Simulating 2D Voronoi Diagrams with C# and Unity

A-level Computer Science Project

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Chapter 1

Introduction

”In the beginning, the universe was created. This has made a lot of people very angry and has been widely regarded as a bad move” [Adams, 1980]. Within this universe, and specifically this paper, I shall create a program that can simulate Voronoi tessellations using various algorithms with C# and the Unity framework with the aim of being user-friendly and allowing for some determination of the optimum method in terms of programming complexity, processing requirements and quality of simulation.

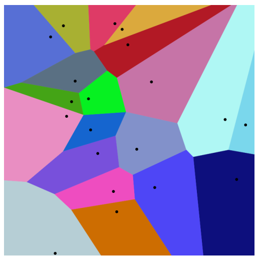
I am undertaking this investigation for a variety of reasons:

- The Voronoi diagram is seemingly ubiquitous throughout the world and appears in a vast array of fields of study including (but not limited to) biology, astrophysics, ecology, meteorology, engineering and bakery. Despite this I am yet to find anyone within my social circle who has ever heard of them and so through creating this project I hope to draw attention to this, in my opinion, underappreciated and fascinating field of mathematics.
- Upon my first introduction to the Voronoi diagram I soon found there are a variety of ways to compute it but nowhere I looked provided any form of explanation as to which method was best to use. This lapse in comparative usefulness is bound to only hinder further progression in the field and fields which use it. Thus I believe it to be very useful to determine the strengths and weaknesses of each computation method.
- There are not many user friendly programs for simulating Voronoi diagrams and so I would like to make one which is both powerful and very easy to use for someone with no programming knowledge.
- For a more personal reason I have always found Voronoi diagrams to be somewhat mesmerising ever since I first encountered them in a video by Sally Le Page [Page, 2019]. The ability to expand radially from points and to form straight line intersections has been stuck in my brain for about 2 years now and this project is a good way to scratch that itch.

Chapter 2

Voronoi Diagrams

A Voronoi diagram (also known as: Dirichlet tessellations or Thiessen polygon maps) is a diagram that shows the regions for which a point in a given set of points is the closest (an example is seen right [Ertl, 2015]). It is named after Georgy Voronoy who is primarily credited for the introduction of the concept for n number of dimensions in 1908 [Voronoi, 1908]. As such they are a relatively new analytical tool and are yet widely used, as shall be shown, in a variety of academic disciplines with a variety of functions.



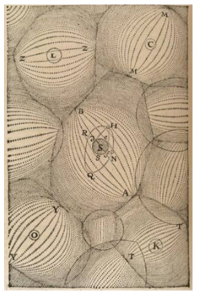
There are two types of voronoi diagram, Euclidean and Manhattan.
Euclidean distance refers to the straight line distance between two points, It is the square root of the sum of the squares of the relative x and y distances between two points, using Pythagoras.
Manhattan distance instead uses the sum of the absolute x and y distances between two points.
For example, imagine a square grid with two points A and B. Point B is 3 squares above and 4 squares to the right relative to A. Their Euclidean distance uses Pythagoras ($a^2 + b^2 = c^2$) to give a distance of 5 along the diagonal between them.
Manhattan distance instead gives a distance of 7, not allowing any diagonal movement.

For the purposes of our program we will only be considering Euclidean based voronoi diagrams although it is worth noting that both exist.

2.1 A Brief History of Voronoi Diagrams

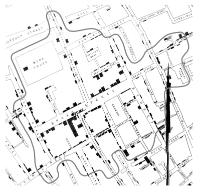
The earliest generally accepted case of a Voronoi diagram is traced to the work of René Descartes in 1644 during his work on his vortices theory. This theory surrounded the description of the universe as matter rubbing against each other with the types of matter grouped into luminous, transparent and opaque [loc.gov]. Descartes's diagrams showing this theory are comparable to later Voronoi diagrams (although they were not defined the same way) as they showed regions surrounding a central point forming boundaries where they met.

The first major attempt at formalising the concept was undertaken by Peter Gustav Lejeune Dirichlet in 1850 when he was investigating positive quadratic forms [Weisstein]. He formalised the definition of these diagrams for 2 and 3 dimensions, however, it wasn't until Georgy Voronoy in 1907 that the concept was formalised to n number of dimensions.



A depiction of Rene Descartes's vortices [loc.gov]

Even before their formal definition, Voronoi or Voronoi-esque diagrams have been used in many contexts. One very famous example is the work of John Snow in his "Report on the Cholera Outbreak in the Parish of St. James, Westminster, During the Autumn of 1854" which included a map showing a continuous broken line called the "Boundary of equal distance between Broad Street Pump and other Pumps", which creates a voronoi diagram around the Broad Street Pump [Okabe et al., 2009, p. 9]. This diagram was used to trace the localised cholera outbreak to the Broad Street pump and eventually resulted in the handle of the pump being removed (although it was later replaced).


John Snow's map with his line made solid for clarity [Snow]

In the 1960s there was a rediscovery of the Voronoi diagram within ecology with reference to estimating the intensity of trees in a forest by describing each tree as a voronoi region for area potentially available. The next year another paper was written using the same concept for plants labelling the regions as "plant polygons" The usage is now commonplace throughout the field [Okabe et al., 2009, p. 9].

2.2 Applications

2.2.1 The Post Office Problem

"Given a finite set P of distinct points, find the nearest neighbour point among P from a given point p (p is not necessarily a point in P)." [Okabe et al., 2009, p. 61]

A famous example of the usage of Voronoi diagrams is what Donald Knuth described as the *post office problem*. It describes a problem in which, given a set of points P, you must find the closest point with reference to another point p. The post office analogy describes this by imagining a scenario in which the closest post office needs to be found with reference to a particular house or post box.

This problem can be solved through the usage of a Voronoi tessellation with the nodes representing the post offices and the regions representing the regions for which that post office is the closest.

This principle of using Voronoi diagrams to find the closest node in a set is a common theme throughout uses with similar examples including ambulance stations in relation to an injury and cell phone towers in relation to a phone.

2.2.2 Astronomy

In a paper by M.N. Drozdovskaya at Leiden University (nl), Voronoi tessellation has been used via Brown's algorithm within the field of Astronomy to aid in modelling the Cosmic web (I promise that's a real thing) [Drozdovskaya, 2010].

2.2.3 Baking

Voronoi diagrams are often seen in baking where borders are found between items baked on one tray. A common example of this is burger buns in a pack, the buns are connected together with straight lines because they were baked next to one another resulting in them expanding and fusing together.

2.2.4 Biology

It is becoming increasingly common to model the arrangement of cells with voronoi tessellations such as that investigated by Martin Bock et al in their paper [Bock et al., 2009]. A slightly less complicated use of a similar nature surrounds the growth of cell colonies in controlled environments. The cell colonies expand radially and when they meet (assuming they can safely coexist) they will form a voronoi tessellation with each other.

Voronoi tessellation has also been used to model bone architecture, in particular by Hui Li et al at the University of New York (usa) [Li et al., 2012].

2.2.5 Ecology

In 1997 F. Mercier from the Claude Bernard University and O. Baujard from the University of Geneva used a modified version of the Green and Sibson algorithm to generate a voronoi tessellation for the canopies of trees allowing them to investigate population dynamics of trees in a Guianan forest. Using the Green and Sibson algorithm allowed them to look at canopy openings and the formation of trees within these openings over time with trees being added and removed as required [Mercier and Baujard, 1997].

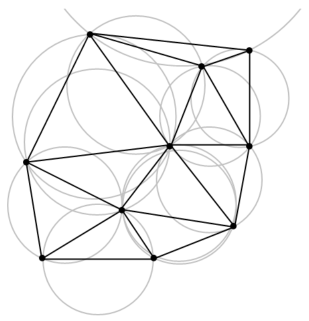
2.3 Methods of Generating Voronoi Diagrams

Below I shall be listing some of the methods of computing Voronoi diagrams that are commonly used. This is by no means an exhaustive list (not even close) and new algorithms are being developed frequently, the ones described here are those which I deem to be the ones what appear most frequently and are the ones I shall attempt to implement later on[1].

2.3.1 Via Delaunay Triangulation

2.3.1.1 What is a Delaunay triangulation

A Delaunay triangulation for a set of points is a triangulation such that none of the points are within the circumcircle of any triangles within the triangulation (A circumcircle being a circle for which all vertices on the polygon are also on the circumference of the circle). It is frequently used as "a technique for creating a method of contiguous, non-overlapping triangles from a dataset of points" [Duh]. The process favours short, wider triangles with long, thin ones being generally avoided. The process was invented by Boris Nikolaevich Delaunay in 1934, Georgy Voronoi was one of his doctoral advisors.


A Delaunay triangulation with circumcircles [es]

2.3.1.2 Using Delaunay for Voronoi

The Delaunay triangulation and the Voronoi diagram are mathematical duals or each other.
Duality in mathematics refers to a relationship in which two states can be switched between by applying the same function.
$f(x) = b \quad \& \quad f(b) = x \quad \& \quad ff(x) = x \quad \& \quad ff(b) = b$
The transformation that converts between Voronoi and Delaunay (v.v.) is the same for all dual graphs. It takes the centre-points of the regions of the graph and draws lines between each centre-point and the centre-points of all adjacent regions.[Weisstein]

2.3.2 The Green & Sibson Algorithm

The Green & Sibson algorithm is a recursive method that adds the seeds one at a time. I shall attempt to summarise the process below but for a more detailed (and probably more coherent) explanation I refer you to their original paper cited here [Green and Sibson, 1977][2].

The Green and Sibson algorithm treats a voronoi diagram as a set of points (seeds) and connections with adjacent regions, on the premise that once those are known all else can be extrapolated.
The diagram is generated by adding points one at a time in an anticlockwise fashion, "recursively modifying

[1]Some other algorithms not covered that you may be interested in:
Shamos and Hoey http://euro.ecom.cmu.edu/people/faculty/mshamos/1975ClosestPoint.pdf
Brown https://www.math.wustl.edu/ victor/classes/pmf/KQBrown.pdf
Yan et al. https://www.microsoft.com/en-us/research/wp-content/uploads/2016/12/Efficient-Computation-of-Clipped-Voronoi-Diagram-and-Applications.pdf

[2]As a quick aside, Green and Sibson win my award for best paper I've had to read. It's clear, concise, doesn't go too deep and actually stops after 3 dimensions whilst also covering problems like adding boundaries AND discussing implementations not simply leaving it in the realms of theory. And the only word I had to google was "contiguous". Utter heroes.

the contiguities as each point is added"[Green and Sibson, 1977]. When a new point is added the first point to check is its nearest, already added neighbour as it is guaranteed that there will be a connection with that point's region. The perpendicular bisector of the new point and its nearest neighbour is found and is followed clockwise to the edge of the nearest neighbour's tile. The edge at which it meets gives the next point for which there is a connection which is stored as a connection and the process is repeated, creating a perpendicular bisector with the next point and following it clockwise until the next tile is met, and so on until returning to the nearest neighbour.

The algorithm has a big O of O(n^2) and (if the amount of times I've found it referenced is any indication) is considered to be a very effective algorithm.

2.3.3 Steven J. Fortune's Algorithm

Fortune's algorithm is a very common method of computing a Voronoi diagram in O(nlogn) time although the theory behind the algorithm is somewhat complicated to understand[3]. I have written an explanation but I would recommend looking at other sources such as [Heunis] which helped me understand the concept.

