Spherical Wave Operators and the Translation Formulas

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Abstract—**Translation formulas for both scalar and vector spherical wave solutions of the Helmholtz equation are developed in a straightforward manner using differential operator representations for the modal functions and well-known expressions for the scalar and dyadic free-space Green's functions. The expansion coefficients are given in compact integral or differential operator forms useful for analytic investigation.**

I. Introduction

SPHERICAL wave functions, i.e., canonical solutions of the Helmholtz equation in spherical coordinates, form complete sets that can be used as bases to expand more general solutions of the Helmholtz equation. The translation formulas are examples of such expansions since they express translated spherical waves as superpositions of untranslated spherical waves.

Translational formulas are useful when it is necessary to expand the fields of a source about some point other than a natural center of symmetry. A typical case is the scattering of electromagnetic waves by multiple spheres [1], [2]. Another example occurs in probe-compensated spherical near-field scanning [3], [4] where the probe receiving function must be written in the test antenna reference frame.

Friedman and Russek [5] were among the first to give explicit statements of translation formulas for scalar spherical waves. (See Danos and Maximon [6] for a more complete historical discussion.) Their derivation employed integral representations (3) and (19) which permitted them to exploit the simple translational properties of plane waves. Various improvements and a generalization to include vector waves were made by Stein [7] and Cruzan [8]. The vector translation formulas resulted from an involved computation based on the corresponding scalar formulas and the vector wave definitions (32) and (44). The expansion coefficients were given by complicated expressions that are not easily simplified.

Motivated by previous work on the spherical scanning algorithm [9]–[11], the translation formula development given in this paper employs representations for spherical waves in terms of linear differential (translationally invariant) operators of finite order ((5), (17), (36), and (45)). While largely complementary to the integral representation approach of Friedman and Russek, this method offers an attractive feature in the fact that the translation formulas are almost trivial consequences of the differential operator representations and familiar spherical wave expansions for the scalar and dyadic free-space Green's functions. Vector translation formulas are developed directly and easily by generalizing the techniques applied in the scalar case. Expansion coefficients are obtained as compact integral or differential operator expressions that can be evaluated to obtain the conventional results.

In Section II translation formulas are derived for scalar spherical waves, beginning in Section II-A with the relatively simple standing wave case. The translation formula follows from the matching of untranslated and translated functions and derivatives at equivalent positions. Traveling waves are discussed in Section II-B where it it necessary to divide space into two regions depending on the distances ρ and r from the origin to the source and observation points. The interior formula ($r < \rho$) is found through a procedure analogous to that used in Section II-A. This leads to the familiar spherical wave expansion for the free-space scalar Green's function. The exterior ($r > \rho$) or interior translation formula can be obtained from the differential operator representation and the Green's function expansion. Section III largely parallels Section II, establishing translation formulas for both standing and traveling vector waves (Sections III-A and III-B). Section IV illustrates how the vector expansion coefficients can be computed in terms of the scalar coefficients. Appendix I gives explicit forms for the spherical wave operators. Appendix II discusses the evaluation of (14), an integral involving the product of three spherical harmonics.

Most of the results collected here are not new. A major goal of this work has been to devise a simple self-contained development of the spherical wave translation formulas that represents an improvement over those otherwise available. The spherical wave operator techniques demonstrated are very powerful and are perhaps worthy of consideration for other applications.

II. Scalar Waves

A. Standing Waves

The function

$$u_{nm}^{(1)}(\mathbf{r}) = j_n(kr)\, Y_{nm}(\hat{\mathbf{r}}) \tag{1}$$

is a scalar solution of the Helmholtz equation

$$(\nabla^2 + k^2)u = 0. \tag{2}$$

Radial dependence is given by the spherical Bessel function

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$j_n(kr)$, and the angular dependence is given by the scalar spherical harmonic $Y_{nm}(\hat{\mathbf{r}})$ (a caret denotes a unit vector). Jackson [12] may be consulted for the elementary properties of these functions.

The integral representation

$$u_{nm}^{(1)}(\mathbf{r}) = \frac{1}{4\pi i^n} \int Y_{nm}(\hat{\mathbf{k}}) \exp(i\mathbf{k} \cdot \mathbf{r})\, d\hat{\mathbf{k}} \tag{3}$$

is a consequence of a familiar expansion of a plane wave into spherical waves [12, eq. 16.127]. The magnitude of $\mathbf{k}$ is $k = \omega/c$, and the integration is over 4π steradians:

$$\int d\hat{\mathbf{k}} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta$$

where θ and ϕ are the polar angles of $\mathbf{k}$. Equation (3) expresses the spherical wave as a superposition of plane waves traveling in all directions.

The scalar spherical wave operator $\mathcal{P}_{nm}$, defined in Appendix I, has the property

$$\mathcal{P}_{nm} \exp(i\mathbf{k} \cdot \mathbf{r}) = Y_{nm}(\hat{\mathbf{k}}) \exp(i\mathbf{k} \cdot \mathbf{r}). \tag{4}$$

$\mathcal{P}_{nm}$ is a linear differential operator of nth order involving only the spatial derivatives ∂_x, ∂_y, and ∂_z. In addition, $\mathcal{P}_{nm}$ is translationally invariant since it is unaltered by the transformation $\mathbf{r} \rightarrow \mathbf{r} - \boldsymbol{\rho}$.

A combination of (3) and (4) leads to a differential operator representation for $u_{nm}^{(1)}$:

$$\begin{aligned} u_{nm}^{(1)}(\mathbf{r}) &= \frac{1}{4\pi i^n} \int \mathcal{P}_{nm} \exp(i\mathbf{k} \cdot \mathbf{r})\, d\hat{\mathbf{k}} \\ &= i^{-n}\mathcal{P}_{nm} \frac{1}{4\pi} \int \exp(i\mathbf{k} \cdot \mathbf{r})\, d\hat{\mathbf{k}} \\ &= i^{-n}\mathcal{P}_{nm} \frac{\sin(kr)}{kr} = i^{-n}\mathcal{P}_{nm} j_0(kr). \end{aligned} \tag{5}$$

Equation (5) can be verified independently with an inductive argument using the explicit form (70) for $\mathcal{P}_{nm}$. The integral representation (3) can then be established by reversing the steps leading to the differential operator representation (5).

If $u_{nm}^{(1)}(\mathbf{r})$ is translated by a displacement $\boldsymbol{\rho}$, the result is a function $u_{nm}^{(1)\prime}(\mathbf{r})$ that is given by the relation

$$u_{nm}^{(1)\prime}(\mathbf{r}) = u_{nm}^{(1)}(\mathbf{r} - \boldsymbol{\rho}). \tag{6}$$

Transformations are taken in the active sense here. For example, a function $f(\mathbf{r})$ becomes a new function of the old coordinates $f'(\mathbf{r})$. (In the passive sense the function is re-expressed in a translated coordinate system $f(\mathbf{r}')$.) Since $u_{nm}^{(1)\prime}(\mathbf{r})$ is a solution of the Helmholtz equation, it can be written in terms of untranslated spherical waves:

$$u_{nm}^{(1)\prime}(\mathbf{r}) = u_{nm}^{(1)}(\mathbf{r} - \boldsymbol{\rho}) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} A(\nu\mu | nm; \boldsymbol{\rho}) u_{\nu\mu}^{(1)}(\mathbf{r}). \tag{7}$$

This is the desired form for the translation formula. Such an expansion is convergent for all $\mathbf{r}$ [13, theorem 13, p. 83]. The $A(\nu\mu | nm; \boldsymbol{\rho})$ can be found using the requirement that the translated and untranslated functions and their derivatives must agree at equivalent points. Instead of working with the familiar gradient operator, it is more convenient to apply the spherical wave operators defined earlier.

