Overview and Learning Objectives

Overview

This chapter covers the basic elements of portfolio theory and asset pricing. In Chapter 10, we study historic returns and statistics for an individual security. This chapter uses a single-period 'state-of-the-world' framework to study statistics for individual securities and portfolios, specifically the effects of covariance on diversification and portfolio risk. This chapter continues to develop an asset pricing model that relates the required return to the risk of assets. The Capital Asset Pricing Model (CAPM) allows us to estimate the (ex ante) appropriate discount rate given a project's risk in capital budgeting analysis and valuation of financial securities.

Note that this chapter utilizes basic statistical concepts and measures that you have learned in one of the admission prerequisite requirement – basic statistics.

Learning Objectives

After reading course materials on this chapter, students should be able to:

- Compute the measures for different types of return (expected, possible and required) and risk (total, systematic and unsystematic) for both individual securities and portfolios.
- Explain the difference between the stand-alone and well-diversified portfolio environments and choose the corresponding type of risk in the analysis.
- Explain the relation between the diversification effect and how securities interrelate with one another, and apply the diversification concept in portfolio management.
- Explain the Separation Theorem.
- Apply the CAPM to determine the appropriate discount rate in valuation of investment projects and financial securities, and identify mispriced securities.

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Individual Securities & their Return & Risk Measures (Ref: Sections 11.1 & 11.2)

Expected Rate of Return (Slide)

The return on a risky security that investors expect to earn over the next period.

- The probability-weighted average of possible returns on a risky security.
  Expected Return of a Security =

$$E(R) = \sum_{s=1}^{K} p_s R_s$$

where K is the number of states of the economy (or possible returns).
p_s is the probability of the realization of the possible return if state s occurs.
R_s is the possible return on the risky security if state s occurs.

Variance and Standard Deviation – Measures of TOTAL Risk (Slide)

- They measure the dispersion of possible returns around the expected return, i.e., the variability of returns.
- The probability-weighted average of squared deviations of possible returns from the expected return.
  Variance of a Single Security,

$$\sigma^2 = \sum_{s=1}^{K} p_s (R_s - E(R))^2$$

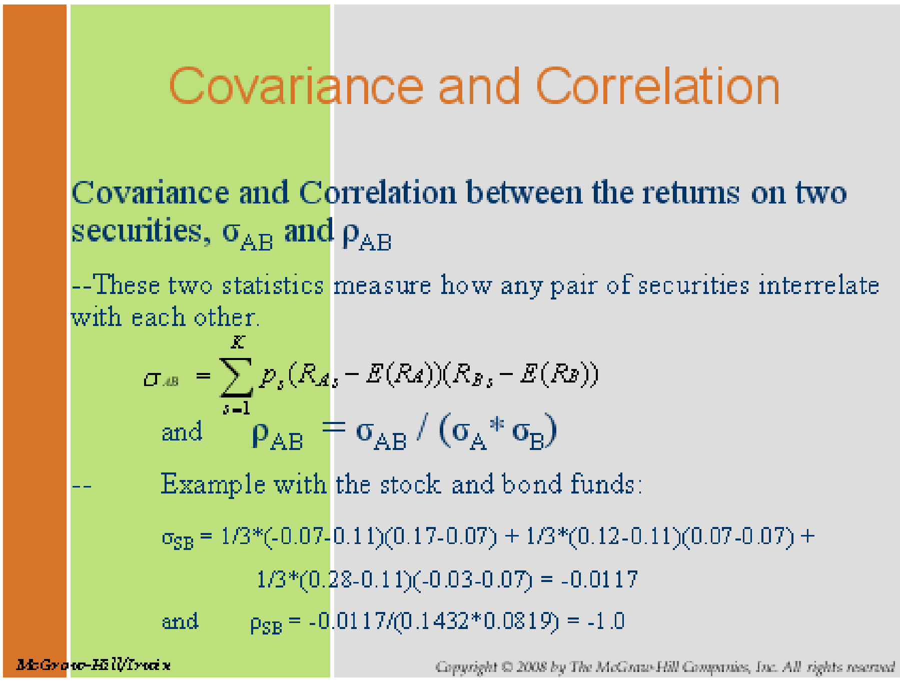
Standard Deviation of a Single Security, σ = √Variance

In a stand-alone investment environment, i.e., investors invest in only one security, the appropriate type of risk to be considered is the total risk, and variance and standard deviation are the appropriate risk measures.

Covariance and Correlation Coefficient

Consider the following two risky asset world. There is a 1/3 chance of each state of the economy, and the only assets are a stock fund and a bond fund.

		Rate of Return	
Scenario	Probability	Stock Fund	Bond Fund
Recession	33.3%	-7%	17%
Normal	33.3%	12%	7%
Boom	33.3%	28%	-3%



Both statistics measure how any pair of securities interrelate with each other, and as such they are proxy measures of systematic risk.

Covariance between two securities =

$$\sigma_{AB} = \sum_{s=1}^{K} p_s (R_{A_s} - E(R_A))(R_{B_s} - E(R_B))$$

The sign of the covariance indicates the direction of the co-movement between the two securities.

- A positive value suggests that when the return on one security is above (or below) its expected level, so is the other security.
- A negative value suggests that the two securities tend to move in opposite directions relative to their respective expected levels.

Correlation Coefficient Between Two Securities = $\rho_{AB} = \sigma_{AB} / (\sigma_A * \sigma_B)$

- It is the standardized version of covariance.
- Similar to covariance, the sign of correlation coefficient is informative of the direction of the co-movement between the two securities.
- Unlike covariance, the magnitude of correlation coefficient indicates the degree of interrelation within a pair of securities when comparing among different pairs of securities.
  - The greater the magnitude the correlation coefficient, the closer the interrelation between the pair of securities. For example, $\rho_{AB} = 0.3$ and $\rho_{CD} = -0.8$ suggest that there is a stronger interrelation between the pair of C and D than that between the pair of A and B.

