Light scattering from a sphere on or near a surface

Gorden Videen

Optical Sciences Center, University of Arizona, Tucson, Arizona 85721

Received April 27, 1990; accepted October 20, 1990

The light-scattering problem of a sphere on or near a plane surface is solved by using an extension of the Mie theory. The approach taken is to solve the boundary conditions at the sphere and at the surface simultaneously and to develop the scattering amplitude and Mueller scattering matrices. This is performed by projecting the fields in the half-space region not including the sphere multiplied by an appropriate Fresnel reflection coefficient onto the half-space region including the sphere. An assumption is that the scattered fields from the sphere, reflecting off the surface and interacting with the sphere, are incident upon the surface at near-normal incidence. The exact solution is asymptotically approached when either the sphere is a large distance from the surface or the refractive index of the surface approaches infinity.

INTRODUCTION

Light scattering from a cylinder was solved independently by Lord Rayleigh[1] in 1881 and by von Ignatowsky[2] in 1905. A few years later, Mie[3] solved the scattering from a sphere. Liang and Lo[4] extended this theory for two spheres. Kattawar and Dean[5] confirmed the existence of resonances discovered by Wang *et al.*,[6] who measured microwave scattering from two dielectric spheres. Bobbert and Vlieger[7] used operators to solve for the scattering of a small sphere near a surface. Yousif[8] solved for the scattering from two parallel cylinders and recently solved for the scattering from a cylinder on a surface. Yousif's results have been verified experimentally by comparing his theory to the light-scattering Mueller matrix elements measured for a cylinder on a surface.[9] His approach to the problem of a cylinder on a surface has been adopted to solve the scattering of a sphere on or near a surface. The approach is similar to that taken by Rao and Barakat,[10] who calculated the scatter by a conducting cylinder partially buried in a conducting medium.

The method used to calculate the electromagnetic radiation scattered by the system is to address how the radiation may interact with the sphere. The incident plane wave strikes the sphere either directly or after interacting with the medium at the surface, in which case it is an image plane wave. Far-field radiation is a result of a superposition of fields either directly from the sphere or from the image sphere. The fields emanating from the sphere may also reflect off the surface and interact with the sphere again. The magnitude and phase of the fields about the image sphere are altered by the reflections that take place at the surface by the Fresnel coefficients. One assumption is that the scattered fields from the sphere, reflecting off the surface and interacting with the sphere, are incident on the surface at near-normal incidence. The exact solution is asymptotically approached when either the sphere is a large distance from the surface or the refractive index of the surface approaches infinity.

Figure 1 shows the geometry of the scattering system. A sphere of radius a is located on the z axis a distance d below a plane surface bounding two media of different refractive indices oriented perpendicular to the z axis. The incident radiation is a plane wave traveling in the x–z plane, oriented at angle α with respect to the z axis. The wavelength and wave vector for the plane wave in the nonabsorbing, nonmagnetic incident medium below the substrate are λ and $\mathbf{k}$, respectively. The complex wave vectors for a plane wave of the same frequency in the sphere and the medium above the surface are $\mathbf{k}_{\text{sph}}$ and $\mathbf{k}_{\text{med}}$, respectively.

SOLUTION

The solution to the light scattering from a sphere on or near a surface is treated here in several subsections. In Subsection 1 the scalar wave equation is solved, and the vector wave functions and Debye potentials are developed. In Subsection 2 the scattering coefficients are solved for a general case. In Subsection 3 the specific case of a sphere-surface system illuminated with plane-wave incident radiation is solved. Finally, in Subsection 4 the Mueller matrix elements are derived from the scattering coefficients.

1. Wave Equations

The starting point for this scattering problem is the vector wave equation, which is derived in almost any text on electromagnetic theory[11]:

$$\nabla^2\mathbf{E} - \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} - \mu\sigma\frac{\partial\mathbf{E}}{\partial t} = 0,$$

$$\nabla^2\mathbf{H} - \mu\epsilon\frac{\partial^2\mathbf{H}}{\partial t^2} - \mu\sigma\frac{\partial\mathbf{H}}{\partial t} = 0, \tag{1.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field vectors, respectively. With the assumption of a time dependence of the form $\exp(-i\omega t)$ these equations can be used for deriving the scalar wave equation, which in spherical polar coordinates may be expressed as

$$\frac{1\partial}{r^2\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2} + k^2u = 0. \tag{1.2}$$

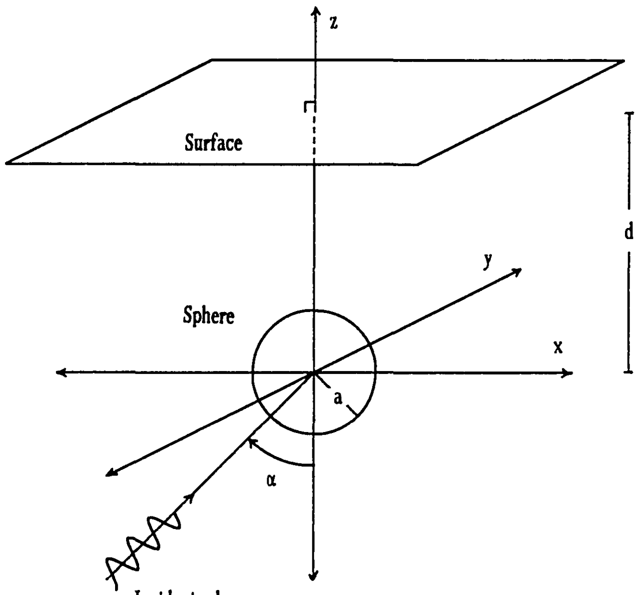

Fig. 1. Geometry of scattering system. A sphere of radius a is located a distance d from a surface. A plane wave travels in the x–z plane at angle α with respect to the z axis.