Fig. 1 presents a physical picture from a measurement point of view. In Fig. 1(a) a probe at the location $-\boldsymbol{\rho}$ interacts with a test antenna located at the origin. In Fig. 1(b) the entire experimental apparatus is translated by a displacement $\boldsymbol{\rho}$ so that the probe and test antenna are located at $\mathbf{0}$ and $\boldsymbol{\rho}$, respectively. Clearly, the results of measurements made in both orientations should be identical.

Mathematically, this translational invariance requirement is equivalent to the statement

$$[\mathcal{P}_{\nu\mu} u_{nm}^{(1)}](-\boldsymbol{\rho}) = [\mathcal{P}_{\nu\mu} u_{nm}^{(1)\prime}](\mathbf{0}) \tag{8}$$

where for present purposes the test antenna is identified with the modal function $u_{nm}^{(1)}(\mathbf{r})$ and the probe is represented by the operator $\mathcal{P}_{\nu\mu}$. Combining (7) and (8),

$$[\mathcal{P}_{\nu\mu} u_{nm}^{(1)}](-\boldsymbol{\rho}) = \sum_{\alpha\beta} A(\alpha\beta | nm; \boldsymbol{\rho}) [\mathcal{P}_{\nu\mu} u_{\alpha\beta}^{(1)}](\mathbf{0}) \tag{9}$$

where $\mathcal{P}_{\nu\mu}$ has been taken inside the summation. Although a formal justification will not be attempted here, it can be shown by generalizing the arguments of Müller [13, sec. 4] that (7) is arbitrarily often differentiable, term by term, with respect to x, y, and z. As a consequence of (3), (4) and the orthonormality of the spherical harmonics,

$$[\mathcal{P}_{\nu\mu} u_{\alpha\beta}^{(1)}](\mathbf{0}) = \frac{1}{4\pi i^{\alpha}} \int Y_{\nu\mu}(\hat{\mathbf{k}}) Y_{\alpha\beta}(\hat{\mathbf{k}})\, d\hat{\mathbf{k}} = \frac{(-)^{\mu}}{4\pi i^{\nu}} \delta_{\nu\alpha}\delta_{-\mu\beta}. \tag{10}$$

Thus the right side of (9) reduces to a single term, and

$$\begin{aligned} A(\nu\mu | nm; \boldsymbol{\rho}) &= (-)^{\mu} 4\pi i^{\nu} [\mathcal{P}_{\nu,-\mu} u_{nm}^{(1)}](-\boldsymbol{\rho}) \\ &= (-)^{\mu} 4\pi i^{-\nu+n} \mathcal{P}_{\nu,-\mu}\mathcal{P}_{nm} j_0(k\rho). \end{aligned} \tag{11}$$

(When $\mathbf{r} \rightarrow -\mathbf{r}$, $\mathcal{P}_{nm} \rightarrow (-)^n\mathcal{P}_{nm}$; see (70) in Appendix I.) The integral form

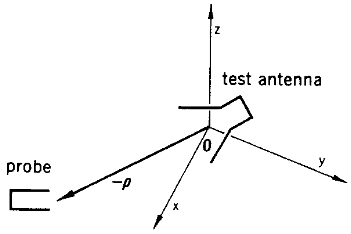
$$A(\nu\mu | nm; \boldsymbol{\rho}) = (-)^{\mu} i^{-\nu+n} \int Y_{\nu,-\mu}(\hat{\mathbf{k}}) \cdot Y_{nm}(\hat{\mathbf{k}}) \exp(i\mathbf{k} \cdot \boldsymbol{\rho})\, d\hat{\mathbf{k}} \tag{12}$$

follows from (3), (4), and (11).

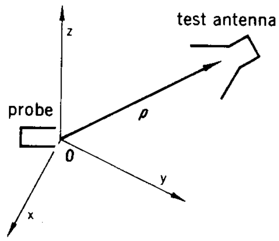
Equations (11) and (12) are convenient analytic expressions for the coefficients in the translation formula (7). These can be evaluated using an expansion of the product of two spherical harmonics:

$$Y_{\nu\mu} Y_{nm} = \sum_{\alpha\beta} a(\alpha\beta | \nu\mu nm) Y_{\alpha\beta}. \tag{13}$$

(This formula also holds with $\mathcal{P}$ substituted for Y with the restriction that the domain be limited to functions that are



(a)



(b)

Fig. 1. Probe and test antenna are shown in equivalent geometries. (a) Test antenna at origin, probe at $-\rho$. (b) Test antenna at ρ, probe at origin.

solutions of the Helmholtz equation.) Calculation of the a's

$$a(\alpha\beta|\nu\mu nm)=\int Y^*_{\alpha\beta}(\hat{\mathbf{k}})Y_{\nu\mu}(\hat{\mathbf{k}})Y_{nm}(\hat{\mathbf{k}})\,d\hat{\mathbf{k}} \tag{14}$$

is sketched in Appendix II. Note that $a(\alpha\beta|\nu\mu nm) = 0$ if $\alpha > \nu + n$ or $\alpha < |\nu - n|$, if $\alpha + \nu + n$ is odd, or if $\beta \neq \mu + m$. Therefore, the number of terms in (13) is finite; in particular, the sum over β only has one nonzero contribution. Substitution of (13) into (12) and application of (3) yields the conventional result [7]:

$$\begin{aligned}A(\nu\mu|nm;\rho)&=(-)^{\mu}i^{-\nu+n}\sum_{\alpha\beta}a(\alpha\beta|\nu,-\mu nm)\\&\quad\cdot\int Y_{\alpha\beta}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\rho)\,d\hat{\mathbf{k}}\\&=(-)^{\mu}4\pi i^{-\nu+n}\sum_{\alpha\beta}i^{\alpha}a(\alpha\beta|\nu,-\mu nm)u^{(1)}_{\alpha\beta}(\rho).\end{aligned} \tag{15}$$

There is considerable simplification for translation along the z axis since, as can be seen from (15), the A's are zero unless $\mu = m$.

Equation (11) can also be evaluated directly using symbolic algebra programs on a computer [14]. This can provide a useful cross check of computations based on (15).

B. Traveling Waves

Next, consider the function

$$u^{(3)}_{nm}(\mathbf{r})=h^{(1)}_n(kr)Y_{nm}(\hat{\mathbf{r}}) \tag{16}$$

where $h^{(1)}_n(kr)$ is the spherical Hankel function of the first kind. If an $\exp(-i\omega t)$ time dependence is assumed, the $u^{(3)}_{nm}(\mathbf{r})$ are appropriate modal functions for describing waves emanating from finite source regions. The operator representation

$$u^{(3)}_{nm}(\mathbf{r})=i^{-n}\mathcal{P}_{nm}h^{(1)}_0(kr) \tag{17}$$

is analogous to (5) and also can be demonstrated by an inductive argument.

A well-known result [15, pp. 338–339]

$$h^{(1)}_0(kr)=\frac{\exp(ikr)}{ikr}=\frac{1}{2\pi}\int\exp(\pm i\mathbf{k}\cdot\mathbf{r})\frac{d\mathbf{K}}{\gamma k},\qquad z\gtrless 0 \tag{18}$$

is the key to obtain an integral representation for $u^{(3)}_{nm}(\mathbf{r})$. Equation (18) is a two-dimensional Fourier transform on transverse $\mathbf{k}$:

$$\int d\mathbf{K}=\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y,\qquad \mathbf{K}=k_x\hat{\mathbf{x}}+k_y\hat{\mathbf{y}},\quad K=|\mathbf{K}|.$$

Due to the restriction $k = |\mathbf{k}|$, the z component of the propagation vector is not independent but is given by

$$\gamma=k_z=\begin{cases}\sqrt{k^2-K^2}, & k\geq K\\ i\sqrt{K^2-k^2}, & k<K\end{cases}.$$

Substitution of (18) into (17) yields [16]

$$u^{(3)}_{nm}(\mathbf{r})=\frac{1}{2\pi i^n}\int Y_{nm}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\mathbf{r})\frac{d\mathbf{K}}{\gamma k},\qquad z>0. \tag{19}$$

Interchange of derivative and integral is valid since the factor $\exp(i\gamma z)$ ensures uniform convergence. Because $d\mathbf{K}/\gamma k$ corresponds to $d\hat{\mathbf{k}}$ by change of variables, (3) and (19) are identical except for the integration range. Notably, (19) is a superposition of plane waves propagating into the $z > 0$ hemisphere (γ real) plus a contribution from evanescent waves (γ imaginary). For $z < 0$ a formula corresponding to (19) follows from (18) with the choice of the lower sign. In the interest of economy, this paper will give such integral results for $z > 0$ only. The reader can easily modify these for $z < 0$ by using the appropriate form of (18).

