Analysis of Electromagnetic Scattering from an Eccentric Multilayered Sphere

Kyutae Lim and Sang Seol Lee

***Abstract*—An exact analytic solution of a plane electromagnetic (EM) wave scattered by eccentric multilayered sphere (EMS) is obtained. It is assumed that the layers are perfect dielectrics and that the innermost core is a perfectly conducting sphere. Each center of a layer is translated along the incident axis. All fields are expanded in terms of the spherical vector wave functions with unknown expansion coefficients. The addition theorem for spherical wave functions is used prior to applying the boundary conditions. The unknown coefficients are determined by solving a system of linear equations derived from the boundary conditions. Numerical results of the scattering cross sections are presented on the plane of $\phi = 0$ degrees and $\phi = 90$ degrees. The convergence of modal solutions and the characteristics of patterns are examined with various geometries and permittivity distributions.**

I. Introduction

Scattering by a spherical scatterer has been studied over many years as a canonical problem for the three-dimensional object with a surface curvature. The scattering properties of the multilayered scatterer also have been investigated for different types of boundaries. The exact analytic solutions for these problems play an important role not only in developing radar systems and in designing antenna systems, but also in estimating the accuracy of solutions obtained by approximate or numerical methods.

Mie has found the exact solution of scattering from a homogeneous dielectric sphere by using the Hertzian vectors [1]. Aden and Kerker have treated the scattering by a single layered dielectric sphere using the spherical vector wave functions [2]. Wait has generalized the Lorentz–Mie solution to find the exact solution of scattering by a radially inhomogeneous sphere [3], while Medgyesi-Mitschang and Putnam have solved the problem of the dielectric-coated concentric sphere by the method of moment [4]. Hamid *et al.* presented many papers on the scattering of array of spheres with various natures, such as the dielectric, the conducting, and the dielectric-coated conducting spheres [5]–[8].

Due to their complexity, scattering properties of eccentric multilayered objects have not been largely studied. Roumeliotis and Fikioris have analyzed the scattering property of an eccentrically coated metallic sphere [9]. Lee *et al.* have studied the transmission of a spherical wave through a dielectric shell by using geometrical optics and applied this result to the analysis of dielectric lenses and radomes [10]. Recently, Kishk *et al.* obtained a rigorous solution of scattering from an eccentric multilayered cylinder by the mode-matching approach [11].

In this paper, an exact analytic solution of plane EM wave scattering from an eccentric multilayered sphere is developed. To overcome the difficulty of imposing boundary conditions on the eccentric interfaces, the addition theorem for spherical vector wave functions is used [12]–[15]. In Section II, geometry of EMS with M-dielectric layers is given, and some geometrical parameters are defined. In Section III, the EM fields in dielectric and free-space regions are expressed in terms of the spherical vector wave functions with unknown expansion coefficients, and a $(2+4M) \times (2+4M)$ matrix equation is derived from appropriate boundary conditions to the field expressions. As the centers of the dielectric spheres are not coincidental, the addition theorem for the spherical vector wave functions is adopted prior to application of boundary conditions. In Section IV, the far-field scattering cross sections are computed for various geometries and permittivity distributions. The convergence of modal solutions is also examined for several dimensions and permittivity distributions of the EMS.

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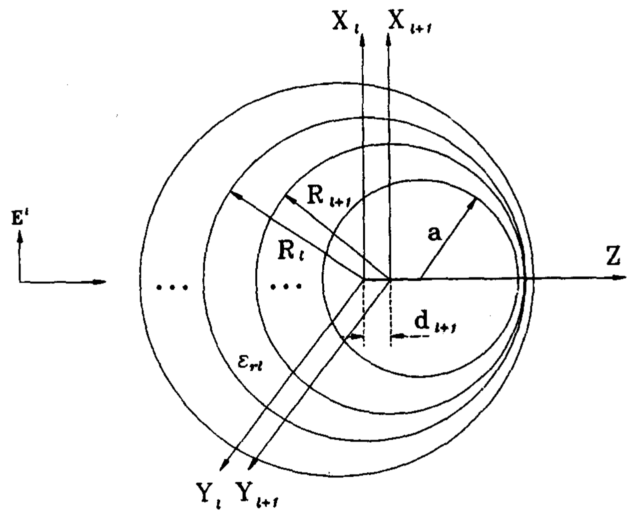


Fig. 1. Geometry of the eccentric multilayered sphere.

