COMPSCI 590N
Lecture 7: Complexity and Numerical Linear Algebra 1

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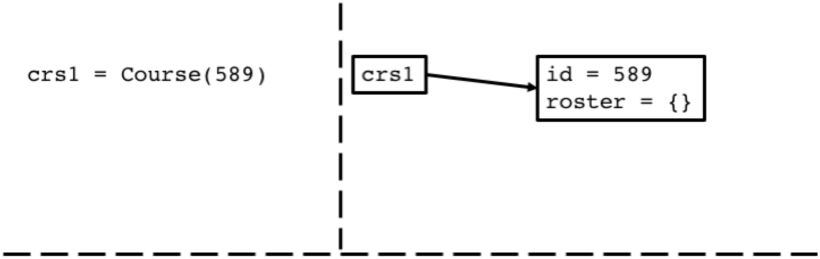
Outline

Problem 1



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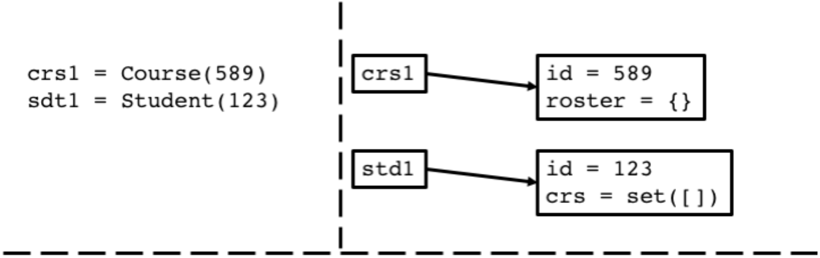
crs1 = Course(589)
sdt1 = Student(123)
crs1
id = 589
roster = {}
std1
id = 123
crs = set([])

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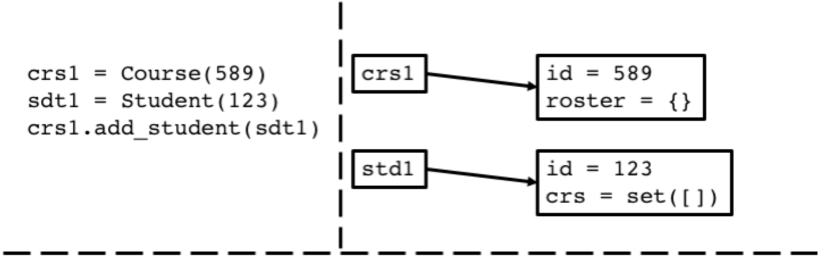
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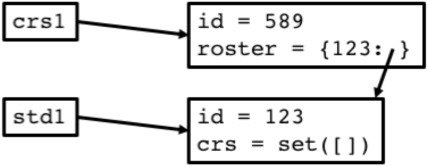


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self.roster[student.id] = student
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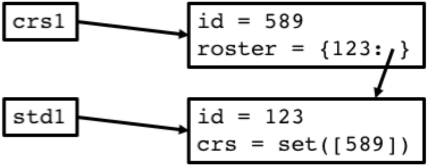


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self.roster[student.id] = student
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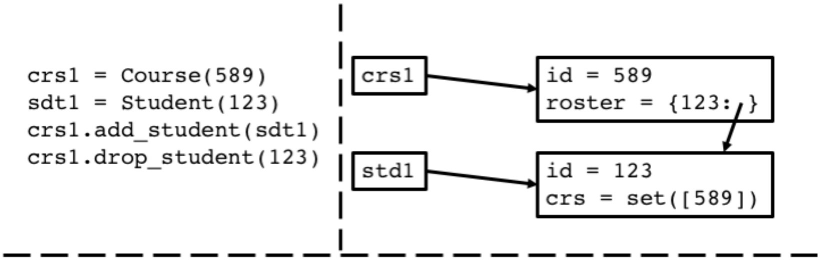
crs1 = Course(589)
sdt1 = Student(123)
crs1.add_student(sdt1)
crs1.drop_student(123)
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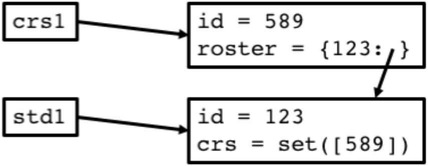


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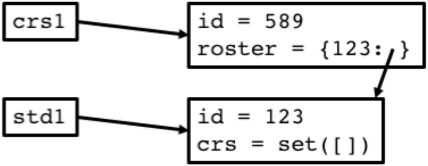


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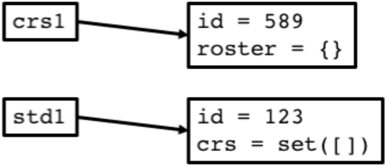


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student = self.roster[student_id]
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del self.roster[student_id]
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Outline

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- `A` and `B` are not the same object, hence `A is B` returns `False`.
- The attribute `base` points to the array from which a view was created, hence `B.base is A` returns `True`.

