COMPSCI 590N
Lecture 4: Computing Special Functions

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Outline

Miscellaneous Python Stuff

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- Mixing tabs and spaces when indenting code will cause an error.

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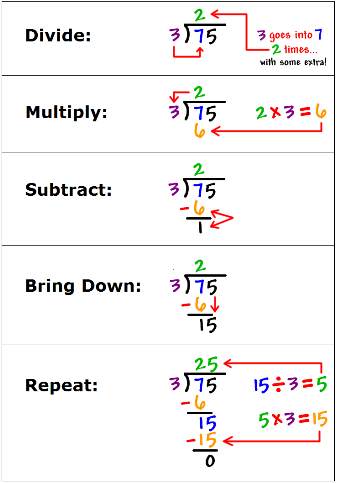
- In numerical computing we must also manipulate numbers, which can introduce further errors.
- **Numerical algorithms** approximate mathematical functions under the restriction of finite precision and finite time.

Long Division



Types of Numerical Algorithms

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A direct method is guaranteed to terminate in a finite number of steps and would return the exact answer if we had infinite precision.

Iterative

An iterative method is never guaranteed to terminate, but instead builds a sequence of increasingly accurate approximations and terminates when a desired accuracy is achieved.

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DEMO

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- Examples:
  1. If an algorithm allows representability errors to accumulate through addition, then we would call this algorithm unstable.
  2. Most iterative algorithms start with an initial guess and refine it. If the accuracy of a method is highly sensitive to the initial guess, then we would call this method unstable.

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- While you will likely never need to implement these algorithms (most are implemented in hardware!), it is important to know how they are implemented as it can have a large impact on the speed of your programs.
- In the following slides we will talk about some general techniques for approximating these functions and then look at a few examples.

Bit-by-bit Methods

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1	3	5	7
9	11	13	15
17	19	21	23
25	27	29	31

Bit-by-bit Methods

2	3	6	7
10	11	14	15
18	19	22	23
26	27	30	31

Bit-by-bit Methods

4	5	6	7
12	13	14	15
20	21	22	23
28	29	30	31

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- Can often be implemented using only simple operations: add, shift (multiplying/dividing by two), comparison.
- Bit-by-bit methods exist for many standard math functions.
- In particular, trigonometric functions (sine, cosine, etc.) are often calculated using a bit-by-bit algorithm called the CORDIC (**CO**ordinate **R**otation **DI**gital **C**omputer) method.

Bit-by-bit Methods

Bit-by-bit method for calculating $f(x) = \sqrt{x}$:

```
Input: x, n_bits   Output: y = sqrt(x)
base = 2**(n_bits-1)
y = 0
for i = 1,...,n_bits:
    if (y + base)**2 <= x:
        y += base
    base = base / 2
return y
```

Iterative Methods

Assume we want to evaluate the function $f(x)$. One strategy is to pick a number a that is close to x such that we know $f(a)$. Using $f(a)$ as our initial guess we can refine it using a number of methods:

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Taylor Series

For a given value a any infinitely differentiable function can be rewritten as:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

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- Values for $f(a)$ can be precomputed and stored in a table.

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Newton's Method

Newton's Method is used to find the solution to problems of the form $g(x) = 0$. Newton's method repeatedly applies the following update:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

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- Can be unstable depending on the initialization.

Newton's Method

For example: Let $f(x) = \sqrt{x}$. Note that if we find $1/\sqrt{x}$, we can find $\sqrt{x}$ as $\sqrt{x} = x/\sqrt{x}$. So, let

$$y = \frac{1}{\sqrt{x}}$$
$$0 = y^{-2} - x$$

If we apply Newton's method to solve for y and apply some algebraic simplifications we get updates of the form:

$$y_{n+1} = \frac{y_n}{2}\left(3 - xy_n^2\right)$$