While the standing wave translation formula (7) holds for all $\mathbf{r}$, it is necessary to divide space into two domains when considering traveling waves. In the interior region ($r < \rho$) the translated function must be expanded in terms of the standing wave basis $u^{(1)}_{nm}(\mathbf{r})$ to avoid singularities at the origin. In the exterior region ($r > \rho$) the translated function must be expanded in terms of the traveling wave basis $u^{(3)}_{nm}(\mathbf{r})$ to satisfy radiation conditions.

Interior Region: In the interior region the translated function $u^{(3)\prime}_{nm}$ can be written

$$u^{(3)\prime}_{nm}(\mathbf{r})=u^{(3)}_{nm}(\mathbf{r}-\rho)=\sum_{\nu\mu}A_i(\nu\mu|nm;\rho)u^{(1)}_{\nu\mu}(\mathbf{r}),\qquad r<\rho. \tag{20}$$

Again, this is the desired form for the translation formula.

The translational invariance requirement (8) becomes

$$[\mathcal{P}_{\nu\mu}u^{(3)}_{nm}](-\rho)=[\mathcal{P}_{\nu\mu}u^{(3)\prime}_{nm}](\mathbf{0}). \tag{21}$$

A combination of (20) and (21) gives

$$[\mathcal{P}_{\nu\mu}u_{nm}^{(3)}](-\boldsymbol{\rho})=\sum_{\alpha\beta}A_i(\alpha\beta|nm;\,\boldsymbol{\rho})[\mathcal{P}_{\nu\mu}u_{\alpha\beta}^{(1)}](\mathbf{0}). \quad (22)$$

Finally, an application of (10) collapses the sum on the right side of (22) to a single term and

$$A_i(\nu\mu|nm;\,\boldsymbol{\rho})=(-)^{\mu}4\pi i^{-\nu+n}\mathcal{P}_{\nu,-\mu}\mathcal{P}_{nm}h_0^{(1)}(k\rho). \quad (23)$$

The integral form

$$A_i(\nu\mu|nm;\,\boldsymbol{\rho})=(-)^{\mu}2i^{-\nu+n}\int Y_{\nu,-\mu}(\hat{\mathbf{k}})Y_{nm}(\hat{\mathbf{k}}) \cdot \exp(i\mathbf{k}\cdot\boldsymbol{\rho})\frac{d\mathbf{K}}{\gamma k}, \qquad \boldsymbol{\rho}\cdot\hat{\mathbf{z}}>0 \quad (24)$$

follows from (4), (18), and (23).

The compact expressions (23) or (24) can be evaluated using (13) and (19):

$$A_i(\nu\mu|nm;\,\boldsymbol{\rho})=(-)^{\mu}2i^{-\nu+n}\sum_{\alpha\beta}a(\alpha\beta|\nu,\,-\mu nm) \cdot\int Y_{\alpha\beta}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\boldsymbol{\rho})\frac{d\mathbf{K}}{\gamma k}$$

$$=(-)^{\mu}4\pi i^{-\nu+n}\sum_{\alpha\beta}i^{\alpha}a(\alpha\beta|\nu,\,-\mu nm)u_{\alpha\beta}^{(3)}(\boldsymbol{\rho}). \quad (25)$$

As in the case of (15), considerable simplification results for translation along the z axis since the A_i's are zero unless $\mu = m$.

When $n = m = 0$, (25) reduces to

$$A_i(\nu\mu|00;\,\boldsymbol{\rho})=(-)^{\mu}\sqrt{4\pi}\,u_{\nu,-\mu}^{(3)}(\boldsymbol{\rho}). \quad (26)$$

Substitution of (26) into (20) gives the spherical wave expansion for the scalar free-space Green's function [12, eq. 16.22]:

$$h_0^{(1)}(k|\mathbf{r}-\boldsymbol{\rho}|)=\frac{\exp(ik|\mathbf{r}-\boldsymbol{\rho}|)}{ik|\mathbf{r}-\boldsymbol{\rho}|}$$

$$=4\pi\sum_{\nu\mu}(-)^{\mu}u_{\nu\mu}^{(1)}(\mathbf{r})u_{\nu,-\mu}^{(3)}(\boldsymbol{\rho}), \qquad r<\rho \quad (27)$$

$$=4\pi\sum_{\nu\mu}(-)^{\mu}u_{\nu\mu}^{(3)}(\mathbf{r})u_{\nu,-\mu}^{(1)}(\boldsymbol{\rho}), \qquad r>\rho. \quad (28)$$

Equation (28) follows from (27) since the left side is unchanged when $\mathbf{r}$ and $\boldsymbol{\rho}$ are exchanged.

Exterior Region: The derivation given for the interior region does not apply in the exterior region since $r > \rho \geq 0$; therefore, (10) cannot be used to project out the expansion coefficients. However, the exterior formula can be obtained from the differential operator representation and the spherical wave expansion for the scalar Green's function: From (17)

$$u_{nm}^{(3)}(\mathbf{r}-\boldsymbol{\rho})=i^{-n}\mathcal{P}_{nm}h_0^{(1)}(k|\mathbf{r}-\boldsymbol{\rho}|),$$

and with (27)

$$u_{nm}^{(3)}(\mathbf{r}-\boldsymbol{\rho})=4\pi i^{-n}\sum_{\nu\mu}(-)^{\mu}u_{\nu,-\mu}^{(3)}(\boldsymbol{\rho})[\mathcal{P}_{nm}u_{\nu\mu}^{(1)}](\mathbf{r}), \qquad r<\rho.$$

Interchanging $\mathbf{r}$ and $\boldsymbol{\rho}$,

$$u_{nm}^{(3)}(\boldsymbol{\rho}-\mathbf{r})=(-)^{n}u_{nm}^{(3)}(\mathbf{r}-\boldsymbol{\rho})$$

$$=4\pi i^{-n}\sum_{\nu\mu}(-)^{\mu}[\mathcal{P}_{nm}u_{\nu,-\mu}^{(1)}](\boldsymbol{\rho})u_{\nu\mu}^{(3)}(\mathbf{r}), \qquad r>\rho.$$

Thus

$$u_{nm}^{(3)\prime}(\mathbf{r})=\sum_{\nu\mu}A_e(\nu\mu|nm;\,\boldsymbol{\rho})u_{\nu\mu}^{(3)}(\mathbf{r}), \qquad r>\rho \quad (29)$$

where

$$A_e(\nu\mu|nm;\,\boldsymbol{\rho})=(-)^{\mu}4\pi i^{-\nu+n}\mathcal{P}_{\nu,-\mu}\mathcal{P}_{nm}j_0(k\rho). \quad (30)$$

Equations (29) and (30) constitute the exterior region translation formula. $A_e(\nu\mu|nm;\,\boldsymbol{\rho})$ is identical to $A(\nu\mu|nm;\,\boldsymbol{\rho})$, the expansion coefficient for the standing wave case given in (11).

Equation (28) could have been used as a starting point for a similar derivation of the interior formulas (20) and (23). The traveling wave translation formulas are therefore simple consequences of the differential operator representation (17) and the familiar expansions (27) and (28). Furthermore, since $j_0(kr)$ is the real part of $h_0^{(1)}(kr)$, the standing wave formulas (7) and (11) also follow from (5) and (27) or (28).