The diagram is conceptually generated by way of a sweep, from top to bottom although it is in practice event based. Before getting onto the details of the algorithm however, a small amount of mathematical background is required. There are multiple ways of describing a parabola, the most common is in the form $y = ax^2 + bx + c$ however there is another way. If there is a point, notated as p_1, and a line, notated as l, a set of all points that have the same distance to p_1 as they do to the closest point on l will form a parabola. If there is a second point, p_2, using the same line l there will be two parabolas that meet at a single point. As these parabolas are designed such that the distance to their respective point is the distance to l, the intersection of these parabolas will be perfectly equidistant between the points. This is the principle Fortune's algorithm is based on.

As the sweep line in our diagram progresses (the line l in our previous example), there are two events that can occur: site events and edge-intersection events.

A site event is where the sweep line encounters a new point to be added to the diagram. When this occurs the section of the beachline directly above the point is going to be divided into two with a new arc (the arc from the new point) appearing in the middle.

An edge-intersection event is where an arc on the beachline gets "squeezed" which is to say that the arcs either side of it expand and meet thus rendering the central arc non existent.
At this event you have three arcs meeting at a common point which thus means that point is equidistant to three points and thus will appear as a three way intersection on the finished diagram. That three way intersection will consist of the two lines that joined together and a new line expanding perpendicular to the arc that got squeezed.

The algorithm itself runs by queuing up all of the events, passing through them one by one and cleaning up the end of the beachline.

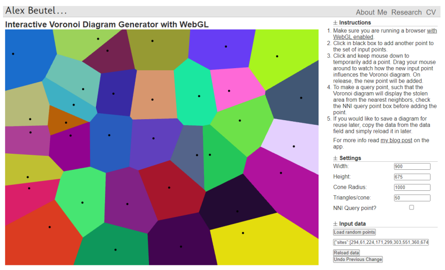
[3]As a quick aside, Steven J. Fortune wins my award for most pain-in-the-ass algorithm to understand. This took me a solid week to get my head around why are all the articles explaining it utter garbage what the actual hell???

Chapter 3

Current Solutions

3.1 Alex Beutel WebGL Program

On his site http://alexbeutel.com/webgl/voronoi.html , Alex Beutel has created a 2D voronoi generator in JavaScript with WebGL that is able to generate 2D voronoi diagrams from seeds added by clicking within the display box.
Alex's site has some features of interest, namely the ability to enter "sites" (seed locations) via a text input box in the UI, although it does have to be be entered with JavaScript Object Notation (JSON) formatting. The data in the text input box is also generated automatically when seeds are added by clicking thus allowing it to also be used as a save feature, with the user able to copy the data out of the box and save it locally.
One thing which has bugged me while using his program is the inability to move seeds after they have been placed, especially given you do not know what the diagram will look like until the seed is placed. This is something I will hope to solve in my program.
Annoyingly, nowhere on his site does he state what algorithm he uses for his program, and the blog post he cites "For more info" no longer exists.



± Instructions

1. Make sure you are running a browser with WebGL enabled.
2. Click in black box to add another point to the set of input points.
3. Click and keep mouse down to temporarily add a point. Drag your mouse around to watch how the new input point influences the Voronoi diagram. On release, the new point will be added.
4. To make a query point, such that the Voronoi diagram will display the stolen area from the nearest neighbors, check the NNI query point box before adding the point.
5. If you would like to save a diagram for reuse later, copy the data from the data field and simply reload it in later.

For more info read my blog post on the app.

± Settings

Width: 900
Height: 675
Cone Radius: 1000
Triangles/cone: 50
NNI Query point?

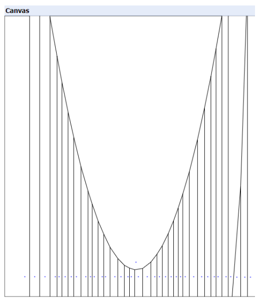
± Input data

{"sites":[294,61,224,171,299,303,551,360,674

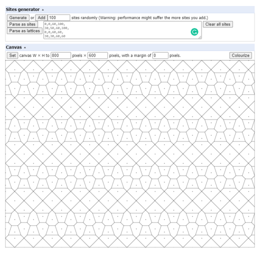
An example diagram created with Alex Beutel's site

3.2 RaymondHill.net JavaScript Program

In a similar vein to Alex Beutel's program, Raymond Hill has created a set of Voronoi generators on his site that allow users to create their own Voronoi diagrams by clicking on the display box. Raymond Hill uses Fortune's algorithm to generate the diagrams and introduces some interesting features such as the ability to "Colourize" the regions when a button is pressed, however the colour that is assigned to each region is random, uncontrollable and removed when an edit is made.
Another feature I am quite a fan of is the ability to easily add lattices with an input box which allows the user to describe a point pattern as a set of 4 numbers (startx, starty, deltax, deltay). The box can accept multiple patterns at once to create remarkably complex patterns and I think if possible some form of template or pattern loading feature could be useful for my program.



(a) A simple parabola I made with Raymond Hill's site



Generate or Add 100 sites randomly (Warning: performance might suffer the more sites you add.)

```
0,0,60,100,
30,50,60,100,
0,0,60,60,
30,30,60,60
```

Set canvas W × H to 800 pixels × 600 pixels, with a margin of 0 pixels.

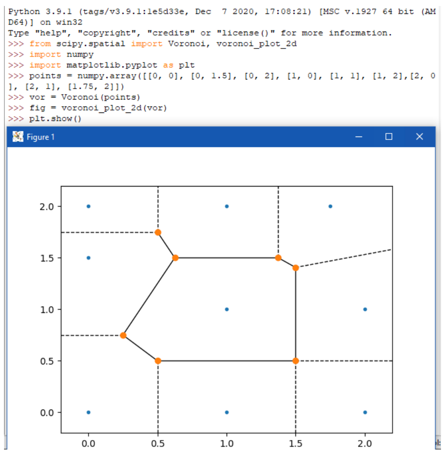
(b) A lattice made with Raymond Hill's generator

3.3 MATLAB

MATLAB offers support for Voronoi generation in two dimensions. However, as a MATLAB subscription costs £760/y and the funding for this project currently consists of 50 pence plus however much I can gain from begging and pleading on my knees in front of my headteacher (£0.00), a full investigation into the capabilities of this software is, for me, currently impossible. I shall attempt to summarise some information below that I have gleaned from examples I have found on the internet:
MATLAB uses a programming interface to allow users to create their own 2D voronoi diagrams by calling the voronoi() subroutine passing arrays of x and y values for seed locations [MATLAB]. As it is MATLAB, it is relatively easy to assign labels to the seed locations if required which is an idea I like and with the voronoin() function you can gain a matrix output of the diagram that allows for relatively straightforward colour-coding of regions [MathWorks]. It does this by using the QHull library (elaborated on below) to compute its outputs. MATLAB suffers from the same problem as Scipy will demonstrate below (in large part due to its dependence on QHull) which is its requirement that its users have proficiency in programming to use it.

3.4 Scipy

Scipy is a module for python which provides a broad array of tools for modelling various scientific problems including optimisation, linear algebra and, crucially for us, Voronoi diagram generation in 2D[Virtanen et al., 2020]. Scipy achieves this by using the Qhull library (elaborated on below) to take inputs in the form of a numpy array and produce a *scipy.spatial.qhull.Voronoi object* which contains the vertices, ridge-points, ridge-vertices, regions and the point-regions for the diagram in question. Although the Voronoi() function does work for n dimensions, its accompanying function voronoi_plot_2d() which plots the Voronoi diagrams with matplotlib only accepts 2D Voronoi diagrams and has no equivalent function for higher order diagrams[Scipy]. As such anyone wanting to use this to create a (N>2) diagram would need to write their own plotting algorithm.



```
Python 3.9.1 (tags/v3.9.1:1e5d33e, Dec  7 2020, 17:08:21) [MSC v.1927 64 bit (AM
D64)] on win32
Type "help", "copyright", "credits" or "license()" for more information.
>>> from scipy.spatial import Voronoi, voronoi_plot_2d
>>> import numpy
>>> import matplotlib.pyplot as plt
>>> points = numpy.array([[0, 0], [0, 1.5], [0, 2], [1, 0], [1, 1], [1, 2],[2, 0
], [2, 1], [1.75, 2]])
>>> vor = Voronoi(points)
>>> fig = voronoi_plot_2d(vor)
>>> plt.show()
```

A basic example of a voronoi tessellation with scipy

My takeaways from this are honestly relatively limited, it's ability to take n dimensional points is impressive but not a reflection on this specific program and not explicitly applicable to us as we are working in 2 dimensions. My main takeaway is predominantly that usability is a large priority and requiring the user write their own code to be able to interpret the output is undesirable.

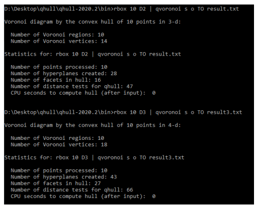
3.5 Qhull

Qhull is a set of packages that perform n-dimensional computations to generate convex hulls, halfspace intersections, Delaunay triangulations and Voronoi diagrams [Qhull, a]. Qhull uses the dual of the Delaunay triangulation to calculate its Voronoi diagrams [Qhull, c][1] and has no theoretical limit on the number of dimensions it can go to although the documentation does not recommend exceeding 7 as "Qhull does not work well with virtual memory" [Qhull, a].

Installing the most basic form of Qhull is trivial, you download the zip file from their downloads page [Qhull, b] and extract it into a directory. From here running the *QHULL-GO* program opens a terminal from which you can operate Qhull.

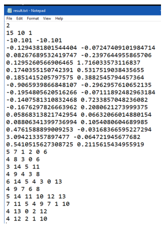
[1]see addenda "Qhull's Method of Voronoi Computation" pg38.

On the right I have demonstrated the abilities of this form of the program.
I have entered two commands, the first of which is
"rbox 10 D2 | qvoronoi s o TO result.txt"
"rbox 10" generates 10 randomly spaced points
"D2" specifies 2 dimensions
"qvoronoi" specifies that I want it to generate a voronoi diagram
"s" requests a summary to be printed to the console
"o TO result.txt" formats and saves the result to a file called *result.txt* in the bin folder
The second command does the same but in 3 dimensions and saves the result in *result3.txt*.



```
D:\Desktop\qhull\qhull-2020.2\bin>rbox 10 D2 | qvoronoi s o TO result.txt

Voronoi diagram by the convex hull of 10 points in 3-d:

  Number of Voronoi regions: 10
  Number of Voronoi vertices: 14

Statistics for: rbox 10 D2 | qvoronoi s o TO result.txt

  Number of points processed: 10
  Number of hyperplanes created: 28
  Number of facets in hull: 16
  Number of distance tests for qhull: 47
  CPU seconds to compute hull (after input):  0


D:\Desktop\qhull\qhull-2020.2\bin>rbox 10 D3 | qvoronoi s o TO result3.txt

Voronoi diagram by the convex hull of 10 points in 4-d:

  Number of Voronoi regions: 10
  Number of Voronoi vertices: 18

Statistics for: rbox 10 D3 | qvoronoi s o TO result3.txt

  Number of points processed: 10
  Number of hyperplanes created: 43
  Number of facets in hull: 27
  Number of distance tests for qhull: 66
  CPU seconds to compute hull (after input):  0
```

(a) Inputs into the Qhull console



```
2
15 10 1
-10.101 -10.101
-0.1294381801544404 -0.07247409101984714
0.08267689532419747 -0.2397644955865706
0.1295260566906465 1.716033573116837
0.1740355150742391 0.5317519038435655
0.1851415205797575 0.3882545794457364
-0.9065939866848107 -0.2962957610652135
-0.1954805620516266 -0.07111892482963184
-0.1407581310832468 0.7233857048236082
-0.1676297826663962 0.2080621273999375
0.05868313821742954 0.06632066014880154
0.08806341399736994 0.1054080604689985
0.4761588899009253 -0.03168366595227294
3.094213357897477 -0.064721945677682
0.5410515627308725 0.2115615434955919
5 7 1 2 0 6
4 8 3 0 6
3 14 5 11
4 9 4 3 8
6 14 5 4 3 0 13
4 9 7 6 8
5 14 11 10 12 13
7 11 5 4 9 7 1 10
4 13 0 2 12
4 12 2 1 10
```

(b) result1.txt

The contents of the resulting files are... less than immediately useful, they have no intuitive graphical meaning and require a GUI to display them. Qhull recommends using an external program called Geomviewer[Qhull, a]. This continues our trend of programs not providing adequate facilities for actually viewing the output without technical skill. The Qhull program alone is not hugely friendly to use let alone having to install Geomviewer (which requires a Linux installation) to understand what it is producing. Qhull is an amazing piece of software but it falls far too short on usability.