II. Geometry of the Eccentric Multilayered Sphere

Fig. 1 shows the geometry of EMS. The innermost core is a perfectly conducting sphere of radius a and coated with M-dielectric layers. Each center of a dielectric layer is translated along the z-axis of the wave incidence.

Now we introduce l to represent regions of EMS: $l = 0, 1 \leq l \leq M$, and $l = M+1$ represent free space, the dielectric layers, and the innermost core regions, respectively. Except for the conducting sphere region ($l = M+1$), the relative permittivity, intrinsic impedance, and wave numbers are given as ϵ_{rl}, η_l, and k_l, respectively. We assume that the media of all layers are nonmagnetic perfectly dielectric.

The EM fields in the lth layer are expressed with respect to their own spherical coordinate systems (r_l, θ_l, ϕ_l), while those in the free space are represented in the (r_1, θ_1, ϕ_1) coordinate system. The radius of a spherical dielectric layer which determines the boundary between $l-1$th and lth layer is given as R_l. The center of coordinate (r_{l+1}, θ_{l+1}, ϕ_{l+1}) is translated from that of (r_l, θ_l, ϕ_l) along z-axis by d_l. (It is not necessary to define d_1.) When an x-polarized uniform plane wave is incident on this geometry, the far-scattered field pattern can be found as shown in the next section.

III. Formulation of the Scattering Problem

A. Field Representation

A plane wave incident upon EMS induces EM fields in all regions except the conducting region. Since all EM fields must satisfy the vector Helmholtz equation, the electric and magnetic fields in lth region can be expanded in terms of the orthogonal functions as follows

$$E_l = E_0 \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} (a_{nl}\, \mathrm{m}_{o1n^{(1)}} + b_{nl}\, \mathrm{m}_{o1n^{(3)}} + jc_{nl}\, \mathrm{n}_{e1n^{(1)}} + jd_{nl}\, \mathrm{n}_{e1n^{(3)}}) \tag{1}$$

$$H_l = -\frac{E_0}{\eta_l} \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} (c_{nl}\, \mathrm{m}_{e1n^{(1)}} + d_{nl}\, \mathrm{m}_{e1n^{(3)}} - ja_{nl}\, \mathrm{n}_{o1n^{(1)}} - jb_{nl}\, \mathrm{n}_{o1n^{(3)}}) \tag{2}$$

where a_{nl}, b_{nl}, c_{nl}, and d_{nl} are coefficients representing EM fields in the lth region. To represent the incident plane wave, we assign the values of a_{n0} and c_{n0} as 1. m_{omn}^{e} and n_{omn}^{e} are the spherical vector wave functions defined in [16].

Now that eccentricity of the boundaries makes it impossible to apply the boundary conditions strictly, we have to express the EM field in $l-1$th region with respect to lth coordinate by using the addition theorem. After the addition theorem for spherical vector-wave functions is applied to (1) and (2), we obtain

$$\begin{aligned} E_{l-1} = E_0 \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} &\cdot \sum_{\nu=1}^{\infty} [a_{nl-1}(A_{1\nu}^{1n}\, \mathrm{m}_{o1n^{(1)}} + B_{1\nu}^{1n}\, \mathrm{n}_{e1n^{(1)}}) \\ &+ b_{nl-1}(A_{1\nu}^{1n}\, \mathrm{m}_{o1n^{(3)}} + B_{1\nu}^{1n}\, \mathrm{n}_{e1n^{(3)}}) \\ &+ jc_{nl-1}(A_{1\nu}^{1n}\, \mathrm{n}_{e1n^{(1)}} + B_{1\nu}^{1n}\, \mathrm{m}_{o1n^{(1)}}) \\ &+ jd_{nl-1}(A_{1\nu}^{1n}\, \mathrm{n}_{e1n^{(3)}} + B_{1\nu}^{1n}\, \mathrm{m}_{o1n^{(3)}})] \end{aligned} \tag{3}$$