III. Vector Waves

A. Standing Waves

Vector solutions of the Helmholtz equation, $\mathbf{m}_{nm}^{(1)}$ and $\mathbf{n}_{nm}^{(1)}$, can be defined to obey the symmetric relationships [17]

$$\mathbf{m}_{nm}^{(1)}(\mathbf{r})=\frac{1}{k}\nabla\times\mathbf{n}_{nm}^{(1)} \qquad \mathbf{n}_{nm}^{(1)}(\mathbf{r})=\frac{1}{k}\nabla\times\mathbf{m}_{nm}^{(1)}$$

$$\nabla\cdot\mathbf{m}_{nm}^{(1)}=\nabla\cdot\mathbf{n}_{nm}^{(1)}=0 \quad (31)$$

where

$$\mathbf{m}_{nm}^{(1)}(\hat{\mathbf{r}}) \equiv \mathbf{L}u_{nm}^{(1)}(\hat{\mathbf{r}})/\sqrt{n(n+1)}=j_n(kr)\mathbf{X}_{nm}(\hat{\mathbf{r}})$$

$$\mathbf{X}_{nm}(\hat{\mathbf{r}}) \equiv \mathbf{L}Y_{nm}(\hat{\mathbf{r}})/\sqrt{n(n+1)}. \quad (32)$$

$\mathbf{L}$ is the orbital angular momentum operator of quantum mechanics:

$$\mathbf{L} \equiv \frac{1}{i}\,\mathbf{r}\times\nabla = i\left[\hat{\boldsymbol{\theta}}\,\frac{1}{\sin\theta}\,\partial_\phi - \hat{\boldsymbol{\phi}}\,\partial_\theta\right] \tag{33}$$

and the $\mathbf{X}$'s are vector spherical harmonics as defined by Jackson [12, ch. 16]. $\mathbf{m}_{nm}^{(1)}$ and $\mathbf{n}_{nm}^{(1)}$ are useful, for example, in expanding electromagnetic fields in source-free regions. (A third function with nonzero divergence is necessary to complete the set. It will be discussed briefly in Section III-C.)

In Appendix I the vector spherical wave operator

$$\mathcal{P}_{nm} \equiv [\mathbf{L}\mathcal{P}_{nm} - \mathcal{P}_{nm}\mathbf{L}]/\sqrt{n(n+1)} \tag{34}$$

is shown to have the property

$$\mathcal{P}_{nm}\exp(i\mathbf{k}\cdot\mathbf{r}) = \mathbf{X}_{nm}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\mathbf{r}). \tag{35}$$

Equation (5) together with (31)–(34) leads to the differential operator representations

$$\mathbf{m}_{nm}^{(1)}(\mathbf{r}) = i^{-n}\mathcal{P}_{nm}j_0(kr) \tag{36a}$$

$$\mathbf{n}_{nm}^{(1)}(\mathbf{r}) = i^{-n}\,\frac{1}{k}\,\nabla\times\mathcal{P}_{nm}j_0(kr). \tag{36b}$$

From the standpoint of (36), the second term on the right side of (34) seems unimportant since it gives no contribution. However, this term is necessary to ensure the translational invariance of $\mathcal{P}_{nm}$. A derivation of the vector translation formulas based on the corresponding scalar formulas and (32) is complicated by the fact that $\mathbf{L}$ depends explicitly on $\mathbf{r}$ [11].

The vector spherical waves have the integral representations

$$\mathbf{m}_{nm}^{(1)}(\mathbf{r}) = \frac{1}{4\pi i^n}\int \mathbf{X}_{nm}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\mathbf{r})\,d\hat{\mathbf{k}} \tag{37a}$$

$$\mathbf{n}_{nm}^{(1)}(\mathbf{r}) = \frac{1}{4\pi i^n}\int i\hat{\mathbf{k}}\times\mathbf{X}_{nm}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\mathbf{r})\,d\hat{\mathbf{k}} \tag{37b}$$

as can be seen if (35) and (36) are applied to (3). Though equivalent, (37) is not of the form given by Stratton [17]. (Stratton basically applies the definition (32) to (3), effectively ignoring the second term on the right side of (34).)

Because $\mathbf{m}_{nm}^{(1)}(\mathbf{r}-\boldsymbol{\rho})$ and $\mathbf{n}_{nm}^{(1)}(\mathbf{r}-\boldsymbol{\rho})$ are solutions of the Helmholtz equation with zero divergence, the translation formulas have the form

$$\mathbf{m}_{nm}^{(1)\prime}(\mathbf{r}) = \sum_{\nu=1}^{\infty}\sum_{\mu=-\nu}^{\nu}[B(\nu\mu|nm;\boldsymbol{\rho})\mathbf{m}_{\nu\mu}^{(1)}(\mathbf{r}) + C(\nu\mu|nm;\boldsymbol{\rho})\mathbf{n}_{\nu\mu}^{(1)}(\mathbf{r})]. \tag{38a}$$

$$\mathbf{n}_{nm}^{(1)\prime}(\mathbf{r}) = \sum_{\nu=1}^{\infty}\sum_{\mu=-\nu}^{\nu}[B(\nu\mu|nm;\boldsymbol{\rho})\mathbf{n}_{\nu\mu}^{(1)}(\mathbf{r}) + C(\nu\mu|nm;\boldsymbol{\rho})\mathbf{m}_{\nu\mu}^{(1)}(\mathbf{r})]. \tag{38b}$$

Equations (38a) and (38b) can be related by (31): For example, since the gradient operator is translationally invariant,

$$\mathbf{n}_{nm}^{(1)\prime}(\mathbf{r}) = \frac{1}{k}\,\nabla\times\mathbf{m}_{nm}^{(1)\prime}(\mathbf{r}).$$

The coefficients in (38) can be found with the translational invariance argument used in Section II. Corresponding to (8),

$$[\mathcal{P}_{\nu\mu}\cdot\mathbf{m}_{nm}^{(1)}](-\boldsymbol{\rho}) = [\mathcal{P}_{\nu\mu}\cdot\mathbf{m}_{nm}^{(1)\prime}](\mathbf{0}) \tag{39a}$$

$$[\mathcal{P}_{\nu\mu}\cdot\mathbf{n}_{nm}^{(1)}](-\boldsymbol{\rho}) = [\mathcal{P}_{\nu\mu}\cdot\mathbf{n}_{nm}^{(1)\prime}](\mathbf{0}). \tag{39b}$$

From (38a) and (39a)

$$[\mathcal{P}_{\nu\mu}\cdot\mathbf{m}_{nm}^{(1)}](-\boldsymbol{\rho}) = \sum_{\alpha\beta}\{B(\nu\mu|nm;\boldsymbol{\rho})[\mathcal{P}_{\nu\mu}\cdot\mathbf{m}_{\alpha\beta}^{(1)}](\mathbf{0}) + C(\nu\mu|nm;\boldsymbol{\rho})[\mathcal{P}_{\nu\mu}\cdot\mathbf{n}_{\alpha\beta}^{(1)}](\mathbf{0})\}. \tag{40}$$