The correlation between two securities falls into one of the following cases:

- Positively correlated
  - $0 < \rho_{AB} < 1$
- Perfectly Positively correlated
  - $\rho_{AB} = 1$
- Negatively correlated
  - $-1 < \rho_{AB} < 0$
- Perfectly Negatively correlated
  - $\rho_{AB} = -1$

- Uncorrelated
  - $\rho_{AB} = 0$

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Numerical Illustration

We use the following example to illustrate calculation of return and risk statistics depending on various "states of the world":

Outcomes	Probability (p_s)	R_A	R_B
Boom	0.25	20%	5%
Normal	0.50	10%	10%
Bust	0.25	0%	15%

$E(R_A) = 0.25 * 0.20 + 0.50 * 0.10 + 0.25 * 0.0 = 0.10 = 10\%$
$E(R_B) = 0.25 * 0.05 + 0.50 * 0.10 + 0.25 * 0.15 = 0.10 = 10\%$

$\sigma_A^2 = 0.25*(0.20-0.10)^2 + 0.50*(0.10-0.10)^2 + 0.25*(0.0-0.10)^2 = 0.005$
$\sigma_B^2 = 0.25*(0.05-0.10)^2 + 0.50*(0.10-0.10)^2 + 0.25*(0.15-0.10)^2 = 0.00125$

$\sigma_A = (0.005)^{(1/2)} = 0.07071 = 7.071\%$
$\sigma_B = (0.00125)^{(1/2)} = 0.03536 = 3.536\%$

$\sigma_{AB} = 0.25*[(0.20-0.10)*(0.05-0.10)] + 0.50*[(0.10-0.10)*(0.10-0.10)] + 0.25*[(0.0-0.10)*(0.15-0.10)]$
$= -0.0025$
$\rho_{AB} = -0.0025 / (0.07071 * 0.03536) = -1.0$

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The Return and Risk for Portfolios (Ref: Section 11.3)

Portfolio Weight and Rate of Return on a Porfolio

NOTE - Both "X" and "W" are used as notation for portfolio weights in this chapter. "X" is used in the images of selected slides and equations that were carried from previous editions of the text by the technical supoprt team in course revision.

Portfolio weight of security i in Portfolio P, W_i

The proportion of a portfolio's total value, V_p, that is in security i.

i.e., $W_i = V_i / V_p$ where $V_p = \Sigma^N_{i=1} V_i$

where V_i is the value of security i
N is the number of securities in the portfolio

Rate of Return on a Portfolio, R_P (Slide)

	Stock Fund		Bond Fund	
Scenario	Rate of Return	Squared Deviation	Rate of Return	Squared Deviation
Recession	-7%	0.0324	17%	0.0100
Normal	12%	0.0001	7%	0.0000
Boom	28%	0.0289	-3%	0.0100
Expected return	11.00%		7.00%	
Variance	0.0205		0.0067	
Standard Deviation	14.3%		8.2%	

Note that stocks have a higher expected return than bonds and higher risk. Let us turn now to the risk-return tradeoff of a portfolio that is 50% invested in bonds and 50% invested in stocks, i.e, $W_B = W_S = ½ = 0.5$.

The return on a portfolio is the value-weighted average of the returns on the individual securities in the portfolio.

i.e., $R_p = \Sigma^N_{i=1} W_i * R_i$

where R_i is the (possible) rate of return on security i

Expected Rate of Return, $E(R_P)$, and Variance of a Portfolio, σ_P^2 (Slide)

The expected return on a portfolio is the value-weighted average of the expected returns on the individual securities in the portfolio.

i.e., $$\mathbf{E(R_p)} = \sum_{i=1}^{N} X_i E(R_i)$$

The variance of a portfolio is a function of both the variances of individual securities, and the covariances (or correlations) among these securities.

i.e. $$\sigma_p^2 = \sum_{i=1}^{N} X_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j=1, i\neq j}^{N} X_i X_j \sigma_{ij}$$ **OR**

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j=1, i\neq j}^{N} X_i X_j \rho_{ij}\sigma_i\sigma_j$$

Note that the above formulas apply to portfolios with many securities. However, we only consider the case of a 2-security portfolio in the numerical illustrations.

For a 2-security portfolio, the expected return and variance formulas are (**Slide**)

Expected return: $\mathbf{E(R_p) = W_A * E(R_A) + W_B * E(R_B)}$

Variance: $\sigma_p^2 = W_A2 * \sigma_A^2 + W_B^2 * \sigma_B^2 + 2 * W_A * W_B * \rho_{AB} * \sigma_A * \sigma_B$

OR $\sigma_p^2 = W_A^2 * \sigma_A^2 + W_B^2 * \sigma_B^2 + 2 * W_A * W_B * \sigma_{AB}$

Applying the portfolio formulas to the numerical illustration **(presented in the previous page, i.e., 1.1.2)** with securities A and B by assuming that we invest \$100 in A and \$200 in B:

W_A = \$100/\$300 = 1/3 and W_B = \$200/\$300 = 2/3 (= 1 - W_A)

$E(R_P) = (1/3)*(0.10) + (2/3)*(0.10) = 0.10$

$\sigma_P^2 = W_A^2\sigma_A^2 + W_B^2\sigma_B^2 + 2\, W_A W_B\, r_{AB}\sigma_A\sigma_B$

$= (1/3)^2(0.005) + (2/3)^2(0.00125) + (2)(1/3)(2/3)(-1)(0.0707)(0.0354) = 0$

OR $\sigma_P^2 = W_A^2\sigma_A^2 + W_B^2\sigma_B^2 + 2\, W_A W_B \sigma_{AB}$

$= (1/3)^2(0.005) + (2/3)^2(0.00125) + (2)(1/3)(2/3)(-0.0025) = 0$

Note that when the two risky securities, A and B, are perfectly negatively correlated ($\rho_{AB} = -1$), we can construct a riskless portfolio ($\sigma_P = 0$) by putting them together with the right proportions (or portfolio weights, W_A and W_B). This is the magic of hedging!

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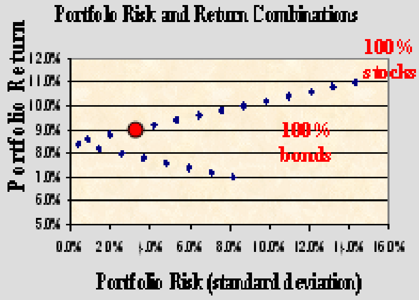
The Efficient Set & The Diversification Effect (Ref: Sections 11.4-11.7)

Building on the previous numerical examples, we can construct many different portfolios of the two risky securities by varying the portfolio weights on each security. When we plot the expected returns against the standard deviations of the portfolios on a graph, we can construct the investment opportunity set or feasible set that represents all possible portfolios of the two risky securities. Among the possible portfolios in the opportunity set, the one with the lowest possible risk is called the minimum variance (MV) portfolio. The MV portfolio denotes the beginning point of the efficient frontier on the opportunity set.