$$\begin{aligned} H_{l-1} = -\frac{E_0}{\eta_{l-1}} \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} &\cdot \sum_{\nu=1}^{\infty} [c_{nl-1}(A_{1\nu}^{1n}\, \mathrm{m}_{e1n^{(1)}} + B_{1\nu}^{1n}\, \mathrm{n}_{o1n^{(1)}}) \\ &+ d_{nl-1}(A_{1\nu}^{1n}\, \mathrm{m}_{e1n^{(3)}} + B_{1\nu}^{1n}\, \mathrm{n}_{o1n^{(3)}}) \\ &- ja_{nl-1}(A_{1\nu}^{1n}\, \mathrm{n}_{o1n^{(1)}} + B_{1\nu}^{1n}\, \mathrm{m}_{e1n^{(1)}}) \\ &- jb_{nl-1}(A_{1\nu}^{1n}\, \mathrm{n}_{o1n^{(3)}} + B_{1\nu}^{1n}\, \mathrm{m}_{e1n^{(3)}})] \end{aligned} \tag{4}$$

where $A_{1\nu}^{1n}$, $B_{1\nu}^{1n}$ are coefficients of the translational addition theorem for the spherical vector-wave functions. For the case of translation along z-axis, the coefficients $A_{m\nu}^{mn}$, $B_{m\nu}^{mn}$ are given by Bruning and Lo [14].

B. Application of the Boundary Conditions

EM fields in all regions are expressed in terms of the spherical vector wave functions with four unknown coefficients. Since a_{n0} and c_{n0} have been given as 1, the total number of unknown coefficients is $2+4M$ for M-layered EMS.

To obtain the coefficients in all regions, a system of linear equations should be derived by applying the boundary conditions. Since the tangential fields were expressed in different ways depending on properties of boundaries, the boundary conditions should be applied separately. At the boundary between free space and the first layer, the boundary conditions can be applied to (1) and (2) directly. At the boundary between dielectric layers, we have to equate (1) and (3) for electric fields and (2) and (4) for magnetic fields. At the boundary on the surface of the conducting core, only (3) is used.

After applying the boundary conditions, we can obtain a $(2+4M) \times (2+4M)$ matrix equation as follows. In (5), shown at the bottom of the page, $[T_{ln}]$ are given by

$$[T_{1n}] = \begin{bmatrix} h_n^{(2)}(k_0R_1) & 0 \\ \partial_r r h_n^{(2)}(k_0R_1) & 0 \\ 0 & h_n^{(2)}(k_0R_1) \\ 0 & \partial_r r h_n^{(2)}(k_0R_1) \end{bmatrix} \quad \text{for } l=1 \tag{6a}$$

and $[S_{ln}]$ is also given as where

$$U_l = \frac{\eta_{l-1}}{\eta_l} = \sqrt{\frac{\mu_{l-1}\epsilon_l}{\mu_l\epsilon_{l-1}}}, \qquad V_l = \frac{k_{l-1}}{k_l} = \sqrt{\frac{\mu_{l-1}\epsilon_{l-1}}{\mu_l\epsilon_l}}$$

and $\partial_x f$ represents $\partial f/\partial x$. In (6) and (7) [(6b), (6c), and (7) are shown at the bottom of the next page], $j_n(\cdot)$ and $h_n(\cdot)$ are the spherical Bessel function of the first kind and the spherical Hankel function of the second kind, respectively. A_{1n}^{1n} and B_{1n}^{1n} on lth boundary have been replaced by A_l, and B_l, respectively.

By solving (5), one can obtain the unknown coefficients. The far-zone scattered field is then determined by the coefficients b_{n0} and d_{n0}. The scattering cross section is given by

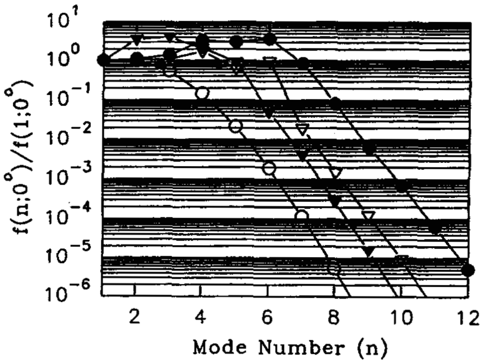
$$\sigma = \frac{\lambda^2}{\pi}[\cos^2\phi|A_\theta|^2 + \sin^2\phi|A_\phi|^2] \tag{8}$$

where

$$|A_\theta|^2 = \left|\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}\left[b_{n0}\frac{P_n^1(\cos\theta)}{\sin\theta} + d_{n0}\partial_\theta P_n^1(\cos\theta)\right]\right|^2 \tag{9a}$$

