section 9 of [KUY18].

References

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