A separable solution of the following form is assumed:

$$u(r, \vartheta, \varphi) = R(r)\Theta(\vartheta)\Phi(\varphi). \tag{1.3}$$

The particular functions satisfying Eqs. (1.2) and (1.3) are

$$\Phi(\varphi) = \exp(im\varphi),$$
$$\Theta(\vartheta) = \tilde{P}_n^m(\cos\vartheta) = \left[\frac{(2n+1)(n-m)!}{2(n+m)!}\right]^{1/2} P_n^m(\cos\vartheta),$$
$$R(r) = z_n(kr) = \left(\frac{\pi}{2r}\right)^{1/2} Z_{n+1/2}(kr), \tag{1.4}$$

where $P_n^m(\cos\vartheta)$ are associated Legendre polynomials and $z_n(kr)$ are either spherical Bessel functions of the first kind, $j_n(kr)$, or spherical Hankel functions of the first kind, $h_n^{(1)}(kr)$. The electric and magnetic fields may be expanded in terms of the vector wave functions $\mathbf{M}_{nm}^{(i)}$ and $\mathbf{N}_{nm}^{(i)}$ [$i = 1$ corresponds to $j_n(kr)$'s being used and $i = 3$ corresponds to $h_n^{(1)}(kr)$'s being used]:

$$\mathbf{E} = \sum_{n,m,} p_{nm}\mathbf{M}_{nm}^{(i)} + q_{nm}\mathbf{N}_{nm}^{(i)},$$
$$\mathbf{H} = \frac{k}{i\omega\mu}\sum_{n,m} p_{nm}\mathbf{N}_{nm}^{(i)} + q_{nm}\mathbf{M}_{nm}^{(i)}. \tag{1.5}$$

The vector wave functions $\mathbf{M}_{nm}^{(i)}$ and $\mathbf{N}_{nm}^{(i)}$ assume the following form:

$$\mathbf{M}_{nm}^{(i)} = \hat{\varphi}\left[\frac{im}{\sin\theta} z_n(kr)\tilde{P}_n^m(\cos\vartheta)\exp(im\varphi)\right] - \hat{\varphi}\left\{z_n(kr)\frac{\mathrm{d}}{\mathrm{d}\vartheta}[\tilde{P}_n^m(\cos\vartheta)]\exp(im\vartheta)\right\},$$

$$\mathbf{N}_{nm}^{(i)} = \hat{\mathbf{r}}\left[\frac{1}{kr} z_n(kr)n(n+1)\tilde{P}_n^m(\cos\vartheta)\exp(im\varphi)\right] + \hat{\varphi}\left\{\frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}[rz_n(kr)]\frac{\mathrm{d}}{\mathrm{d}\vartheta}[\tilde{P}_n^m(\cos\vartheta)]\exp(im\varphi)\right\} + \hat{\varphi}\left\{\frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}[rz_n(kr)]\frac{im}{\sin\vartheta}\tilde{P}_n^m(\cos\vartheta)\exp(im\varphi)\right\}. \tag{1.6}$$

Rather than expanding the electric and magnetic fields in terms of the vector wave functions as in Eq. (1.2), it is often convenient to expand the fields in terms of the Debye potentials:

$$\mathbf{E} = \sum_{n,m} \nabla\times(\mathbf{r}p_{nm}u_{nm}^{(i)}) + \frac{1}{k}\nabla\times\nabla\times(\mathbf{r}q_{nm}u_{nm}^{(i)}),$$
$$\mathbf{H} = \frac{k}{i\omega\mu}\sum_{n,m} \nabla\times(\mathbf{r}q_{nm}u_{nm}^{(i)}) + \frac{1}{k}\nabla\times\nabla\times(\mathbf{r}p_{nm}u_{nm}^{(i)}). \tag{1.7}$$

The Debye potentials are $p_{nm}u_{nm}^{(i)}$ and $q_{nm}u_{nm}^{(i)}$. The coefficients p_{nm} and q_{nm} are common to both potentials and vector wave functions, making it convenient to consider the scattering problem from either perspective.