% in stocks	Risk	Return
0%	8.2%	7.0%
5%	7.0%	7.2%
10%	5.9%	7.4%
15%	4.8%	7.6%
20%	3.7%	7.8%
25%	2.6%	8.0%
30%	1.4%	8.2%
35%	0.4%	8.4%
40%	0.9%	8.6%
45%	2.0%	8.8%
50%	3.1%	9.0%
55%	4.2%	9.2%
60%	5.3%	9.4%
65%	6.4%	9.6%
70%	7.6%	9.8%
75%	8.7%	10.0%
80%	9.8%	10.2%
85%	10.9%	10.4%
90%	12.1%	10.6%
95%	13.2%	10.8%
100%	14.3%	11.0%

10.4 The Efficient Set for Two Assets

% in stocks	Risk	Return
0%	8.2%	7.0%
5%	7.0%	7.2%
10%	5.9%	7.4%
15%	4.8%	7.6%
20%	3.7%	7.8%
25%	2.6%	8.0%
30%	1.4%	8.2%
35%	0.4%	8.4%
40%	0.9%	8.6%
45%	2.0%	8.8%
50.00%	**3.08%**	**9.00%**
55%	4.2%	9.2%
60%	5.3%	9.4%
65%	6.4%	9.6%
70%	7.6%	9.8%
75%	8.7%	10.0%
80%	9.8%	10.2%
85%	10.9%	10.4%
90%	12.1%	10.6%
95%	13.2%	10.8%
100%	14.3%	11.0%



We can consider other portfolio weights besides 50% in stocks and 50% in bonds …

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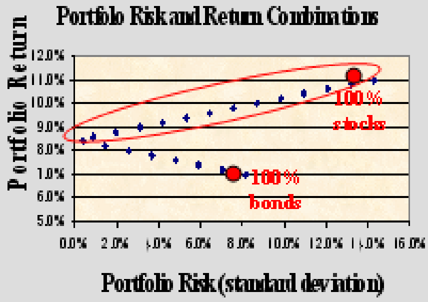
The Efficient Set for Two Assets

% in stocks	Risk	Return
0%	8.2%	7.0%
5%	7.0%	7.2%
10%	5.9%	7.4%
15%	4.8%	7.6%
20%	3.7%	7.8%
25%	2.6%	8.0%
30%	1.4%	8.2%
35%	0.4%	8.4%
40%	0.9%	8.6%
45%	2.0%	8.8%
50%	3.1%	9.0%
55%	4.2%	9.2%
60%	5.3%	9.4%
65%	6.4%	9.6%
70%	7.6%	9.8%
75%	8.7%	10.0%
80%	9.8%	10.2%
85%	10.9%	10.4%
90%	12.1%	10.6%
95%	13.2%	10.8%
100%	14.3%	11.0%

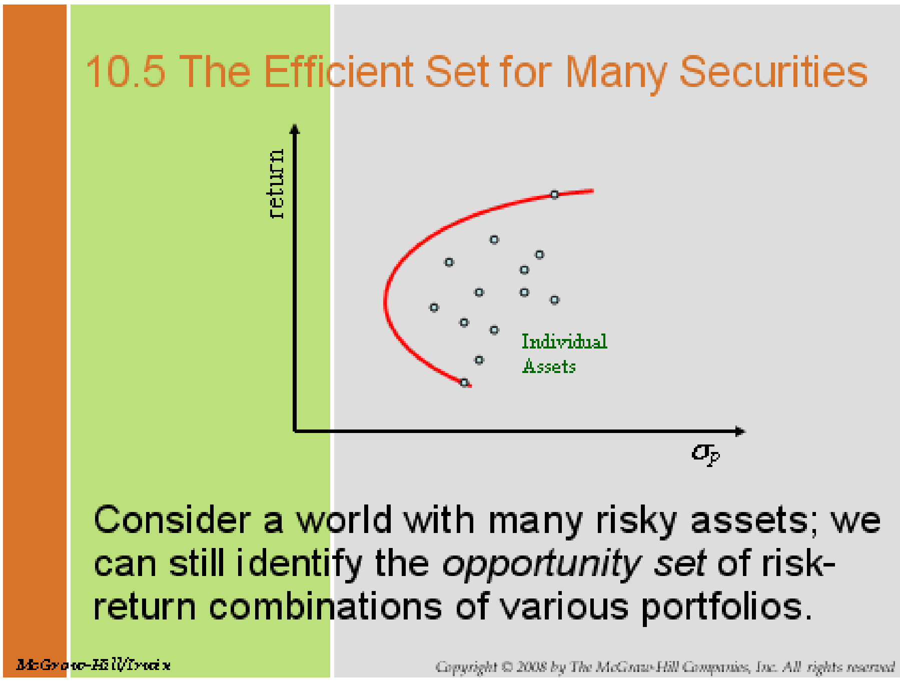


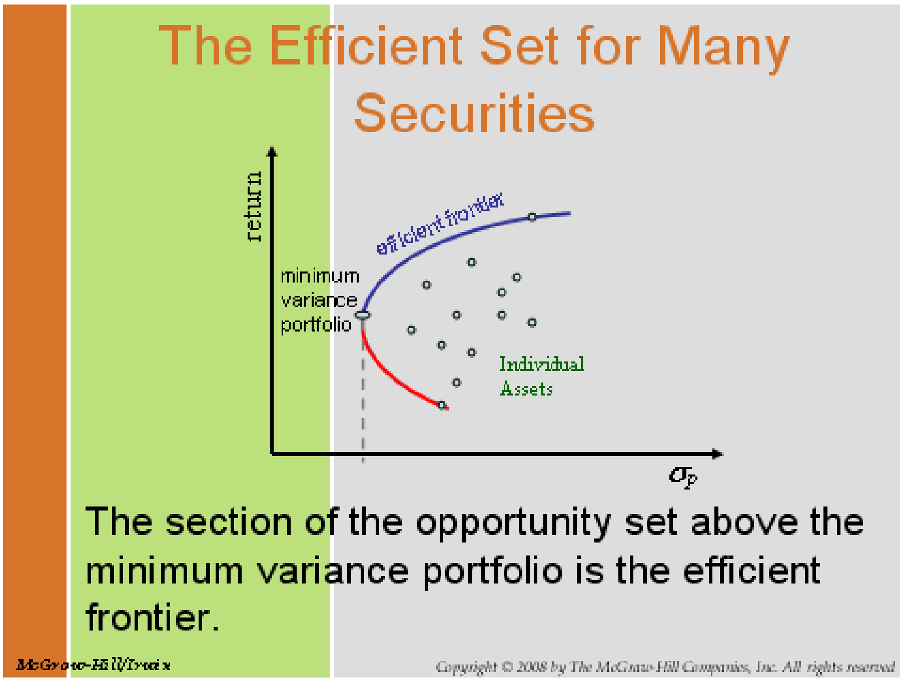
Note that some portfolios are "better" than others. They have higher returns for the same level of risk or less. These portfolios form the *efficient frontier*.