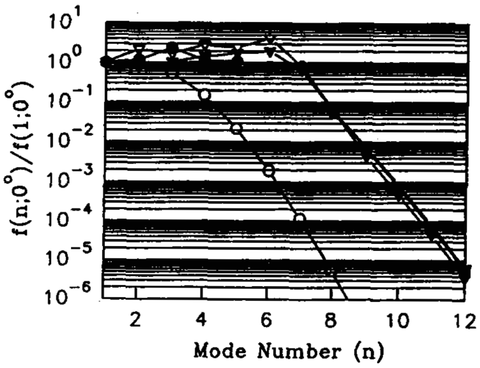
$$|A_\phi|^2 = \left|\sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}\left[b_{n0}\partial_\theta P_n^1(\cos\theta) + d_{n0}\frac{P_n^1(\cos\theta)}{\sin\theta}\right]\right|^2 \tag{9b}$$

$$\begin{bmatrix} -j_n(k_0R_1) \\ -\partial_r j_n(k_0R_1) \\ -j_n(k_0R_1) \\ -\partial_r j_n(k_0R_1) \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} [T_{1n}] & [S_{1n}] & [0] & \cdots & \cdots & \cdots & \cdots & [0] \\ [0] & [T_{2n}] & [S_{2n}] & [0] & \cdots & \cdots & \cdots & [0] \\ \vdots & & \ddots & \ddots & & & & \vdots \\ [0] & \cdots & [0] & [T_{1n}] & [S_{1n}] & [0] & \cdots & [0] \\ \vdots & & & & \ddots & \ddots & & \vdots \\ \vdots & & & & & \ddots & \ddots & \vdots \\ [0] & \cdots & \cdots & \cdots & \cdots & [0] & [T_{Mn}] & [S_{Mn}] \\ [0] & \cdots & \cdots & \cdots & \cdots & \cdots & [0] & [T_{M+1n}] \end{bmatrix} \cdot \begin{bmatrix} b_{n0} \\ d_{n0} \\ a_{n1} \\ b_{n1} \\ c_{n1} \\ d_{n1} \\ \vdots \\ a_{nl} \\ b_{nl} \\ c_{nl} \\ d_{nl} \\ \vdots \\ a_{nM} \\ b_{nM} \\ c_{nM} \\ d_{nM} \end{bmatrix} \tag{5}$$



	$(R_1,R_2,R_3,a)/\lambda$	$(\varepsilon_{r1},\varepsilon_{r2},\varepsilon_{r3})$	$(d_2,d_3,d_4)/\lambda$	type
●	1.0, 2.5/3, 2/3, 0.5	2, 4, 6	0.5/3, 0.5/3, 0.5/3	three-layered eccentric sphere
▽	0.8, 0.7, 0.6, 0.5	2, 4, 6	0.1, 0.1, 0.1	
▼	0.65, 0.6, 0.55, 0.5	2, 4, 6	0.05, 0.05, 0.05	
○	perfect conducting sphere (a=0.5λ)			

Fig. 2. The convergence test function $f(n;\,\phi)$ on the various dimensions of EMS.



	$(R_1,R_2,R_3,a)/\lambda$	$(\varepsilon_{r1},\varepsilon_{r2},\varepsilon_{r3})$	$(d_2,d_3,d_4)/\lambda$	type
●	1.0, 2.5/3, 2/3, 0.5	2, 2, 2	0, 0, 0	single-layered concentric sphere
▽	1.0, 2.5/3, 2/3, 0.5	2, 4, 6	0, 0, 0	three-layered concentric sphere
▼	1.0, 2.5/3, 2/3, 0.5	2, 4, 6	0.5/3, 0.5/3, 0.5/3	three-layered eccentric sphere
○	perfect conducting sphere (a=0.5λ)			

Fig. 3. The convergence test function $f(n;\,\phi)$ on the various dielectric distributions and the eccentricities of EMS.

IV. Numerical Results

If the dimension of a scatterer is much larger or smaller than a wavelength (in the optical or Rayleigh region), certain approximations can give quite accurate results. But in the resonance region, these approximations produce too many errors. To show the validity of our analysis, we consider a three-layered EMS in the resonance region.