2. Scattering Coefficients

The general case for scattering from a sphere on or near a surface can be treated by representing the incident field on the sphere as a superposition of terms in the expansion given by Eq. (1.5):

$$\mathbf{E}^{\mathrm{inc}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm}\mathbf{M}_{nm}^{(1)} + b_{nm}\mathbf{N}_{nm}^{(1)}$$
$$\mathbf{H}^{\mathrm{inc}} = \frac{k}{i\omega\mu}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} b_{nm}\mathbf{M}_{nm}^{(1)} + a_{nm}\mathbf{N}_{nm}^{(1)}. \tag{2.1}$$

The coefficients a_{nm} and b_{nm} will be solved for the special case of an incident plane wave in Subsection 3. The radial functions used in expanding the vector functions for the incident radiation are the spherical Bessel functions $j_n(kr)$. In this section the scattering coefficients will be solved in terms of these coefficients. The scattered field is also expanded:

$$\mathbf{E}^{\mathrm{sca}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} e_{nm}\mathbf{M}_{nm}^{(3)} + f_{nm}\mathbf{N}_{nm}^{(3)},$$
$$\mathbf{H}^{\mathrm{sca}} = \frac{k}{i\omega\mu}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_{nm}\mathbf{M}_{nm}^{(3)} + e_{nm}\mathbf{N}_{nm}^{(3)}. \tag{2.2}$$

The radial functions are the spherical Hankel functions of the first kind $h_n^{(1)}(kr)$. The fields inside the sphere are also expanded:

$$\mathbf{E}^{\mathrm{sph}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} c_{nm}\mathbf{M}_{nm}^{(1)} + d_{nm}\mathbf{N}_{nm}^{(1)},$$
$$\mathbf{H}^{\mathrm{sph}} = \frac{k_{\mathrm{sph}}}{i\omega\mu_{\mathrm{sph}}}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} d_{nm}\mathbf{M}_{nm}^{(1)} + c_{nm}\mathbf{N}_{nm}^{(1)}. \tag{2.3}$$

The radial functions for the internal fields are the spherical Bessel functions $j_n(k_{\mathrm{sph}}r)$. To this point, the analysis is identical to those given by Refs. 11–14 for the case of a single sphere (Mie theory), except that here the coefficients a_{nm} and b_{nm} will include a reflection from the surface. In addition to the three fields described by Eqs. (2.1)–(2,3), a fourth field is incident upon the sphere. The field is a result of the scattered field of the sphere's reflecting off the surface and striking the sphere. This interacting field is also expanded:

$$\mathbf{E}^{\mathrm{int}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} g_{nm}\mathbf{M}_{nm}^{(1)} + h_{nm}\mathbf{N}_{nm}^{(1)},$$

$$\mathbf{H}^{\mathrm{int}} = \frac{k}{i\omega\mu}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} h_{nm}\mathbf{M}_{nm}^{(1)} + g_{nm}\mathbf{N}_{nm}^{(1)}. \tag{2.4}$$

The radial functions used for the interacting field are the spherical Bessel functions $j_n(kr)$.

The next step is to apply the boundary conditions at the surface of the sphere. The coefficients for the internal fields, c_{nm} and d_{nm}, can be eliminated, and the scattering coefficients can be solved:

$$\begin{aligned} e_{nm} &= -(a_{nm} + g_{nm}) \\ &\quad\times \frac{k_{\mathrm{sph}}\mu\psi_n'(k_{\mathrm{sph}}a)\psi_n(ka) - k\mu_{\mathrm{sph}}\psi_n'(ka)\psi_n(k_{\mathrm{sph}}a)}{k_{\mathrm{sph}}\mu\psi_n'(k_{\mathrm{sph}}a)\xi_n(ka) - k\mu_{\mathrm{sph}}\xi_n'(ka)\psi_n(k_{\mathrm{sph}}a)} \\ &= (a_{nm} + g_{nm})Q_e^n, \\ f_{nm} &= -(b_{nm} + h_{nm}) \\ &\quad\times \frac{k_{\mathrm{sph}}\mu\psi_n'(ka)\psi_n(k_{\mathrm{sph}}a) - k\mu_{\mathrm{sph}}\psi_n'(k_{\mathrm{sph}}a)\psi_n(ka)}{k_{\mathrm{sph}}\mu\xi_n'(ka)\psi_n(k_{\mathrm{sph}}a) - k\mu_{\mathrm{sph}}\psi_n'(k_{\mathrm{sph}}a)\xi_n(ka)} \\ &= (b_{nm} + h_{nm})Q_f^n, \end{aligned} \tag{2.5}$$

where $\psi(r)$ and $\xi(r)$ are the Riccati–Bessel functions defined by

$$\phi_n(r) = rj_n(r), \qquad \xi_n(r) = rh_n^{(1)}(r), \tag{2.6}$$

and the primes denote derivatives with respect to the argument.

The interaction coefficients can be solved by using the relationship between the scattered field and the interaction field. The interaction field is due to the scattered field's reflecting off the surface and striking the sphere. The interaction field is the image of the scattered field. Identically, the interaction potential is the image of the scattered potential.

