Physica 137A (1986) 209–242
North-Holland, Amsterdam

LIGHT SCATTERING BY A SPHERE ON A SUBSTRATE

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Received 13 January 1986

The problem of light scattering by a sphere on a substrate is treated using Mie's solution for scattering by a sphere in a homogeneous medium and an extension of Weyl's method for the calculation of the reflection of dipole radiation by a flat surface. The developed theory can be applied to spheres with a radius of the order of the wavelength of the incident light.

Particular solutions are given for the case of a perfectly conducting substrate and for the so-called static limit. In the latter case it is shown that the solution is equivalent to that obtained by Wind, Vlieger and Bedeaux for small spheres.

1. Introduction

In 1908, Mie[1]) obtained a solution for the diffraction of a plane monochromatic wave by a homogeneous sphere in a homogeneous medium. Shortly afterwards, Debye[2]) published an equivalent solution, introducing the concept of Debye potentials.

In order to solve the light scattering problem of a sphere on a substrate one has to take into account the electromagnetic interaction between the sphere and the substrate. In 1919 Weyl[3]) developed a method to describe the propagation of dipole radiation along a flat surface. In the present paper we shall use an extension of this method to calculate this interaction.

The abovementioned problem arises in surface physics. In particular the theory developed in the present paper can account for results obtained in ellipsometric experiments by Greef[4]). In these experiments mercury particles are deposited electrochemically onto a carbon substrate. The density of these particles is sufficiently low, so that their interaction can be neglected which means that one essentially has a one-particle scattering problem. The radius of the mercury particles is in general of the order of the wavelength of the light, so that the (quasi-) static theory developed by Wind, Vlieger and Bedeaux[5]) cannot be applied.

After giving a survey of the theory of Debye potentials in sections 2 and 3, and

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(North-Holland Physics Publishing Division)

of the Mie solution in section 4, we present the formal solution of the scattering problem of a sphere on a substrate in section 5. In this section we introduce a matrix **A** characterizing the reflection of spherical waves by the substrate. After generalizing an integral expression first used by Weyl in section 6, we calculate this matrix in section 8, using so-called Hertz vectors, which are introduced in section 7. The matrix elements of **A** are obtained as integrals over a complex angle. In general, these integrals can only be evaluated numerically. However, in the case of a perfectly conducting substrate they can be performed analytically, which is shown in section 9. In section 10 it is proved that in the static limit (radius of sphere $\ll$ wavelength) the results of Wind, Vlieger and Bedeaux[5]) are reproduced. Finally, in section 11 we give an expression for the far-away scattered electric field above the substrate.

2. The Debye potentials

Throughout this paper a time dependence $e^{-i\omega t}$ is assumed but always omitted, ω being the angular frequency of the incident light. Physical quantities can be obtained by multiplying the time-independent quantities by this factor and taking real parts. The wavelength λ and the wavevector k in the non-absorbing non-magnetic medium above the substrate (ambient) with refractive index n_- and dielectric constant $\epsilon_- = n_-^2$ are related to ω by

$$\lambda = \frac{2\pi v}{\omega}, \tag{2.1}$$

$$k = \frac{\omega}{v}, \tag{2.2}$$

v being the speed of light in this medium. We use rationalized Gaussian units.

For the use of the Debye potentials, we refer to a paper of Bouwkamp and Casimir[6])*. In a completely analogous way as in that paper one can prove that the electromagnetic fields in a region D between two concentric spheres, $0 < r_1 \leqslant r \leqslant r_2 < \infty, 0 \leqslant \phi \leqslant 2\pi, 0 \leqslant \theta \leqslant \pi$ (r, θ, ϕ spherical coordinates) in which there are no free charges and currents, are completely determined by two scalar functions ${}^{e}\Pi(\boldsymbol{r})$ and ${}^{h}\Pi(\boldsymbol{r})$, the so-called electric and magnetic Debye potential. The electric and magnetic fields can be derived from these potentials by

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{r}\,{}^{e}\Pi(\boldsymbol{r})) + ik\,\boldsymbol{\nabla} \times (\boldsymbol{r}\,{}^{h}\Pi(\boldsymbol{r})), \tag{2.3a}$$

$$\boldsymbol{H}(\boldsymbol{r}) = n_-\{-ik\,\boldsymbol{\nabla} \times (\boldsymbol{r}\,{}^{e}\Pi(\boldsymbol{r})) + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{r}\,{}^{h}\Pi(\boldsymbol{r}))\}. \tag{2.3b}$$

* Bouwkamp and Casimir assume the ambient to be vacuum.

These potentials are called electric and magnetic because the corresponding fields have vanishing magnetic and electric radial components respectively.

Maxwell's equations can be written in the form:

$$\begin{aligned} &\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}) - \mathrm{i}k\,\boldsymbol{H}(\boldsymbol{r})/n_- = \boldsymbol{0}\,, \quad \boldsymbol{\nabla}\cdot \boldsymbol{E}(\boldsymbol{r}) = \tilde{\rho}(\boldsymbol{r})\,, \\ &\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r})/n_- + \mathrm{i}k\,\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{v}\,\tilde{\boldsymbol{i}}(\boldsymbol{r})\,, \quad \boldsymbol{\nabla}\cdot \boldsymbol{H}(\boldsymbol{r}) = 0\,, \end{aligned} \tag{2.4}$$

with $\tilde{\rho}(\boldsymbol{r})$ the total charge density and $\tilde{\boldsymbol{i}}(\boldsymbol{r})$ the total current density, both relative to the ambient:

$$\begin{aligned} &\tilde{\rho}(\boldsymbol{r}) = (\rho(\boldsymbol{r}) - \boldsymbol{\nabla}\cdot \boldsymbol{P}_{\mathrm{ex}}(\boldsymbol{r}))/\epsilon_-\,, \\ &\tilde{\boldsymbol{i}}(\boldsymbol{r}) = (\boldsymbol{i}(\boldsymbol{r}) - \mathrm{i}\omega\,\boldsymbol{P}_{\mathrm{ex}}(\boldsymbol{r}))/\epsilon_-\,, \end{aligned} \tag{2.5}$$

$\rho(\boldsymbol{r})$ and $\boldsymbol{i}(\boldsymbol{r})$ being the free charge and current density respectively, and $\boldsymbol{P}_{\mathrm{ex}}(\boldsymbol{r})$ the excess polarization density which is related to the total polarization density $\boldsymbol{P}(\boldsymbol{r})$ by

$$\boldsymbol{P}_{\mathrm{ex}}(\boldsymbol{r}) = \boldsymbol{P}(\boldsymbol{r}) - (\epsilon_- - 1)\boldsymbol{E}(\boldsymbol{r})\,. \tag{2.6}$$

The densities $\tilde{\rho}(\boldsymbol{r})$ and $\tilde{\boldsymbol{i}}(\boldsymbol{r})$ satisfy the continuity equation

$$\frac{1}{v}\,\boldsymbol{\nabla}\cdot \tilde{\boldsymbol{i}}(\boldsymbol{r}) = \mathrm{i}k\,\tilde{\rho}(\boldsymbol{r})\,. \tag{2.7}$$

In the case that the fields in D are the result of currents $\tilde{\boldsymbol{i}}(\boldsymbol{r})$ flowing in the region inside of D ($0 \leq r < r_1$) we can derive, in an analogous way as in ref. 6, the following expansions for ${}^{\mathrm{e}}\Pi(\boldsymbol{r})$ and ${}^{\mathrm{h}}\Pi(\boldsymbol{r})$ in D:

$${}^{\mathrm{e}}\Pi(\boldsymbol{r}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} {}^{\mathrm{e}}w_l^m\, \Pi_l^m(\boldsymbol{r})\,, \tag{2.8a}$$

$${}^{\mathrm{h}}\Pi(\boldsymbol{r}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} {}^{\mathrm{h}}w_l^m\, \Pi_l^m(\boldsymbol{r})\,, \tag{2.8b}$$

in which the coefficients ${}^{\mathrm{e}}w_l^m$ and ${}^{\mathrm{h}}w_l^m$ are given by

$${}^{\mathrm{e}}w_l^m = -\frac{1}{4\pi v}\,\frac{1}{l(l+1)}\,(-1)^m \int \tilde{\boldsymbol{i}}(\boldsymbol{r})\cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\boldsymbol{r}\Psi_l^{-m}(\boldsymbol{r}))\,\mathrm{d}\boldsymbol{r}\,, \tag{2.9a}$$

$${}^{\mathrm{h}}w_l^m = \frac{\mathrm{i}k}{4\pi v}\,\frac{1}{l(l+1)}\,(-1)^m \int \tilde{\boldsymbol{i}}(\boldsymbol{r})\cdot \boldsymbol{\nabla} \times (\boldsymbol{r}\Psi_l^{-m}(\boldsymbol{r}))\,\mathrm{d}\boldsymbol{r}\,, \tag{2.9b}$$

with the functions $\Pi_l^m(\mathbf{r})$ and $\Psi_l^m(\mathbf{r})$:

$$\Pi_l^m(\mathbf{r}) \equiv h_l^{(1)}(kr) Y_l^m(\theta, \phi) , \tag{2.10a}$$

$$\Psi_l^m(\mathbf{r}) \equiv j_l(kr) Y_l^m(\theta, \phi) . \tag{2.10b}$$

Here $h_l^{(1)}(\rho)$ and $j_l(\rho)$ are spherical Hankel and Bessel functions of order l respectively:

$$h_l^{(1)}(\rho) \equiv \sqrt{\frac{\pi}{2\rho}}\, H_{l+1/2}^{(1)}(\rho) , \tag{2.11a}$$

$$j_l(\rho) \equiv \sqrt{\frac{\pi}{2\rho}}\, J_{l+1/2}(\rho) , \tag{2.11b}$$

in the notation of Watson[7]), and $Y_l^m(\theta, \phi)$ the surface harmonics of degree l and order m:

$$Y_l^m(\theta, \phi) \equiv \left[(2l+1)\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\, \mathrm{e}^{\mathrm{i}m\phi} , \tag{2.12}$$

with the associated Legendre function $P_l^m(\cos\theta)$ in the notation of Stratton[8]):

$$\begin{aligned} P_l^m(\eta) &\equiv \frac{(1-\eta^2)^{m/2}}{2^l l!} \frac{\mathrm{d}^{l+m}}{\mathrm{d}\eta^{l+m}} (\eta^2-1)^l \quad (m \geq 0) , \\ P_l^m(\eta) &\equiv (-1)^{-m} \frac{(l+m)!}{(l-m)!} P_l^{-m}(\eta) \qquad (m<0) . \end{aligned} \tag{2.13}$$

It can be shown[6]), with the relations (2.8)–(2.13), that the electromagnetic fields as given by eqs. (2.3a) and (2.3b) indeed satisfy Maxwell's equations (2.4).

Similar expressions can be derived for the case that the fields in D are the result of currents flowing in the region outside of D ($r > r_2$). Denoting the Debye potentials now by ${}^{\mathrm{e}}\Psi(\mathbf{r})$ and ${}^{\mathrm{h}}\Psi(\mathbf{r})$ to indicate the difference, one obtains:

$${}^{\mathrm{e}}\Psi(\mathbf{r}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} {}^{\mathrm{e}}v_l^m \Psi_l^m(\mathbf{r}) , \tag{2.14a}$$

$${}^{\mathrm{h}}\Psi(\mathbf{r}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} {}^{\mathrm{h}}v_l^m \Psi_l^m(\mathbf{r}) , \tag{2.14b}$$

with

$$ {}^{\mathrm{e}}v_l^m = -\frac{1}{4\pi v}\frac{1}{l(l+1)}(-1)^m \int \tilde{i}(\boldsymbol{r})\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(\boldsymbol{r}\Pi_l^{-m}(\boldsymbol{r}))\,\mathrm{d}\boldsymbol{r}\,, \tag{2.15a} $$

$$ {}^{\mathrm{h}}v_l^m = \frac{\mathrm{i}k}{4\pi v}\frac{1}{l(l+1)}(-1)^m \int \tilde{i}(\boldsymbol{r})\cdot\boldsymbol{\nabla}\times(\boldsymbol{r}\Pi_l^{-m}(\boldsymbol{r}))\,\mathrm{d}\boldsymbol{r}\,. \tag{2.15b} $$

Because $\{\Pi_l^m(\boldsymbol{r})\}$ and $\{\Psi_l^m(\boldsymbol{r})\}$ are orthonormal sets of solutions of Helmholtz's equation:

$$ (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}+k^2)\Pi_l^m(\boldsymbol{r})=0\,,\quad (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}+k^2)\Psi_l^m(\boldsymbol{r})=0\,, \tag{2.16} $$

it follows from eqs. (2.8) and (2.14) that the Debye potentials also satisfy this equation in the region D.