As a consequence of (35), (37), and the orthonormality of vector spherical harmonics,

$$[\mathcal{P}_{\nu\mu}\cdot\mathbf{m}_{\alpha\beta}^{(1)}](\mathbf{0}) = \frac{1}{4\pi i^\alpha}\int \mathbf{X}_{\nu\mu}(\hat{\mathbf{k}})\cdot\mathbf{X}_{\alpha\beta}(\hat{\mathbf{k}})\,d\hat{\mathbf{k}} = \frac{(-)^{\mu+1}}{4\pi i^\nu}\,\delta_{\nu\alpha}\delta_{-\mu\beta} \tag{41a}$$

$$[\mathcal{P}_{\nu\mu}\cdot\mathbf{n}_{\alpha\beta}^{(1)}](\mathbf{0}) = \frac{1}{4\pi i^\alpha}\int \mathbf{X}_{\nu\mu}(\hat{\mathbf{k}})\cdot i\hat{\mathbf{k}}\times\mathbf{X}_{\alpha\beta}(\hat{\mathbf{k}})\,d\hat{\mathbf{k}} = 0 \tag{41b}$$

and the summation in (40) reduces to a single term. Thus

$$B(\nu\mu|nm;\boldsymbol{\rho}) = (-)^{\mu+1}4\pi i^{-\nu+n}\mathcal{P}_{\nu,-\mu}\cdot\mathcal{P}_{nm}j_0(k\rho). \tag{42a}$$

Similarly, beginning with (38b) and (39b),

$$C(\nu\mu|nm;\boldsymbol{\rho}) = (-)^{\mu+1}4\pi i^{-\nu+n}\,\frac{1}{k}\,\nabla\times\mathcal{P}_{\nu,-\mu}\cdot\mathcal{P}_{nm}j_0(k\rho). \tag{42b}$$

The integral forms

$$B(\nu\mu|nm;\boldsymbol{\rho}) = (-)^{\mu+1}i^{-\nu+n}\int \mathbf{X}_{\nu,-\mu}(\hat{\mathbf{k}})\cdot\mathbf{X}_{nm}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\boldsymbol{\rho})\,d\hat{\mathbf{k}} \tag{43a}$$

$$C(\nu\mu|nm;\boldsymbol{\rho}) = (-)^{\mu+1}i^{-\nu+n}\int i\hat{\mathbf{k}}\times\mathbf{X}_{\nu,-\mu}(\hat{\mathbf{k}})\cdot\mathbf{X}_{nm}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\boldsymbol{\rho})\,d\hat{\mathbf{k}} \tag{43b}$$

follow from (3), (4), and (42).

B. Traveling Waves

For traveling waves, appropriate modal functions are

$$\mathbf{m}_{nm}^{(3)}(\mathbf{r}) \equiv \mathbf{L}u_{nm}^{(3)}(\mathbf{r})/\sqrt{n(n+1)} = h_n^{(1)}(kr)\mathbf{X}_{nm}(\hat{\mathbf{r}})$$

$$\mathbf{n}_{nm}^{(3)}(\mathbf{r}) = \frac{1}{k}\,\nabla\times\mathbf{m}_{nm}^{(3)}. \tag{44}$$

The differential operator representations

$$\mathbf{m}_{nm}^{(3)}(\mathbf{r}) = i^{-n}\mathcal{P}_{nm}h_0^{(1)}(kr) \tag{45a}$$

$$\mathbf{n}_{nm}^{(3)}(\mathbf{r}) = i^{-n}\frac{1}{k}\nabla\times\mathcal{P}_{nm}h_0^{(1)}(kr) \tag{45b}$$

follow from (17), (34), and (44), while the plane-wave representations

$$\mathbf{m}_{nm}^{(3)}(\mathbf{r}) = \frac{1}{2\pi i^n}\int \mathbf{X}_{nm}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\mathbf{r})\frac{d\mathbf{K}}{\gamma k}, \tag{46a}$$

$$z>0$$

$$\mathbf{n}_{nm}^{(3)}(\mathbf{r}) = \frac{1}{2\pi i^n}\int i\hat{\mathbf{k}}\times\mathbf{X}_{nm}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\mathbf{r})\frac{d\mathbf{K}}{\gamma k}, \tag{46b}$$

follow from applying (35) and (45) to (18). Equations (46) are consistent with a well-known relationship between spectrum and far field [18, eq. 1.2-16b].

Interior Region: In the interior region ($r < \rho$) the translation formulas can be written

$$\mathbf{m}_{nm}^{(3)\prime}(\mathbf{r}) = \sum_{\nu\mu}[B_i(\nu\mu|nm;\boldsymbol{\rho})\mathbf{m}_{\nu\mu}^{(1)}(\mathbf{r}) + C_i(\nu\mu|nm;\boldsymbol{\rho})\mathbf{n}_{\nu\mu}^{(1)}(\mathbf{r})] \tag{47a}$$

$$\mathbf{n}_{nm}^{(3)\prime}(\mathbf{r}) = \sum_{\nu\mu}[B_i(\nu\mu|nm;\boldsymbol{\rho})\mathbf{n}_{\nu\mu}^{(1)}(\mathbf{r}) + C_i(\nu\mu|nm;\boldsymbol{\rho})\mathbf{m}_{\nu\mu}^{(1)}(\mathbf{r})], \qquad r<\rho. \tag{47b}$$

This is the most general form for a source-free region that includes the origin.

Vector translation formulas for the interior (or exterior) region can be obtained directly by applying (45) to (27) and (28). While these might be useful in certain problems, they are not immediately in the form (47) and it is a remarkably tedious job to demonstrate the equivalence. The coefficients B_i and C_i are better obtained using the translational invariance condition

$$[\mathcal{P}_{\nu\mu}\cdot\mathbf{m}_{nm}^{(3)}](-\boldsymbol{\rho}) = [\mathcal{P}_{\nu\mu}\cdot\mathbf{m}_{nm}^{(3)\prime}](\mathbf{0}) \tag{48a}$$

$$[\mathcal{P}_{\nu\mu}\cdot\mathbf{n}_{nm}^{(3)}](-\boldsymbol{\rho}) = [\mathcal{P}_{\nu\mu}\cdot\mathbf{n}_{nm}^{(3)\prime}](\mathbf{0}). \tag{48b}$$

Following the procedure of Section III-A, (47) is substituted into (48) and then (41) is used to collapse the summations into single terms. The coefficients are

$$B_i(\nu\mu|nm;\boldsymbol{\rho}) = (-)^{\mu+1}4\pi i^{-\nu+n}\mathcal{P}_{\nu,-\mu}\cdot\mathcal{P}_{nm}h_0^{(1)}(k\rho) \tag{49a}$$

$$C_i(\nu\mu|nm;\boldsymbol{\rho}) = (-)^{\mu+1}4\pi i^{-\nu+n}\frac{1}{k}\nabla\times\mathcal{P}_{\nu,-\mu}\cdot\mathcal{P}_{nm}h_0^{(1)}(k\rho) \tag{49b}$$

or, in integral form,

$$B_i(\nu\mu|nm;\boldsymbol{\rho}) = (-)^{\mu+1}2i^{-\nu+n}\int \mathbf{X}_{\nu,-\mu}(\hat{\mathbf{k}})\cdot\mathbf{X}_{nm}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\boldsymbol{\rho})\frac{d\mathbf{K}}{\gamma k} \tag{50a}$$

$$C_i(\nu\mu|nm;\boldsymbol{\rho}) = (-)^{\mu+1}2i^{-\nu+n}\int i\hat{\mathbf{k}}\times\mathbf{X}_{\nu,-\mu}(\hat{\mathbf{k}})\cdot\mathbf{X}_{nm}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\boldsymbol{\rho})\frac{d\mathbf{K}}{\gamma k}, \qquad \boldsymbol{\rho}\cdot\hat{\mathbf{z}}>0. \tag{50b}$$