McGraw-Hill/Irwin

The upper portion of the opportunity set, i.e., the portion above and includes the MV portfolio, is the efficient frontier or efficient set, representing the set of efficient portfolios of the best return-risk combinations. An efficient portfolio is defined as the portfolio that has the highest level of expected return for a given level of risk. We assume that rational investors, who are mean-variance optimizers, prefer higher return given the level of risk (greed), and prefer less risk given the level of return (risk aversion). Hence, they will invest only in efficient portfolios.

Note that the above discussions are also applicable to the general case of portfolios of N securities.





The return-risk characteristics of portfolios:

1. The expected return on a portfolio is the weighted average of the expected returns on the individual securities, and the relationship is linear.
2. In general, the variance of a portfolio is a function of both the variances of individual securities, and the covariances (or correlations) among these securities. As such, the variance of a portfolio is NOT a linear function of the variances of individual securities. As the number of securities increases in a portfolio, the covariance terms outnumber the variance terms. In general, there are $N^2 - N$ covariance terms and N variance terms in a portfolio. Consequently, the variance of a portfolio with many securities is more dependent on the covariances between the individual securities than on the variances of the individual securities. This concept is central to the Modern Portfolio Theory.
3. In the extreme case that all securities in a portfolio are perfectly positively correlated, i.e., $\rho = 1$, the standard deviation of a portfolio is the weighted average of the standard deviations of the individual securities.

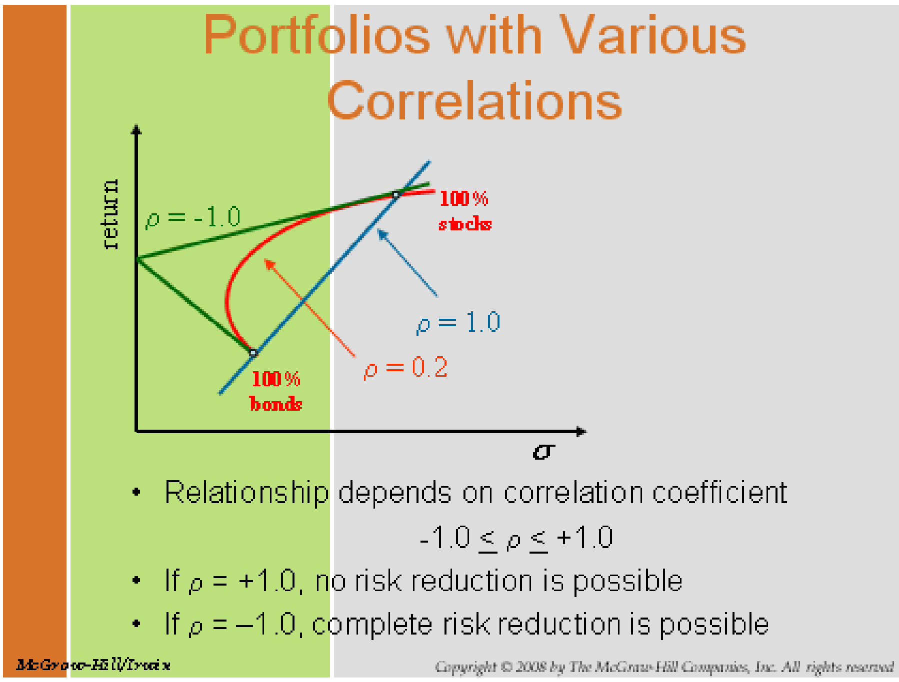
$$\sigma_P = \sum_{i=1}^{N} X_i \sigma_i$$

As long as the securities in a portfolio are less than perfectly positively correlated, i.e., $\rho < 1$, the standard deviation of a portfolio is less than the weighted average of the standard deviations of the individual securities. In other words, some of the risk of individual securities can be diversified away in a well-diversified portfolio.

4. The diversification effect, i.e., the reduction in the risk of a portfolio, depends on how the returns on the individual securities correlate with one another. When the securities are perfectly positively correlated, there is no diversification effect, i.e., no risk reduction is possible. As shown in the portfolio variance formula,

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + 2\sum_{i=1}^{N} \sum_{j=1, i \neq j}^{N} X_i X_j \rho_{ij} \sigma_i \sigma_j$$

the lower the correlation, i.e., the less positive its value, the greater the risk reduction potential (or diversification benefit). Hence, we can lower the risk of the portfolio by adding securities that have less positive correlations with the securities that are already included in the portfolio.



Consider the total risk of the equally weighted portfolio of the stock and bond funds with different correlation coefficients:

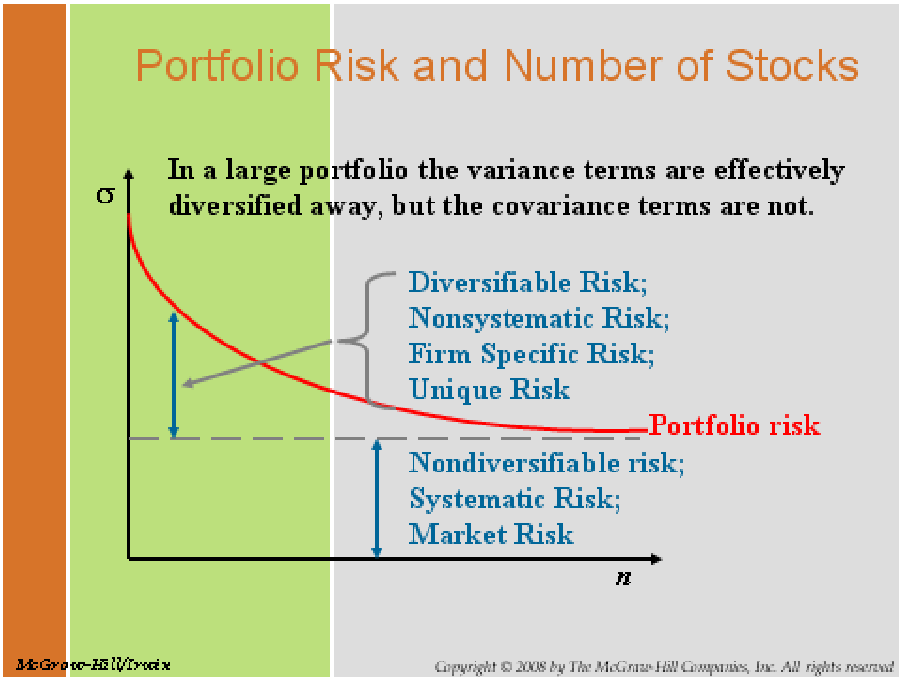
For $\rho_{SB} = +1.0$, $\sigma_P = 0.1125$
$\rightarrow$ Diversification benefit = 0