The Debye potentials are, except for the addition of terms proportional to Π_0^0 or Ψ_0^0, uniquely determined by the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$. Inserting Π_0^0 or Ψ_0^0 in eq. (2.3) gives $\boldsymbol{E}=\boldsymbol{0}, \boldsymbol{H}=\boldsymbol{0}$. We shall take the coefficients of Π_0^0 and Ψ_0^0 always equal to zero.

As a concluding remark we would like to emphasize the fact that any electromagnetic field in D is completely determined by the coefficients ${}^{\mathrm{e}}v_l^m$, ${}^{\mathrm{h}}v_l^m$ and ${}^{\mathrm{e}}w_l^m$, ${}^{\mathrm{h}}w_l^m$ given by eqs. (2.8) and (2.15). These coefficients can formally be considered as the components of two infinite vectors $\boldsymbol{V}$ and $\boldsymbol{W}$. So an electromagnetic field in D caused by currents flowing in the region inside of D is described by a vector $\boldsymbol{V}$, if it is caused by currents flowing in the region outside of D it is described by a vector $\boldsymbol{W}$. This notation will be used below.

3. The Debye potentials of a plane wave

In the case of a plane wave propagating in the z-direction of our coordinate system with the electric vector in the x-direction and with amplitude $|\mathbf{E}| = |\mathrm{e}^{\mathrm{i}kz}| = 1$, the Debye potentials are*

$$ {}^{\mathrm{e}}\Psi(\boldsymbol{r}) = \frac{1}{k}\sum_{l=1}^{\infty}\mathrm{i}^{l-1}\left[\frac{2l+1}{l(l+1)}\right]^{1/2}\left(\frac{1}{2}\,\Psi_l^1(\boldsymbol{r}) - \frac{1}{2}\,\Psi_l^{-1}(\boldsymbol{r})\right), \tag{3.1a} $$

$$ {}^{\mathrm{h}}\Psi(\boldsymbol{r}) = -\frac{1}{k}\sum_{l=1}^{\infty}\mathrm{i}^{l}\left[\frac{2l+1}{l(l+1)}\right]^{1/2}\left(\frac{1}{2}\,\Psi_l^1(\boldsymbol{r}) + \frac{1}{2}\,\Psi_l^{-1}(\boldsymbol{r})\right), \tag{3.1b} $$

and with the electric vector in the y-direction:

* See e.g. Born and Wolf[9]).

$$^{\mathrm{e}}\Psi(\boldsymbol{r}) = -\mathrm{i}\,\frac{1}{k}\sum_{l=1}^{\infty} \mathrm{i}^{l-1}\left[\frac{2l+1}{l(l+1)}\right]^{1/2}\left(\frac{1}{2}\,\Psi_l^1(\boldsymbol{r}) + \frac{1}{2}\,\Psi_l^{-1}(\boldsymbol{r})\right), \tag{3.2a}$$

$$^{\mathrm{h}}\Psi(\boldsymbol{r}) = \mathrm{i}\,\frac{1}{\mathrm{k}}\sum_{\mathrm{l}=1}^{\infty} \mathrm{i}^{l}\left[\frac{2l+1}{l(l+1)}\right]^{1/2}\left(\frac{1}{2}\,\Psi_l^1(\boldsymbol{r}) - \frac{1}{2}\,\Psi_l^{-1}(\boldsymbol{r})\right). \tag{3.2b}$$

In these formulae the functions $\Psi_l^m(\boldsymbol{r})$ appear because the electromagnetic field of a plane wave is the result of currents flowing at infinity, i.e. outside of the region D of section 2.

To calculate the Debye potentials of a plane wave propagating in another direction than the z-direction we need to express the surface harmonics in a rotated coordinate system (r, θ', ϕ') in terms of those in the coordinate system (r, θ, ϕ). This can be done by a formula given e.g. by Rose[10])*:

$$Y_l^{m'}(\theta', \phi') = \sum_{m=-l}^{l} (-1)^{m-m'} D_{m,m'}^{l}(\beta, \alpha, \gamma) Y_l^m(\theta, \phi). \tag{3.3}$$

Here β, α and γ are the so-called Euler angles, through which we must rotate the coordinate-system (r, θ, ϕ) to obtain the coordinate system (r, θ', ϕ'). These angles are defined by performing this rotation in three steps, starting with the coordinate system (x, y, z). First a rotation about the z-axis through an angle β, then a rotation about the new y-axis through an angle α and finally a rotation about the new z-axis through an angle γ, eventually giving the coordinate system (x', y', z'). The rotation matrix $D_{m,m'}^{l}(\beta, \alpha, \gamma)$ can be written as

$$D_{m,m'}^{l}(\beta, \alpha, \gamma) = \mathrm{e}^{-\mathrm{i}m\beta} d_{m,m'}^{l}(\alpha) \mathrm{e}^{-\mathrm{i}m'\gamma}, \tag{3.4}$$

with

$$d_{m,m'}^{l}(\alpha) \equiv [(l+m')!(l-m')!(l+m)!(l-m)!]^{1/2} \times \sum_k (-1)^k \frac{\left(\cos\frac{\alpha}{2}\right)^{2l+m'-m-2k}\left(-\sin\frac{\alpha}{2}\right)^{m-m'+2k}}{(l-m-k)!(l+m'-k)!(k+m-m')!k!}, \tag{3.5}$$

where the sum is over the values of the integer k for which the factorial arguments are greater than or equal to zero.

Some useful properties of $d_{m,m'}^{l}(\alpha)$ are:

$$d_{m',m}^{l}(\alpha) = (-1)^{m'-m} d_{m,m'}^{l}(\alpha), \tag{3.6}$$

* Our definition eq. (2.13) of the Legendre function of order m differs from that of Rose by a factor $(-1)^m$.

$$d^l_{m',m}(\alpha) = d^l_{-m,-m'}(\alpha)\,, \tag{3.7}$$

$$d^l_{m,1}(\pi - \alpha) = (-1)^{l-m} d^l_{m,-1}(\alpha)\,, \tag{3.8a}$$

$$d^l_{m,-1}(\pi - \alpha) = (-1)^{l-m} d^l_{m,1}(\alpha)\,. \tag{3.8b}$$

Now for a plane wave propagating in the z'-direction of some rotated coordinate system, with electric vector in the x'-direction and with unit amplitude, the Debye potentials become

$$\begin{aligned} {}^{\mathrm{e}}\Psi(\boldsymbol{r}) &= \frac{1}{k} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \mathrm{i}^{l-1} \left[\frac{2l+1}{l(l+1)}\right]^{1/2} (-1)^{m-1} \\ &\quad \times \left(\frac{1}{2} D^l_{m,1}(\beta, \alpha, \gamma) - \frac{1}{2} D^l_{m,-1}(\beta, \alpha, \gamma)\right) \Psi^m_l(\boldsymbol{r})\,, \end{aligned} \tag{3.9a}$$

$$\begin{aligned} {}^{\mathrm{h}}\Psi(\boldsymbol{r}) &= -\frac{1}{k} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \mathrm{i}^{l} \left[\frac{2l+1}{l(l+1)}\right]^{1/2} (-1)^{m-1} \\ &\quad \times \left(\frac{1}{2} D^l_{m,1}(\beta, \alpha, \gamma) + \frac{1}{2} D^l_{m,-1}(\beta, \alpha, \gamma)\right) \Psi^m_l(\boldsymbol{r})\,, \end{aligned} \tag{3.9b}$$

where eqs. (3.1) and (3.3) have been used. For a plane wave propagating in the same direction but with the electric vector in the y'-direction, we obtain with eqs. (3.2) and (3.3):

$$\begin{aligned} {}^{\mathrm{e}}\Psi(\boldsymbol{r}) &= -\mathrm{i}\,\frac{1}{k} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \mathrm{i}^{l-1} \left[\frac{2l+1}{l(l+1)}\right]^{1/2} (-1)^{m-1} \\ &\quad \times \left(\frac{1}{2} D^l_{m,1}(\beta, \alpha, \gamma) + \frac{1}{2} D^l_{m,-1}(\beta, \alpha, \gamma)\right) \Psi^m_l(\boldsymbol{r})\,, \end{aligned} \tag{3.10a}$$

$$\begin{aligned} {}^{\mathrm{h}}\Psi(\boldsymbol{r}) &= \mathrm{i}\,\frac{1}{k} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \mathrm{i}^{l} \left[\frac{2l+1}{l(l+1)}\right]^{1/2} (-1)^{m-1} \\ &\quad \times \left(\frac{1}{2} D^l_{m,1}(\beta, \alpha, \gamma) - \frac{1}{2} D^l_{m,-1}(\beta, \alpha, \gamma)\right) \Psi^m_l(\boldsymbol{r})\,. \end{aligned} \tag{3.10b}$$

4. The Mie solution

Let us again consider the region D of section 2. If there is an object in the region inside of D and we have an incident wave $\boldsymbol{V}$ (according to the notation of section 2), e.g. a plane wave, then the electromagnetic fields of this wave will cause currents flowing in this object, resulting in a scattered wave $\boldsymbol{W}$. The vectors $\boldsymbol{W}$ and $\boldsymbol{V}$ will be linearly related by a matrix $\mathbf{B}$:

$$\boldsymbol{W} = \mathbf{B} \cdot \boldsymbol{V} . \tag{4.1}$$

The matrix $\mathbf{B}$ will depend on the geometry and the constitutive electromagnetic properties of the substance of the object.

In the case that the object is a homogeneous non-magnetic sphere with refractive index n, centred at the origin of the coordinate system, Mie[1]) in fact has calculated this matrix. The result is, in our notation (cf. ref. 9):

$$\begin{aligned}
B_{l',m',\mathrm{e};l,m,\mathrm{e}} &= -\delta_{ll'}\delta_{mm'} \frac{\tilde{n}\psi_l'(q)\psi_l(\tilde{n}q) - \psi_l(q)\psi_l'(\tilde{n}q)}{\tilde{n}\zeta_l'(q)\psi_l(\tilde{n}q) - \zeta_l(q)\psi_l'(\tilde{n}q)} ,\\
B_{l',m',\mathrm{h};l,m,\mathrm{h}} &= -\delta_{ll'}\delta_{mm'} \frac{\tilde{n}\psi_l(q)\psi_l'(\tilde{n}q) - \psi_l'(q)\psi_l(\tilde{n}q)}{\tilde{n}\zeta_l(q)\psi_l'(\tilde{n}q) - \zeta_l'(q)\psi_l(\tilde{n}q)} ,\\
B_{l',m',\mathrm{e};l,m,\mathrm{h}} &= B_{l',m',\mathrm{h};l,m,\mathrm{e}} = 0 ,
\end{aligned} \tag{4.2}$$

with $\tilde{n}$ the relative refractive index of the sphere:

$$\tilde{n} = \frac{n}{n_-} , \tag{4.3}$$

and q defined by

$$q \equiv \frac{2\pi}{\lambda} a , \tag{4.4}$$

where a is the radius of the sphere. The functions $\psi_l(\rho)$ and $\zeta_l(\rho)$ are defined as

$$\psi_l(\rho) \equiv \rho j_l(\rho) , \quad \zeta_l(\rho) \equiv \rho h_l^{(1)}(\rho) . \tag{4.5}$$

The addition of a prime to these functions denotes differentiation with respect to their argument.

The practical importance of this solution is that one can show[2]) that when $q \gg 1$ the matrix elements in eq. (4.2) fall off rapidly to zero as soon as $l + \frac{1}{2}$ exceeds q. This is intuitively clear if one observes that the function $j_l(\rho)$ falls off rapidly to zero as soon as its argument is smaller than l. So the electromagnetic fields of an incident partial electric wave (${}^{\mathrm{e}}\Psi = \Psi_l^m$, ${}^{\mathrm{h}}\Psi = 0$) or a partial magnetic wave (${}^{\mathrm{e}}\Psi = 0$, ${}^{\mathrm{h}}\Psi = \Psi_l^m$) of order $l > q$ will be approximately zero in the neighbourhood of the sphere and there will be no scattered wave. This is also clear from eq. (2.9): $\Psi_l^m(\boldsymbol{r})$ will be approximately zero in the region where $\tilde{\imath}(\boldsymbol{r})$ is different from zero.

5. Formal solution of the scattering by a sphere on a substrate

Let us now consider a sphere on a substrate. For clarity we shall assume the sphere to be at a distance δ above the substrate so that we can introduce a region D as defined in section 2 (see fig. 1). At the end of the argument we can take the limit of δ going to zero. One should note that the solutions found for the electromagnetic fields in D, which are caused by currents flowing in the region inside of D, are valid in the whole region above the substrate and outside the sphere.