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- transpose
- reshape

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```
>>> A = np.ones((4,5))
>>> B = A.transpose()
>>> B.base is A
True
>>> C = B.reshape((5,4)) # Creates another view
>>> C.base is A
True
>>> D = B.reshape((2,10)) # Creates a copy!
>>> D.base is A
False
```

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- The moral of the story is: be careful when modifying the contents of an array directly.
- When in doubt, create a copy!

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- In our discussion of numerical algorithms so far, we have obliquely discussed the speed of our algorithms.
- We need a formal language for discussing the speed of an algorithm.
- The mathematical language used by computer scientists and mathematicians is called *complexity*.

Computational Complexity

High level idea: The complexity of an algorithm is the number of *basic operations* used by the algorithm as a function of the input size.

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Examples include: arithmetic operations, basic functions, logical comparisons, basic indexing, and assignment.

Array Scanning

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- If `A` has length n, how many basic operations will be performed by `max`?
- Aside from n, does the run time of `max` depend on the contents of `A`?

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- What are the basic operations used in this algorithm?
- If `A` has length n, how many basic operations will be performed by `find_first_zero`?
- It is standard practice to assume the worst possible contents of `A`. This is called **worst-case analysis**.

Binary Search

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- If `A` has length n, what is the worst case number of operations performed by `linear_search`?
- What information is not being used by `linear_search`?

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- We know that everything before m is less than m and everything after is greater than m.
- Therefore, if $x > m$, we can eliminate the first half of the list and if $x < m$ we can eliminate the second half.
- Apply this idea recursively.

Binary Search

Animation

Binary Search

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```
def binary_search(A,x):
    cur_min_idx = 0
    cur_max_idx = len(A) - 1
    while cur_min_idx <= cur_max_idx:
        cur_split = cur_min_idx + np.floor((cur_max_idx
            - cur_min_idx)/2)
        if A[cur_split] > x:
            cur_min_idx = cur_split + 1
        elif A[cur_split] < x:
            cur_max_idx = cur_split - 1
        else:
            return cur_split
    return None
```

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- If `A` has length n, what is the worst case number of operations performed by `binary_search`?
  - In each iteration, we are dividing the list in half so `binary_search` will take $\log(n)$ iterations.

Big O Notation

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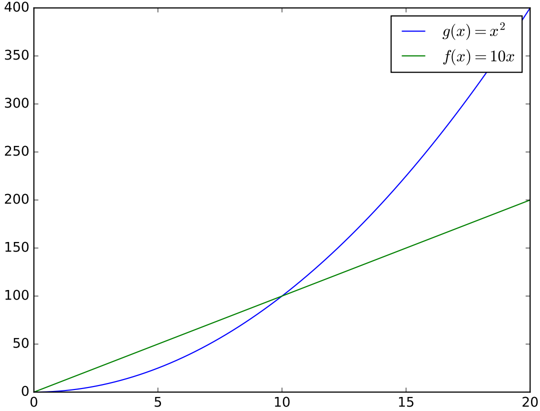
Let f and g be functions, then $f(x) = \mathcal{O}(g(x))$ if and only if there exists a constant M and x_0 such that

$$|f(x)| \leq M|g(x)| \text{ for all } x \geq x_o$$

Intuitively, this means that for large numbers $g(x)$ is always greater than $f(x)$, up to a constant.

Big O Notation



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- It is typical to consider the **worst-case** input of a fixed size.
- Big-O notation is used to formalize this concept so that algorithms can be rigorously compared.

Outline

Intro

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- Linear algebra computations are at the core **many** algorithms. For example:
  - Linear regression uses matrix multiplication and inversion.
  - PageRank, the algorithm that spawned Google, was fundamentally calculating eigenvalues.
  - Many of the recommendation systems that power sites like Netflix, Amazon, Google Ads, etc. are based on matrix decomposition.

Element-wise Operations

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Scalar/matrix operations have complexity $\mathcal{O}(n^2)$ for an $n \times n$ matrix.

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Element-wise matrix/matrix operations have complexity $\mathcal{O}(n^2)$ for two $n \times n$ matrices.