If B_i and C_i from (49) are substituted into (47), the resulting expression can be rearranged to give

$$\mathbf{m}_{nm}^{(3)\prime}(\mathbf{r}) = i^{-n}\mathcal{P}_{nm}\cdot\bar{\mathbf{G}}(\mathbf{r}|\boldsymbol{\rho}) \tag{51a}$$

$$\mathbf{n}_{nm}^{(3)\prime}(\mathbf{r}) = i^{-n}\frac{1}{k}\nabla\times\mathcal{P}_{nm}\cdot\bar{\mathbf{G}}(\mathbf{r}|\boldsymbol{\rho}) \tag{51b}$$

where

$$\bar{\mathbf{G}}(\mathbf{r}|\boldsymbol{\rho}) = 4\pi\sum_{\nu\mu}(-)^{\mu+1}[\mathbf{m}_{\nu\mu}^{(3)}(\mathbf{r})\mathbf{m}_{\nu,-\mu}^{(1)}(\boldsymbol{\rho}) + \mathbf{n}_{\nu\mu}^{(3)}(\mathbf{r})\mathbf{n}_{\nu,-\mu}^{(1)}(\boldsymbol{\rho})], \qquad r>\rho \tag{52}$$

$$= 4\pi\sum_{\nu\mu}(-)^{\mu+1}[\mathbf{m}_{\nu\mu}^{(1)}(\mathbf{r})\mathbf{m}_{\nu,-\mu}^{(3)}(\boldsymbol{\rho}) + \mathbf{n}_{\nu\mu}^{(1)}(\mathbf{r})\mathbf{n}_{\nu,-\mu}^{(3)}(\boldsymbol{\rho})], \qquad r<\rho \tag{53}$$

$$= \left(\bar{\mathbf{I}} + \frac{1}{k^2}\nabla\nabla\right)\frac{\exp(ik|\mathbf{r}-\boldsymbol{\rho}|)}{ik|\mathbf{r}-\boldsymbol{\rho}|}. \tag{54}$$

$\bar{\mathbf{I}}$ is the unit dyadic.

Equations (52) and (53) imply each other since from (54) $\bar{\mathbf{G}}(\mathbf{r}|\boldsymbol{\rho}) = \bar{\mathbf{G}}(\boldsymbol{\rho}|\mathbf{r})$ and $G_{ij} = G_{ji}$. To establish (54), note that according to (45) and (51),

$$\mathcal{P}_{nm}\cdot\bar{\mathbf{G}}(\mathbf{r}|\boldsymbol{\rho}) = \mathcal{P}_{nm}h_0^{(1)}(k|\mathbf{r}-\boldsymbol{\rho}|). \tag{55}$$

(Equation (55) is clearly consistent with (54).) With (55) and the dipole operators

$$\mathcal{P}_{11} = -\frac{1}{k}\sqrt{\frac{3}{16\pi}}(\hat{\mathbf{x}}+i\hat{\mathbf{y}})\times\nabla$$

$$\mathcal{P}_{10} = \frac{1}{k}\sqrt{\frac{3}{8\pi}}\hat{\mathbf{z}}\times\nabla$$

$$\mathcal{P}_{1,-1} = \frac{1}{k}\sqrt{\frac{3}{16\pi}}(\hat{\mathbf{x}}-i\hat{\mathbf{y}})\times\nabla$$

it can be shown that

$$\nabla\times\bar{\mathbf{G}}(\mathbf{r}|\boldsymbol{\rho}) = \nabla\times\bar{\mathbf{I}}h_0^{(1)}(k|\mathbf{r}-\boldsymbol{\rho}|). \tag{56}$$

Because

$$\bar{G}(\mathbf{r}|\boldsymbol{\rho})=\frac{1}{k^2}\nabla\times\nabla\times\bar{G}(\mathbf{r}|\boldsymbol{\rho}),$$

(56) leads directly to (54). $\bar{G}(\mathbf{r}|\boldsymbol{\rho})$ is a free-space dyadic Green's function [19], [20]. Spherical wave expansions equivalent to (52) and (53) are given by Tai [19, p. 174].

Exterior Region: From (51a) and (53)

$$\begin{aligned}\mathbf{m}_{nm}^{(3)\prime}(\mathbf{r})&=i^{-n}\mathcal{P}_{nm}\cdot\bar{G}(\mathbf{r}|\boldsymbol{\rho})\\&=4\pi i^{-n}\sum_{\nu\mu}(-)^{\mu+1}\{[\mathcal{P}_{nm}\cdot\mathbf{m}_{\nu\mu}^{(1)}](\mathbf{r})\mathbf{m}_{\nu,-\mu}^{(3)}(\boldsymbol{\rho})\\&\quad+[\mathcal{P}_{nm}\cdot\mathbf{n}_{\nu\mu}^{(1)}](\mathbf{r})\mathbf{n}_{\nu,-\mu}^{(3)}(\boldsymbol{\rho})\},\qquad r<\rho\end{aligned}$$

An interchange of $\mathbf{r}$ and $\boldsymbol{\rho}$ gives

$$\begin{aligned}\mathbf{m}_{nm}^{(3)\prime}(\mathbf{r})=\sum_{\nu\mu}&[B_e(\nu\mu|nm;\boldsymbol{\rho})\mathbf{m}_{\nu\mu}^{(3)}(\mathbf{r})\\&+C_e(\nu\mu|nm;\boldsymbol{\rho})\mathbf{n}_{\nu\mu}^{(3)}(\mathbf{r})]\end{aligned}\tag{57a}$$

$$\begin{aligned}\mathbf{n}_{nm}^{(3)\prime}(\mathbf{r})=\sum_{\nu\mu}&[B_e(\nu\mu|nm;\boldsymbol{\rho})\mathbf{n}_{\nu\mu}^{(3)}(\mathbf{r})\\&+C_e(\nu\mu|nm;\boldsymbol{\rho})\mathbf{m}_{\nu\mu}^{(3)}(\mathbf{r})],\qquad r>\rho\end{aligned}\tag{57b}$$

where

$$B_e(\nu\mu|nm;\boldsymbol{\rho})=(-)^{\mu+1}4\pi i^{-\nu+n}\mathcal{P}_{\nu,-\mu}\cdot\mathcal{P}_{nm}j_0(k\rho)\tag{58a}$$

$$C_e(\nu\mu|nm;\boldsymbol{\rho})=(-)^{\mu+1}4\pi i^{-\nu+n}\frac{1}{k}\nabla\times\mathcal{P}_{\nu,-\mu}\cdot\mathcal{P}_{nm}j_0(k\rho).\tag{58b}$$

B_e and C_e are identical to the standing wave coefficients B and C given in (42).

Equation (52) could have been used as a starting point for a similar derivation of (38) and (42) or (47) and (49). Thus the vector translation formulas are simple consequences of the differential operator representation (51) and the well-known expansions (52) and (53).

A derivation of the exterior region translation formula is also possible using the integral representations (46) [11]. For example,

$$\mathbf{m}_{nm}^{(3)\prime}(\mathbf{r})=\frac{1}{2\pi i^n}\int\mathbf{X}_{nm}(\hat{\mathbf{k}})\exp[i\mathbf{k}\cdot(\mathbf{r}-\boldsymbol{\rho})]\frac{d\mathbf{K}}{\gamma k},\qquad \hat{\mathbf{z}}\cdot(\mathbf{r}-\boldsymbol{\rho})>0.$$

From (43) and the orthonormality relations implied in (41) it follows that B and C may be interpreted as coefficients in the spherical harmonic expansion

$$\begin{aligned}i^{-n}\mathbf{X}_{nm}(\hat{\mathbf{k}})\exp(-i\mathbf{k}\cdot\boldsymbol{\rho})=\sum_{\nu\mu}i^{-\nu}&[B(\nu\mu|nm;\boldsymbol{\rho})\mathbf{X}_{\nu\mu}(\hat{\mathbf{k}})\\&+C(\nu\mu|nm;\boldsymbol{\rho})i\hat{\mathbf{k}}\times\mathbf{X}_{\nu\mu}(\hat{\mathbf{k}})].\end{aligned}$$

(Since $\mathbf{k}\cdot\mathbf{X}_{nm}(\hat{\mathbf{k}})=0$, there are no terms involving the third independent vector spherical harmonic $\hat{\mathbf{k}}Y_{\nu\mu}(\hat{\mathbf{k}})$.) After substitution of the last equation into the previous one, an application of (46) yields the translation formula (57a). This method also is useful in the scalar case [5]. The integral representation approach complements the differential operator approach by providing direct derivations of the translation formulas in exterior regions.