Let a wave V^{I} (e.g. a plane wave) be incident on this system. If the sphere were absent we could satisfy the boundary conditions at the interface between the ambient and the substrate by adding a wave V^{IR} (just Fresnel reflection in the case of a plane wave). In the presence of the sphere there will be an additional wave W^{S} as a result of the currents flowing inside the sphere. But this wave will also be reflected by the substrate – i.e. induce currents flowing inside the substrate – and will give rise to a wave V^{SR}. V^{SR} and W^{S} will be linearly related by some matrix $\mathbf{A}$ characterizing the reflection of spherical waves by the substrate:

$$V^{\mathrm{SR}} = \mathbf{A} \cdot W^{\mathrm{S}} . \tag{5.1}$$

We therefore have the following set of equations:

$$W^{\mathrm{S}} = \mathbf{B} \cdot (V^{\mathrm{I}} + V^{\mathrm{IR}} + V^{\mathrm{SR}}) , \quad V^{\mathrm{SR}} = \mathbf{A} \cdot W^{\mathrm{S}} , \tag{5.2}$$

with the matrix $\mathbf{B}$ given by eq. (4.2). From these equations W^{S} can be solved in terms of V^{I} and V^{IR}, giving

$$W^{\mathrm{S}} = (\mathbf{1} - \mathbf{B} \cdot \mathbf{A})^{-1} \cdot \mathbf{B} \cdot (V^{\mathrm{I}} + V^{\mathrm{IR}}) , \tag{5.3}$$

and this is the formal solution of our scattering problem.

The following sections will be concerned with the calculation of the matrix $\mathbf{A}$, resulting in an integral expression with integration over a complex angle.

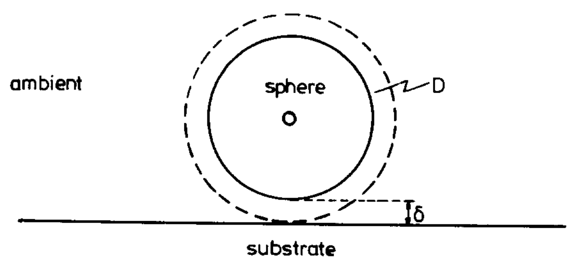

Fig. 1. Sphere on a substrate. The sphere is drawn at a small distance δ above the substrate. The region D is indicated. The origin is at the centre of the sphere.

6. Weyl's method

In his paper about the propagation of dipole radiation along a flat surface[3]) Weyl derives a formula expressing the spherical wave e^{ikr}/ikr as an integral over plane waves travelling into various directions*:

$$\frac{e^{ikr}}{ikr} = h_0^{(1)}(kr) = \Pi_0^0(\boldsymbol{r}) = \frac{1}{2\pi}\int_0^{2\pi} d\beta \int_0^{\pi/2-i\infty} \sin\alpha \, d\alpha \, e^{ikr\cos\gamma} , \tag{6.1}$$

in which $\cos\gamma$ is the angle between $\boldsymbol{r} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ and $\boldsymbol{k} = (k\sin\alpha\cos\beta, k\sin\alpha\sin\beta, k\cos\alpha)$:

$$\cos\gamma = \frac{\boldsymbol{k}\cdot\boldsymbol{r}}{kr} = \sin\theta\sin\alpha\cos(\phi-\beta) + \cos\theta\cos\alpha . \tag{6.2}$$

The integral $\int_0^{\pi/2-i\infty} d\alpha$ is an integral along a path in the complex plane as indicated in fig. 2. The integral expression (6.1) is only valid in the halfspace $z > 0$, since only then it converges. A similar expression is valid in the halfspace $z < 0$. With the help of eq. (6.1) Weyl is now able to give an expression for the reflected wave using Fresnel reflection for each plane wave.

Because we want to calculate the reflection of spherical waves of higher order we need a generalization of eq. (6.1). This generalization is

$$\begin{aligned} h_l^{(1)}(kr)Y_l^m(\theta,\phi) &= \Pi_l^m(\boldsymbol{r}) \\ &= \frac{i^{-l}}{2\pi}\int_0^{2\pi} d\beta \int_0^{\pi/2-i\infty} \sin\alpha \, d\alpha \, e^{ikr\cos\gamma} Y_l^m(\alpha,\beta) . \end{aligned} \tag{6.3}$$

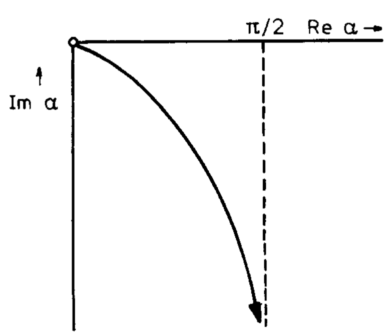

Fig. 2. Path in the complex plane along which the integration over the azimuthal angle α is performed.

* Weyl uses the wave $e^{-ikr}/-ikr$ instead of e^{ikr}/ikr.

This integral equation is easily proved if one notes that

$$\frac{\partial}{\partial z}\,\Pi_l^m(\boldsymbol{r}) = k\left\{\left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{1/2}\Pi_{l-1}^m(\boldsymbol{r})\right.$$
$$\left.-\left[\frac{(l+m+1)(l-m+1)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^m(\boldsymbol{r})\right\},$$
$$\left(\frac{\partial}{\partial x}+\mathrm{i}\,\frac{\partial}{\partial y}\right)\Pi_l^m(\boldsymbol{r}) = -k\left\{\left[\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}\right]^{1/2}\Pi_{l-1}^{m+1}(\boldsymbol{r})\right.$$
$$\left.+\left[\frac{(l+m+1)(l+m+2)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^{m+1}(\boldsymbol{r})\right\},$$
$$\left(\frac{\partial}{\partial x}-\mathrm{i}\,\frac{\partial}{\partial y}\right)\Pi_l^m(\boldsymbol{r}) = k\left\{\left[\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)}\right]^{1/2}\Pi_{l-1}^{m-1}(\boldsymbol{r})\right.$$
$$\left.+\left[\frac{(l-m+1)(l-m+2)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^{m-1}(\boldsymbol{r})\right\}, \tag{6.4}$$

and

$$\frac{\partial}{\partial z}\left\{\mathrm{i}^{-l}\,\mathrm{e}^{\mathrm{i}kr\cos\gamma}Y_l^m(\alpha,\beta)\right\} = k\,\mathrm{e}^{\mathrm{i}\,kr\cos\gamma}$$
$$\times\left\{\left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{1/2}\mathrm{i}^{-(l-1)}Y_{l-1}^m(\alpha,\beta)\right.$$
$$\left.-\left[\frac{(l+m+1)(l-m+1)}{(2l+3)(2l+1)}\right]^{1/2}\mathrm{i}^{-(l+1)}Y_{l+1}^m(\alpha,\beta)\right\},$$
$$\left(\frac{\partial}{\partial x}+\mathrm{i}\,\frac{\partial}{\partial y}\right)\{\mathrm{i}^{-l}\,\mathrm{e}^{\mathrm{i}kr\cos\gamma}Y_l^m(\alpha,\beta)\} = -k\,\mathrm{e}^{\mathrm{i}kr\cos\gamma}$$
$$\times\left\{\left[\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}\right]^{1/2}\mathrm{i}^{-(l-1)}Y_{l-1}^{m+1}(\alpha,\beta)\right. \tag{6.5}$$
$$\left.+\left[\frac{(l+m+1)(l+m+2)}{(2l+3)(2l+1)}\right]^{1/2}\mathrm{i}^{-(l+1)}Y_{l+1}^{m+1}(\alpha,\beta)\right\},$$
$$\left(\frac{\partial}{\partial x}-\mathrm{i}\,\frac{\partial}{\partial y}\right)\{\mathrm{i}^{-l}\,\mathrm{e}^{\mathrm{i}kr\cos\gamma}Y_l^m(\alpha,\beta)\} = k\,\mathrm{e}^{\mathrm{i}kr\cos\gamma}$$
$$\times\left\{\left[\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)}\right]^{1/2}\mathrm{i}^{-(l-1)}Y_{l-1}^{m-1}(\alpha,\beta)\right.$$
$$\left.+\left[\frac{(l-m+1)(l-m+2)}{(2l+3)(2l+1)}\right]^{1/2}\mathrm{i}^{-(l+1)}Y_{l+1}^{m-1}(\alpha,\beta)\right\}.$$

These relations can be proved using recurrence relations for Legendre and Hankel functions. Eqs. (6.4) and (6.5) are consistent with eq. (6.3). Since for $l = m = 0$ eq. (6.3) reduces to Weyl's integral eq. (6.1), it follows that eq. (6.3) is true for any values of l and m.

7. Debye potentials and Hertz vectors

We cannot apply Weyl's method – by means of eq. (6.3) – directly to the Debye potentials themselves – written in the form (2.8) – since the integrand would not correspond with physical plane waves, to which we could apply Fresnel reflection. This is because of the multiplication by $\boldsymbol{r}$ in eq. (2.3).

Beside the Debye potentials we shall therefore introduce other potentials, the so-called Hertz vectors*, which are more appropriate to Cartesian symmetry, as we have in the case of reflection by a flat surface. Just like the Debye potentials there are two Hertz vectors, an electric Hertz vector ${}^{\mathrm{e}}\boldsymbol{\Pi}(\boldsymbol{r})$ and a magnetic Hertz vector ${}^{\mathrm{h}}\boldsymbol{\Pi}(\boldsymbol{r})$. In the absence of free changes and currents the electric and magnetic fields can be derived from these vectors by means of

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times {}^{\mathrm{e}}\boldsymbol{\Pi}(\boldsymbol{r}) + \mathrm{i}k\, \boldsymbol{\nabla} \times {}^{\mathrm{h}}\boldsymbol{\Pi}(\boldsymbol{r}) , \tag{7.1a}$$

$$\boldsymbol{H}(\boldsymbol{r}) = n_{-}\{-\mathrm{i}k\, \boldsymbol{\nabla} \times {}^{\mathrm{e}}\boldsymbol{\Pi}(\boldsymbol{r}) + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times {}^{\mathrm{h}}\boldsymbol{\Pi}(\boldsymbol{r})\} . \tag{7.1b}$$

From eq. (2.3) it is clear that Hertz vectors can be obtained from the Debye potentials simply by putting ${}^{\mathrm{e}}\boldsymbol{\Pi}(\boldsymbol{r}) = \boldsymbol{r}\,{}^{\mathrm{e}}\Pi(\boldsymbol{r})$ and ${}^{\mathrm{h}}\boldsymbol{\Pi}(\boldsymbol{r}) = \boldsymbol{r}\,{}^{\mathrm{h}}\Pi(\boldsymbol{r})$. However, these vectors do not satisfy Helmholtz's equation, but the equations:

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} + k^2)\boldsymbol{r}\,{}^{\mathrm{e}}\Pi(\boldsymbol{r}) = 2\boldsymbol{\nabla}\,{}^{\mathrm{e}}\Pi(\boldsymbol{r}) , \tag{7.2a}$$

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} + k^2)\boldsymbol{r}\,{}^{\mathrm{h}}\Pi(\boldsymbol{r}) = 2\boldsymbol{\nabla}\,{}^{\mathrm{h}}\Pi(\boldsymbol{r}) . \tag{7.2b}$$

Consequently, the x-, y- and z-components of these vectors are not expressible as a sum over the functions $\Pi_l^m(\boldsymbol{r})$, so we cannot apply eq. (6.3) to these vectors.