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Input: x, y
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out = 0
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A single sum/product can often be translated directly to a loop. This means that the inner product of two length n vectors has complexity $\mathcal{O}(n)$

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The naive method for matrix multiplication has complexity $\mathcal{O}(n^3)$.

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Strassen's Algorithm: Basic Idea

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- Strassen's algorithm recursively divides a matrix into 2×2 matrices and applies Strassen's 2×2 method.

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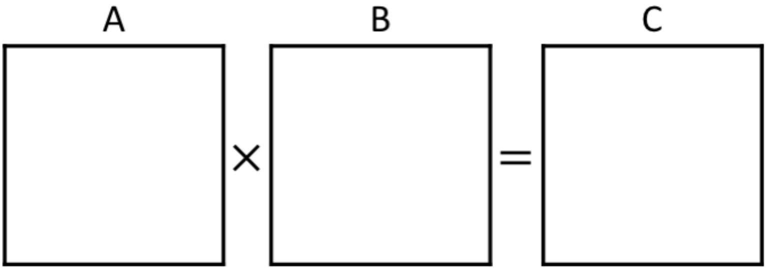

Strassen's Algorithm

$$\overset{A}{\begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}} \times \overset{B}{\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}} = \overset{C}{\begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}}$$

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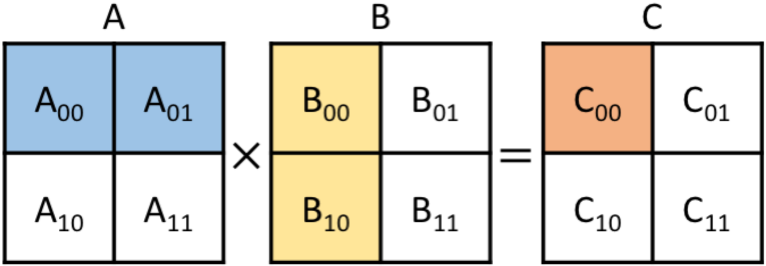

$$C_{00} = A_{00}B_{00} + A_{01}B_{10}$$

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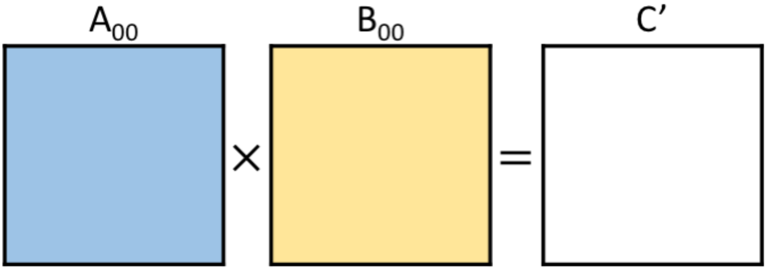
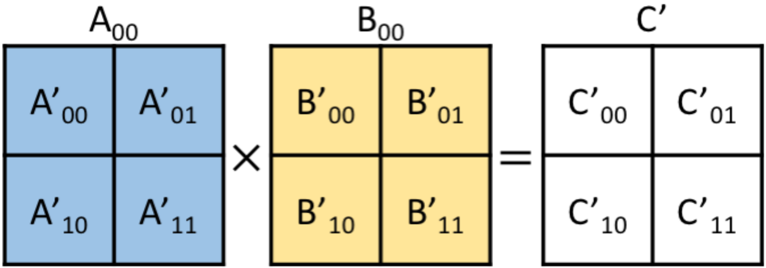
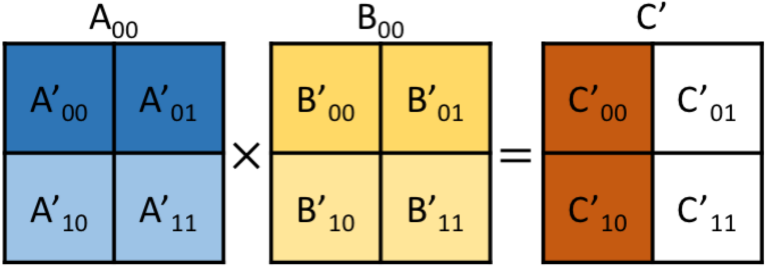


Strassen's Algorithm



Strassen's Algorithm



$$C'_{00} = A'_{00}B'_{00} + A'_{01}B'_{10}$$

Strassen's Algorithm

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- Many extensions to Strassen's have further reduced the complexity. Numpy uses one of these (see Demo).
- Strassen's is typically faster for matrices of size 100×100 and above. Good implementations will switch between naive and Strassen's.

Demo

Demo