Chapter 4

Potential User

4.1 Nathan Easow

The development of this project shall revolve around input from my primary user in the form of Nathan Easow.
Nathan is a 17 year old biology student and prospective medic who is interested in the applications of Voronoi patterns to simulate bacteria colony growth within boundaries. He is interested in analysing which method is most efficient for his purposes and would then wish to use the program for his own purposes after the analysis has occurred.

4.2 Interview

I did an interview with Nathan to assess priorities for the program, the transcript is below:

James: Okay interview time. Question one, what types of control do you want over the simulation setup?
Nathan: I guess being able to adjust the points after placement... because obviously that's a uh feature that a lot of creators don't allow at the moment.
James: Yeah.
Nathan: And also the ability to alter the shading of the different regions of the result.
James: Yep, that makes sense aight. Okay question 2, do you want any form of bounding box system so a way to restrict where it goes as such, put a limit on how far the Voronoi diagram goes?
Nathan: Yes that would definitely be useful yep.
James: Okay, cool. Question 3, do you need the program to be able to use both Euclidean and Manhattan distance or just Euclidean?
Nathan: Just Euclidean is fine.
James: Cool... 4, are you worried about watching the development of the diagram as an animation?
Nathan: It would be nice, if the method you use allows for it but it's not a key priority.
James: Right, right got ya. Suppose at the end of the day the end result is the important bit... so then yeah I've got what control do you want over the simulation progression but... suppose assuming their is an animation what control do you want over that?
Nathan: Yeah I mean I suppose speed control but that's not necessary that's obviously assuming the animation exists in the first place.
James: Cool. Question 6, Ed Balls?
Nathan: Ed Balls.
James: 7. Would you like to be able to save the diagram conditions?
Nathan: Yes that would definitely be useful yeah
James: and actually continuing on that would you, if you just had the output be able to save a screenshot of that? Do you want that?

Nathan: Yes that would also be great yeah rather than having to load up the program each time yeah.
James: Yeah. Easier to export. 8, how do you feel about having to run a few lines of code in order to run it?
Nathan: I'd prefer to avoid that and have an interface that I can interact with directly.
James: Yeah, not having to mess around with code is probably useful. Loading points from a file, would that be useful?
Nathan: Yeah that would be great rather than having to enter each one manually yeah.
James: Yeah, and finally question 10, what sort of information do you want regarding looking at the various algorithms, looking at how they compare?
Nathan: So I suppose processing time, memory efficiency and how well each one scales.
James: Yeah, yeah I suppose you don't want to be using one that works really badly with large amounts of data.
Nathan: Yeah.
James: Right ok that's all I've got, thank you very much.

4.3 Analysis

The interview with Nathan was able to confirm and clarify some assumptions I had. His emphasis on user experience and an intuitive UI "an interface I can interact with directly" was largely to be expected, as was the requests for features such as point location adjustment and saving. I was surprised by the lack of interest in an animation that shows the development of the diagram although with retrospect that might've been my own personal interest in those animations rather than their actual use. Loading data points from a file is a fairly obvious requirement and *should* be simple enough.

Chapter 5

Goals and Requirements

Based on the interview with Nathan and my analysis of other people's solutions, these are some general and specific goals for the end program.

5.1 General Requirements

- An intuitive user interface requiring no programming skill to interact with
- The ability to generate 2D Euclidean Voronoi diagrams via Delaunay transformations, the Green & Sibson algorithm and Steven J. Fortune's algorithm
- Performance data on the various algorithms

5.2 Specific Goals

CRITICAL
- The ability to load data points from a file
- The ability to alter seed positions after they have been placed by selecting them and either replacing them or altering their coordinates via input boxes
- Ed Balls
- The ability to adjust the RGBA values of the voronoi regions associated with each seed via an input box
- The ability to save the setup of the diagram (x,y positions and RGBA values)
- The ability to save the output of the program as a .png file.
- The UI must be fully graphical with no programming required
- The processing time and memory usage must be displayed when the program is run.

OPTIONAL
- The ability to restrict the maximum and minimum x and y values of the diagram via input boxes
- The ability to view an animation of the diagram developing if the algorithm in question allows it
- If above implemented, the ability to control the speed of the animation
- The ability to add a series of points conforming to a template or pattern by inputting information about the pattern into an input box.

Chapter 6

Testing Algorithms with Python

6.1 Delaunay

6.1.1 Problem Outline

Using the Delaunay triangulation to generate Voronoi diagrams has 2 stages, generation of the Delaunay triangulation, and construction of the Voronoi diagram using the dual of the former.

There is a well established algorithm for constructing Delaunay triangulations which is the Bower-Watson algorithm. This algorithm generates the Delaunay triangulation in O(n^2) time and is rather simple to create once you have a formula for the circumcircle centres.

After the Delaunay triangulation is acquired, finding the dual is relatively trivial. For each triangle in the triangulation you draw a line between all of the circumcentres and then for the triangles on the diagram edge add a line parallel to the perpendicular bisector of the outer edge of the triangle heading out (or multiple if there are multiple outer edges).

6.1.2 Circumcircle Derivation

In order to perform the Bower-Watson algorithm, I am required to be able to generate the circumcircle for any given triangle. Mathematically speaking this is trivial, the centre of the circle is given by the intersection point of two of the perpendicular bisectors of the triangle's sides. The difficulty comes in the fact that this must all be done algebraically as I do not have any of the constants. My derivation of a formula for the (x, y) coordinates of the circumcentre is shown below:

$$let\, ABC = triangle\, where\, A = (x_1, y_1),\ B = (x_2, y_2),\ C = (x_3, y_3)$$

$$let\, (x_{ab}, y_{ab}) = midpoint\, of\, line\, AB$$

$$let\, (x_{ac}, y_{ac}) = midpoint\, of\, line\, AC$$

$$let\, l_1 = bisector\, of\, AB$$

$$let\, l_2 = bisector\, of\, AC$$

$$y - y_n = m(x - x_n)\, where\, m\, is\, gradient\, of\, the\, line$$

$$l_1 = -\frac{x_1 - x_2}{y_1 - y_2}(x - x_{ab}) + y_{ab}$$

$$l_2 = -\frac{x_1 - x_3}{y_1 - y_3}(x - x_{ac}) + y_{ac}$$

$$let\ l_1 = l_2$$

$$-\frac{x_1 - x_2}{y_1 - y_2}(x - x_{ab}) + y_{ab} = -\frac{x_1 - x_3}{y_1 - y_3}(x - x_{ac}) + y_{ac}$$

expand brackets

$$x\left(-\frac{x_1 - x_2}{y_1 - y_2}\right) - x_{ab}\left(-\frac{x_1 - x_2}{y_1 - y_2}\right) + y_{ab} = x\left(-\frac{x_1 - x_3}{y_1 - y_3}\right) - x_{ac}\left(-\frac{x_1 - x_3}{y_1 - y_3}\right) + y_{ac}$$

move x onto one side

$$x\left(-\frac{x_1 - x_2}{y_1 - y_2}\right) - x\left(-\frac{x_1 - x_3}{y_1 - y_3}\right) = x_{ab}\left(-\frac{x_1 - x_2}{y_1 - y_2}\right) - x_{ac}\left(-\frac{x_1 - x_3}{y_1 - y_3}\right) + y_{ac} - y_{ab}$$

factorise out x

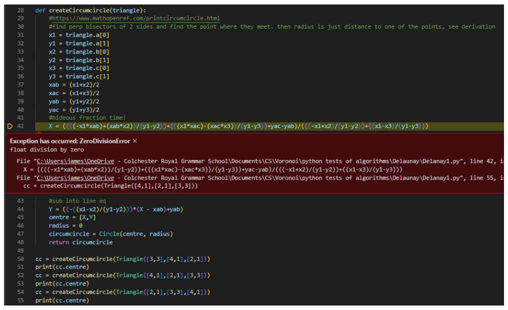
$$x\left(-\frac{x_1 - x_2}{y_1 - y_2} + \frac{x_1 - x_3}{y_1 - y_3}\right) = \frac{-x_1x_{ab} + x_{ab}x_2}{y_1 - y_2} + \frac{x_1x_{ac} - x_{ac}x_3}{y_1 - y_3} + y_{ac} - y_{ab}$$

divide to find x

$$x = \frac{\frac{-x_1x_{ab}+x_{ab}x_2}{y_1-y_2} + \frac{x_1x_{ac}-x_{ac}x_3}{y_1-y_3} + y_{ac} - y_{ab}}{\frac{-x_1+x_2}{y_1-y_2} + \frac{x_1-x_3}{y_1-y_3}}$$

Substitute value of x back into l_1 or l_2 to get y

Upon implementation the algorithm initially worked but encountered a problem; namely that if points A and B or points A and C had the same y coordinate then it would cause a $ZeroDivisionError$. So I got a refill of tea and a friend of mine pointed out that fortunately, in this scenario, the x coordinate will just be half way between the x coordinates of the two corresponding points. After which the y coordinate can be obtained by substituting this value of x into the equation that does not have both offending points.