However, from eq. (7.1) it is clear that we can add to the Hertz vectors the gradient of an arbitrary function without changing the electromagnetic fields. This freedom can be used to construct Hertz vectors that do satisfy Helmholtz's equation. If we put

* See e.g. Born and Wolf[9]).

$$^{\mathrm{e}}\boldsymbol{\Pi}(\boldsymbol{r}) = \boldsymbol{r}\,^{\mathrm{e}}\Pi(\boldsymbol{r}) + \frac{1}{k^2}\,\boldsymbol{\nabla}[\boldsymbol{\nabla}\cdot(\boldsymbol{r}\,^{\mathrm{e}}\Pi(\boldsymbol{r})) - 4\,^{\mathrm{e}}\Pi(\boldsymbol{r})]\,, \tag{7.3a}$$

$$^{\mathrm{h}}\boldsymbol{\Pi}(\boldsymbol{r}) = \boldsymbol{r}\,^{\mathrm{h}}\Pi(\boldsymbol{r}) + \frac{1}{k^2}\,\boldsymbol{\nabla}[\boldsymbol{\nabla}\cdot(\boldsymbol{r}\,^{\mathrm{h}}\Pi(\boldsymbol{r})) - 4\,^{\mathrm{h}}\Pi(\boldsymbol{r})]\,, \tag{7.3b}$$

it can easily be checked that this condition is now satisfied. Eq. (7.3) can be rewritten as

$$^{\mathrm{e}}\boldsymbol{\Pi}(\boldsymbol{r}) = \boldsymbol{r}\,^{\mathrm{e}}\Pi(\boldsymbol{r}) + \frac{1}{k^2}\,r\,\frac{\partial}{\partial r}\,\boldsymbol{\nabla}\,^{\mathrm{e}}\Pi(\boldsymbol{r})\,, \tag{7.4a}$$

$$^{\mathrm{h}}\boldsymbol{\Pi}(\boldsymbol{r}) = \boldsymbol{r}\,^{\mathrm{h}}\Pi(\boldsymbol{r}) + \frac{1}{k^2}\,r\,\frac{\partial}{\partial r}\,\boldsymbol{\nabla}\,^{\mathrm{h}}\Pi(\boldsymbol{r})\,. \tag{7.4b}$$

The components of these vectors should be expressible in terms of the functions $\Pi_l^m(\boldsymbol{r})$. Indeed, taking $^{\mathrm{e}}\Pi = \Pi_l^m$ and $^{\mathrm{h}}\Pi = \Pi_l^m$ respectively, we find after a lengthy calculation involving recurrence relations for Legendre and Hankel functions:

$$\begin{aligned}
^{\mathrm{f}}\Pi_x ={}& \frac{(l-1)}{2k}\left\{\left[\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)}\right]^{1/2}\Pi_{l-1}^{m-1}\right.\\
&\left.-\left[\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}\right]^{1/2}\Pi_{l-1}^{m+1}\right\}\\
&-\frac{(l+2)}{2k}\left\{\left[\frac{(l-m+1)(l-m+2)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^{m-1}\right.\\
&\left.-\left[\frac{(l+m+1)(l+m+2)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^{m+1}\right\},\\
^{\mathrm{f}}\Pi_y ={}& \mathrm{i}\,\frac{(l-1)}{2k}\left\{\left[\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)}\right]^{1/2}\Pi_{l-1}^{m-1}\right.\\
&\left.+\left[\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}\right]^{1/2}\Pi_{l-1}^{m+1}\right\}\\
&-\mathrm{i}\,\frac{(l+2)}{2k}\left\{\left[\frac{(l-m+1)(l-m+2)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^{m-1}\right.\\
&\left.+\left[\frac{(l+m+1)(l+m+2)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^{m+1}\right\},\\
^{\mathrm{f}}\Pi_z ={}& \frac{(l-1)}{k}\left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{1/2}\Pi_{l-1}^{m}\\
&+\frac{(l+2)}{k}\left[\frac{(l+m+1)(l-m+1)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^{m}\,,
\end{aligned} \tag{7.5}$$

where we have introduced an index f, being e and h respectively. We have left out the argument $\boldsymbol{r}$ in all the functions occurring in this equation.

Conversely, given a Hertz vector expressed in terms of the functions Π_l^m, one can ask for the corresponding Debye potentials. The way to find these is also described in the paper of Bouwkamp and Casimir[6]). First, one should calculate the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$, in this case by means of eq. (7.1), then evaluate the dot products $\boldsymbol{r}\cdot\boldsymbol{E}$ and $\boldsymbol{r}\cdot\boldsymbol{H}$, which are solutions of Helmholtz's equation, and write these dot products in terms of the functions Π_l^m:

$$\boldsymbol{r}\cdot\boldsymbol{E}(\boldsymbol{r}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} a_{l,m}\Pi_l^m(\boldsymbol{r}) , \tag{7.6a}$$

$$\boldsymbol{r}\cdot\boldsymbol{H}(\boldsymbol{r}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} b_{l,m}\Pi_l^m(\boldsymbol{r}) . \tag{7.6b}$$

Knowing the coefficients $a_{l,m}$ and $b_{l,m}$, the Debye potentials are

$${}^{\mathrm{e}}\Pi(\boldsymbol{r}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \frac{a_{l,m}}{l(l+1)}\,\Pi_l^m(\boldsymbol{r}) , \tag{7.7a}$$

$${}^{\mathrm{h}}\Pi(\boldsymbol{r}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \frac{b_{l,m}}{l(l+1)}\,\Pi_l^m(\boldsymbol{r}) . \tag{7.7b}$$

Carrying out this programme we find after a lengthy calculation, again involving recurrence relations for Legendre and Hankel functions, for ${}^{\mathrm{e}}_{\mathrm{h}}\boldsymbol{\Pi} = (\Pi_l^m, 0, 0)$* (and ${}^{\mathrm{h}}_{\mathrm{e}}\boldsymbol{\Pi} = (0,0,0)$):

$$\begin{aligned}
{}^{\mathrm{e}}_{\mathrm{h}}\Pi = {} & \frac{k}{2l}\left\{\left[\frac{(l+m-1)(l+m)}{(2l-1)((2l+1)}\right]^{1/2}\Pi_{l-1}^{m-1} - \left[\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}\right]^{1/2}\Pi_{l-1}^{m+1}\right\} \\
& - \frac{k}{2(l+1)}\left\{\left[\frac{(l-m+1)(l-m+2)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^{m-1}\right. \\
& \left. - \left[\frac{(l+m+1)(l+m+2)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^{m+1}\right\} ,
\end{aligned} \tag{7.8a}$$

and

$$\begin{aligned}
{}^{\mathrm{h}}_{\mathrm{e}}\Pi = {} & \mp\frac{k}{2l(l+1)}\left\{[(l+m)(l-m+1)]^{1/2}\Pi_l^{m-1}\right. \\
& \left. + [(l+m+1)(l-m)]^{1/2}\Pi_l^{m+1}\right\} ,
\end{aligned} \tag{7.8b}$$

* This is a shorthand notation for two possibilities, the formulae can be read either with the upper or with the lower superscripts and signs.

for ${}^{\mathrm{e}}_{\mathrm{h}}\boldsymbol{\Pi} = (0, \Pi_l^m, 0)$:

$$
\begin{aligned}
{}^{\mathrm{e}}_{\mathrm{h}}\Pi = \mathrm{i}\,\frac{k}{2l}\left\{\left[\frac{(l+m-1)(l+m)}{(2l-1)((2l+1)}\right]^{1/2}\Pi_{l-1}^{m-1} + \left[\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}\right]^{1/2}\Pi_{l-1}^{m+1}\right\} \\
-\mathrm{i}\,\frac{k}{2(l+1)}\left\{\left[\frac{(l-m+1)(l-m+2)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^{m-1}\right. \\
\left. + \left[\frac{(l+m+1)(l+m+2)}{(2l+3)(2l+1)}\right]^{1/2}\Pi_{l+1}^{m+1}\right\},
\end{aligned}
\tag{7.8c}
$$

and

$$
\begin{aligned}
{}^{\mathrm{h}}_{\mathrm{e}}\Pi = \mp\mathrm{i}\,\frac{k}{2l(l+1)}\,\{[(l+m)(l-m+1)]^{1/2}\Pi_l^{m-1} \\
-[(l+m+1)(l-m)]^{1/2}\Pi_l^{m+1}\},
\end{aligned}
\tag{7.8d}
$$

whereas for ${}^{\mathrm{e}}_{\mathrm{h}}\boldsymbol{\Pi} = (0, 0, \Pi_l^m)$:

$$
\begin{aligned}
{}^{\mathrm{e}}_{\mathrm{h}}\Pi = \frac{k}{l}\left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{1/2}\Pi_{l-1}^{m} \\
+\frac{k}{l+1}\left[\frac{(l+m+1)(l-m+1)}{(2l+3)((2l+1)}\right]^{1/2}\Pi_{l+1}^{m},
\end{aligned}
\tag{7.8e}
$$

and

$$
{}^{\mathrm{h}}_{\mathrm{e}}\Pi = \pm\frac{k}{l(l+1)}\,m\Pi_l^m\,.
\tag{7.8f}
$$

One can finally check, using eq. (7.8), that the Debye potentials, corresponding to the Hertz vectors eq. (7.5), are, for f = e, ${}^{\mathrm{e}}\Pi = \Pi_l^m$, ${}^{\mathrm{h}}\Pi = 0$ and for f = h, ${}^{\mathrm{e}}\Pi = 0$, ${}^{\mathrm{h}}\Pi = \Pi_l^m$, as should be the case.

All formulae in this section are also valid if we replace Π by Ψ.

8. Calculation of the matrix **A**

We choose our coordinate system as indicated in fig. 3. In this section we shall express the electromagnetic wave corresponding to the Debye potential ${}^{\mathrm{f}}\Pi = \Pi_l^m$ after its reflection by the substrate in terms of the Debye potentials ${}^{\mathrm{f}}\Psi = \Psi_l^m$ (f = e, h). This will give us the matrix **A** defined by eq. (5.1).

By means of eqs. (7.5) and (6.3) the Hertz vector corresponding to the Debye

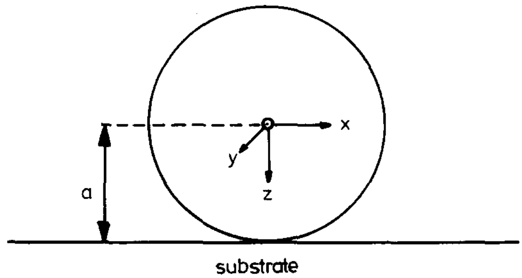

Fig. 3. Choice of the coordinate system, x-axis to the right, y-axis forward, z-axis downward.

potential ${}^{\mathrm{f}}\Pi = \Pi_l^m$ can be written as an integral over plane waves:

$$ {}^{\mathrm{f}}\boldsymbol{\Pi}(r, \theta, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\beta \int_0^{\pi/2 - \mathrm{i}\infty} \sin\alpha \, \mathrm{d}\alpha \, \boldsymbol{\pi}(\alpha, \beta)\, \mathrm{e}^{ikr\cos\gamma} , \tag{8.1} $$

in which $\cos\gamma$ is defined by eq. (6.2) and with

$$ \begin{aligned} \pi_x = {} & \frac{(l-1)}{2k}\, \mathrm{i}^{-(l-1)} \left\{ \left[\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)} \right]^{1/2} Y_{l-1}^{m-1} \right. \\ & \left. - \left[\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)} \right]^{1/2} Y_{l-1}^{m+1} \right\} \\ & - \frac{(l+2)}{2k}\, \mathrm{i}^{-(l+1)} \left\{ \left[\frac{(l-m+1)(l-m+2)}{(2l+3)(2l+1)} \right]^{1/2} Y_{l+1}^{m-1} \right. \\ & \left. - \left[\frac{(l+m+1)(l+m+2)}{(2l+3)(2l+1)} \right]^{1/2} Y_{l+1}^{m+1} \right\} , \\ \pi_y = {} & \mathrm{i}\, \frac{(l-1)}{2k}\, \mathrm{i}^{-(l-1)} \left\{ \left[\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)} \right]^{1/2} Y_{l-1}^{m-1} \right. \\ & \left. + \left[\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)} \right]^{1/2} Y_{l-1}^{m+1} \right\} \\ & - \mathrm{i}\, \frac{(l+2)}{2k}\, \mathrm{i}^{-(l+1)} \left\{ \left[\frac{(l-m+1)(l-m+2)}{(2l+3)(2l+1)} \right]^{1/2} Y_{l+1}^{m-1} \right. \\ & \left. + \left[\frac{(l+m+1)(l+m+2)}{(2l+3)(2l+1)} \right]^{1/2} Y_{l+1}^{m+1} \right\} , \\ \pi_z = {} & \frac{(l-1)}{k}\, \mathrm{i}^{-(l-1)} \left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)} \right]^{1/2} Y_{l-1}^{m} \\ & + \frac{(l+2)}{k}\, \mathrm{i}^{-(l+1)} \left[\frac{(l+m+1)(l-m+1)}{(2l+3)(2l+1)} \right]^{1/2} Y_{l+1}^{m} , \end{aligned} \tag{8.2} $$

where we have left out the arguments α, β in the angular functions.