In modal analysis, it is very important to truncate the infinite series to a finite one, while the difference between the exact solution and the truncated results is insignificant. So the convergence of the series should be investigated to determine the range of summation. We define a convergence test function as below

$$f(n;\,\phi) = \frac{\sqrt{\sum_{n=1}^{N} S(n;\,\theta_n,\,\phi)}}{N}, \qquad \theta_n = \frac{180^\circ}{(N-1)}(n-1) \quad (10)$$

where

$$S(n;\,\theta,\,\phi) = |\text{n-th term of } A_\theta|^2 \cos^2\phi + |\text{n-th term of } A_\phi|^2 \sin^2\phi. \quad (11)$$

In (11), A_θ and A_ϕ have already been defined in (9) and N is the number of sample points on θ. We can clearly see that $f(n;\,\phi)$ is the rms average of the nth term of the scattering cross section defined in (8). By adjusting the lower bound of $f(n;\,\phi = 0$ degrees), we can determine the number of modes necessary to satisfy a desired accuracy.

Fig. 2 shows $f(n;\,\phi = 0$ degrees) [normalized to $f(n = 1;\,\phi = 0$ degrees)] for each different dimension of EMS. If we set the lower bound of the test function to 10^{-5}, we only need to sum the first eight terms of the series for the conducting sphere of radius 0.5λ. Obviously, the larger the dimension of the scatterer becomes, the more modes should be added to meet the accuracy of the computation. For instance, EMS with a radius of 1.0λ needs at least 12 modes.

Fig. 3 shows the various cases of dielectric distributions and eccentricities. As we observe, the convergence of a modal solution is affected by neither the number of layers, dielectric distribution, nor eccentricity.

For verification of this solution, the scattering cross section of a single-layered concentric sphere was calculated and compared with the solution obtained by the method of moment [4]. Our results agreed well with Medgyesi-Mitschang and Putnam.

The scattering patterns of the different eccentricities of EMS are shown in Fig. 4. Although the permittivity distributions are identical, patterns are very different from each other.

$$[T_{ln}] = \begin{bmatrix} A_l j_n(K_{l-1}R_l) & A_l h_n^{(2)}(K_{l-1}R_l) & jB_l j_n(K_{l-1}R_l) & jB_l h_n^{(2)}(K_{l-1}R_l) \\ A_l \partial_r r j_n(K_{l-1}R_l) & A_l \partial_r r h_n^{(2)}(K_{l-1}R_l) & jB_l \partial_r r j_n(K_{l-1}R_l) & jB_l \partial_r r h_n^{(2)}(K_{l-1}R_l) \\ -jB_l j_n(K_{l-1}R_l) & -jB_l h_n^{(2)}(K_{l-1}R_l) & A_l j_n(K_{l-1}R_l) & A_l h_n^{(2)}(K_{l-1}R_l) \\ -jB_l \partial_r r j_n(K_{l-1}R_l) & -jB_l \partial_r r h_n^{(2)}(K_{l-1}R_l) & A_l \partial_r r j_n(K_{l-1}R_l) & A_l \partial_r r h_n^{(2)}(K_{l-1}R_l) \end{bmatrix} \quad \text{for } 2 \le l \le M \quad (6b)$$

$$[T_{M+1n}] = \begin{bmatrix} A_{M+1} j_n(k_M R_{M+1}) & A_{M+1} h_n^{(2)}(k_M R_{M+1}) & jB_{M+1} j_n(k_M R_{M+1}) & jB_{M+1} h_n^{(2)}(k_M R_{M+1}) \\ B_{M+1} \partial_r r j_n(k_M R_{M+1}) & B_{M+1} \partial_r r h_n^{(2)}(k_M R_{M+1}) & jA_{M+1} \partial_r r j_n(k_M R_{M+1}) & jA_{M+1} \partial_r r h_n^{(2)}(k_M R_{M+1}) \end{bmatrix} \quad \text{for } l = M+1 \quad (6c)$$

$$[S_{ln}] = \begin{bmatrix} -j_n(k_l R_l) & -h_n^{(2)}(k_l R_l) & 0 & 0 \\ -U_l V_l \partial_r r j_n(k_l R_l) & -U_l V_l \partial_r r h_n^{(2)}(k_l R_l) & 0 & 0 \\ 0 & 0 & -U_l j_n(k_l R_l) & -U_l h_n^{(2)}(k_l R_l) \\ 0 & 0 & -V_l \partial_r r j_n(k_l R_l) & -V_l \partial_r r h_n^{(2)}(k_l R_l) \end{bmatrix} \quad \text{for } 1 \le l \le M+1 \quad (7)$$