Figure 2 shows the two coordinate systems, one centered about the real sphere and one at an image location of the coordinate system, a distance of $2d$ along the positive z axis. It is easiest to consider potentials when solving for the interaction coefficients. The interaction potentials are the scattering potentials centered about the image coordinate. The interaction coefficients, however, are inverted and undergo a reflection at the surface. The inversion can be explained by taking advantage of the nature of the associated Legendre polynomials that make up the potentials; i.e.,

$$\tilde{P}_n^m[\cos(\pi - \vartheta)] = (-1)^{n+m}\tilde{P}_n^m[\cos(\vartheta)]. \tag{2.7}$$

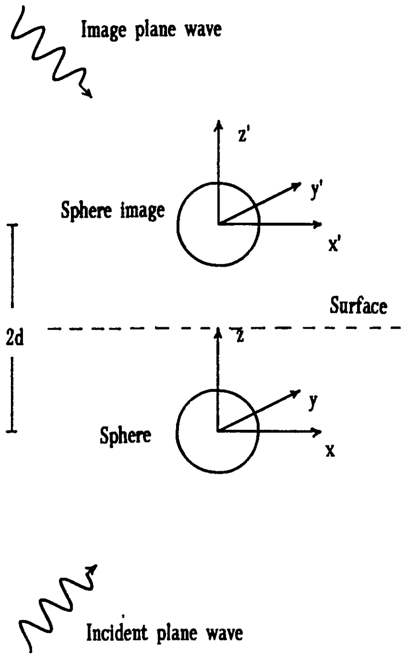

Fig. 2. Image coordinate surface is located a distance $2d$ from the sphere coordinate system along the positive z axis. Fields in the image coordinate system are inverted; e.g., the image of the incident plane wave travels in the x–z plane at angle $\pi - \alpha$ with respect to the z axis.

An approximation is made in order to account for the interaction. The interacting radiation is assumed to strike the surface at normal incidence. A Fresnel reflection at normal incidence is then used to account for the reflection loss at the surface. This same approximation was used by Yousif to solve for the scattering from a cylinder on a surface that has been verified by experiment. Justification for this approximation can be seen by tracing rays from the image coordinate system to the edges of the sphere, as in Fig. 3. The maximum angle at which a ray emanating from the center of the image coordinate system strikes the surface is 30°, and the Fresnel coefficients are fairly constant from normal incidence out to this angle for most optical materials. For highly conducting optical surfaces, such as mirrors, the Fresnel coefficients are nearly constant from 0° to 90°, and, as the refractive index approaches ∞, the Fresnel coefficients are constants and there is no approximation. The Fresnel reflection and transmission coefficients for this system can be written as follows:

$$R_{\mathrm{TE}}(\vartheta_i) = \frac{Z_2\cos\vartheta_i - Z_1[1 - (n_1/n_2)^2\sin^2\vartheta_i]^{1/2}}{Z_2\cos\vartheta_i + Z_1[1 - (n_1/n_2)^2\sin^2\vartheta_i]^{1/2}},$$

$$T_{\mathrm{TE}}(\vartheta_i) = \frac{2Z_2\cos\vartheta_i}{Z_2\cos\vartheta_i + Z_1[1 - (n_1/n_2)^2\sin^2\vartheta_i]^{1/2}},$$

$$R_{\mathrm{TM}}(\vartheta_i) = -\frac{Z_1\cos\vartheta_i - Z_2[1 - (n_1/n_2)^2\sin^2\vartheta_i]^{1/2}}{Z_1\cos\vartheta_i + Z_2[1 - (n_1/n_2)^2\sin^2\vartheta_i]^{1/2}},$$

$$T_{\mathrm{TM}}(\vartheta_i) = \frac{2Z_2\cos\vartheta_i}{Z_1\cos\vartheta_i + Z_2[1 - (n_1/n_2)^2\sin^2\vartheta_i]^{1/2}}, \tag{2.8}$$

where

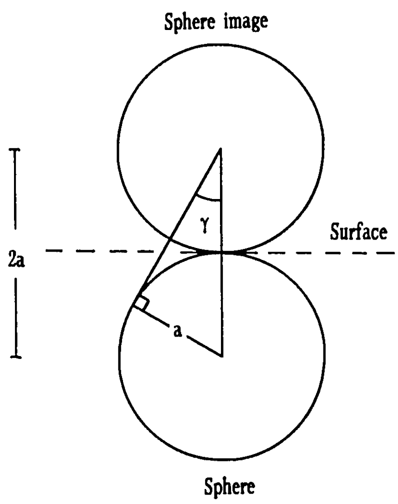

Fig. 3. Maximum angle of incidence on the surface γ for an interacting ray occurs for a ray traced from the image coordinate system (located at the center of the image sphere) to the edge of the sphere and is greatest when the image sphere touches the real sphere (the sphere touches the surface). This angle can be no greater than 30°. The Fresnel equations are fairly constant near normal incidences.

$$\frac{Z_1}{Z_2} = \frac{\mu_1 k_2}{\mu_2 k_1}, \tag{2.9}$$

the TE and TM subscripts pertain to the transverse electric and transverse magnetic cases, and the subscripts 1 and 2 refer to the optical parameters of the media below and above the surface, respectively. The interaction terms can be expressed in the image coordinate system as

$$g'_{nm} u'^{(3)}_{nm} = R(0°)(-1)^{n+m} e_{nm} u'^{(3)}_{nm},$$
$$h'_{nm} u'^{(3)}_{nm} = R(0°)(-1)^{n+m} f_{nm} u'^{(3)}_{nm}, \tag{2.10}$$

where the primes denote quantities in the image coordinate system. Note that $R_{\mathrm{TE}}(0°) = R_{\mathrm{TM}}(0°) = R(0°)$.