C. Irrotational Waves

In addition to $\mathbf{m}$ and $\mathbf{n}$, Stratton [17] also defines an $\mathbf{l}$ function that has nonzero divergence. For example,

$$\mathbf{l}_{nm}^{(1)}(\mathbf{r})\equiv\frac{1}{k}\nabla u_{nm}^{(1)}(\mathbf{r}).\tag{59}$$

Noting that the gradient operator is translationally invariant and using (7) yields [21]

$$\begin{aligned}\mathbf{l}_{nm}^{(1)\prime}(\mathbf{r})&=\frac{1}{k}\nabla u_{nm}^{(1)\prime}(\mathbf{r})\\&=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}A(\nu\mu|nm;\boldsymbol{\rho})\mathbf{l}_{\nu\mu}^{(1)}(\mathbf{r}).\end{aligned}\tag{60}$$

Similarly, the translation formulas for traveling irrotational waves follow directly from the corresponding scalar formulas.

IV. Evaluation of the Coefficients

It remains to evaluate the coefficients given in (49) and (58). Only (49) will be considered here, but (58) can be treated in the same manner (e.g., replace the subscript i with e in (62), (65), and (66)). There are several routes that lead to results of varying complexity. Perhaps the most obvious procedure is to apply the Cartesian representation (72) for $\mathcal{P}_{nm}$ and a similar (but more complicated) representation for $\nabla\times\mathcal{P}_{nm}$ to reduce (49) to a sum of terms involving the scalar translation coefficient A_i evaluated in (25). It is more efficient, however, to condition (49) first.

For example, it follows from the definition (34) of the vector spherical wave operator that

$$\begin{aligned}&2\sqrt{\nu(\nu+1)n(n+1)}\,\mathcal{P}_{\nu\mu}\cdot\mathcal{P}_{nm}\\&=L^2\mathcal{P}_{\nu\mu}\mathcal{P}_{nm}-\mathcal{P}_{nm}L^2\mathcal{P}_{\nu\mu}-\mathcal{P}_{\nu\mu}L^2\mathcal{P}_{nm}+\mathcal{P}_{\nu\mu}\mathcal{P}_{nm}L^2.\end{aligned}\tag{61}$$

Substituting (61) into (49a) gives [2]

$$\begin{aligned}B_i(\nu\mu|nm;\boldsymbol{\rho})&=\frac{\nu(\nu+1)+n(n+1)-L^2}{2\sqrt{\nu(\nu+1)n(n+1)}}A_i(\nu\mu|nm;\boldsymbol{\rho})\\&=(-)^{\mu}4\pi i^{-\nu+n}\\&\quad\cdot\sum_{\alpha\beta}\frac{\nu(\nu+1)+n(n+1)-\alpha(\alpha+1)}{2\sqrt{\nu(\nu+1)n(n+1)}}\\&\quad i^{\alpha}a(\alpha\beta|\nu,-\mu nm)u_{\alpha\beta}^{(3)}(\boldsymbol{\rho})\end{aligned}\tag{62}$$

where (23), (25), and the differential equation

$$L^2Y_{nm}=n(n+1)Y_{nm}\tag{63}$$

have been used.

Similarly, (assuming that $\nabla^2 = -k^2$)

$$\frac{1}{k}\nabla\times\mathcal{P}_{\nu\mu}\cdot\mathcal{P}_{nm}=\frac{ik}{\sqrt{\nu(\nu+1)}}[\mathbf{r}\mathcal{P}_{\nu\mu}-\mathcal{P}_{\nu\mu}\mathbf{r}]\cdot\mathcal{P}_{nm} \tag{64}$$

and, therefore, with (49b) and (72) [7]

$$\begin{aligned}C_i(\nu\mu|nm;\rho)=&\frac{-ik\rho}{\sqrt{\nu(\nu+1)n(n+1)}}\\&\cdot\left\{\frac{\hat{\mathbf{x}}}{2}[\lambda^+_{nm}A_i(\nu\mu|n,m+1;\rho)\right.\\&\qquad+\lambda^-_{nm}A_i(\nu\mu|n,m-1;\rho)]\\&+\frac{\hat{\mathbf{y}}}{2i}[\lambda^+_{nm}A_i(\nu\mu|n,m+1;\rho)\\&\qquad-\lambda^-_{nm}A_i(\nu\mu|n,m-1;\rho)]\\&\left.+m\hat{\mathbf{z}}A_i(\nu\mu|nm;\rho)\right\}.\end{aligned} \tag{65}$$

For translation along the z axis, B_i and C_i are nonzero only if $\mu = m$. Also (65) reduces to

$$C_i(\nu m|nm;\rho\hat{\mathbf{z}})=\frac{-imk\rho}{\sqrt{\nu(\nu+1)n(n+1)}}A_i(\nu m|nm;\rho\hat{\mathbf{z}}). \tag{66}$$

In a practical situation it is often possible to choose a coordinate system so that translation will be along the z axis to take advantage of the simplification. Actually, it is sufficient to consider only z axis translations [6]. A more general translation can be accomplished by first performing a rotation so that the axis of translation coincides with the z axis, then translating in the z direction, and, finally, rotating back to the original orientation.

Asymptotic formulas for large ρ can be obtained easily from integral expressions for the expansion coefficients. For example, it follows from (50) that [1], [2]

$$\begin{aligned}B_i(\nu\mu|nm;\rho)\underset{\rho\to\infty}{\longrightarrow}&(-)^{\mu+1}4\pi i^{-\nu+n}\mathbf{X}_{\nu,-\mu}(\hat{\rho})\\&\cdot\mathbf{X}_{nm}(\hat{\rho})\frac{\exp(ik\rho)}{ik\rho}\end{aligned} \tag{67a}$$

$$\begin{aligned}C_i(\nu\mu|nm;\rho)\underset{\rho\to\infty}{\longrightarrow}&(-)^{\mu+1}4\pi i^{-\nu+n}i\hat{\rho}\times\mathbf{X}_{\nu,-\mu}(\hat{\rho})\\&\cdot\mathbf{X}_{nm}(\hat{\rho})\frac{\exp(ik\rho)}{ik\rho}.\end{aligned} \tag{67b}$$

For translation along the z axis (67) implies that B_i and C_i are zero to order $1/\rho$ unless $\mu = m = \pm 1$. Similar asymptotic formulas for B_e and C_e have the factor $\exp(ik\rho)/(ik\rho)$ replaced by $\sin(k\rho)/(k\rho)$.

Equations (67) are examples of the following rule: If

$$f(\mathbf{r})=\frac{1}{2\pi}\int b(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\mathbf{r})\frac{d\mathbf{K}}{\gamma k},\qquad z>0,$$

then

$$f(\mathbf{r})\underset{r\to\infty}{\longrightarrow}b(\hat{\mathbf{r}})\frac{\exp(ikr)}{ikr}.$$

This rule (which can be motivated by a stationary phase evaluation of the integral) holds for the scalar spherical waves $u^{(3)}_{nm}(\mathbf{r})$; therefore, it is true for (50) which can be expressed as a finite sum of such waves using (62) and (65).

Appendix I

Spherical Wave Operators

Scalar spherical harmonics are given by the formulas [12, ch. 3]

$$\left.\begin{aligned}Y_{nm}(\hat{\mathbf{k}})&=(-)^m K_{nm}(k_x/k+ik_y/k)^m P^{(m)}_n(k_z/k),\\Y_{n,-m}(\hat{\mathbf{k}})&=K_{nm}(k_x/k-ik_y/k)^m P^{(m)}_n(k_z/k),\end{aligned}\right\}\quad m\ge 0 \tag{68}$$

where

$$K_{nm}=\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}} \tag{69}$$

and

$$P^{(m)}_n(x)\equiv\frac{d^m}{dx^m}P_n(x).$$

$P_n(x)$ is a Legendre polynomial.