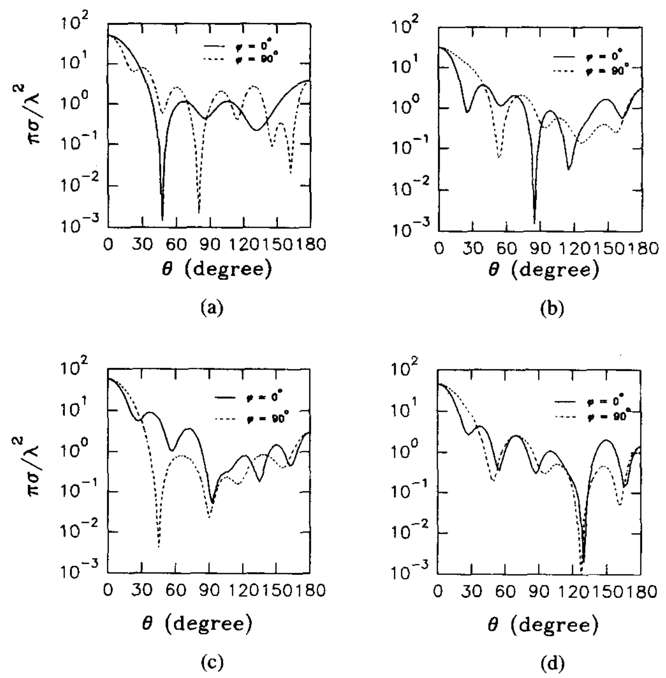


Fig. 4. Scattering cross sections on different eccentricities of EMS with $(\epsilon_{r1}, \epsilon_{r2}, \epsilon_{r3})$ (2, 4, 6) and $(R_1, R_2, R_3)/\lambda = (1.0, 2.5/3, 2/3, 0.5)$. $(d_2, d_3, d_4)/\lambda$ are given as (a) (0, 0, 0), (b) (0.1, 0.1, 0.1), (c) (−0.1, −0.1, −0.1), and (d) (−0.1, 0.1, −0.1)

V. Conclusion

An exact series solution for scattering from EMS has been found when the innermost core was a conducting sphere. EM fields in all regions were expanded by the spherical vector wave functions, and then the addition theorem for the spherical coordinate system was used to apply the boundary conditions. A system of linear equations was derived from the boundary conditions. By solving this equation, the far-zone scattered field patterns have been evaluated for a uniform plane wave incidence. Numerical results for EMS in the resonance region have been presented. Also the convergence of the modal solutions have been investigated with various dielectric distributions, eccentricities, and dimensions. We found that the convergency of the solution only depends on the dimension of the scatterer.

From these results, we can predict fields in the dielectric lens, radome, or resonator with spherical boundary. If the innermost core is dielectric, an exact series solution can be found by a slight modification of the boundary condition on the core.

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Efficient Kernel Calculation of Cylindrical Antennas

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***Abstract*—This paper presents a technique for evaluating analytically and without limitation the singular part of the kernel integral of cylindrical wires due to uniform current distribution. This approach uses the static Green's function expression in cylindrical coordinates. The formula of the singular part converges rapidly and illustrates its usefulness for kernel calculations without loss of accuracy.**

I. Introduction

The practical numerical calculation of the thin-wire kernel of cylindrical wires typically relies on the static or singular part of the double integral $1/R_s$ on the surface of the cylinder [1]–[6]. Butler [4] evaluated the integral with the series form valid only for $\Delta/2a > 1$. Another approach evaluates this problem by approximating a cylindrically curved subsection in the neighborhood of the singularity by a flat rectangular patch [5]. An exact expression for the kernel integration in cylindrical antennas has been obtained recently by Wang [7] and Werner [8]. The double integral $1/R_s$ on the cylindrical surface, however, is not available in the literature.

In this paper, a closed-form expression for the singular part due to uniform current distribution is presented. Although the exact expression is available, this method can be used to evaluate efficiently and accurately the matrix elements in a moment-method solution of thin wires.

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