At this point the scattering coefficients are essentially solved, but they are coefficients of functions in a different coordinate system. The next step is to find a representation of the functions in the unprimed coordinate system. Each of the primed functions can be expanded in terms of functions in the unprimed system; i.e.,

$$g'_{nm} u'^{(3)}_{nm} = R(0°)(-1)^{n+m} e_{nm} \sum_{n'=|m|}^{\infty} c_{n'}^{(n,m)} u_{n'm}^{(1)},$$

$$h'_{nm} u'^{(3)}_{nm} = R(0°)(-1)^{n+m} f_{nm} \sum_{n'=|m|}^{\infty} c_{n'}^{(n,m)} u_{n'm}^{(1)}. \tag{2.11}$$

The interaction coefficients can be solved by extending summations over n and m. Collecting like terms yields

$$g_{nm} = R(0°) \sum_{n'=|m|}^{\infty} (-1)^{n'+m} e_{n'm} c_n^{(n',m)},$$

$$h_{nm} = R(0°) \sum_{n'=|m|}^{\infty} (-1)^{n'+m} f_{n'm} c_n^{(n',m)}. \tag{2.12}$$

Substituting these expressions into Eqs. (2.5) yields

$$e_{nm} = \left[a_{nm} + R(0°) \sum_{n'=|m|}^{\infty} (-1)^{n'+m} e_{n'm} c_n^{(n',m)} \right] Q_e^n,$$

$$f_{nm} = \left[b_{nm} + R(0°) \sum_{n'=|m|}^{\infty} (-1)^{n'+m} f_{n'm} c_n^{(n',m)} \right] Q_e^n. \tag{2.13}$$

The only unknowns are the translation coefficients $c_n^{(n',m)}$. These were solved by Bobbert and Vlieger by way of recursion relations. In this paper the translation is along the positive z axis rather than the negative z axis. This creates some minor differences. The coefficients can be determined by using the following equations:

$$c_n^{(0,0)} = (2n+1)^{1/2} h_n^{(1)}(2kd), \tag{2.14}$$

while

$$c_n^{(-1,0)} = -(2n+1)^{1/2} h_n^{(1)}(2kd), \tag{2.15}$$

$$\begin{aligned}[(n'-m+1)(n'+m)(2n+1)]^{1/2} c_n^{(n',m)} &= [(n-m+1)(n+m)(2n+1)]^{1/2} c_n^{(n',m-1)} \\ &\quad - 2kd \left[\frac{(n-m+2)(n-m+1)}{(2n+3)} \right]^{1/2} c_{n+1}^{(n',m-1)} \\ &\quad - 2kd \left[\frac{(n+m)(n+m-1)}{(2n-1)} \right]^{1/2} c_{n-1}^{(n',m-1)},\end{aligned} \tag{2.16}$$

$$c_n^{(n',m)} = c_n^{(n',-m)}, \tag{2.17}$$

$$\begin{aligned} n' c_n^{(n'-1,0)} \left(\frac{2n+1}{2n'-1} \right)^{1/2} - (n'+1) c_n^{(n'+1,0)} \left(\frac{2n+1}{2n'+3} \right)^{1/2} \\ = (n+1) c_{n+1}^{(n',0)} \left(\frac{2n'+1}{2n+3} \right)^{1/2} - n c_{n-1}^{(n',0)} \left(\frac{2n'+1}{2n-1} \right)^{1/2}. \end{aligned} \tag{2.18}$$

With the use of these equations, the translation coefficients can be determined, and the only remaining unknowns in Eqs. (2.12) are the coefficients for the incident field. In Subsection 3 these coefficients will be solved for the particular case of an arbitrarily incident plane wave.

3. The Case of Plane-Wave Incident Radiation

The incident radiation on the sphere can be separated into two parts. One part strikes the sphere directly, and the other part reflects off the surface before striking the sphere. The part that reflects off the surface will undergo a Fresnel reflection on striking the surface, and it will also be out of phase by an amount $\exp(i2kd\cos\alpha)$. The Fresnel reflection term depends on whether the plane wave is TE or TM with respect to the plane of incidence (the x–z plane). Both cases must be considered separately. Equations (2.1) must be separated into two parts: one for the TE case and one for the TM case. The fields can be expressed by including both parts of the incident radiation as follows:

$$\mathbf{E}_{\mathrm{TE}}^{\mathrm{inc}} = [1 + R_{\mathrm{TE}}(\alpha)\exp(2ikd\cos\alpha)(-1)^{n+m}] \times \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm}^{\mathrm{TE}} \mathbf{M}_{nm}^{(1)} + b_{nm}^{\mathrm{TE}} \mathbf{N}_{nm}^{(1)},$$

$$\mathbf{H}_{\mathrm{TE}}^{\mathrm{inc}} = \frac{k}{i\omega\mu}[1 + R_{\mathrm{TE}}(\alpha)\exp(2ikd\cos\alpha)(-1)^{n+m}] \times \sum_{n=0}^{\infty}\sum_{m=-n}^{n} b_{nm}^{\mathrm{TE}}\mathbf{M}_{nm}^{(1)} + a_{nm}^{\mathrm{TE}}\mathbf{N}_{nm}^{(1)},$$