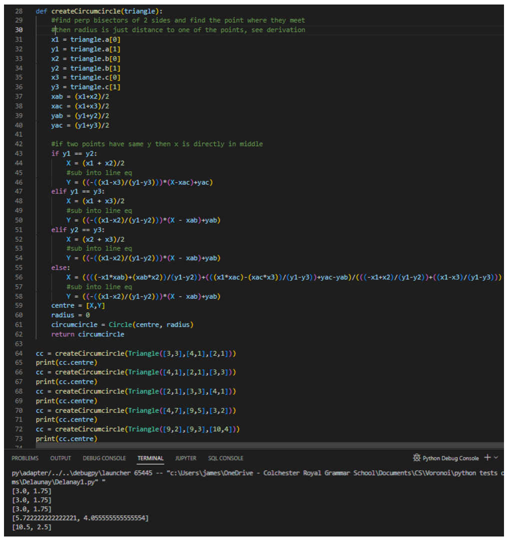
```
def createCircumcircle(triangle):
    #https://www.mathopenref.com/printcircumcircle.html
    #find perp bisectors of 2 sides and find the point where they meet. then radius is just distance to one of the points, see derivation
    x1 = triangle.a[0]
    y1 = triangle.a[1]
    x2 = triangle.b[0]
    y2 = triangle.b[1]
    x3 = triangle.c[0]
    y3 = triangle.c[1]
    xab = (x1+x2)/2
    xac = (x1+x3)/2
    yab = (y1+y2)/2
    yac = (y1+y3)/2
    #hideous fraction time!
    X = ((((-x1*xab)+(xab*x2))/(y1-y2))+(((x1*xac)-(xac*x3))/(y1-y3))+yac-yab)/(((-x1+x2)/(y1-y2))+((x1-x3)/(y1-y3)))
Exception has occurred: ZeroDivisionError ×
float division by zero
  File "C:\Users\james\OneDrive - Colchester Royal Grammar School\Documents\CS\Voronoi\python tests of algorithms\Delaunay\Delanay1.py", line 42, i
    X = ((((-x1*xab)+(xab*x2))/(y1-y2))+(((x1*xac)-(xac*x3))/(y1-y3))+yac-yab)/(((-x1+x2)/(y1-y2))+((x1-x3)/(y1-y3)))
  File "C:\Users\james\OneDrive - Colchester Royal Grammar School\Documents\CS\Voronoi\python tests of algorithms\Delaunay\Delanay1.py", line 55, i
    cc = createCircumcircle(Triangle([4,1],[2,1],[3,3]))
    #sub into line eq
    Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
    centre = [X,Y]
    radius = 0
    circumcircle = Circle(centre, radius)
    return circumcircle

cc = createCircumcircle(Triangle([3,3],[4,1],[2,1]))
print(cc.centre)
cc = createCircumcircle(Triangle([4,1],[2,1],[3,3]))
print(cc.centre)
cc = createCircumcircle(Triangle([2,1],[3,3],[4,1]))
print(cc.centre)
```

$ZeroDivisionError$

After this change was made the circumcentres were being generated reliably. All that remained was to calculate the radius which is equal to the distance between the circumcentre and any of the points.

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

This is using the distance formula that applies Pythagoras to the euclidean coordinates of the points.



```
def createCircumcircle(triangle):
    #find perp bisectors of 2 sides and find the point where they meet
    #then radius is just distance to one of the points, see derivation
    x1 = triangle.a[0]
    y1 = triangle.a[1]
    x2 = triangle.b[0]
    y2 = triangle.b[1]
    x3 = triangle.c[0]
    y3 = triangle.c[1]
    xab = (x1+x2)/2
    xac = (x1+x3)/2
    yab = (y1+y2)/2
    yac = (y1+y3)/2

    #if two points have same y then x is directly in middle
    if y1 == y2:
        X = (x1 + x2)/2
        #sub into line eq
        Y = ((-((x1-x3)/(y1-y3)))*(X-xac)+yac)
    elif y1 == y3:
        X = (x1 + x3)/2
        #sub into line eq
        Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
    elif y2 == y3:
        X = (x2 + x3)/2
        #sub into line eq
        Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
    else:
        X = ((((-x1*xab)+(xab*x2))/(y1-y2))+(((x1*xac)-(xac*x3))/(y1-y3))+yac-yab)/(((-x1+x2)/(y1-y2))+((x1-x3)/(y1-y3)))
        #sub into line eq
        Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
    centre = [X,Y]
    radius = 0
    circumcircle = Circle(centre, radius)
    return circumcircle

cc = createCircumcircle(Triangle([3,3],[4,1],[2,1]))
print(cc.centre)
cc = createCircumcircle(Triangle([4,1],[2,1],[3,3]))
print(cc.centre)
cc = createCircumcircle(Triangle([2,1],[3,3],[4,1]))
print(cc.centre)
cc = createCircumcircle(Triangle([4,7],[9,5],[3,2]))
print(cc.centre)
cc = createCircumcircle(Triangle([9,2],[9,3],[10,4]))
print(cc.centre)
```

```
py\adapter/../..\debugpy\launcher 65445 -- "c:\Users\james\OneDrive - Colchester Royal Grammar School\Documents\CS\Voronoi\python tests o
ms\Delaunay\Delanay1.py" "
[3.0, 1.75]
[3.0, 1.75]
[3.0, 1.75]
[5.722222222222221, 4.055555555555554]
[10.5, 2.5]
```

working circumcircles code without radii

6.1.3 Creating the Dual

The other great hurdle of this algorithm comes with the transformation between the finished Delaunay triangulation and the Voronoi diagram. The first part is simple, connect the circumcentres of all adjacent triangles. The difficulty comes with the unbounded edges. These are the lines that stretch to theoretical infinity on the outside of the diagram. These lines have a gradient equal to that of the perpendicular bisector of their corresponding line in the Delaunay triangulation and connect to their triangle's circumcentre, the problem is deciding in which direction to extend the line. The obvious answer is "away from the diagram"; but how do you calculate that? You could try

comparing it to the average point location, that doesn't work.
You could try looking at the direction of the circumcentre relative to the original line, nope. Ultimately this became a far greater problem than I had anticipated and my solution to it is rather indicative of that.

As I saw it there were two solutions that I or anyone I asked could think of; the first was to use a form of ray casting algorithm to see if it collided with any other voronoi lines, this seemed overly complicated. Especially given the algorithm for that alone would've been O(n^2).
The other was to treat the supertriangle used in the Bowyer-Watson algorithm as part of the diagram. The supertriangle is a triangle that encompasses all of the other points. By keeping this in the diagram instead of filtering it out at the end of the Delaunay code, it will create the unbounded voronoi lines as part of the regular Delaunay-Voronoi dual flip as described earlier. On the theory that the supertriangle is hard-coded to be so large that it's existence would only be noticeable when you are so zoomed out that you can't see the main section of the diagram. You can even then filter out any lines that go between two points that are far away from the main section to remove all evidence of its existence entirely and simply leave cropped lines that act as our unbounded ones.
This to me sounded rather... janky. And in the future I will try to find a better method of doing this. But for now it works, and works reliably. The "unbounded" lines could be cropped to fit any required reference frame by a program that recalculates their gradients and redraws them within required boundaries, or they could just be left hanging off-screen. Either way I will settle with it.

6.1.4 Full Algorithm

Below is my full python code for calculating the Delaunay triangulation and Voronoi diagram from a set of points, it outputs in Geogebra form for my ease of testing and I have not yet filtered out the outer Voronoi lines that relate to the supertriangle [1][2].

```python
import math

class Point():
   def __init__(self, x, y):
      self.x = x
      self.y = y

class Edge():
   def __init__(self, a, b):
      self.a = a
      self.b = b

class Triangle():
   def __init__(self, a, b, c):
      self.a = a
      self.b = b
      self.c = c
      self.edges = [Edge(a,b), Edge(a,c), Edge(b,c)]
      self.circumcircle = self.genCircumcircle()

   def genCircumcircle(self):
      #find perp bisectors of 2 sides and find the point where they meet
      #then radius is just distance to one of the points, see derivation
      x1 = self.a.x
      y1 = self.a.y
      x2 = self.b.x
      y2 = self.b.y
      x3 = self.c.x
      y3 = self.c.y
```

[1]see 6.1.3
[2]Thanks go to Jameson Liu for his invaluable help in bug-fixing and actually making this work

```
        xab = (x1+x2)/2
        xac = (x1+x3)/2
        yab = (y1+y2)/2
        yac = (y1+y3)/2

        #if two points have same y then x is directly in middle
        if y1 == y2:
            X = (x1 + x2) / 2
            #sub into line eq
            Y = ((-((x1-x3)/(y1-y3)))*(X-xac)+yac)
        elif y1 == y3:
            X = (x1 + x3)/2
            #sub into line eq
            Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
        elif y2 == y3:
            X = (x2 + x3)/2
            #sub into line eq
            Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
        else:
            X = ((((-x1*xab)+(xab*x2))/(y1-y2))+(((x1*xac)-(xac*x3))/(y1-y3))+
            yac-yab)/(((-x1+x2)/(y1-y2))+((x1-x3)/(y1-y3)))
            #sub into line eq
            Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
        #Assemble circle object
        centre = Point(X,Y)
        radius = math.sqrt((X-x2)**2 + (Y-y2)**2)
        circumcircle = Circle(centre, radius)
        return circumcircle

class Circle():
    def __init__(self, centre, radius):
        self.centre = centre
        self.radius = radius

def checkEdgeEquality(e1, e2):
    if (e1.a == e2.a and e1.b == e2.b) or (e1.a == e2.b and e1.b == e2.a):
        return True
    return False

def checkIfSharesEdges(edge, currentTri, triangleSet):
    for tri in triangleSet:
        if tri != currentTri:
            for e in tri.edges:
                if checkEdgeEquality(edge, e):
                    return True
    return False

def genDelaunay(points, supertriangle):
    triangulation = []
    triangulation.append(supertriangle)
    #add each point one by one
    for point in points:
        btriangles = []
        #find all triangles that are no longer valid
        for tri in triangulation:
            distToCircumcentre = math.sqrt(((tri.circumcircle.centre.x - point.x)**2) +
                ((tri.circumcircle.centre.y - point.y)**2))
            if tri.circumcircle.radius > distToCircumcentre:
                btriangles.append(tri)
```

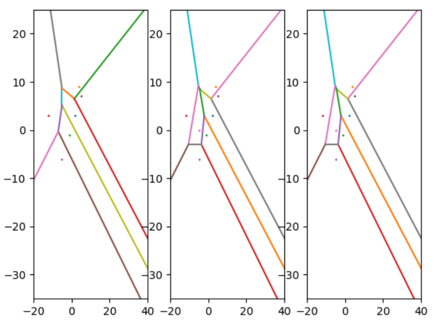
```
      #find boundary of polygonal hole
      polygon = []
      for tri in btriangles:
         for edge in tri.edges:
            if not checkIfSharesEdges(edge, tri, btriangles):
               polygon.append(edge)
      #remove bad triangles from triangulation
      for tri in btriangles:
         triangulation.remove(tri)
      #create new triangles
      for edge in polygon:
         triangulation.append(Triangle(edge.a, edge.b, point))

   return triangulation

supertriangle = Triangle(Point(-100000,-1000000), Point(0,1000000), Point(1000000,-1000000)) #very
    large supertriangle
points = [Point(2,3), Point(4,9), Point(-1,-1), Point(-12, 3), Point(-5,-6), Point(5,7)] #Example
    points
delaunay = genDelaunay(points, supertriangle)


#Convert to Voronoi

voronoiLines = []
for triangle in delaunay:
   for tri in delaunay:
      if triangle != tri:
         if checkEdgeEquality(triangle.edges[0], tri.edges[0]) or
             checkEdgeEquality(triangle.edges[0], tri.edges[1]) or
             checkEdgeEquality(triangle.edges[0], tri.edges[2]) or
             checkEdgeEquality(triangle.edges[1], tri.edges[0]) or
             checkEdgeEquality(triangle.edges[1], tri.edges[1]) or
             checkEdgeEquality(triangle.edges[1], tri.edges[2]) or
             checkEdgeEquality(triangle.edges[2], tri.edges[0]) or
             checkEdgeEquality(triangle.edges[2], tri.edges[1]) or
             checkEdgeEquality(triangle.edges[2], tri.edges[2]):
            voronoiLines.append(Edge(triangle.circumcircle.centre, tri.circumcircle.centre))

#remove duplicates
for line in voronoiLines[:]:
   for l in voronoiLines[:]:
      if line != l:
         if (line.a == l.a and line.b == l.b) or (line.a == l.b and line.b == l.a):
            voronoiLines.remove(line)

print("VORONOI LINES")
for i in voronoiLines:
   print("Segment((", i.a.x, ",", i.a.y, "),(", i.b.x, ",", i.b.y, "))")
```

6.1.5 Incrementability

The Bowyer-Watson algorithm is already built on a theory of incremental construction, i.e. it creates the diagram by adding the points one by one and correcting after each. So to edit the code to edit the Delaunay triangulation would be trivial. By taking the contents of the *for point in points* : loop and putting it in an *addPoint*(*point*) function you could easily allow for the Delaunay triangulation to be updated with new points.