Let us first consider one plane wave in the integral (8.1) characterized by the angles α and β with wave vector $\boldsymbol{k} = (k \sin\alpha \cos\beta,\ k \sin\alpha \sin\beta,\ k\cos\alpha)$ and amplitude $\boldsymbol{\pi}(\alpha, \beta)$. Corresponding to this plane wave we introduce a coordinate system (x', y', z') with unit vectors $\hat{\boldsymbol{x}}'$, $\hat{\boldsymbol{y}}'$, $\hat{\boldsymbol{z}}'$:

$$\begin{aligned}
\hat{\boldsymbol{x}}' &= (\cos\alpha \cos\beta, \cos\alpha \sin\beta, -\sin\alpha)\ , \\
\hat{\boldsymbol{y}}' &= (-\sin\beta, \cos\beta, 0)\ , \\
\hat{\boldsymbol{z}}' &= (\sin\alpha \cos\beta, \sin\alpha \sin\beta, \cos\alpha)\ ,
\end{aligned} \tag{8.3}$$

which can be obtained from the coordinate system (x, y, z) by a rotation through the Euler angles $\beta, \alpha, 0$. We have chosen this coordinate system in such a way that $\hat{\boldsymbol{z}}'$ is the direction of propagation of the plane wave, $\hat{\boldsymbol{x}}'$ the direction parallel with the plane of incidence (p-direction) and $\hat{\boldsymbol{y}}'$ the direction normal to the plane of incidence (s-direction). Using eqs. (7.1a), (8.2) and (8.3) we can calculate the p- and s-components of the electric field of this plane wave. Omitting the phase factor $e^{ikr\cos\gamma}$, we obtain for $\mathrm{f} = \mathrm{e}$:

$$E_{\mathrm{p}} = k\tilde{V}_l^m(\cos\alpha)\, e^{im\beta}\ , \quad E_{\mathrm{s}} = \mathrm{i}k\, \tilde{U}_l^m(\cos\alpha)\, e^{im\beta}\ , \tag{8.4a}$$

and for $\mathrm{f} = \mathrm{h}$:

$$E_{\mathrm{p}} = \mathrm{i}k\, \tilde{U}_l^m(\cos\alpha)\, e^{im\beta}\ , \quad E_{\mathrm{s}} = -k\tilde{V}_l^m(\cos\alpha)\, e^{im\beta}\ , \tag{8.4b}$$

with the abbreviations

$$\begin{aligned}
\tilde{U}_l^m = {} & \frac{(l-1)}{2}\, \mathrm{i}^{-(l-1)} \left\{ \left[\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)} \right]^{1/2} \tilde{P}_{l-1}^{m-1} \right. \\
& \left. + \left[\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)} \right]^{1/2} \tilde{P}_{l-1}^{m+1} \right\} \\
& - \frac{(l+2)}{2}\, \mathrm{i}^{-(l+1)} \left\{ \left[\frac{(l-m+1)(l-m+2)}{(2l+3)(2l+1)} \right]^{1/2} \tilde{P}_{l+1}^{m-1} \right. \\
& \left. + \left[\frac{(l+m+1)(l+m+2)}{(2l+3)(2l+1)} \right]^{1/2} \tilde{P}_{l+1}^{m+1} \right\} ,
\end{aligned} \tag{8.5a}$$

$$\begin{aligned}
\tilde{V}_l^m = {} & \tfrac{1}{2}\, \mathrm{i}^{-(l-1)} \{ [(l-m+1)(l+m)]^{1/2} \tilde{P}_l^{m-1} \\
& - [(l+m+1)(l-m)]^{1/2} \tilde{P}_l^{m+1} \} ,
\end{aligned} \tag{8.5b}$$

and with the normalized Legendre function

$$\tilde{P}_l^m(\cos\alpha) \equiv \left[(2l+1)\,\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\alpha)\,. \tag{8.6}$$

We have left out the argument $\cos\alpha$ in eq. (8.5).

The reflected wave can be calculated using the Fresnel reflection coefficients

$$r_{\mathrm{p}}(\cos\alpha) = \frac{n_+\cos\alpha - n_-\cos\alpha_{\mathrm{t}}}{n_+\cos\alpha + n_-\cos\alpha_{\mathrm{t}}}\,, \tag{8.7a}$$

$$r_{\mathrm{s}}(\cos\alpha) = \frac{n_-\cos\alpha - n_+\cos\alpha_{\mathrm{t}}}{n_-\cos\alpha + n_+\cos\alpha_{\mathrm{t}}}\,, \tag{8.7b}$$

with n_+ the refractive index of the substrate and

$$\cos\alpha_{\mathrm{t}} = \sqrt{1 - \frac{n_-^2}{n_+^2}\,(1-\cos^2\alpha)}\,, \tag{8.8}$$

where we have to choose the sign of the square root in such a way that $n_+\cos\alpha_{\mathrm{t}}$ has a non-negative imaginary part (this excludes the unphysical situation that the transmitted wave increases exponentially with increasing z).

We now introduce a coordinate system (x'', y'', z''), corresponding to the reflected plane wave, with unit vectors $\hat{\boldsymbol{x}}''$, $\hat{\boldsymbol{y}}''$, $\hat{\boldsymbol{z}}''$:

$$\begin{aligned}
\hat{\boldsymbol{x}}'' &= (-\cos\alpha\cos\beta,\, -\cos\alpha\sin\beta,\, -\sin\alpha)\,,\\
\hat{\boldsymbol{y}}'' &= (-\sin\beta, \cos\beta, 0)\,,\\
\hat{\boldsymbol{z}}'' &= (\sin\alpha\cos\beta,\, \sin\alpha\sin\beta,\, -\cos\alpha)\,,
\end{aligned} \tag{8.9}$$

which can be obtained from (x, y, z) by a rotation through the Euler angles $\beta, \pi-\alpha, 0$. We have chosen this coordinate system in such a way that $\hat{\boldsymbol{z}}''$ is the direction of propagation of the reflected plane wave, $\hat{\boldsymbol{x}}''$ its p- and $\hat{\boldsymbol{y}}''$ its s-direction. In this coordinate system the electric field $\boldsymbol{E}^{\mathrm{r}}$ belonging to the reflected plane wave has p- and s-components

$$E_{\mathrm{p}}^{\mathrm{r}} = r_{\mathrm{p}}(\cos\alpha)E_{\mathrm{p}}\,, \quad E_{\mathrm{s}}^{\mathrm{r}} = r_{\mathrm{s}}(\cos\alpha)E_{\mathrm{s}}\,. \tag{8.10}$$

By means of eqs. (3.9) and (3.10) the Debye potentials corresponding to the reflected plane wave can be found:

$$
{}^{e}\Psi_{\alpha,\beta} = e^{2iq\cos\alpha}\,\frac{1}{k}\sum_{l'=1}^{\infty}\sum_{m'=-l'}^{l'} i^{l'-1}\left[\frac{2l'+1}{l'(l'+1)}\right]^{1/2}(-1)^{m'-1}
$$

$$
\times\{E_{p}^{r}(\tfrac{1}{2}D_{m',1}^{l'}(\beta,\pi-\alpha,0)-\tfrac{1}{2}D_{m',-1}^{l'}(\beta,\pi-\alpha,0))
$$

$$
-\,i\,E_{s}^{r}(\tfrac{1}{2}D_{m',1}^{l'}(\beta,\pi-\alpha,0)+\tfrac{1}{2}D_{m',-1}^{l'}(\beta,\pi-\alpha,0))\}\Psi_{l'}^{m'}\,, \quad (8.11a)
$$

$$
{}^{h}\Psi_{\alpha,\beta} = -\,e^{2iq\cos\alpha}\,\frac{1}{k}\sum_{l'=1}^{\infty}\sum_{m'=-l'}^{l'} i^{l'}\left[\frac{2l'+1}{l'(l'+1)}\right]^{1/2}(-1)^{m'-1}
$$

$$
\times\{E_{p}^{r}(\tfrac{1}{2}D_{m',1}^{l'}(\beta,\pi-\alpha,0)+\tfrac{1}{2}D_{m',-1}^{l'}(\beta,\pi-\alpha,0))
$$

$$
-\,i\,E_{s}^{r}(\tfrac{1}{2}D_{m',1}^{l'}(\beta,\pi-\alpha,0)-\tfrac{1}{2}D_{m',-1}^{l'}(\beta,\pi-\alpha,0))\}\Psi_{l'}^{m'}\,. \quad (8.11b)
$$

The subscripts α and β indicate that these Debye potentials correspond to the plane wave characterized by these angles. The phase factor $e^{2iq\cos\alpha}$ arises from the phase difference between the plane wave and its reflected wave in 0. From eq. (3.4) we have

$$
D_{m',\pm1}^{l'}(\beta,\pi-\alpha,0) = e^{-im'\beta}\,d_{m',\pm1}^{l'}(\pi-\alpha)\,. \quad (8.12)
$$

Eq. (8.11) should now be integrated over all angles α and β. Inserting eqs. (8.10), (8.4) and (8.12), and performing the integration over β:

$$
\frac{1}{2\pi}\int_{0}^{2\pi} d\beta\, e^{i(m-m')\beta} = \delta_{mm'}\,, \quad (8.13)
$$

we are left with an integration over α:

$$
{}^{f'}\Psi = \sum_{l'=1}^{\infty}\sum_{m'=-l'}^{l'}\left\{ i^{l'-1}\left[\frac{2l'+1}{l'(l'+1)}\right]^{1/2}(-1)^{m-1}\delta_{mm'}\right.
$$

$$
\left.\times\int_{0}^{\pi/2-i\infty}\sin\alpha\, d\alpha\, e^{2iq\cos\alpha}\, a_{l',f';l,f}^{m}(\alpha)\right\}\Psi_{l'}^{m'}\,, \quad (8.14)
$$

with the abbreviations

$$
a_{l',e;l,e}^{m}(\alpha) = r_{p}(\cos\alpha)\tilde{V}_{l}^{m}(\cos\alpha)d_{m,-}^{l'}(\pi-\alpha)
$$

$$
+\, r_{s}(\cos\alpha)\tilde{U}_{l}^{m}(\cos\alpha)d_{m,+}^{l'}(\pi-\alpha)\,,
$$

$$a^m_{l',\mathrm{h};l,\mathrm{e}}(\alpha) = -\mathrm{i}\{r_\mathrm{p}(\cos\alpha)\tilde{V}^m_l(\cos\alpha)d^{l'}_{m,+}(\pi-\alpha) + r_\mathrm{s}(\cos\alpha)\tilde{U}^m_l(\cos\alpha)d^{l'}_{m,-}(\pi-\alpha)\}\,,$$

$$a^m_{l',\mathrm{e};l,\mathrm{h}}(\alpha) = \mathrm{i}\{r_\mathrm{p}(\cos\alpha)\tilde{U}^m_l(\cos\alpha)d^{l'}_{m,-}(\pi-\alpha) + r_\mathrm{s}(\cos\alpha)\tilde{V}^m_l(\cos\alpha)d^{l'}_{m,+}(\pi-\alpha)\}\,,$$

$$a^m_{l',\mathrm{h};l,\mathrm{h}}(\alpha) = r_\mathrm{p}(\cos\alpha)\tilde{U}^m_l(\cos\alpha)d^{l'}_{m,+}(\pi-\alpha) + r_\mathrm{s}(\cos\alpha)\tilde{V}^m_l(\cos\alpha)d^{l'}_{m,-}(\pi-\alpha)\,, \tag{8.15}$$

and

$$\begin{aligned} d^{l'}_{m,-}(\pi-\alpha) &= \tfrac{1}{2}\,d^{l'}_{m,1}(\pi-\alpha) - \tfrac{1}{2}\,d^{l'}_{m,-1}(\pi-\alpha)\,, \\ d^{l'}_{m,+}(\pi-\alpha) &= \tfrac{1}{2}\,d^{l'}_{m,1}(\pi-\alpha) + \tfrac{1}{2}\,d^{l'}_{m,-1}(\pi-\alpha)\,. \end{aligned} \tag{8.16}$$

From eq. (8.14) we see that the matrix **A**, characterizing the reflection of spherical waves by the substrate, is:

$$A_{l',m',\mathrm{f}';l,m,\mathrm{f}} = \mathrm{i}^{l'-1}\left[\frac{2l'+1}{l'(l'+1)}\right]^{1/2}(-1)^{m-1}\delta_{mm'} \times \int_0^{\pi/2-\mathrm{i}\infty} \sin\alpha\,\mathrm{d}\alpha\, \mathrm{e}^{2\mathrm{i}q\cos\alpha}a^m_{l',\mathrm{f}';l,\mathrm{f}}(\alpha)\,. \tag{8.17}$$

It is noteworthy that the matrix **A** is diagonal with respect to the index m, just as the matrix **B** in the case of a sphere (eq. (4.2)). One could call this "conservation of angular momentum along the z-axis". It is a consequence of the rotational symmetry about the z-axis. This means that the calculation of the scattered field can be done for each m separately.

In general, it will be difficult to perform the integration over α in eq. (8.17) analytically because of the Fresnel factors r_p and r_s appearing in it. However, if the substrate is a perfect conductor, r_p and r_s are 1 and -1 respectively and eq. (8.17) reduces to an integrable form. We shall give the result in the next section.