For $\rho_{SB} = +0.2$, $\sigma_P = 0.0893$
$\rightarrow$ Diversification benefit = 0.1125 − 0.0893 = 0.0232

For $\rho_{SB} = -0.2$, $\sigma_P = 0.0750$
$\rightarrow$ Diversification benefit = 0.1125 − 0.0750 = 0.0375

For $\rho_{SB} = -1.0$, $\sigma_P = 0.0306$
$\rightarrow$ Diversification benefit = 0.1125 − 0.0306 = 0.0819

5. The total risk of a security can be decomposed into two parts,



Total Risk = Systematic Risk + Unsystematic Risk

Systematic Risk: Risk that influences many securities

- Resulted from economy-wide events.
- Since systematic risk cannot be diversified away even when the risky security is part of a well-diversified portfolio, risk averse investors demand a compensation for bearing systematic risk.

Unsystematic Risk: Risk that affects at most a small number of securities

- Resulted from firm-specific events.
- The market will not compensate risk averse investors for bearing this type of risk because they can avoid it by investing risky securities via a well-diversified portfolio.

An Example of Diversification Effect

It is helpful to demonstrate the return-risk characteristics with a portfolio of two assets: IBM and Homestake Mining (HM). Given:

Stock	Investment	X_i	E(Ri)	σ_i
IBM	$6,000	60%	0.09	0.1
HM	$4,000	40%	013	0.2
Total	$10,000	100%		

Recall: For a 2-security portfolio:

Expected return, $E[R_p] = W_{IBM} * E(R_{IBM}) + W_{HM} * E(R_{HM})$

Variance, $\sigma_P^2 = W^2_{IBM}\sigma^2_{IBM} + W^2_{HM}\sigma_{HM}^2 + 2\, W_{IBM}\, W_{HM}\rho_{IBM,HM}\sigma_{IBM}\sigma_{HM}$

Given $\rho_{AB} = 1$: $E[R_p] = (0.6)(0.09) + (0.4)(0.13) = 0.106 = 10.6\%$

$\sigma_p^2 = (0.6)^2(0.1)^2 + (0.4)^2(0.2)^2 + 2(0.6)(0.4)(1)(0.1)(0.2) = 0.0196$

$\rightarrow \sigma_p = (0.0196)^{(1/2)} = 0.14 \rightarrow$ diversification effect = 0

Given $\rho_{AB} = 0$: $E[R_p] = (0.6)(0.09) + (0.4)(0.13) = 0.106 = 10.6\%$

$\sigma_p^2 = (0.6)^2(0.1)^2 + (0.4)^2(0.2)^2 + 2(0.6)(0.4)(0)(0.1)(0.2) = 0.01$

$\rightarrow \sigma_p = (0.01)^{(1/2)} = 0.10 \rightarrow$ diversification effect = 0.14 − 0.10 = 0.04

Given $\rho_{AB} = -1$: $E[R_p] = (0.6)(0.09) + (0.4)(0.13) = 0.106 = 10.6\%$

$\sigma_p^2 = (0.6)^2(0.1)^2 + (0.4)^2(0.2)^2 + 2(0.6)(0.4)(-1)(0.1)(0.2) = 0.0004$

$\rightarrow \sigma_p = (0.0004)^{(1/2)} = 0.02 \rightarrow$ diversification effect = 0.14 − 0.02 = 0.12

Two observations:

1. The less positive (or more negative) the correlation, the lower the risk level of the portfolio, meaning the greater the risk reduction and hence the greater the diversification effect.
2. The correlation has NO impact on the expected return on the portfolio.

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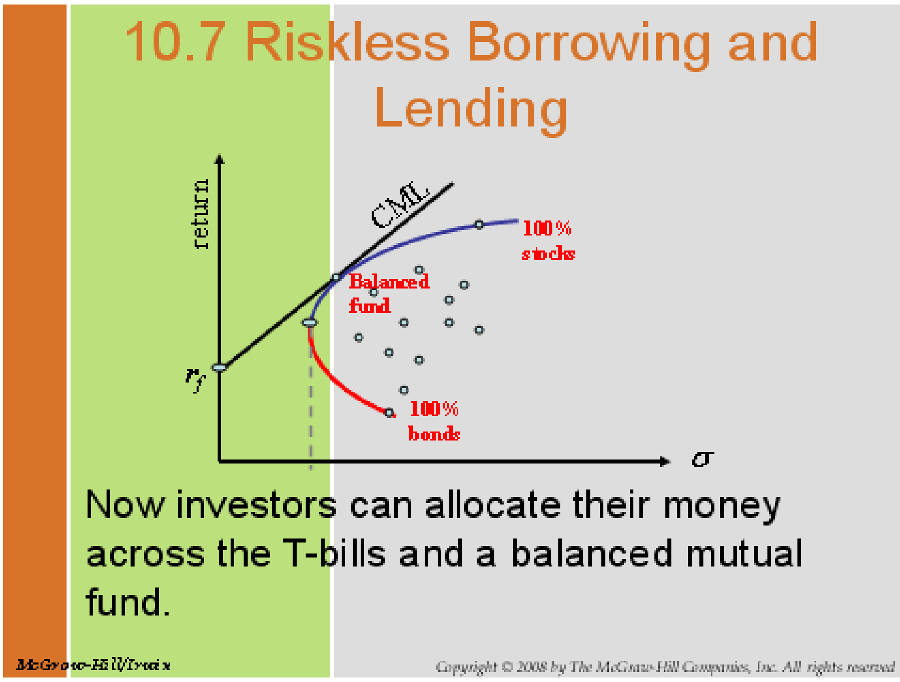
The Separate Theorem (Ref: Sections 11.7 and 11.8)