The Voronoi diagram however is more difficult. The obvious response is to simply call the code that calculates the dual of the graph again after you've added all your new points. This would work but is wildly inefficient. Even if you added all the new points to the Delaunay first it is still less than ideal, especially given in the context of my finished program it will only be adding one point at a time anyway.
In order to optimise this incremental Voronoi flip one would need to keep track of all the edges what are changed in the Delaunay triangulation and only adjust those ones. As such I have now remade my python code with this in mind.

Outside of general structure changes (reorganising my classes to be... better), the first major difference I have is the sectioning off of the $addPoint()$ and $addToVoronoi()$ functions so that they can be called whenever required without running the entire generation. After this a small but very important change is the new Edge property called $dualLine$ which upon initialisation is set to $None$. Whenever a Voronoi line is generated from an edge, that edge's $dualLine$ is set to equal the new Voronoi edge. This is so that when updating the Voronoi diagram later it is easy to find and remove the deleted edges from the list of edited triangles.



The matplotlib output of my incremental code. Left is original, middle is updated with full redo, right is updated with partial redo

In the interest of brevity I shall not include the entirety of my new program here but I will include snippets showing the $findChangedTriangles()$ function which gets lists of the triangles that have been added and removed to the new Delaunay triangulation. And the $updateVoronoi()$ function which is rather self explanatory. I have also set up a rudimentary timing system to see if there are any improvements in computation time.
For 6 points incremented to 7 the time taken by redoing the entire Voronoi diagram was 0.00196695s, the time taken by using my partial redo method was 0.00101352s, giving a difference of 0.00095343s.
With 50 points incremented to 55 the time taken by redoing the entire Voronoi diagram was 0.09634828s, the time taken by using my partial redo method was 0.02517295s, giving a difference of 0.07117533s
With 300 points incremented to 320 the time taken by redoing the entire Voronoi diagram was 3.46091080s, the time taken by using my partial redo method was 0.80913591s giving a difference of 2.651774889s.
So my method is faster although practically speaking it does appear to be somewhat minimal outside of extreme cases. A situational success, if you will.

```python
def findChangedTriangles(delaunay1, delaunay2):
   deletedTriangles = []
   newTriangles = delaunay2.copy()
   for triangle in delaunay1:
      try:
         newTriangles.remove(triangle)
      except:
         deletedTriangles.append(triangle)

   return newTriangles, deletedTriangles

def updateVoronoi(voronoiLines, delaunay, newTriangles, deletedTriangles):
   #add the new stuff
   for triangle in newTriangles:
      voronoiLines = addToVoronoi(triangle, delaunay, voronoiLines)
   #delete the old stuff
   for triangle in deletedTriangles:
      if triangle.edges[0].dualLine in voronoiLines:
         voronoiLines.remove(triangle.edges[0].dualLine)
      if triangle.edges[1].dualLine in voronoiLines:
         voronoiLines.remove(triangle.edges[1].dualLine)
```

```
        if triangle.edges[2].dualLine in voronoiLines:
            voronoiLines.remove(triangle.edges[2].dualLine)
    #remove duplicate lines
    for line in voronoiLines[:]:
        for l in voronoiLines[:]:
            if line != l:
                if (line.a == l.a and line.b == l.b) or (line.a == l.b and line.b == l.a):
                    voronoiLines.remove(line)
    return voronoiLines
```

6.1.6 Potential for animation

The problem with animating the Voronoi generation via the Delaunay triangulation is that the algorithm being used is one which does the entire diagram at once. If I wanted some form of expanding circles algorithm I would need to only calculate the triangulations of points that have grown to the extent that they make a connection and then I would have to shorten the Voronoi lines that come out of it to only extend to the width of the circle at that section. And I would then need to draw the circle with the sections missing around the point. Long story short it's not really feasible with this algorithm.

6.2 Green and Sibson

6.2.1 Problem Outline

6.2.2 Putting points in clockwise order

For the Green and Sibson algorithm to work intentionally, the points list needs to be put in clockwise order; I have achieved this with polar coordinates.
I start by finding the centre of the points by taking the average of the x and y coordinates.

```
sumX = 0
sumY = 0
for point in points:
    sumX += point.x
    sumY += point.y
centre = Point((sumX / len(points)), (sumY / len(points)))
```

From there it's a matter of finding the angle in radians relative to north. I have done this with a series of if elifs to cover all 4 regions around the centre that then use the arcsine of the absolute relative y distance divided by the distance to find the angle.

```
for point in points:
r = math.sqrt(((centre.x - point.x)**2) + ((centre.y - point.y)**2))
relx = point.x - centre.x
rely = point.y - centre.y
if relx > 0 and rely > 0:
    angle = (math.pi / 2) - math.asin(rely/r)
elif relx < 0 and rely < 0:
    angle = ((math.pi * 3) / 2) - math.asin(abs(rely)/r)
elif relx < 0 and rely > 0:
    angle = ((math.pi * 3) / 2) + math.asin(rely/r)
elif relx > 0 and rely < 0:
    angle = (math.pi / 2) + math.asin(abs(rely)/r)
elif relx == 0:
    if rely > 0:
        angle = 0
    else:
        angle = math.pi
#storing polar coords as [rad, angle]
point.polarCoords = [r, angle]
```

Finally the points are ordered by their angle with a simple bubble sort algorithm.

```
n = len(points)
swapped = True
while swapped:
    swapped = False
    for i in range(n - 1):
        if points[i].polarCoords[1] > points[i + 1].polarCoords[1]:
            swapped = True
            points[i], points[i + 1] = points[i + 1], points[i]
    n -= 1
```

6.2.3 A little maths

There are two small bits of maths required to program the Green and Sibson algorithm. They are much simpler than the circumcentre equation for the Delaunay triangulation but it is worth noting them.

First is finding an equation for the perpendicular bisector of two points:
The gradient for this bisector is equal to the negative reciprocal of the gradient of the line that goes between the two points.

$$A = (x_1, y_1)$$

$$B = (x_2, y_2)$$

$$gradient\ AB = \frac{y_1 - y_2}{x_1 - x_2}$$

$$gradientBisector = -\frac{1}{gradient\ AB}$$

$$= \frac{-x_1 + x_2}{y_1 - y_2}$$

This line will pass through a point with coordinates equal to the average of the coordinates of A and B

$$Centrepoint = C = (x_3, y_3)$$

$$x_3 = \frac{x_1 + x_2}{2}$$

$$y_3 = \frac{y_1 + y_2}{2}$$

Substitute into line equation to get equation of bisector

$$y - y_3 = m(x - x_3)$$

$$y = mx - mx_3 + y_3$$

$$where\ m = \frac{-x_1 + x_2}{y_1 - y_2}$$

The other is finding the coordinates of a point that is mirrored across a line $y = mx + c$.

$$given\ equation\ y = m_1 x + c$$

$$let\ A = original\ point\ location = (x_1, x_2)$$

$$gradient\ of\ perpendicular\ line = m_2 = \frac{-1}{m_1}$$

find c value for perpendicular line by rearranging line formula

$$y - y_1 = m_2(x - x_1)$$

$$y = m_2 x - m_2 x_1 + y_1$$

$$-m_2 x_1 + y_1 = y - m_2 x$$

$$thus\ c = -m_2 x_1 + y_1$$

The midpoint between this point and the reflected one will be the intersection between the original line and the perpendicular line derived above. We shall refer to this point as (x_2, y_2) This midpoint will be half way between the two points and is thus their average, reversing this average will give the x,y coordinates of the reflected point (x_3, y_3)

$$x_2 = \frac{x_1 + x_3}{2}$$

$$x_3 = 2x_2 - x_1$$

$$y_2 = \frac{y_1 + y_3}{2}$$

$$y_3 = 2y_2 - y_1$$

6.2.4 Handling Edge Cases

The useful part of the Green and Sibson algorithm is that it is designed with edge cases in mind. That is, the paper which describes how the algorithm works [Green and Sibson, 1977] has a large explanation into ”windows” and how the algorithm interacts with boundary lines.

Essentially, it handles them by pretending they are like any other voronoi line with a corresponding seed. When the algorithm interacts with them it gives them a ”virtual point” such that the bisector of the virtual point and the currently being added seed is the boundary line. Despite the fact that I am evidently terrible at explaining it, this solves all problems surrounding boundaries with the exception of scenarios where there aren't any. In which case I will just assign some arbitrarily large ones.

6.2.5 Full Algorithm

6.2.6 Incrementability

6.2.7 Potential for animation

6.3 Fortune’s Algorithm

6.3.1 Problem Outline

Fortune’s algorithm is conceptually... complex. It operates by way of a sweepline tracking the parabolas formed by the points forming a beachline of the furthest forward arcs and their points of intersection along said beachline.
To illustrate, I shall include some images made with GeoGebra to the right[Geo]:

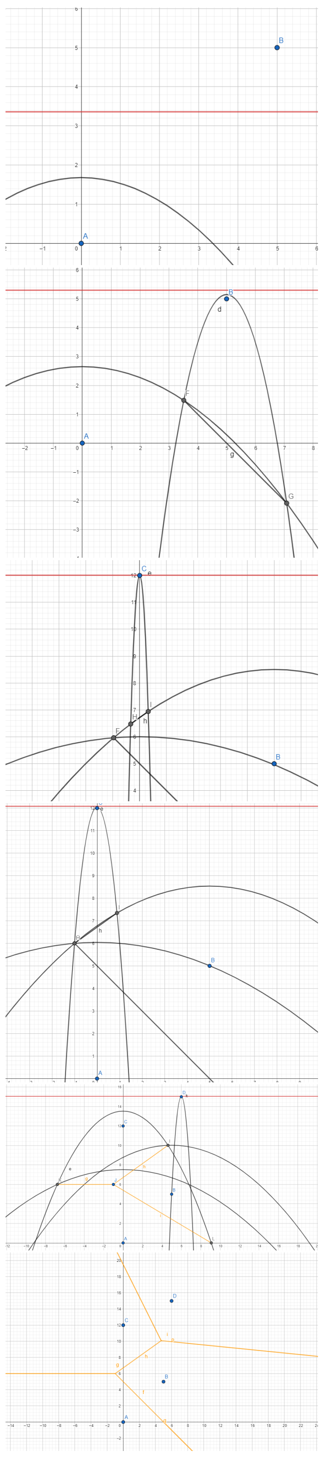
First we see the sweepline having only passed one point, this forms a beachline of one parabola ‘A‘.

Next the sweepline has met a second point, which is where we see the first real instance of a ”site event”. The new parabola ‘B‘ intersects the beachline at points ‘F‘ and ‘G‘ meaning the arc ‘A is split around ‘B‘ making the new beachline ‘ABA‘. This also creates a new edge (stored as 2 for implementation reasons) going between the intersection points ‘F‘ and ‘G‘ and expanding as the sweepline progresses

After this we have another site event as before however this one is slightly more interesting. As before the new parabola ‘C‘ intersects the beachline along the ‘B‘ arc splitting the beachline into ‘ABCBA‘ and forming an edge between the new intersection points ‘H‘ and ‘I‘. However, looking at the beachline we can see the section of ‘B‘ arc to the left of our new arc is getting smaller. This is going to create our first instance of a ”circle event”, where an arc is removed from the beachline.