$$\mathbf{E}_{\mathrm{TM}}^{\mathrm{inc}} = [1 + R_{\mathrm{TM}}(\alpha)\exp(2ikd\cos\alpha)(-1)^{n+m}] \times \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm}^{\mathrm{TM}}\mathbf{M}_{nm}^{(1)} + b_{nm}^{\mathrm{TM}}\mathbf{N}_{nm}^{(1)},$$

$$\mathbf{H}_{\mathrm{TM}}^{\mathrm{inc}} = \frac{k}{i\omega\mu}[1 + R_{\mathrm{TM}}(\alpha)\exp(2ikd\cos\alpha)(-1)^{n+m}] \times \sum_{n=0}^{\infty}\sum_{m=-n}^{n} b_{nm}^{\mathrm{TM}}\mathbf{M}_{nm}^{(1)} + a_{nm}^{\mathrm{TM}}\mathbf{N}_{nm}^{(1)}. \quad (3.1)$$

The factor of $(-1)^{n+m}$ is a result of Eq. (2.7), since the reflected plane wave is traveling in the x–z plane at an angle of $\pi - \alpha$ with respect to the z axis. Coefficients a_{nm}^{TE}, a_{nm}^{TM}, b_{nm}^{TE}, and b_{nm}^{TM} are now the coefficients for a plane wave traveling in the x–z plane at an angle α with respect to the z axis. The path to the solution is outlined by Stratton[10] (the case for $\alpha = 0°$ is solved). The result, after much algebra, is

$$a_{nm}^{\mathrm{TE}} = \frac{i^n}{n(n+1)}\{[(n-m)(n+m+1)]^{1/2}\tilde{P}_n^{m+1}(\cos\alpha) - [(n-m+1)(n+m)]^{1/2}\tilde{P}_n^{m-1}(\cos\alpha)\} = \frac{2i^{n+2}}{n(n+1)}\frac{\partial\tilde{P}_n^m(\cos\alpha)}{\partial\alpha}, \quad (3.2)$$

$$b_{nm}^{\mathrm{TE}} = \frac{i^{n+2}(2n+1)}{n(n+1)}\left\{\tilde{P}_{n+1}^{m-1}(\cos\alpha) \times \left[\frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)}\right]^{1/2} + \tilde{P}_{n+1}^{m+1}(\cos\alpha) \times \left[\frac{(n+m+1)(n+m+2)}{(2n+1)(2n+3)}\right]^{1/2}\right\} = \frac{2i^{n+2}}{n(n+1)}\frac{m\tilde{P}_n^m(\cos\alpha)}{\sin\alpha}, \quad (3.3)$$

$$a_{nm}^{\mathrm{TM}} = ib_{nm}^{\mathrm{TE}}, \quad (3.4)$$

$$b_{nm}^{\mathrm{TM}} = ia_{nm}^{\mathrm{TE}}. \quad (3.5)$$

From these expressions, Eqs. (2.5) take on the following form:

$$e_{nm}^{\mathrm{TE}} = \left\{[1 + R_{\mathrm{TE}}(\alpha)(-1)^{n+m}\exp(2ikd\cos\alpha)]a_{nm}^{\mathrm{TE}} + R_{\mathrm{TE}}(0°)\sum_{n'=|m|}^{\infty}(-1)^{n'+m}e_{n'm}^{\mathrm{TE}}c_n^{(n',m)}\right\}Q_e^n,$$

$$e_{nm}^{\mathrm{TM}} = \left\{[1 + R_{\mathrm{TM}}(\alpha)(-1)^{n+m}\exp(2ikd\cos\alpha)]a_{nm}^{\mathrm{TM}} + R_{\mathrm{TM}}(0°)\sum_{n'=|m|}^{\infty}(-1)^{n'+m}e_{n'm}^{\mathrm{TM}}c_n^{(n',m)}\right\}Q_e^n,$$

$$f_{nm}^{\mathrm{TE}} = \left\{[1 + R_{\mathrm{TE}}(\alpha)(-1)^{n+m}\exp(2ikd\cos\alpha)]b_{nm}^{\mathrm{TE}} + R_{\mathrm{TE}}(0°)\sum_{n'=|m|}^{\infty}(-1)^{n'+m}f_{n'm}^{\mathrm{TE}}c_n^{(n',m)}\right\}Q_f^n,$$

$$f_{nm}^{\mathrm{TM}} = \left\{[1 + R_{\mathrm{TM}}(\alpha)(-1)^{n+m}\exp(2ikd\cos\alpha)]b_{nm}^{\mathrm{TM}} + R_{\mathrm{TM}}(0°)\sum_{n'=|m|}^{\infty}(-1)^{n'+m}f_{n'm}^{\mathrm{TM}}c_n^{(n',m)}\right\}Q_f^n. \quad (3.6)$$

In practice the infinite number of terms in these equations are not needed. Experimental precision determines the number of terms needed. In any case the scattering coefficients may be solved, since there always remains an equivalent number of equations and unknowns. The next step in this analysis is to develop the Mueller scattering matrix.