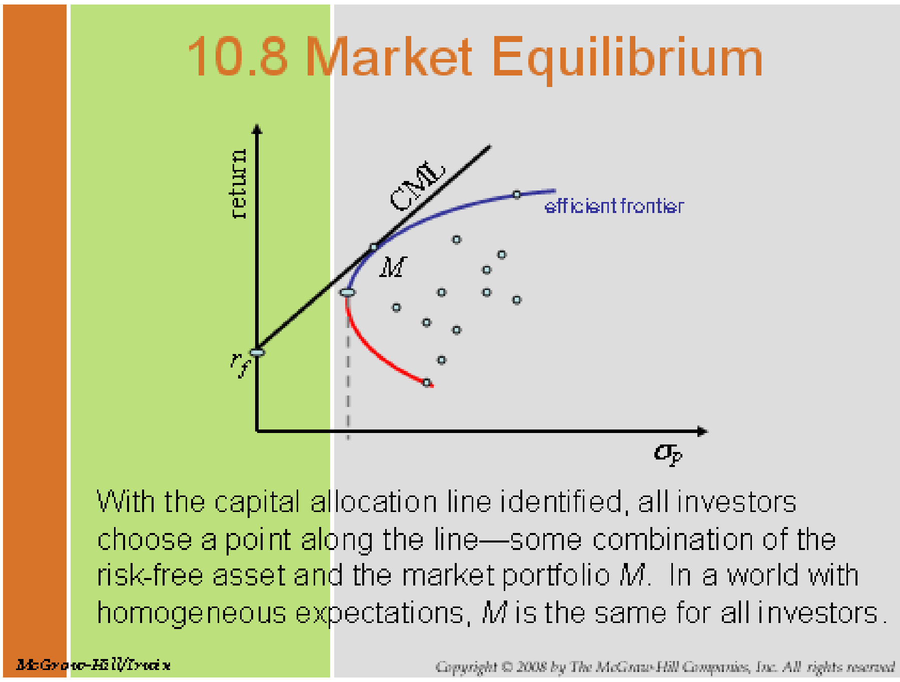
Consider a riskfree asset (F), which is assumed to have zero standard deviation (σ_F=0) and hence zero covariance with any risky asset, is available such that all investors can borrow and lend at the same riskfree rate, R_F. The combination of this riskfree asset and the opportunity set of risky securities is represented by a line that projects from the riskfree rate, i.e., the y-intercept, and tangent to the efficient set of risky securities.

Riskless Borrowing and Lending



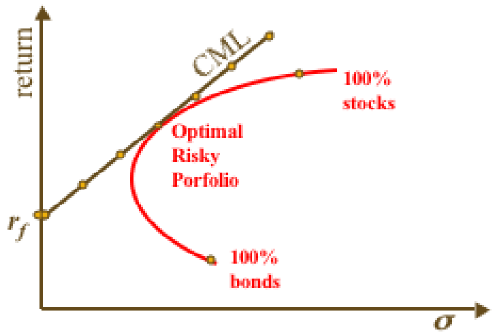
This line provides investors with the highest return at any given standard deviation. As such, this tangent line replaces the efficient set of risk portfolios and becomes the new efficient frontier. Under the assumption that investors are greedy and risk averse, investors will prefer combinations along the tangent line to any other risky portfolios. The portfolio of risky assets (M) that lies on the tangent line is the optimal risky portfolio and the tangent line is the Capital Market Line (CML). In a world with homogeneous expectations, all investors will identify the same efficient set and the same capital market line and hold the same portfolio of risky assets (M). In equilibrium, this portfolio of risky asset (M) is a market-value-weighted portfolio of all existing securities, i.e. the market portfolio. A more conservative investor will choose a point lower on this line and a more aggressive investor will choose a point higher on this line.

Market Equilibrium



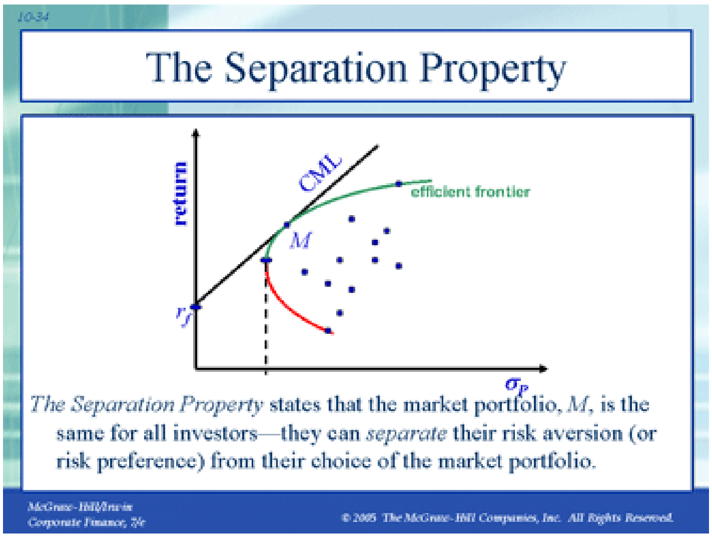
This insight leads to the Separation Theorem of investment decision-making. The Separation Theorem states that the decision involves two independent steps:

The Separation Property

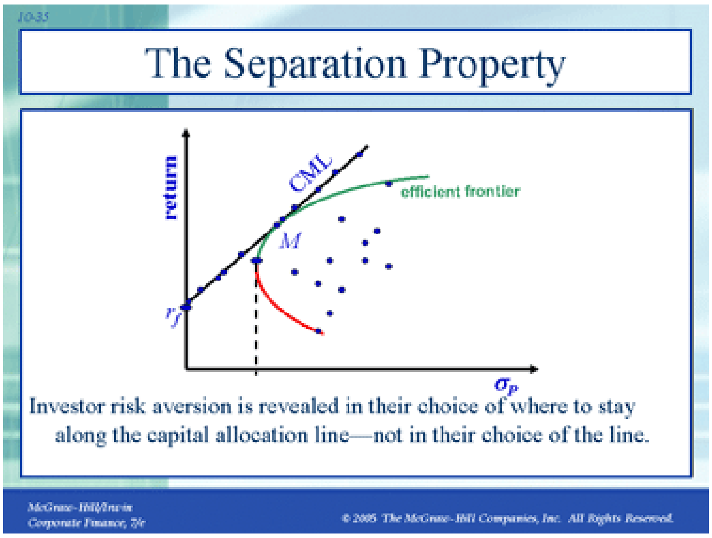


The separation property implies that portfolio choice can be separated into two tasks: (1) determine the optimal risky portfolio, and (2) selecting a point on the CML.

1. The construction of the common optimal risky portfolio (M) for all investors is independent of (or separated from) their risk preferences;



2. The allocation of investment between the riskfree asset (F) and the optimal risky portfolio (M), i.e., choosing a point along the CML, depends on the risk preference of the individual investor.