Now the circle event has occurred we can look at its implications.
At any given point on a parabola it is the same distance to the nearest point on the sweepline as it is to the point which forms the parabola. if two parabolas intersect at a point then at that point the distance to either parabola forming point is equal, which is why we form voronoi edges as two parabolas intersect. Thus, when three parabolas intersect at a point, as they do during a circle event, that point is equidistant to three parabola creating points and is therefore a vertex in our voronoi diagram.
With that in mind we bind the two edges that were being formed from the arcs to that point, we remove the section of ‘B‘ arc from our beachline (leaving a beachline of ‘ACBA‘) and start a new edge between the ‘A‘ and ‘C‘ arcs that are now intersecting.

We can now see the effects of that circle event as we progress, a new edge is propagating to the left while the old two continue to expand away from the vertex although the top edge is about to be capped by a new circle event caused by the addition of ‘D‘ to the sweepline (‘ACBDBA‘ soon to be ‘ACDBA‘). It is by these processes that the entire diagram is completed (seen in final image). The minutia of implementation complicates matters somewhat as we have to predetermine where the 3 arcs will intersect in advance and create an event for it. As sweepline algorithms do not actually simulate the entire advance of the sweepline, they determine at what y values of the sweepline will things occur at and then just through them in order which is why I have previously said that Fortune’s algorithm is in fact event based. These events are handled by a priority queue and when that queue is empty then any edges which are missing an end node (vertex) are unbounded and continue forever on the finished diagram.



The development of a voronoi diagram of vertices (0,0),(5,5),(0,12),(6,15) made with GeoGebra[Geo]

6.3.2 A stupid amount of maths

Fortune's algorithm is a very maths heavy algorithm and, whilst none of
the maths is particularly difficult it is worth noting:
Alongside the circumcircle derivation from 6.1.2 I also need to convert a parabola stored as a point and a horizontal line into one in the form $ax^2 + bx + c$. This is done by way of a system of encoding equations shown below:

$$let\,A\,be\,the\,point\,(x, y)$$
$$let\,B\,be\,the\,line\,y = c$$
$$k = \frac{y + c}{2}$$
$$p = \frac{y - c}{2}$$
$$a = \frac{1}{4p}$$
$$b = -\frac{x}{2p}$$
$$c = \frac{x^2}{4p} + k$$
$$parabola = ax^2 + bx + c$$

I also need to be able to intersect two arcs, an arc with an edge and two edges:

$$Arc\,with\,Arc$$
$$ax^2 + bx + c = dx^2 + ex + f$$
$$(a - d)x^2 + (b - e)x + (c - f) = 0$$
$$using\,quadratic\,formula \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x_1, x_2 = \frac{-(b - e) \pm \sqrt{(b - e)^2 - 4(a - d)(c - f)}}{2(a - d)}$$

$$Arc\,with\,Edge$$
$$ax^2 + bx + c = dx + e$$
$$ax^2 + (b - d)x + (c - e)$$
$$using\,quadratic\,formula \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x_1, x_2 = \frac{-(b - d) \pm \sqrt{(b - d)^2 - 4a(c - e)}}{2a}$$

$$Edge\,with\,Edge$$
$$ax + b = cx + d$$
$$(a - c)x = d - b$$
$$x = \frac{d - b}{a - c}$$

6.3.3 Full Algorithm

Below is my entire python program for generating a Voronoi diagram with Fortune's algorithm[3]. It outputs in GeoGebra form for ease of testing and it prints each unbounded edge twice.

```
import math
import copy

class Point():#(x,y) point in 2D space
   def __init__(self, x, y):
      self.x = x
      self.y = y

class Arc():#parabola represented by a point and a line
   def __init__(self, point, minX = None, maxX = None):
      self.point = point
      self.minX = minX
      self.maxX = maxX
      self.leftEdge = None
      self.rightEdge = None

   def calcVals(self, sweeplineY):
      k = (self.point.y + sweeplineY)/2
      p = (self.point.y - sweeplineY)/2

      self.a = 1/(4*p)
      self.b = (-1 * self.point.x) / (2*p)
      self.c = ((self.point.x**2) / (4*p)) + k

   def calcY(self, x):
      return ((self.a * (x**2)) + (self.b * x) + self.c)

class Edge():#linear line
   def __init__(self, start, directionVector, m = None, vertical = False, propDirection = None):
      self.start = start
      self.directionVector = directionVector #(dx,dy)
      self.propDirection = propDirection

      if m == None and not vertical:
         self.m = (directionVector[1]/directionVector[0]) #dy/dx
      else:
         self.m = m

      self.vertical = vertical

      if not self.vertical:
         self.c = start.y - (self.m * start.x)
         self.xVal = None
      else:
         self.xVal = start.x

      self.end = None

   def calcX(self, y):
```

[3]For this program I am forever indebted to Paul Reed[Reed] who's explanation about the implementation of Fortune's algorithm acted as the basis for my understanding and, whilst I did not stick to his methods entirely, I have him to thank for this code working. I would also not have been able to complete this if it were not for my ability to manually create the diagram with GeoGebra[Geo]. I also had an experiment with attempting to get help from ChatGPT which I will talk about in addenda "The use of AI in bugfixing" pg38.

```
        dy = y - self.start.y
        dx = dy * self.directionVector[0]
        x = self.start.x + dx
        return x
    def calcY(self, x):
        if self.vertical:
            return None
        dx = x - self.start.x
        dy = dx * self.directionVector[1]
        y = self.start.y + dy
        return y

class SiteEvent(Point):
    def __init__(self, x, y, site):
        self.eType = "site"
        self.x = x
        self.y = y
        self.site = site

class CircleEvent(Point):
    def __init__(self, x, y, arcs, side):
        self.eType = "circle"
        self.x = x
        self.y = y
        self.arcs = arcs
        self.side = side
        self.circumcentre = self.calcCircumcentre()

    def calcCircumcentre(self):
        point1 = self.arcs[0].point
        point2 = self.arcs[1].point
        point3 = self.arcs[2].point
        #find perp bisectors of 2 sides and find the point where they meet
        #then radius is just distance to one of the points, see derivation
        x1 = point1.x
        y1 = point1.y
        x2 = point2.x
        y2 = point2.y
        x3 = point3.x
        y3 = point3.y
        xab = (x1+x2)/2
        xac = (x1+x3)/2
        yab = (y1+y2)/2
        yac = (y1+y3)/2
        #if two points have same y then x is directly in middle
        if y1 == y2:
            X = (x1 + x2) / 2
            #sub into line eq
            if y1 - y3 == 0:
                Y = yac
            else:
                Y = ((-((x1-x3)/(y1-y3)))*(X-xac)+yac)
        elif y1 == y3:
            X = (x1 + x3)/2
            #sub into line eq
            if y1 - y2 == 0:
                Y = yab
            else:
                Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
        elif y2 == y3:
            X = (x2 + x3)/2
            #sub into line eq
            if y1 - y2 == 0:
                Y = yab
            else:
                Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
        else:
            X =
                ((((-x1*xab)+(xab*x2))/(y1-y2))+(((x1*xac)-(xac*x3))/(y1-y3))+yac-yab)/(((-x1+x2)/(y1-y2))+((x1-x3)/(y
            #sub into line eq
            if y1 - y2 == 0:
                Y = yab
            else:
                Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
        #Assemble circle object
        cc = Point(X,Y)
        return cc

class Queue():
    def __init__(self):
        self.queue = []

    def insert(self, pos, item):
        self.queue = self.queue[:pos] + [item] + self.queue[pos:]

    def append(self, item):
        self.queue.append(item)

    def pop(self):
        item = self.queue[0]
        self.queue = self.queue[1:]
        return item

    def delete(self, pos):
        self.queue = self.queue[:pos] + self.queue[pos + 1:]

class Sweepline():
    def __init__(self, y):
        self.y = y
        self.arcs = []

    def insert(self, pos, item):
        self.queue = self.queue[:pos] + [item] + self.queue[pos:]

def orderPoints(points):
    newPoints = []
    for point in points:
        y = point.y
        if len(newPoints) == 0:
            newPoints.append(point)
        else:
            for i in range(len(newPoints)):
                if y < newPoints[i].y:
                    newPoints = newPoints[:i] + [point] + newPoints[i:]
                    break
                else:
                    if i == len(newPoints) - 1:
                        newPoints.append(point)
                break
    return newPoints

def intersectArcs(arc1, arc2):
    a1 = arc1.a
    b1 = arc1.b
    c1 = arc1.c
    a2 = arc2.a
    b2 = arc2.b
    c2 = arc2.c
    #a1x^2 + b1x + c1 = a2x^2 + b2x + c2
    a = (a1 - a2)
    b = (b1 - b2)
    c = (c1 - c2)
    #(a1-a2)x^2 + (b1-b2)x + (c1-c2) = 0
    discriminant = (b**2) - (4*a*c)
    if discriminant < 0:
        return None #no intersection
    x1 = (-b + math.sqrt(discriminant)) / (2*a)
    x2 = (-b - math.sqrt(discriminant)) / (2*a)
    av = (x1 + x2) / 2 #direct above
    return av

def arcEdgeIntersection(arc, edge):
    if edge.vertical:
        return edge.xVal

    a = arc.a
    b = arc.b - edge.m
    c = arc.c - edge.c
    try:
        x1 = (-b + math.sqrt(b**2 - (4*a*c))) / (2*a)
        x2 = (-b - math.sqrt(b**2 - (4*a*c))) / (2*a)
    except:
        return None
    if edge.directionVector[0] < 0:
        if x1 > x2:
            return x1
        else:
            return x2
    else:
        if x1 < x2:
            return x1
        else:
            return x2

def intersectEdges(edge1, edge2):
    if edge1.vertical:
        X = edge1.xVal
        Y = edge2.calcY(X)
    elif edge2.vertical:
        X = edge2.xVal
        Y = edge1.calcY(X)
    else:
        X = (edge1.c - edge2.c) / (edge2.m + edge1.m)
        Y = edge1.calcY(X)
    return Point(X,Y)

def splitArc(sweep, queue, origArcIdx, newArc):
    origArc = sweep.arcs[origArcIdx]
    left = copy.deepcopy(origArc)
    right = copy.deepcopy(origArc)
    #update circle events
    for i in queue.queue:
        if i.eType == "circle":
            if origArc == i.arcs[0]:
                i.arcs[0] = right
            elif origArc == i.arcs[1]:
                del i
            elif origArc == i.arcs[2]:
                i.arcs[2] = left
    #remake sweepline
    sweep.arcs = sweep.arcs[:origArcIdx] + [left, newArc, right] + sweep.arcs[origArcIdx + 1:]
    #add new edges
    x = newArc.point.x
    start = Point(x, origArc.calcY(x))

    direction = ((-1 * (origArc.point.y - newArc.point.y), origArc.point.x - newArc.point.x))
    vertical = False
    if direction[0] == 0:
        vertical = True
    edge1 = Edge(start, direction, vertical = vertical, propDirection = "left")
    sweep.arcs[origArcIdx].rightEdge = edge1
    sweep.arcs[origArcIdx + 1].leftEdge = edge1

    direction = ((-1 * (newArc.point.y - origArc.point.y), newArc.point.x - origArc.point.x))
    vertical = False
    if direction[0] == 0:
        vertical = True
    edge2 = Edge(start, direction, vertical = vertical, propDirection = "right")
    sweep.arcs[origArcIdx + 1].rightEdge = edge2
    sweep.arcs[origArcIdx + 2].leftEdge = edge2

def getCircumcentreFrom3Arcs(Arcs):
    #find perp bisectors of 2 sides and find the point where they meet
    #then radius is just distance to one of the points, see derivation
    x1 = Arcs[0].point.x
    y1 = Arcs[0].point.y
    x2 = Arcs[1].point.x
    y2 = Arcs[1].point.y
    x3 = Arcs[2].point.x
    y3 = Arcs[2].point.y
    xab = (x1+x2)/2
    xac = (x1+x3)/2
    yab = (y1+y2)/2
    yac = (y1+y3)/2

    #if two points have same y then x is directly in middle
    if y1 == y2:
        X = (x1 + x2) / 2
        #sub into line eq
        if y1 - y3 == 0:
            Y = yac
        else:
            Y = ((-((x1-x3)/(y1-y3)))*(X-xac)+yac)
    elif y1 == y3:
        X = (x1 + x3)/2
        #sub into line eq
```