9. Perfectly conducting substrate

If the substrate is a perfect electric conductor, $r_\mathrm{p} = 1$ and $r_\mathrm{s} = -1$, and eq. (8.15) becomes

$$
\begin{aligned}
a^m_{l',e;l,e}(\alpha) &= \tilde{V}^m_l(\cos\alpha) d^{l'}_{m,-}(\pi-\alpha) - \tilde{U}^m_l(\cos\alpha) d^{l'}_{m,+}(\pi-\alpha)\,,\\
a^m_{l',h;l,e}(\alpha) &= -\mathrm{i}\{\tilde{V}^m_l(\cos\alpha) d^{l'}_{m,+}(\pi-\alpha) - \tilde{U}^m_l(\cos\alpha) d^{l'}_{m,-}(\pi-\alpha)\}\,,\\
a^m_{l',e;l,h}(\alpha) &= \mathrm{i}\{\tilde{U}^m_l(\cos\alpha) d^{l'}_{m,-}(\pi-\alpha) - \tilde{V}^m_l(\cos\alpha) d^{l'}_{m,+}(\pi-\alpha)\}\,,\\
a^m_{l',h;l,h}(\alpha) &= \tilde{U}^m_l(\cos\alpha) d^{l'}_{m,+}(\pi-\alpha) - \tilde{V}^m_l(\cos\alpha) d^{l'}_{m,-}(\pi-\alpha)\,.
\end{aligned}
\tag{9.1}
$$

Using eqs. (8.5) and (3.5) one can show that in this case the integral over α in eq. (8.17) can be written in the form:

$$
\int_1^{\mathrm{i}\infty} \mathrm{d}(\cos\alpha)\, \mathrm{e}^{2\mathrm{i}q\cos\alpha} P(\cos\alpha)\,,
$$

where $P(\cos\alpha)$ is a polynomial in $\cos\alpha$. This integral can be performed analytically with the help of

$$
\begin{aligned}
\int_1^{\mathrm{i}\infty} \mathrm{d}(\cos\alpha)\, \mathrm{e}^{2\mathrm{i}q\cos\alpha}(\cos\alpha)^n &= \frac{\mathrm{d}^n}{\mathrm{d}(2\mathrm{i}q)^n} \int_1^{\mathrm{i}\infty} \mathrm{d}(\cos\alpha)\, \mathrm{e}^{2\mathrm{i}q\cos\alpha}\\
&= \frac{\mathrm{d}^n}{\mathrm{d}(2\mathrm{i}q)^n}\left\{-\frac{1}{2\mathrm{i}q}\,\mathrm{e}^{2\mathrm{i}q}\right\}.
\end{aligned}
\tag{9.2}
$$

Here, however, we shall use a quite different method of calculating **A**. Beside (x, y, z) we introduce a coordinate system (x', y', z') that can be obtained from (x, y, z) by shifting the origin a distance $2a$ along the positive z-axis:

$$
x' = x\,, \quad y' = y\,, \quad z' = z - 2a\,. \tag{9.3}
$$

We can express the electromagnetic wave corresponding to the Debye potentials ${}^{\mathrm{f}}\Pi = \Pi^m_l$ in terms of the Debye potentials ${}^{\mathrm{f}}\Psi = \Psi^m_l$ in the shifted coordinate system in the region $r' < 2a$. In this way a matrix **T** can be introduced. The calculation of **T** can be performed in a completely analogous way to that of **A**. The only difference is that there is no reflection, so that

$$
\begin{aligned}
T_{l',m',\mathrm{f}';l,m,\mathrm{f}} = \mathrm{i}^{l'-1}\left[\frac{2l'+1}{l'(l'+1)}\right]^{1/2} (-1)^{m-1}\delta_{mm'}\\
\times \int_0^{\pi/2-\mathrm{i}\infty} \sin\alpha\, \mathrm{d}\alpha\, \mathrm{e}^{2\mathrm{i}q\cos\alpha}\, t^m_{l',\mathrm{f}';l,\mathrm{f}}(\alpha)\,,
\end{aligned}
\tag{9.4}
$$

with

$$
\begin{aligned}
t^m_{l',\mathrm{e};l,\mathrm{e}}(\alpha) &= \tilde{V}^m_l(\cos\alpha)d^{l'}_{m,-}(\alpha) + \tilde{U}^m_l(\cos\alpha)d^{l'}_{m,+}(\alpha)\,, \\
t^m_{l',\mathrm{h};l,\mathrm{e}}(\alpha) &= -\mathrm{i}\{\tilde{V}^m_l(\cos\alpha)d^{l'}_{m,+}(\alpha) + \tilde{U}^m_l(\cos\alpha)d^{l'}_{m,-}(\alpha)\}\,, \\
t^m_{l',\mathrm{e};l,\mathrm{h}}(\alpha) &= \mathrm{i}\{\tilde{U}^m_l(\cos\alpha)d^{l'}_{m,-}(\alpha) + \tilde{V}^m_l(\cos\alpha)d^{l'}_{m,+}(\alpha)\}\,, \\
t^m_{l',\mathrm{h};l,\mathrm{h}}(\alpha) &= \tilde{U}^m_l(\cos\alpha)d^{l'}_{m,+}(\alpha) + \tilde{V}^m_l(\cos\alpha)d^{l'}_{m,-}(\alpha)\,.
\end{aligned}
\tag{9.5}
$$

Using eq. (3.8) we see that there is a simple relation between **A** and **T**:

$$
\begin{aligned}
A_{l',m',\mathrm{e};l,m,\mathrm{e}} &= (-1)^{l'-m-1}T_{l',m',\mathrm{e};l,m,\mathrm{e}}\,, \\
A_{l',m',\mathrm{h};l,m,\mathrm{e}} &= (-1)^{l'-m}T_{l',m',\mathrm{h};l,m,\mathrm{e}}\,, \\
A_{l',m',\mathrm{e};l,m,\mathrm{h}} &= (-1)^{l'-m-1}T_{l',m',\mathrm{e};l,m,\mathrm{h}}\,, \\
A_{l',m',\mathrm{h};l,m,\mathrm{h}} &= (-1)^{l'-m}T_{l',m',\mathrm{h};l,m,\mathrm{h}}\,.
\end{aligned}
\tag{9.6}
$$

But there is also a direct way of calculating **T** using an addition theorem:

$$
\Pi^m_l(\boldsymbol{r}) = \sum_{l'=|m|}^{\infty} c^{(l,m)}_{l'}\Psi^m_{l'}(\boldsymbol{r}')\,,
\tag{9.7}
$$

valid in the region $r' < 2a$. In the appendix we prove that the following recurrence relations can be derived for the coefficients $c^{(l,m)}_{l'}$:

$$
\begin{aligned}
\sqrt{(2l'+1)(l-m)(l+m+1)}\,c^{(l,m)}_{l'} &= \sqrt{(2l'+1)(l'-m)(l'+m+1)}\,c^{(l,m+1)}_{l'} \\
&+ 2q\left\{\sqrt{\frac{(l'+m+1)(l'+m+2)}{2l'+3}}\;c^{(l,m+1)}_{l'+1}\right. \\
&\left.+\sqrt{\frac{(l'-m)(l'-m-1)}{2l'-1}}\;c^{(l,m+1)}_{l'-1}\right\},
\end{aligned}
\tag{9.8a}
$$

$$
\begin{aligned}
\sqrt{(2l'+1)(l+m)(l-m+1)}\,c^{(l,m)}_{l'} &= \sqrt{(2l'+1)(l'+m)(l'-m+1)}\,c^{(l,m-1)}_{l'} \\
&+ 2q\left\{\sqrt{\frac{(l'-m+1)(l'-m+2)}{2l'+3}}\;c^{(l,m-1)}_{l'+1}\right. \\
&\left.+\sqrt{\frac{(l'+m)(l'+m+1)}{2l'-1}}\;c^{(l,m-1)}_{l'-1}\right\},
\end{aligned}
\tag{9.8b}
$$

$$\sqrt{2l'+1}\left\{\frac{l}{\sqrt{2l-1}}\,c_{l'}^{(l-1,0)}-\frac{(l+1)}{\sqrt{2l+3}}\,c_{l'}^{(l+1,0)}\right\}$$
$$=\sqrt{2l+1}\left\{\frac{(l'+1)}{\sqrt{2l'+3}}\,c_{l'+1}^{(l,0)}-\frac{l'}{\sqrt{2l'-1}}\,c_{l'-1}^{(l,0)}\right\}. \tag{9.9}$$

From the addition theorem*

$$\Pi_0^0(\boldsymbol{r})=\sum_{l'=0}^{\infty}\sqrt{2l'+1}(-1)^{l'}h_{l'}^{(1)}(2q)\Psi_{l'}^0(\boldsymbol{r}')\,, \tag{9.10}$$

we have

$$c_{l'}^{(0,0)}=\sqrt{2l'+1}(-1)^{l'}h_{l'}^{(1)}(2q)\,. \tag{9.11}$$

By means of eqs. (9.8)–(9.11) $c_{l'}^{(l,m)}$ can be calculated for all l, l' and m.

Again, one should not apply the addition theorem eq. (9.7) directly to the Debye potentials but to the corresponding Hertz vectors, given by eq. (7.5). One then finally returns to the Debye potentials using eq. (7.8). In this way the following expressions for the matrix elements of $\mathbf{T}$ are found:

$$T_{l',m',\mathrm{e};l,m,\mathrm{e}}=T_{l',m',\mathrm{h};l,m,\mathrm{h}}=\frac{1}{2}\,\frac{1}{\sqrt{(2l+1)(2l'+1)}}\,\delta_{mm'}$$
$$\times\left[\frac{1}{l'}\left((l-1)\left\{\left[\frac{(l+m-1)(l+m)(l'+m-1)(l'+m)}{(2l-1)(2l'-1)}\right]^{1/2}c_{l'-1}^{(l-1,m-1)}\right.\right.\right.$$
$$+\left[\frac{(l-m-1)(l-m)(l'-m-1)(l'-m)}{(2l-1)(2l'-1)}\right]^{1/2}c_{l'-1}^{(l-1,m+1)}$$
$$\left.+2\left[\frac{(l-m)(l+m)(l'-m)(l'+m)}{(2l-1)(2l'-1)}\right]^{1/2}c_{l'-1}^{(l-1,m)}\right\}$$
$$-(l+2)\left\{\left[\frac{(l-m+1)(l-m+2)(l'+m-1)(l'+m)}{(2l+3)(2l'-1)}\right]^{1/2}c_{l'-1}^{(l+1,m-1)}\right.$$
$$+\left[\frac{(l+m+1)(l+m+2)(l'-m-1)(l'-m)}{(2l+3)(2l'-1)}\right]^{1/2}c_{l'-1}^{(l+1,m+1)}$$
$$\left.\left.-2\left[\frac{(l-m+1)(l+m+1)(l'-m)(l'+m)}{(2l+3)(2l'-1)}\right]^{1/2}c_{l'-1}^{(l+1,m)}\right\}\right)-\frac{1}{l'+1}$$
$$\times\left((l-1)\left\{\left[\frac{(l+m-1)(l+m)(l'+m+1)(l'+m+2)}{(2l-1)(2l'+3)}\right]^{1/2}c_{l'+1}^{(l-1,m-1)}\right.\right.$$

* See e.g. ref. 8.

$$+\left[\frac{(l-m-1)(l-m)(l'+m+1)(l'+m+2)}{(2l-1)(2l'+3)}\right]^{1/2} c_{l'+1}^{(l-1,m+1)}$$

$$-2\left[\frac{(l-m)(l+m)(l'-m+1)(l'+m+1)}{(2l-1)(2l'+3)}\right]^{1/2} c_{l'+1}^{(l-1,m)}\Bigg\}$$

$$-(l+2)\Bigg\{\left[\frac{(l-m+1)(l-m+2)(l'-m+1)(l'-m+2)}{(2l+3)(2l'+3)}\right]^{1/2} c_{l'+1}^{(l+1,m-1)}$$

$$+\left[\frac{(l+m+1)(l+m+2)(l'+m+1)(l'+m+2)}{(2l+3)(2l'+3)}\right]^{1/2} c_{l'+1}^{(l+1,m+1)}$$

$$+2\left[\frac{(l-m+1)(l+m+1)(l'-m+1)(l'+m+1)}{(2l+3)(2l'+3)}\right]^{1/2} c_{l'+1}^{(l+1,m)}\Bigg\}\Bigg)\Bigg],$$

(9.12a)

$$T_{l',m',\mathrm{h};l,m,\mathrm{e}} = -T_{l',m,\mathrm{e};l,m,\mathrm{h}} = \frac{1}{2}\,\frac{1}{\sqrt{2l+1}}\,\delta_{mm'}\,\frac{1}{l'(l'+1)}$$

$$\times\Bigg(-(l-1)\Bigg\{\left[\frac{(l+m-1)(l+m)(l'-m+1)(l'+m)}{(2l-1)}\right]^{1/2} c_{l'}^{(l-1,m-1)}$$

$$-\left[\frac{(l-m-1)(l-m)(l'-m)(l'+m+1)}{(2l-1)}\right]^{1/2} c_{l'}^{(l-1,m+1)}$$

$$-2m\left[\frac{(l-m)(l+m)}{(2l-1)}\right]^{1/2} c_{l'}^{(l-1,m)}\Bigg\}$$

$$+(l+2)\Bigg\{\left[\frac{(l-m+1)(l-m+2)(l'-m+1)(l'+m)}{(2l+3)}\right]^{1/2} c_{l'}^{(l+1,m-1)}$$

$$-\left[\frac{(l+m+1)(l+m+2)(l'-m)(l'+m+1)}{(2l+3)}\right]^{1/2} c_{l'}^{(l+1,m+1)}$$

$$+2m\left[\frac{(l-m+1)(l+m+1)}{(2l+3)}\right]^{1/2} c_{l'}^{(l+1,m)}\Bigg\}\Bigg). \qquad (9.12b)$$

The matrix **A** is finally found using eq. (9.6).