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The Capital Asset Pricing Model (CAPM) (Ref: Sections 11.8 & 11.9)

In a well-diversified (market) portfolio, the riskiness of a security depends on how this security contributes to the risk of the portfolio. In other words, in a well-diversified portfolio environment, the relevant risk of a security is its systematic risk, NOT total risk!

It has been shown that the **beta (β)** of a security, which measures the responsiveness of a security to movements in the market portfolio, is the best measure of the contribution of the security to the risk of the portfolio, i.e., the systematic risk of a security.

$$\beta_i = \sigma_{iM} / \sigma^2_M$$

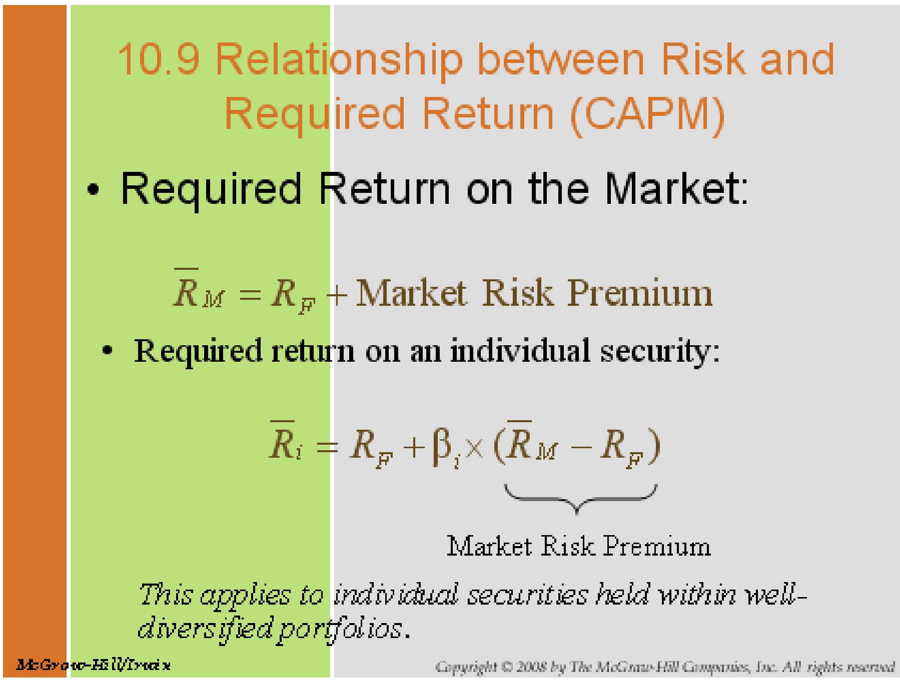
where σ_{iM} is the covariance between the security and the market portfolio
σ^2_M is the variance of the market portfolio.

Reference page 1.1.2 for numerical illustrations on the computations of covariance and variance!

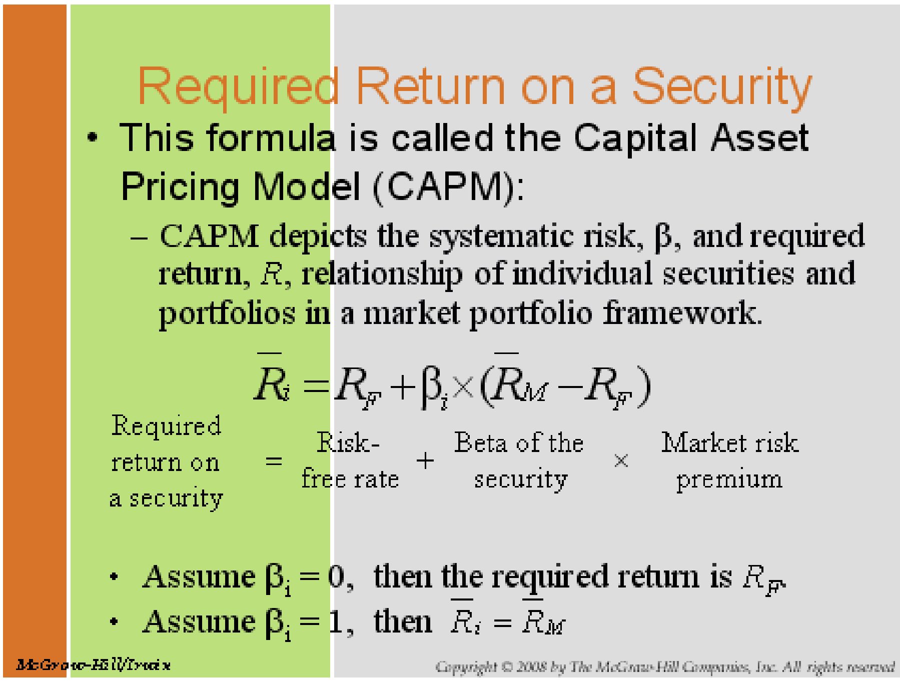
The relation between the beta of a portfolio and the betas of the individual securities is similar to that of the expected return, i.e., a linear relation. The beta of portfolio is just the weighted average of the betas of the individual securities in the portfolio.

i. e., $\boldsymbol{\beta_p = \sum^N_{i=1} X_i * \beta_i}$

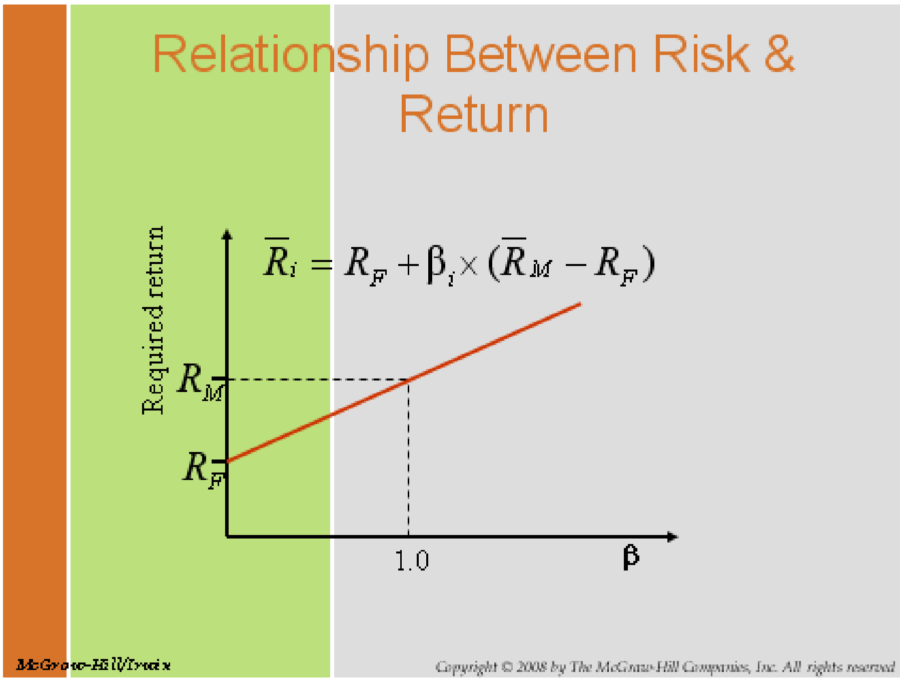
Recall that we assume that risk averse investors dislike risk and demand a risk premium for bearing risk in their investment, they will require a return on any risky security that is in proportion to its systematic risk level. The Capital Asset Pricing Model (CAPM) is a formal model for the risk-return relation of securities and portfolios in the well-diversified market portfolio framework.