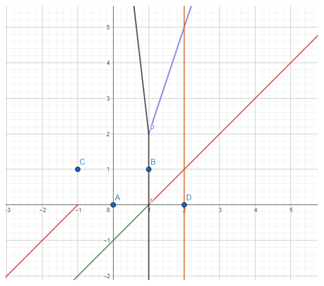
```
      if y1 - y2 == 0:
         Y = yab
      else:
         Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
   elif y2 == y3:
      X = (x2 + x3)/2
      #sub into line eq
      if y1 - y2 == 0:
         Y = yab
      else:
         Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
   else:
      X =
          ((((-x1*xab)+(xab*x2))/(y1-y2))+(((x1*xac)-(xac*x3))/(y1-y3))+yac-yab)/(((-x1+x2)/(y1-y2))+((x1-x3)/(y1-y
      #sub into line eq
      if y1 - y2 == 0:
         Y = yab
      else:
         Y = ((-((x1-x2)/(y1-y2)))*(X - xab)+yab)
   #Assemble circle object
   centre = Point(X,Y)
   return centre

def checkNewCircleEvents(sweep, index, queue, subEvent = False):
   #to the left
   if index - 1>= 1:
   arcs = sweep.arcs[index - 2:index + 1]
   if (arcs[0].point.x != arcs[1].point.x or arcs[0].point.y != arcs[1].point.y) and
       (arcs[0].point.x != arcs[2].point.x or arcs[0].point.y != arcs[2].point.y) and
       (arcs[1].point.x != arcs[2].point.x or arcs[1].point.y != arcs[2].point.y):#doesn't share 2
       points
      circumcentre = getCircumcentreFrom3Arcs(arcs)
      if not circumcentre.x > arcs[2].point.x: #covered by other one
         radius = math.sqrt((arcs[0].point.y - circumcentre.y)**2 + (arcs[0].point.x -
             circumcentre.x)**2)
         if not circumcentre.y + radius <= sweep.y: #if above current sweep ignore
            newEvent = CircleEvent(circumcentre.x, circumcentre.y + radius, arcs, "left")
            if len(queue.queue) == 0:
               queue.append(newEvent)
            else:
               added = False
               for i in range(len(queue.queue)):
                  if queue.queue[i].y > newEvent.y:
                     queue.insert(i, newEvent)
                     added = True
                     break
               if not added:
                  queue.append(newEvent)
   #to the right
   if index <= len(sweep.arcs) - 3:
      arcs = sweep.arcs[index : index + 3]
      if (arcs[0].point.x != arcs[1].point.x or arcs[0].point.y != arcs[1].point.y) and
          (arcs[0].point.x != arcs[2].point.x or arcs[0].point.y != arcs[2].point.y) and
          (arcs[1].point.x != arcs[2].point.x or arcs[1].point.y != arcs[2].point.y):#doesn't
          share 2 points
         circumcentre = getCircumcentreFrom3Arcs(arcs)
         if not circumcentre.x < arcs[0].point.x: #covered by other one
            radius = math.sqrt((arcs[0].point.y - circumcentre.y)**2 + (arcs[0].point.x -
                circumcentre.x)**2)
            if not circumcentre.y + radius <= sweep.y: #if above current sweep ignore
               newEvent = CircleEvent(circumcentre.x, circumcentre.y + radius, arcs, "right")
            if len(queue.queue) == 0:
               queue.append(newEvent)
            else:
               added = False
               for i in range(len(queue.queue)):
                  if queue.queue[i].y > newEvent.y:
                     queue.insert(i, newEvent)
                     added = True
                     break
               if not added:
                  queue.append(newEvent)

minX = -50
maxX = 50
leftMostEdge = Edge(Point(minX, -50), (0,1), vertical = True)
rightMostEdge = Edge(Point(maxX, -50), (0,1), vertical = True)
points = [Point(0,0), Point(5,5), Point(0,12), Point(6,15)]
points = orderPoints(points)
queue = Queue()

for point in points:
   queue.append(SiteEvent(point.x, point.y, point))

#setup first arc with boundary edges
firstPoint = queue.pop()
sweep = Sweepline(firstPoint.y)
firstArc = Arc(firstPoint)
firstArc.leftEdge = leftMostEdge
firstArc.rightEdge = rightMostEdge
sweep.arcs.append(firstArc)

vertices = []
edges = []
while len(queue.queue) > 0:
   event = queue.pop()
   sweep.y = event.y + 0.0000001

   #site events
   if event.eType == "site":
      arc = Arc(event.site)
      arc.calcVals(sweep.y)
      #add arc
      highest = [-1,-50000]
      for i in range(len(sweep.arcs)):
         otherArc = sweep.arcs[i]
         otherArc.calcVals(sweep.y)
         yIntersection = otherArc.calcY(event.site.x)
         if yIntersection > highest[1]:
            highest = [i, yIntersection]
      splitArc(sweep, queue, highest[0], arc)

      #add circle events
      checkNewCircleEvents(sweep, highest[0] + 1, queue)



   #handling circle events
```

```
    if event.eType == "circle":
        #mark down the circumcentre and edge
        vertices.append(event.circumcentre)
        event.arcs[1].leftEdge.end = event.circumcentre
        event.arcs[1].rightEdge.end = event.circumcentre
        edges.append(event.arcs[1].leftEdge)
        edges.append(event.arcs[1].rightEdge)
        #remove arc
        arc1 = event.arcs[1]
        idx = sweep.arcs.index(arc1)
        del sweep.arcs[idx]
        #remove other circle events using that arc
        toRemove = []
        for i in range(len(queue.queue)):
            if queue.queue[i].eType == "circle":
                if arc1 in queue.queue[i].arcs:
                    toRemove.append(i)
        for i in range(len(toRemove)):
            del queue.queue[toRemove[i] - i]
        #assign new edge to left and right arc
        direction = (-1 * (event.arcs[0].point.y - event.arcs[2].point.y), event.arcs[0].point.x -
            event.arcs[2].point.x)
        newEdge = Edge(event.circumcentre, direction, propDirection = event.side)
        sweep.arcs[idx - 1].rightEdge = newEdge
        sweep.arcs[idx].leftEdge = newEdge

        #if adjacent arcs are same arc
        if idx != len(sweep.arcs) - 1:
            if sweep.arcs[idx - 1].point == sweep.arcs[idx + 1].point:
                sweep.arcs[idx - 1].rightEdge = sweep.arcs[idx + 1].rightEdge
                del sweep.arcs[idx + 1]


        if event.side == "left":
            checkNewCircleEvents(sweep, idx, queue, True)
        else:
            checkNewCircleEvents(sweep, idx - 2, queue, True)




print("\n\n\n")
for i in edges:
    print("Segment((", i.start.x, ",", i.start.y, "),(", i.end.x, ",", i.end.y, "))")
print("unfinished Edges")
for i in sweep.arcs:
    if i.leftEdge not in edges:
        if not i.leftEdge.vertical:
            if i.leftEdge.propDirection == "left":
                print(f"If(x<{i.leftEdge.start.x},{i.leftEdge.m}x + {i.leftEdge.c})")
            else:
                print(f"If(x>{i.leftEdge.start.x},{i.leftEdge.m}x + {i.leftEdge.c})")
        else:
            print("vertical", i.leftEdge.xVal)
    if i.rightEdge not in edges:
        if i.rightEdge != None:
            if not i.rightEdge.vertical:
                if i.rightEdge.propDirection == "left":
                    print(f"If(x<{i.rightEdge.start.x},{i.rightEdge.m}x + {i.rightEdge.c})")
                else:
                    print(f"If(x>{i.rightEdge.start.x},{i.rightEdge.m}x + {i.rightEdge.c})")
            else:
                print("vertical ", i.rightEdge.xVal)
```

6.3.4 Limitations/Degeneracies

My implementation of Fortune's algorithm works in most cases however as of current it is completely incapable of handling two points that lie on the same y value. I have no idea why but the affects of it are shown on the right.

If I had to guess it would be something to do with the implication of which arc the lines land on when two are added at once, but in all truth I don't know how this happened and the easiest solution for everyone will most likely be to warn the user about this degeneracy and suggest a different algorithm for diagrams of this type. I could even implement a check for this situation to alert the user automatically.



An attempt at the voronoi diagram of vertices (0,0),(1,1),(-1,1),(2,0) made with my program and plotted with GeoGebra[Geo]

6.3.5 Incrementability

Unlike the other two algorithms, Fortune's algorithm is not incremental in construction. Instead it requires a full list of points before being run and as such if the user has Fortune's algorithm selected and decides to add an extra point to the diagram I have no choice but to run the entire generation again. This is less than ideal but unfortunately (hehe, fortune) it is unavoidable.

6.3.6 Potential for animation

I am in fact able to provide an animation for Fortune's algorithm although it is not one which is particularly enlightening. I can provide an animation of the sweepline progressing through the points to show the algorithm at work. This could be done (at extreme expense of computation time) by actually tracking progression of a sweepline and waiting for it to reach events before triggering them. I can track/plot the actual development of edges and parabolas in real time. It'll be incredibly inefficient but can/will be done.

Chapter 7

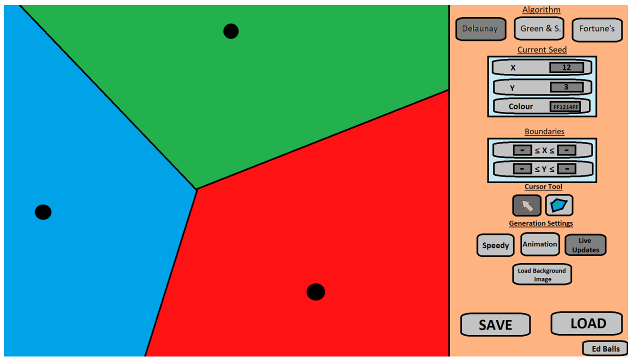
Designing an Interface

7.1 Interface requirements

The interface requirements in large part draw from the specific objectives in section 5.2. From which I can glean that my interface needs to be able to support:
- Loading data points
- Altering seed positions
- Saving conditions and images
- Performance displayed
- Bounding boxes
- Animations
- Animation speed control
- Lattices/pattern generation/templates

7.2 Version 1

7.2.1 Overview



On the right is my first interface design that I've put together in digital art station software.
It consists of a main voronoi viewing section with a right hand side panel that contains the controls.