10. The static limit

In this section we shall consider the static limit, i.e. $a \ll \lambda/2\pi$ or $q \ll 1$, and show that in this limit the solution presented in this paper is equivalent to the solution given by Wind, Vlieger and Bedeaux[5]). Because we assume the sphere and the substrate to be non-magnetic we only have to deal with the electric field and the electric Debye-potentials, so that we may drop the index f, assuming it to be e. In the static limit the matrix **B**, eq. (4.2), becomes*

$$B_{l',m';l,m} = \mathrm{i}\delta_{ll'}\delta_{mm'} \frac{l+1}{2l+1} \frac{q^{2l+1}}{((2l-1)!!)^2} \frac{n^2 - n_-^2}{ln^2 + (l+1)n_-^2}, \tag{10.1}$$

with the convention

$$(2l-1)!! \equiv (2l-1)(2l-3)\cdots 3\cdot 1\,. \tag{10.2}$$

If a plane wave is incident on the system, then for $r \ll \lambda/2\pi$ or $kr \ll 1$, in eqs. (3.9a) and (3.10a) only the term with $l = 1$ is important. This is clear from the expansion of $j_l(\rho)$† into powers of ρ:

$$j_l(\rho) = 2^l \rho^l \sum_{k=0}^{\infty} \frac{(-1)^k (l+k)!}{k!(2l+2k+1)!}\, \rho^{2k}\,. \tag{10.3}$$

So in the the neighbourhood of the sphere only the partial waves with $l = 1$ and $m = 0, \pm 1$ are important in eqs. (3.9a) and (3.10a). Because of conservation of m we only have to consider the case $m = 0, \pm 1$ in the whole analysis.

In order to calculate the static limit of **A** we develop $r_p(\cos\alpha)$ and $r_s(\cos\alpha)$, given by eq. (8.7), into powers of $1/\cos^2\alpha$:

$$r_\mathrm{p}(\cos\alpha) = \sum_{n=0}^{\infty} r_{\mathrm{p},n} \frac{1}{(\cos^2\alpha)^n}, \tag{10.4a}$$

$$r_\mathrm{s}(\cos\alpha) = \sum_{n=0}^{\infty} r_{\mathrm{s},n} \frac{1}{(\cos^2\alpha)^n}\,. \tag{10.4b}$$

From eq. (9.2) we see that the static limit is obtained by integrating the term with the highest power of $\cos\alpha$. So in the integral eq. (8.17) we only have to know the coefficient of the highest power of $\cos\alpha$. Using this fact and the fact that

$$r_{\mathrm{p},0} = \frac{n_+^2 - n_-^2}{n_+^2 + n_-^2}, \tag{10.5a}$$

$$r_{\mathrm{s},0} = 0\,, \tag{10.5b}$$

* See Born and Wolf[9]).
† See e.g. Stratton[8]).

we can write eq. (8.17) in the static limit, for $m, m' = 0, \pm 1$, as

$$A_{l',m';l,m} = \mathrm{i}^{l'-1}\left[\frac{2l'+1}{l'(l'+1)}\right]^{1/2}(-1)^{m-1}\delta_{mm'}$$
$$\times \frac{n_+^2 - n_-^2}{n_+^2 + n_-^2} \int_0^{\pi/2-\mathrm{i}\infty} \sin\alpha \, d\alpha \, \mathrm{e}^{2\mathrm{i}q\cos\alpha}\, \tilde{V}_l^m(\cos\alpha) d^{l'}_{m,-}(\pi-\alpha) . \qquad (10.6)$$

Furthermore, one can show that the expansion of $\tilde{U}_l^m(\cos\alpha)\, d^{l'}_{m,+}(\pi-\alpha)$ into powers of $\cos\alpha$ has a lower highest power than the expansion of $\tilde{V}_l^m(\cos\alpha) \times d^{l'}_{m,-}(\pi-\alpha)$ in the case $m = 0, \pm 1$, so that in the static limit we can identify the static solution eq. (10.6) with the solution for a perfectly conducting substrate, eqs. (8.17) and (9.1), apart from the factor $(n_+^2 - n_-^2)/(n_+^2 + n_-^2)$. Using this solution, eqs. (9.6) and (9.12), and the expansion of $h_l^{(1)}(\rho)$ (see also ref. 8):

$$h_l^{(1)}(\rho) = j_l(\rho) - \frac{\mathrm{i}}{2^l \rho^{l+1}} \sum_{k=0}^{\infty} \frac{\Gamma(2l-2k+1)}{k!\Gamma(1-k+1)} \rho^{2k} , \qquad (10.7)$$

one can show that in the limit $q \to 0$ **A** becomes, for $m = 0, \pm 1$,

$$A_{l',0;l,0} = -\mathrm{i}\frac{n_+^2 - n_-^2}{n_+^2 + n_-^2}\sqrt{(2l'+1)(2l+1)}$$
$$\times \frac{(2l'-1)!!(l'+l)!(2l-1)!!}{(l'+1)!(l-1)!} \frac{1}{(2q)^{l'+l+1}} , \qquad (10.8a)$$

$$A_{l',\pm 1;l,\pm 1} = -\mathrm{i}\frac{n_+^2 - n_-^2}{n_+^2 + n_-^2}\sqrt{\frac{l'(2l'+1)l(2l+1)}{(l'+1)(l+1)}}$$
$$\times \frac{(2l'-1)!!(l'+l)!(2l-1)!!}{(l'+1)!(l-1)!} \frac{1}{(2q)^{l'+l+1}} . \qquad (10.8b)$$

In order to compare the results obtained above with those found by Wind, Vlieger and Bedeaux[5]), we first remark that in the static limit the electric potential V used by these authors is related to the electric Debye potential ${}^{\mathrm{e}}\Pi$ by (see also ref. 9)

$$V = -\frac{\partial(r\,{}^{\mathrm{e}}\Pi)}{\partial r} . \qquad (10.9)$$

Comparing their expansion of the potential V, corresponding to the scattered field, in terms of the functions $r^{-j-1}P_j^0(\cos\theta)$ $(m = 0)$ and $r^{-j-1}P_j^1(\cos\theta)\cos\phi$ $(m = 1)$ with coefficients $A_{1,j}$ and $B_{1,j}$ respectively, with the expansion eq.

(2.8a) of the electric Debye potential ${}^{e}\Pi$ in terms of the functions Π_l^m in which the coefficients ${}^{e}w_l^m$ are obtained by eq. (5.3), we find, using eqs. (10.7) and (10.9), that in the static limit we have to make the identifications

$$S_{j,0;1,0} = -\mathrm{i}\,\frac{2}{\sqrt{3}}\,\frac{1}{\sqrt{2j+1}}\,\frac{1}{j}\,\frac{1}{(2j-1)!!}\,q^{j+2}A_{1,j}\,, \tag{10.10a}$$

$$S_{j,\pm1;1,\pm1} = -\mathrm{i}\sqrt{\frac{2}{3}}\,\sqrt{\frac{j(j+1)}{2j+1}}\,\frac{1}{j}\,\frac{1}{(2j-1)!!}\,q^{j+2}B_{1,j}\,, \tag{10.10b}$$

with the matrix $\mathbf{S}$ defined by

$$\mathbf{S} \equiv (\mathbf{1} - \mathbf{B}\cdot\mathbf{A})^{-1}\cdot\mathbf{B}\,. \tag{10.11}$$

This relation can also be written as

$$(\mathbf{1} - \mathbf{B}\cdot\mathbf{A})\cdot\mathbf{S} = \mathbf{B}\,, \tag{10.12}$$

which means that $S_{j,0;1,0}$ and $S_{j,\pm1;1,\pm1}$ satisfy the relations

$$\sum_{j=1}^{\infty} \{\delta_{j,k} - B_{k,0;k,0}A_{k,0;j,0}\}S_{j,0;1,0} = \delta_{1,k}B_{1,0;1,0}\,, \tag{10.13a}$$

$$\sum_{j=1}^{\infty} \{\delta_{j,k} - B_{k,\pm1;k,\pm1}A_{k,\pm1;j,\pm1}\}S_{j,\pm1;1,\pm1} = \delta_{1,k}B_{1,\pm1;1,\pm1}\,, \tag{10.13b}$$

where we have used the fact that $\mathbf{B}$ is diagonal in the index l. Using eqs. (10.1), (10.8) and (10.10) it can easily be shown that in the static limit eq. (10.13) is equivalent to the relations given by Wind, Vlieger and Bedeaux[5]) for $A_{1,j}$ and $B_{1,j}$:

$$\sum_{j=1}^{\infty}\left\{\delta_{j,k} + \frac{(\epsilon_2-\epsilon_1)j(\epsilon_1-\epsilon_3)}{(\epsilon_2+\epsilon_1)((j+1)\epsilon_1+j\epsilon_3)}\,\frac{(k+j)!}{k!j!2^{k+j+1}}\right\}\frac{((j+1)\epsilon_1+j\epsilon_3)}{j(\epsilon_1-\epsilon_3)}\,A_{1,j} = \delta_{1,k}\,, \tag{10.14a}$$

$$\sum_{j=1}^{\infty}\left\{\delta_{j,k} + \frac{(\epsilon_2-\epsilon_1)j(\epsilon_1-\epsilon_3)}{(\epsilon_2+\epsilon_1)((j+1)\epsilon_1+j\epsilon_3)}\,\frac{(k+j)!}{(k+1)!(j-1)!2^{k+j+1}}\right\}$$
$$\times\frac{((j+1)\epsilon_1+\epsilon_3)}{j(\epsilon_1-\epsilon_3)}\,B_{1,j} = \delta_{1,k}\,, \tag{10.14b}$$

with $\epsilon_1 = n_-^2$, $\epsilon_2 = n_+^2$ and $\epsilon_3 = n^2$.

Furthermore, it can be shown, using eq. (10.6) and the results of the preceding section, that the reflected scattered wave can be identified with the field of an image scattering centre located at the image point $(0, 0, 2a)$. With this image a scattering matrix $\mathbf{S}'$ can be related having the relevant matrix elements

$$S'_{j,0;1,0} = \frac{n_+^2 - n_-^2}{n_+^2 + n_-^2} (-1)^{j-1} S_{j,0;1,0} , \tag{10.15a}$$

$$S'_{j,\pm 1;1,\pm 1} = \frac{n_+^2 - n_-^2}{n_+^2 + n_-^2} (-1)^{j} S_{j,\pm 1;1,\pm 1} . \tag{10.15b}$$

This is equivalent to the relations

$$A'_{1,j} = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} (-1)^{j-1} A_{1,j} , \tag{10.16a}$$

$$B'_{1,j} = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} (-1)^{j} B_{1,j} , \tag{10.16b}$$

found by Wind, Vlieger and Bedeaux[5]).

11. The far-away scattered field

In most practical situations we shall be interested in the scattered electric field at large distances, i.e. $r \gg \lambda/2\pi$ and $r \gg a$. At such distances we are in the so-called radiation zone, where the fields have no radial components. Above the substrate $(\theta > \pi/2)$ we have two contributions to the scattered field: the contribution of $\boldsymbol{W}^{\mathrm{S}}$ (see section 5) and that of the reflection of $\boldsymbol{W}^{\mathrm{S}}$ by the substrate.