According to the Capital Asset Pricing Model (CAPM),



The Security Market Line (SML) is the graphical representation of the CAPM.



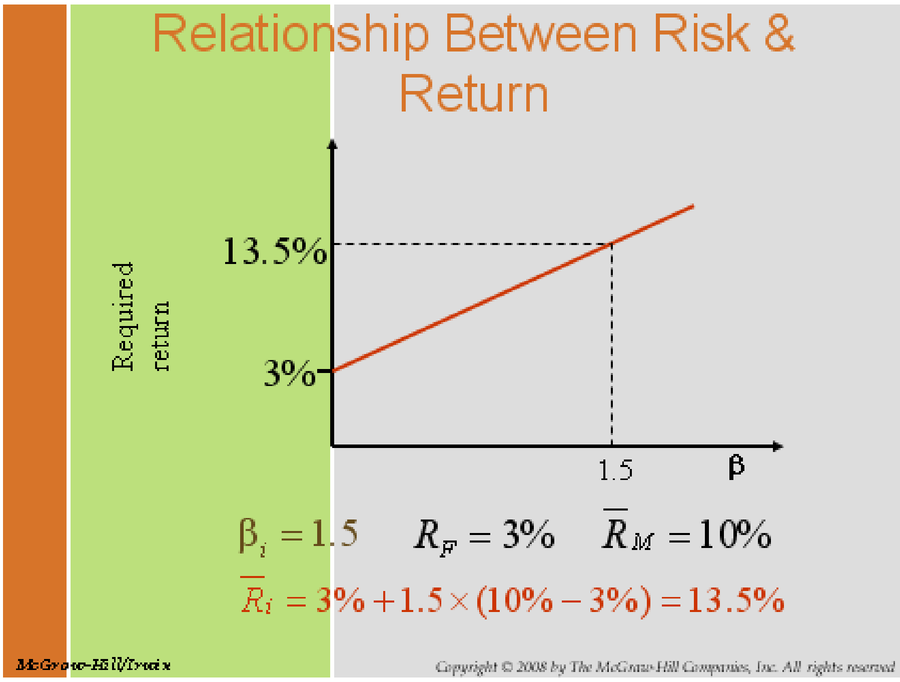
Note: The Capital Market Line (CML) applies to well-diversified portfolios and therefore, the risk in the CML is represented by the standard deviation, i.e., total risk measure, of the portfolios. The Security Market Line (SML) applies to securities and portfolios that are held within the well-diversified market portfolio and the risk in the SML is represented by beta, i.e., systematic risk measure.

CAPM Analysis

Another approach to identify mispriced assets and securities!

- REQUIRED (or CAPM) return > Expected return → OVER-priced!
- REQUIRED (or CAPM) return < Expected return → UNDER-priced!

An Example



Given: $B_i = 1.5$; $R_F = 3\%$; $R_M = 10\%$; and $E(R_i) = 15\%$
According to CAPM, Required return, R = 3% + 1.5 * (10% − 3%) = 13.5%

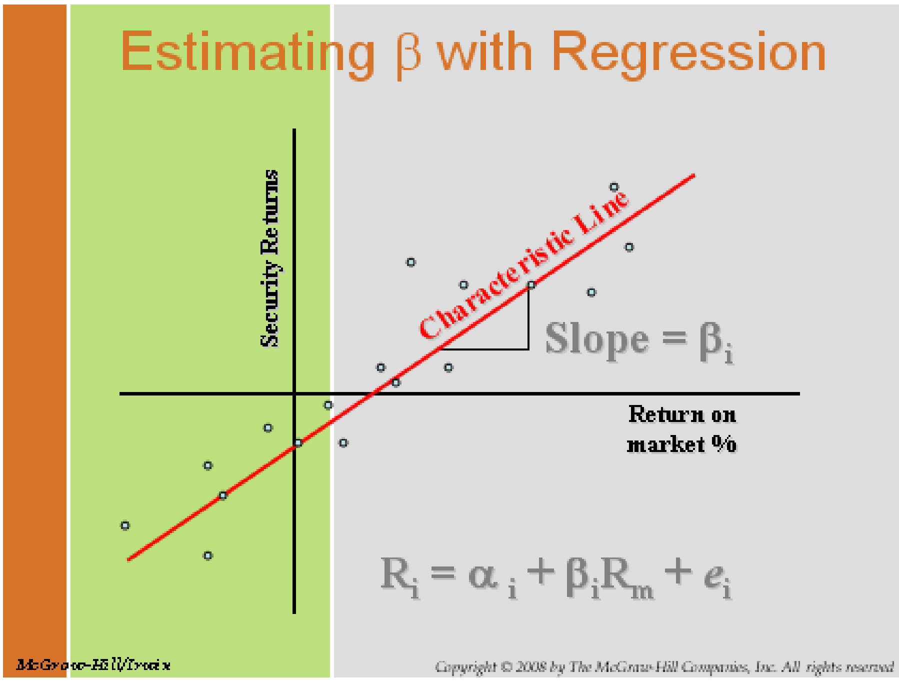
Since the required return of 13.5% is less than the expected return of 15%, this security is underpriced and a buy recommendation follows.

Estimation of the CAPM

Input Parameters:

- R_F: Yield on short-term Treasury securities.
- $(R_M - R_F)$: Historical market risk premium (Chapter 10)

- B_i: Historical beta, the slope coefficient of the market model (or the Characteristic Line)



Financial Management I (FIN 531)