4. The Mueller Scattering Matrix

Two axes are used for determining the Mueller scattering matrix, one perpendicular to the plane of incidence ($\hat{\varphi}$) and one parallel to the plane of incidence ($\hat{\varphi}$). The matrix will also be derived for the far-field solution, where $kr \gg n^2$. In this limit the spherical Hankel functions reduce to spherical waves:

$$h_n^{(1)}(kr) \sim \frac{(-i)^n}{ikr}e^{ikr}. \quad (4.1)$$

The field below the surface is composed of two parts. One part results from the sphere directly. The other part results from the field reflecting off the surface. This part is centered about the image coordinate system and is, in fact, the interaction term. In the far field the contributions of spherical waves can be added without translating the image coordinate system. This image field will contain a Fresnel reflection term, a phase term equivalent to $\exp(-2ikd\cos\vartheta)$, and an inversion term equivalent to $(-1)^{n+m}$. The field above the surface results only from the scattering directly from the sphere that is reduced by the Fresnel transmission factor when passing through the surface.

The starting point for calculating the Mueller scattering matrix is the amplitude scattering matrix, which is defined by

$$\begin{bmatrix} E_\vartheta^{\mathrm{sca}} \\ E_\phi^{\mathrm{sca}} \end{bmatrix} = \frac{e^{ikr}}{-ikr}\begin{bmatrix} S_2 & S_3 \\ S_4 & S_1 \end{bmatrix}\begin{bmatrix} E_{\mathrm{TM}}^{\mathrm{inc}} \\ E_{\mathrm{TE}}^{\mathrm{inc}} \end{bmatrix}. \quad (4.2)$$

The scattering amplitude matrix elements are solved by expanding the scattered electric fields in terms of the vector wave functions and then expanding the vector wave functions in terms of the polarization directions by using Eqs. (1.6). The scattering amplitude matrix elements for $|\vartheta| > \pi/2$ assume the following form:

$$S_1 = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-i)^n e^{im\varphi} \times [1 + R_{\mathrm{TE}}(\pi - \vartheta)(-1)^{n+m}\exp(-2ikd\cos\vartheta)] \times \left[\frac{f_{nm}^{\mathrm{TE}}m}{\sin\vartheta}\tilde{P}_n^m(\cos\vartheta) + e_{nm}^{\mathrm{TE}}\frac{\partial}{\partial\vartheta}\tilde{P}_n^m(\cos\vartheta)\right],$$

$$S_2 = -i \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-i)^n e^{im\varphi} \times [1 + R_{\mathrm{TM}}(\pi - \vartheta)(-1)^{n+m} \exp(-2ikd \cos \vartheta)] \times \left[\frac{e_{nm}^{\mathrm{TM}} m}{\sin \vartheta} \tilde{P}_n^m(\cos \vartheta) + f_{nm}^{\mathrm{TM}} \frac{\partial}{\partial \vartheta} \tilde{P}_n^m(\cos \vartheta)\right],$$

$$S_3 = -i \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-i)^n e^{im\varphi} \times [1 + R_{\mathrm{TM}}(\pi - \vartheta)(-1)^{n+m} \exp(-2ikd \cos \vartheta)] \times \left[\frac{e_{nm}^{\mathrm{TE}} m}{\sin \vartheta} \tilde{P}_n^m(\cos \vartheta) + f_{nm}^{\mathrm{TE}} \frac{\partial}{\partial \vartheta} \tilde{P}_n^m(\cos \vartheta)\right],$$

$$S_4 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-i)^n e^{im\varphi} \times [1 + R_{\mathrm{TE}}(\pi - \vartheta)(-1)^{n+m} \exp(-2ikd \cos \vartheta)] \times \left[\frac{f_{nm}^{\mathrm{TM}} m}{\sin \vartheta} \tilde{P}_n^m(\cos \vartheta) + e_{nm}^{\mathrm{TM}} \frac{\partial}{\partial \vartheta} \tilde{P}_n^m(\cos \vartheta)\right]. \quad (4.3)$$

The elements behind the surface ($|\vartheta| < \pi/2$) take on the form

$$S_1 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-i)^n e^{im\varphi} T_{\mathrm{TE}}(\vartheta) \times \left[\frac{f_{nm}^{\mathrm{TE}} m}{\sin \vartheta} \tilde{P}_n^m(\cos \vartheta) + e_{nm}^{\mathrm{TE}} \frac{\partial}{\partial \vartheta} \tilde{P}_n^m(\cos \vartheta)\right],$$

$$S_2 = -i \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-i)^n e^{im\varphi} T_{\mathrm{TM}}(\vartheta) \times \left[\frac{e_{nm}^{\mathrm{TM}} m}{\sin \vartheta} \tilde{P}_n^m(\cos \vartheta) + f_{nm}^{\mathrm{TM}} \frac{\partial}{\partial \vartheta} \tilde{P}_n^m(\cos \vartheta)\right],$$

$$S_3 = -i \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-i)^n e^{im\varphi} T_{\mathrm{TM}}(\vartheta) \times \left[\frac{e_{nm}^{\mathrm{TE}} m}{\sin \vartheta} \tilde{P}_n^m(\cos \vartheta) + f_{nm}^{\mathrm{TE}} \frac{\partial}{\partial \vartheta} \tilde{P}_n^m(\cos \vartheta)\right],$$

$$S_4 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-i)^n e^{im\varphi} T_{\mathrm{TE}}(\vartheta) \times \left[\frac{f_{nm}^{\mathrm{TM}} m}{\sin \vartheta} \tilde{P}_n^m(\cos \vartheta) + e_{nm}^{\mathrm{TM}} \frac{\partial}{\partial \vartheta} \tilde{P}_n^m(\cos \vartheta)\right]. \quad (4.4)$$

For the transmission case, ϑ is the angle incident upon the surface. The actual field inside the surface is determined from Snell's law. Most measurements are taken by placing a detector in air on the opposite side of a thin slab of the bulk material. If the rear surface of the bulk material is antireflection coated, these are the elements that would result. The Mueller scattering matrix can be determined directly from these elements by using relations given by Bohern and Huffman.