Let us first consider $\boldsymbol{W}^{\mathrm{S}}$. The electric field can be obtained from eq. (2.3a). Written in spherical coordinates this equation becomes

$$\begin{aligned} E_r &= \frac{\partial^2 (r\,{}^{\mathrm{e}}\Pi)}{\partial r^2} + k^2 r\,{}^{\mathrm{e}}\Pi , \\ E_\theta &= \frac{1}{r} \frac{\partial^2 (r\,{}^{\mathrm{e}}\Pi)}{\partial r \partial \theta} + \mathrm{i}k \frac{1}{r \sin\theta} \frac{\partial (r\,{}^{\mathrm{h}}\Pi)}{\partial \phi} , \\ E_\phi &= \frac{1}{r\sin\theta} \frac{\partial^2 (r\,{}^{\mathrm{e}}\Pi)}{\partial r \partial \phi} - \mathrm{i}k \frac{1}{r} \frac{\partial (r\,{}^{\mathrm{h}}\Pi)}{\partial \theta} , \end{aligned} \tag{11.1}$$

with the Debye potentials given by eq. (2.8):

$$ {}^{\mathrm{e}}\Pi=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}{}^{\mathrm{e}}w_l^m\Pi_l^m=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}{}^{\mathrm{e}}w_l^m h_l^{(1)}(kr)\tilde{P}_l^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}\,, \tag{11.2a} $$

$$ {}^{\mathrm{h}}\Pi=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}{}^{\mathrm{h}}w_l^m\Pi_l^m=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}{}^{\mathrm{h}}w_l^m h_l^{(1)}(kr)\tilde{P}_l^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}\,, \tag{11.2b} $$

where ${}^{\mathrm{e}}w_l^m$ and ${}^{\mathrm{h}}w_l^m$ are found from eq. (5.3). When $r \gg \lambda/2\pi$ we have*

$$ rh_l^{(1)}(kr)\sim\frac{1}{k}\,(-\mathrm{i})^{l+1}\,\mathrm{e}^{\mathrm{i}kr}\,, \tag{11.3} $$

where the symbol $\sim$ means that this relation is an equality in the limit $kr\rightarrow\infty$. Inserting this relation and eq. (11.2) into eq. (11.1) we find the asymptotic formulae

$$ \begin{aligned} &E_r\sim 0\,,\\ &E_\theta\sim\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left\{{}^{\mathrm{e}}w_l^m(-\mathrm{i})^l\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\,\frac{\mathrm{d}}{\mathrm{d}\theta}\,\tilde{P}_l^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}\right.\\ &\qquad\left.-\,{}^{\mathrm{h}}w_l^m(-\mathrm{i})^{l+1}\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\,\frac{1}{\sin\theta}\,\tilde{P}_l^m(\cos\theta)m\,\mathrm{e}^{\mathrm{i}m\phi}\right\},\\ &E_\phi\sim\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left\{-\,{}^{\mathrm{e}}w_l^m(-\mathrm{i})^{l+1}\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\,\frac{1}{\sin\theta}\,\tilde{P}_l^m(\cos\theta)m\,\mathrm{e}^{\mathrm{i}m\phi}\right.\\ &\qquad\left.-\,{}^{\mathrm{h}}w_l^m(-\mathrm{i})^{l}\,\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\,\frac{\mathrm{d}}{\mathrm{d}\theta}\,\tilde{P}_l^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}\right\}. \end{aligned} \tag{11.4} $$

In order to calculate the electric field corresponding to the reflection of $\boldsymbol{W}^{\mathrm{S}}$ by the substrate at a point (r, θ, ϕ) $(\theta > \pi/2)$, we introduce a rotated coordinate system (x', y', z') with the z'-axis pointing to the image point (r', θ', ϕ) of (r, θ, ϕ) (see fig. 4). In an analogous way as in section 8 we can express the electric field corresponding to the wave $\boldsymbol{W}^{\mathrm{S}}$ as an integral over plane waves in the coordinate system (x', y', z'). The electric field corresponding to the reflection of $\boldsymbol{W}^{\mathrm{S}}$ then becomes the sum of an integral over p- and an integral over s-polarized plane waves, each wave multiplied by its corresponding Fresnel factor. But, according to eq. (6.3), at the point (r, θ, ϕ) only the reflected wave of the plane wave propagating in the z'-direction ($\cos\gamma = \cos\alpha = 1$) is not damped. So at large distances ($r \gg \lambda/2\pi$) only the reflected waves of the plane waves propagating in directions within a small solid angle about the z'-direction will contribute to the integrals. Because the Fresnel factors occurring in these

* See e.g. ref. 8.

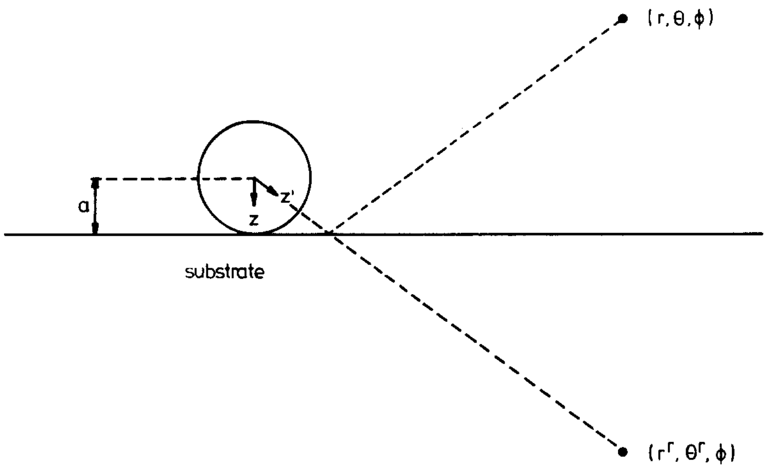

Fig. 4. Choice of the z'-axis for the calculation of the scattered wave after its reflection by the substrate.

integrals will vary slowly within this solid angle we may replace them by the Fresnel factors corresponding to the z'-direction. But the resulting integrals are precisely the plane wave expansions for the θ- and ϕ-components of the electric field at the point (r^r, θ^r, ϕ) corresponding to $\boldsymbol{W}^S$ itself. So at large distances the θ- and ϕ-components of the reflected wave are

$$\begin{aligned} E_\theta^r(r, \theta, \phi) &\sim r_p(\cos\theta^r) E_\theta(r^r, \theta^r, \phi) , \\ E_\phi^r(r, \theta, \phi) &\sim r_s(\cos\theta^r) E_\phi(r^r, \theta^r, \phi) . \end{aligned} \tag{11.5}$$

When in addition to $r \gg \lambda/2\pi$ also $|r\cos\theta| \gg a$ we have asymptotically:

$$\theta^r \sim \pi - \theta , \quad r^r \sim r + 2a\cos(\pi - \theta) \tag{11.6}$$

and eq. (11.5) becomes

$$\begin{aligned} E_\theta^r(r, \theta, \phi) &\sim r_p(\cos(\pi - \theta))\, e^{2iq\cos(\pi-\theta)} E_\theta(r, \pi - \theta, \phi) , \\ E_\phi^r(r, \theta, \phi) &\sim r_s(\cos(\pi - \theta))\, e^{2iq\cos(\pi-\theta)} E_\phi(r, \pi - \theta, \phi) . \end{aligned} \tag{11.7}$$

The total scattered field at (r, θ, ϕ) is obtained by adding eqs. (11.4) and (11.7).

We finally want to remark that the far-away scattered field can also be calculated for $\theta < \pi/2$, i.e. in the substrate. Because in future work we intend to apply the theory only to reflection problems, we shall not consider this more complicated case in this paper.

12. Concluding remarks

We have shown that the problem of light scattering by a sphere on a substrate can be reduced to the problem of scattering by a sphere in a homogeneous medium and that of the reflection of spherical waves by the substrate. The first problem is solved using the Mie theory[1]), for the second we have used an extension of Weyl's method[3]) for the propagation of dipole radiation along a flat surface. The formal solution is given by eq. (5.3). For the matrix **A** appearing in this equation we have derived an integral expression eq. (8.17). In two cases this integral can be performed analytically, namely in the case of a perfectly conducting substrate and in the so-called static limit. In the latter case we have shown that the result is equivalent to that obtained by Wind, Vlieger and Bedeaux[5]).

In general, however, the calculation of **A** has to be performed numerically. In all cases, the matrix inversion in eq. (5.3) has to be done by computer after truncation of this infinite matrix at a certain value of l. In a future paper we shall publish computer results related to the ellipsometric experiments by Greef[4]).

Acknowledgement

The authors want to express their gratitude to Dr. R. Greef of the Department of Chemistry of the University of Southampton, England, and to Mr. M.M. Wind, Prof. Dr. D. Bedeaux and Dr. R.M.J. van Damme of the Leiden University, for stimulating discussions and useful remarks. The international collaboration between Dr. R. Greef and the authors has been made possible by support of the NATO Scientific Affairs Division (grant nr. 908/83).

Appendix

Proof of the recurrence relations eqs. (9.8) *and* (9.9)

Let us consider the coordinate system (x', y', z') which is related to the coordinate system (x, y, z) by a shift 2a along the positive z-axis:

$$x' = x\,, \quad y' = y\,, \quad z' = z - 2a\,. \tag{A.1}$$

The function $\Pi_l^m(\boldsymbol{r})$ should be expressible as a sum over the functions $\Psi_l^m(\boldsymbol{r}')$ in the region $r' < 2a$ because it has no singularities in this region. Furthermore, m is conserved because of symmetry, so

$$\Pi_l^m(\boldsymbol{r}) = \sum_{l'=|m|}^{\infty} c_{l'}^{(l,m)} \Psi_{l'}^m(\boldsymbol{r}') \quad (r' < 2a). \tag{A.2}$$

Or, with the definition eq. (2.10) of these functions:

$$h_l^{(1)}(kr) Y_l^m(\theta, \phi) = \sum_{l'=|m|}^{\infty} c_{l'}^{(l,m)} j_{l'}(kr) Y_{l'}^m(\theta', \phi'). \tag{A.3}$$

We shall evaluate this expression at the point $(r, \theta, 0)$, with $r > 2a$, in the limit $\theta \to 0$. Before taking this limit we divide both sides by $\sin^m\theta$ (we assume m to be non-negative). In this limit we have

$$\lim_{\theta\to 0} \frac{Y_l^m(\theta, 0)}{\sin^m\theta} = \frac{1}{2^m} \frac{1}{m!} \left[(2l+1) \frac{(l+m)!}{(l-m)!} \right]^{1/2}, \tag{A.4}$$

$$\sin\theta \simeq \frac{r'}{r' + 2a} \sin\theta' \quad (\theta \to 0, r > 2a). \tag{A.5}$$

Using these expressions, we obtain:

$$\begin{aligned} &\left[(2l+1) \frac{(l+m)!}{(l-m)!} \right]^{1/2} h_l^{(1)}(kr) \\ &= \left(1 + \frac{2a}{r'}\right)^m \sum_{l'=m}^{\infty} c_{l'}^{(l,m)} \left[(2l'+1) \frac{(l'+m)!}{(l'-m)!} \right]^{1/2} j_{l'}(kr'), \end{aligned} \tag{A.6}$$

where r' is now just $r - 2a$. This equation is valid for all m, in particular for $m+1$:

$$\begin{aligned} &\left[(2l+1) \frac{(l+m+1)!}{(l-m-1)!} \right]^{1/2} h_l^{(1)}(kr) \\ &= \left(1 + \frac{2a}{r'}\right)^{m+1} \sum_{l'=m+1}^{\infty} c_{l'}^{(l,m+1)} \left[(2l'+1) \frac{(l'+m+1)!}{(l'-m-1)!} \right]^{1/2} j_{l'}(kr'). \end{aligned} \tag{A.7}$$

Using the recurrence relation

$$\frac{1}{\rho} j_l(\rho) = \frac{1}{2l+1} (j_{l-1}(\rho) + j_{l+1}(\rho)) \tag{A.8}$$

and equating the coefficients of $j_{l'}$ in eqs. (A.6) and (A.7), the following recurrence relation is obtained:

$$\sqrt{(2l'+1)(l-m)(l+m+1)}c_{l'}^{(l,m)} = \sqrt{(2l'+1)(l'-m)(l'+m+1)}c_{l'}^{(l,m+1)}$$
$$+2q\left\{\sqrt{\frac{(l'+m+1)(l'+m+2)}{2l'+3}}\, c_{l'+1}^{(l,m+1)}\right.$$
$$\left.+\sqrt{\frac{(l'-m)(l'-m-1)}{2l'-1}}\, c_{l'-1}^{(l,m+1)}\right\}, \tag{A.9}$$

with $q = ak$ (see eq. (4.4)). This relation is also valid for negative m. In an analogous way one can derive the recurrence relation

$$\sqrt{(2l'+1)(l+m)(l-m+1)}c_{l'}^{(l,m)} = \sqrt{(2l'+1)(l'+m)(l'-m+1)}c_{l'}^{(l,m-1)}$$
$$+2q\left\{\sqrt{\frac{(l'-m+1)(l'-m+2)}{2l'+3}}\, c_{l'+1}^{(l,m-1)}\right.$$
$$\left.+\sqrt{\frac{(l'+m)(l'+m-1)}{2l'-1}}\, c_{l'-1}^{(l,m-1)}\right\}. \tag{A.10}$$

Furthermore, putting $m = 0$ in eq. (A.6) yields

$$\sqrt{2l+1}h_l^{(1)}(kr) = \sum_{l'=0}^{\infty} c_{l'}^{(l,0)}\sqrt{2l'+1}j_{l'}(kr') . \tag{A.11}$$

Differentiating this relation with respect to kr or kr', using the recurrence relation

$$\frac{\mathrm{d}}{\mathrm{d}\rho} z_l(\rho) = \frac{1}{2l+1}\{l z_{l-1}(\rho) - (l+1)z_{l+1}(\rho)\} , \tag{A.12}$$

where $z_l(\rho)$ may be either $j_l(\rho)$ or $h_l^{(1)}(\rho)$, and applying eq. (A.11) again to the obtained expression, yields the recurrence relation

$$\sqrt{2l'+1}\left\{\frac{l}{\sqrt{2l-1}}\, c_{l'}^{(l-1,0)} - \frac{(l+1)}{\sqrt{2l+3}}\, c_{l'}^{(l+1,0)}\right\}$$
$$= \sqrt{2l+1}\left\{\frac{(l'+1)}{\sqrt{2l'+3)}}\, c_{l'+1}^{(l,0)} - \frac{l'}{\sqrt{2l'-1}}\, c_{l'-1}^{(l,0)}\right\}. \tag{A.13}$$

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