Since

$$\exp(i\mathbf{k}\cdot\mathbf{r})=\exp i[k_x x+k_y y+k_z z],$$

it follows by inspection that

$$\mathcal{P}_{nm}\exp(i\mathbf{k}\cdot\mathbf{r})=Y_{nm}(\hat{\mathbf{k}})\exp(i\mathbf{k}\cdot\mathbf{r})$$

with

$$\begin{aligned}\mathcal{P}_{nm}&=(-)^m K_{nm}(\alpha+i\beta)^m P^{(m)}_n(\eta),\qquad m\ge 0\\\mathcal{P}_{n,-m}&=(-)^{n+m}\mathcal{P}^*_{nm}\\\alpha\equiv\frac{1}{ik}\partial_x\qquad&\beta\equiv\frac{1}{ik}\partial_y\qquad\eta\equiv\frac{1}{ik}\partial_z.\end{aligned} \tag{70}$$

$\mathcal{P}_{nm}$ is translationally invariant because the spatial derivatives ∂_x, ∂_y and ∂_z are translationally invariant. The $\mathcal{P}$'s were introduced by Erdélyi [16] in establishing the integral representation (19).

The vector spherical wave operators can be found using the Cartesian form

$$\begin{aligned}\mathbf{X}_{nm}(\hat{\mathbf{k}})=&\left\{\frac{\hat{\mathbf{x}}}{2}[\lambda^+_{nm}Y_{n,m+1}(\hat{\mathbf{k}})+\lambda^-_{nm}Y_{n,m-1}(\hat{\mathbf{k}})]\right.\\&+\frac{\hat{\mathbf{y}}}{2i}[\lambda^+_{nm}Y_{n,m+1}(\hat{\mathbf{k}})-\lambda^-_{nm}Y_{n,m-1}(\hat{\mathbf{k}})]\\&\left.+m\hat{\mathbf{z}}Y_{nm}(\hat{\mathbf{k}})\right\}\Big/\sqrt{n(n+1)}\end{aligned} \tag{71}$$

where

$$\lambda_{nm}^{\pm}=\sqrt{(n \mp m)(n \pm m+1)}.$$

With (4), (35), and (71)

$$\mathcal{P}_{nm}=\left\{\frac{\hat{\mathbf{x}}}{2}\,[\lambda_{nm}^{+}\mathcal{P}_{n,m+1}+\lambda_{nm}^{-}\mathcal{P}_{n,m-1}]+\frac{\hat{\mathbf{y}}}{2i}\,[\lambda_{nm}^{+}\mathcal{P}_{n,m+1}-\lambda_{nm}^{-}\mathcal{P}_{n,m-1}]+m\hat{\mathbf{z}}\mathcal{P}_{nm}\right\}\Big/\sqrt{n(n+1)} \quad (72)$$

$$=[\mathbf{L}\mathcal{P}_{nm}-\mathcal{P}_{nm}\mathbf{L}]/\sqrt{n(n+1)}. \quad (73)$$

The translational invariance of $\boldsymbol{\mathcal{P}}$ follows from (72) and the translational invariance of $\mathcal{P}$.

In obtaining (73) the following formulas are helpful:

$$L_{\pm}\mathcal{P}_{nm}-\mathcal{P}_{nm}L_{\pm}=\lambda_{nm}^{\pm}\mathcal{P}_{n,m\pm 1}, \qquad L_{\pm} \equiv L_x \pm iL_y$$

$$L_z\mathcal{P}_{nm}-\mathcal{P}_{nm}L_z=m\mathcal{P}_{nm}. \quad (74)$$

These can be found by direct computation with (70). (Operator identities in this paper require a domain limited to solutions of the Helmholtz equation. This ensures adequate differentiability as well as the fact that $\nabla^2 = -k^2$.) The commutation relations (74) also mark the $\mathcal{P}$'s as spherical tensor operators [22, p. 82]; i.e., they transform under rotation in the same manner as the spherical harmonics Y_{nm}.

Vector spherical wave operators were apparently first introduced in [23], [10] where they were used to represent a receiving antenna as a differential operator that acts on the incident field.

Appendix II

Evaluation of $a(\alpha\beta\,|\,\nu\mu nm)$

While the underlying theory is quite involved, the needed results are well known and are summarized here. The a's are given by [22, eq. 4.34]

$$a(\alpha\beta\,|\,\nu\mu nm)=\sqrt{\frac{(2\nu+1)(2n+1)}{4\pi(2\alpha+1)}}\,\langle\alpha\beta\,|\,\nu\mu nm\rangle\langle\alpha 0\,|\,\nu 0 n 0\rangle \quad (75)$$

where the quantities with angle brackets are Clebsch–Gordan coefficients. (This relation is often written in terms of the more symmetric Wigner 3-j symbols.) $\langle\alpha\beta\,|\,\nu\mu nm\rangle$ is nonzero only when $|\nu - n| \le \alpha \le \nu + n$ and $\mu + m = \beta$. Also, $\langle\alpha 0\,|\,\nu 0 n 0\rangle$ is zero if $\nu + n + \alpha$ is odd.

There are closed-form expressions for the Clebsch–Gordan coefficients but these are complicated and ill suited for numerical evaluation. A better computational approach is to use a recursion formula such as [22, eq. I.8, p. 224]

$$\zeta_\alpha\langle\alpha\beta\,|\,\nu\mu nm\rangle=\sqrt{\xi_\alpha}\,\langle\alpha-1,\,\beta\,|\,\nu\mu nm\rangle+\sqrt{\xi_{\alpha+1}}\,\langle\alpha+1,\,\beta\,|\,\nu\mu nm\rangle \quad (76)$$

where

$$\zeta_\alpha=(\mu-m)-\beta\,\frac{\nu(\nu+1)-n(n+1)}{\alpha(\alpha+1)}$$

$$\xi_\alpha=\frac{(\alpha^2-\beta^2)[\alpha^2-(\nu-n)^2][(\nu+n+1)^2-\alpha^2]}{\alpha^2(4\alpha^2-1)}.$$

When $\mu = m = \beta = 0$, (76) reduces to

$$0=\sqrt{\xi_\alpha}\,\langle\alpha-1,\,0\,|\,\nu 0 n 0\rangle+\sqrt{\xi_{\alpha+1}}\,\langle\alpha+1,\,0\,|\,\nu 0 n 0\rangle. \quad (77)$$

Equations (76) and (77) can be used independently (probably best in the general case) or they can be combined to produce a three-term recursion formula for the a's. For translation along the z axis, $\beta = \mu + m = 0$ and (76) and (77) lead to the particularly simple relationship [1], [2]

$$\xi_{\alpha-1}\langle\alpha-2\rangle-(\xi_\alpha+\xi_{\alpha+1}-4m^2)\langle\alpha\rangle+\xi_{\alpha+2}\langle\alpha+2\rangle=0 \quad (78)$$

where $\langle\alpha\rangle = \langle\alpha 0\,|\,\nu,\ -mnm\rangle\,\langle\alpha 0\,|\,\nu 0 n 0\rangle$.

Starting values for the recursion formulas (76)–(78) are given by

$$a(\alpha\beta\,|\,\nu\mu nm)=0, \qquad \alpha>\nu+n \quad (79)$$

$$a(\nu+n,\ \mu+m\,|\,\nu\mu nm)=\frac{G_{\nu\mu}G_{nm}}{G_{\nu+n,\mu+m}},$$

$$G_{nm}=K_{nm}\,\frac{(2n)!}{n!(n-m)!}. \quad (80)$$

K_{nm} is given in (69). Equation (80) can be established by comparing the highest order terms in (13) or by using explicit expressions for the Clebsch–Gordan coefficients [24, eq. 3.6.12] in (75).

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