A few symmetries that exist simplify the calculations. A careful examination of Eqs. (3.6) reveals that

$$\begin{aligned} e_{n\bar{m}}^{\mathrm{TE}} &= (-1)^m e_{nm}^{\mathrm{TE}}, \\ e_{n\bar{m}}^{\mathrm{TM}} &= (-1)^{m+1} e_{nm}^{\mathrm{TM}}, \\ f_{n\bar{m}}^{\mathrm{TE}} &= (-1)^{m+1} f_{nm}^{\mathrm{TE}}, \\ f_{n\bar{m}}^{\mathrm{TM}} &= (-1)^m f_{nm}^{\mathrm{TM}}, \end{aligned} \quad (4.5)$$

where $\bar{m} = -m$. If the Mueller scattering matrix is measured in the plane of incidence ($\varphi = 0°$), then from these symmetries and Eqs. (4.3) and (4.4), S_3 and S_4 reduce to zero, since for the normalized associated Legendre polynomials

$$\tilde{P}_n^{-m}(x) = (-1)^m \tilde{P}_n^m(x). \quad (4.6)$$

Of the Mueller scattering matrix elements, only four remain that are not zero or simple multiples of another element. These may be determined with the following relations:

$$\begin{aligned} S_{11} &= \tfrac{1}{2}(|S_1|^2 + |S_2|^2), \\ S_{12} &= \tfrac{1}{2}(|S_2|^2 - |S_1|^2), \\ S_{33} &= \mathrm{Re}(S_1 S_2^*), \\ S_{34} &= \mathrm{Im}(S_2 S_1^*). \end{aligned} \quad (4.7)$$

In the plane of incidence these four matrix elements are sufficient to catagorize the light scattering from a sphere on or near a plane surface.

CONCLUSION

Relations for calculating the light-scattering Mueller matrix for a sphere of arbitrary size and optical parameters on or near a surface of arbitrary complex refractive index, illuminated by light of arbitrary wavelength and incident angle, are provided. When the matrix is determined in the plane of incidence, the Mueller matrix can be characterized by four elements.

The particular functions used in the derivation facilitate the calculation of the matrix elements. Incorporating Eqs. (2.7), (4.5), and (4.6) into a scattering program will significantly reduce the number of calculations and the run time.

ACKNOWLEDGMENTS

The author thanks William L. Wolfe and William S. Bickel for their support and encouragement. He also thanks Mary Turner, Paul Spyak, and Lowell Lamb for many helpful discussions. This research was partly supported by a fellowship from Itek Optical Systems.

REFERENCES

1. Lord Rayleigh, "On the electromagnetic theory of light," Philos. Mag. 12, 81–101 (1881).
2. W. von Ignatowsky, "Reflexion elektromagnetischer Wellen an einem Draht," Ann. Phys. (Leipzig) 18, 495–522 (1905).
3. G. Mie, "Beitrage zer Optik truber Meiden speziell kolloidaler Metallosungen," Ann. Phys. (Leipzig) 25, 377–445 (1908).
4. C. Liang and Y. T. Lo, "Scattering by two spheres," Radio Sci. 2, 1481–1495 (1967).
5. G. W. Kattawar and C. E. Dean, "Electromagnetic scattering from two dielectric spheres: comparison between theory and experiment," Opt. Lett. 8, 48–52 (1983).
6. R. T. Wang, J. M. Greenberg, and D. W. Schuerman, "Experimental results of dependent light scattering by two spheres," Opt. Lett. 8, 543–545 (1981).

7. P. A. Bobbert and J. Vlieger, "Light scattering by a sphere on a substrate," Physica **137A,** 209–241 (1986).
8. H. Yousif, "Light scattering from parallel tilted fibers," Ph.D. dissertation (Department of Physics, University of Arizona, Tucson, Ariz., 1987).
9. L. Lamb, Department of Physics, University of Arizona, Tucson, Ariz. 85721 (personal communication).
10. T. C. Rao and R. Barakat, "Plane-wave scattering by a conducting cylinder partially buried in a ground plane. 1. TM case," J. Opt. Soc. Am. A **6,** 1270–1280 (1989).
11. J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
12. C. F. Bohren and D. R. Huffman, *Absorption and Scattering of Light by Small Particles* (Wiley, New York, 1983).
13. H. C. van de Hulst, *Light Scattering by Small Particles* (Wiley, New York, 1957).
14. M. Kerker, *The Scattering of Light and Other Electromagnetic Radiation* (Academic, New York, 1969).