From top to bottom:
The Algorithm buttons (Delaunay, Green & S., Fortune's) refer to what algorithm is going to be used to generate the diagrams. These are mutually exclusive and will disable others when picked.
The current seed input box contains input boxes for x, y coordinates and colour for whatever seed is selected (currently that is the bottom right red seed).
The boundaries input boxes allow the user to input maximum and minimum values for which the diagram is displayed with regards to the x and y axis. The boxes currently include a hyphen to indicate no limits.
The cursor tool buttons are another set of mutually exclusive buttons that choose between the cursor clicks selecting which seed is selected and adding a new seed.
The generation settings buttons are mutually exclusive and dictate how the diagram will be generated. Whether it will be generated as fast as possible, with an accompanying animation, or in a form where you

can add to the diagram after generation.
Load Background Image will open a window that will let the user select an image to put behind the voronoi diagram being displayed.

7.2.2 Initial Thoughts

As a first draft I think this layout has some merits; the organisation is relatively neat with all the buttons and inputs being tucked away to the side and grouped by category. However, it does feel slightly cluttered.
I also have a problem with my colour inputs. The value represented in the image is $FF1214FF$, this uses an 8-digit RGBA colour code to represent the colour and transparency of the tile. Personally I think having a separate transparency slider would likely be better although I shall confer with my user on this.
I also think after looking at this it would likely be beneficial to have some form of help button if my documentation is insufficient (and it is also likely true that most users would prefer a simple help menu to a clunky documentation file).
Finally, I am not currently fulfilling the requirement that I display performance data, I have no template loading system and for the non-incremental running modes there is currently no ”Run” button. This will obviously require significant remedying based on this and any pointers from my user.

7.2.3 User's Thoughts

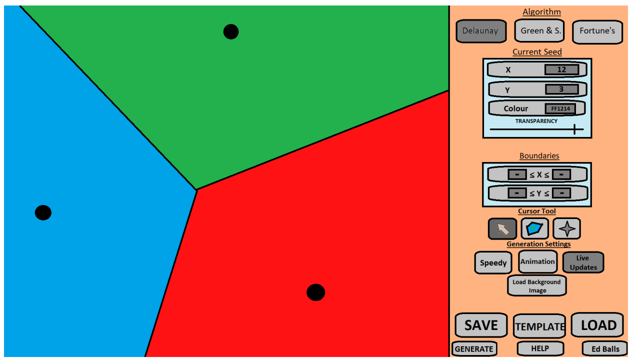
Upon consultation with my user, they first agreed with my points about the lacking features (performance data, run button, template system) and said that a transparency slider would be better than a longer colour code.
Outside of this they suggested some basic rearrangement of the button locations (symmetry, centering, etc.) alongside some zoom controls and a panning option under cursor tools. Admittedly these now seem obvious omissions and I'm paranoid about what else I've missed.
Finally some form of help button was suggested which I had forgotten to propose to my user so that clears that up.

7.2.4 Conclusions

In the next design version I need to:
- Move colour transparency to a separate slider
- Add a button linking to a help window
- Display performance data
- Add a template loading button
- Add a run button
- Do some basic rearrangement of button locations
- Add a panning option to the cursor tools
- Add zoom controls

7.3 Version 2

7.3.1 Overview



On the right is my updated design created with the same digital art station software as before.
The general layout is the same as before consisting of a main voronoi viewing section with a right hand side panel containing the controls.

From top to bottom:
The algorithm buttons which are unchanged.
The current seed options with an added transparency slider.
The boundary options which are unchanged.
The cursor tool options with a new panning option on the right (denoted by the 4-pointed star).
The generation settings which are unchanged.
A button that will open a window to enter a template design.
A generate button that will run the selected algorithm with the current points and selected settings.
A help button that will open a window with operation instructions.

MAYBE DESIGN THE NEW WINDOWS IDK JUST A THOUGHT DUMBASS

7.3.2 Initial Thoughts

Personally I think this is a significant improvement from the previous design if just from a feature perspective. I am not aware of any glaring omissions with regards to functionality although I'm sure my user will be all too happy to point any out.

7.3.3 User's Thoughts

My user was remarkably reserved when commenting upon the reference image I send him: "I don't have loads to say other than I think the organisation is cool visually speaking". So I'm calling this an absolute win and green-lit for the next step of creating a version in Unity

7.3.4 Conclusions

The new design is an improvement on the old and it is now time to work on a Unity implementation.

7.4 Unity Draft

Acknowledgements

Miss Baker	For teaching me my proficiency in C#
Nathan Easow	Agreeing to be my user
My friends who are forced to listen to me	For putting up with my excited rambling and proof reading some of my grammar
Lydia Gee	For suggesting a risk assessment
The unofficial school cat	Mental health support
The Flying Spaghetti Monster	All are welcome into the loving embrace of His Noodly Appendage
Newton's first law	Great support
Entropy	Providing an unassailable deadline
Connor Fox Cunningham	Pointing out that my acknowledgements are just another form of procrastination
The "free science lessons" guy	Acting as a true role model
Sam the snake plant	Making the air in my room slightly less toxic

Addenda

Clarifications

Qhull's Method of Voronoi Computation

I stated in 3.5 that Qhull computes Voronoi diagrams via the dual of the Delaunay triangulation, and this is what is stated on their documentation page for the function. However, when the function is run for a 2D diagram it displays the line ”Voronoi diagram by the convex hull of 10 points in 3-d). This would refer to some variation on Brown's algorithm which I did not cover here. I did not comment on this discrepancy in the main section as I did not consider it relevant however I would like to discuss it here.

Qhull is, at its core, not a Voronoi program. It is designed to generate what are known as convex hulls which are the smallest shape for a set of points which contains the entire linear line between any two points within it. As such I believe the creators are simply generating Voronoi diagrams based on what they already have. In their paper it occurs to me that their main priorities are in generating convex hulls and Delaunay triangulations from said hulls. It does not specify how it makes the Voronoi diagrams, and thus it could be that Qhull uses the convex hulls directly (such as with Brown's algorithm) or what I believe is more likely is that Qhull is using convex hulls to generate Delaunay triangulations and is then using the dual of those to generate the Voronoi diagrams. That theory agrees with both statements that it is generated from convex hulls and from Delaunay. I only wish that if this is the case then Qhull would just say so.

The use of AI in bugfixing

Artificial intelligence (AI) can be used to assist in the bugfixing process by identifying and prioritizing bugs, suggesting potential solutions, and even automatically fixing certain types of bugs. For example, machine learning algorithms can analyze the codebase and identify patterns that are commonly associated with bugs, allowing them to prioritize the most critical issues and suggest solutions based on similar issues that have been fixed in the past. AI can also be used to automate certain parts of the bugfixing process, such as regression testing, which involves re-testing the code after a bug has been fixed to ensure that it has not introduced any new issues. Overall, the use of AI in bugfixing can help to improve the efficiency and accuracy of the debugging process, ultimately leading to a higher quality and more reliable software product.

That above paragraph was written by an AI called ChatGPT made by OpenAI[OpenAI, 2022] when prompted ”write a paragraph about the use of AI in bugfixing”. As it correctly says, artificial intelligences of some form or another have long been used to aid in the bugfixing process to varying degrees. However, one of the selling points of ChatGPT is it's ability to write code based on a prompt and look at code to try and spot errors. Whilst I was working on my Fortune's algorithm python program I ran into some difficulties and decided I would see how well ChatGPT held up.

I began by simply giving the program my entire work in progress code, saying it was attempting to generate a Voronoi diagram using Fortune's algorithm and asking it why it wasn't working.
The response it gave initially surprised me (right), it first correctly identified that there were no syntax errors in the code which was a relief. But then it started suggesting fixes and this is where the limitations come in. Instead of being able to analyse the code and point to logic errors (that was always a very optimistic hope but still) it instead pointed to areas of my code basically saying "have you got this bit right, have you got this bit right, have you got this bit right" and on every occasion the answer was, yes, that bit is correct. It was able to pull the class and function names from the code and was even able to describe what they were meant to do which greatly surprised me, but as for pointing out errors, not exactly useful.

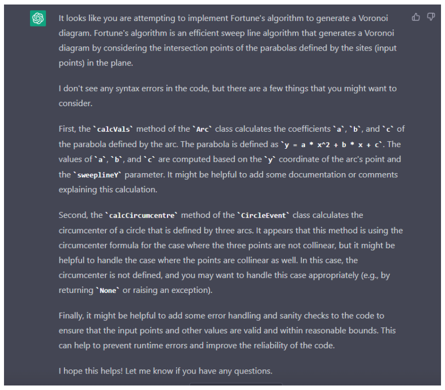
This first attempt pointed me to a great strength of ChatGPT however, its ability to describe what is *meant* to happen. So instead of asking what was wrong with the code, I started asking how I was meant to achieve the correct outcome so I could check my code's logic was correct.
This had some strong initial success as it was able to correctly tell me how (in semi-abstract terms) I am meant to handle point events (shown right). It was even able to describe it within the context of my own code as if I were making edits.

This optimism died however when after one request it started confidently talking about "point events". I am 99% certain that there is no such thing as a point event within Fortune's algorithm. Upon being asked what a point event is it described one as "an event that is triggered when the sweep line intersects the circumcircle of a voronoi edge" which is to my eyes total nonsense. For a start how can an edge have a circumcircle. It then went on to talk about it being when the sweepline passes through the circumcentre of 3 points, this whole description sounded somewhat like a circle event, so I asked if it was distinct from a circle event and ChatGPT said that, whilst similar, the events are distinct. This actually got me very worried that I had misunderstood a significant part of Fortune's algorithm, but as far as I have been able to find anywhere, "point events" do not exist.

This was... disappointing as it means my hope about AIs like ChatGPT being able to help even on the description end of programming is severely hampered by the fact that it is perfectly capable of making up utter rubbish and talking about it very confidently. When asked in a separate chat what a point event was in the context of Fortune's algorithm for Voronoi diagrams I simply got a description of a site event.

So overall I cannot see much help from AIs such as ChatGPT at the moment with regards to logic error fixing, as they do not possess the sufficient insight to correct your code or to tell you things that aren't complete horse manure and throw you off track.

Initial attempts



It looks like you are attempting to implement Fortune's algorithm to generate a Voronoi diagram. Fortune's algorithm is an efficient sweep line algorithm that generates a Voronoi diagram by considering the intersection points of the parabolas defined by the sites (input points) in the plane.

I don't see any syntax errors in the code, but there are a few things that you might want to consider.

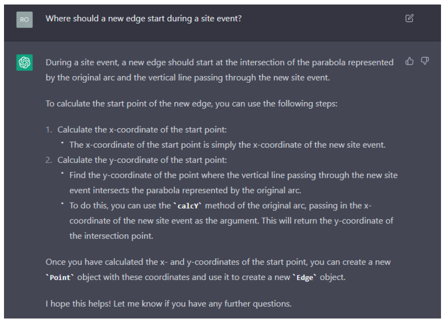
First, the `calcVals` method of the `Arc` class calculates the coefficients `a`, `b`, and `c` of the parabola defined by the arc. The parabola is defined as `y = a * x^2 + b * x + c`. The values of `a`, `b`, and `c` are computed based on the `y` coordinate of the arc's point and the `sweeplineY` parameter. It might be helpful to add some documentation or comments explaining this calculation.

Second, the `calcCircumcentre` method of the `CircleEvent` class calculates the circumcenter of a circle that is defined by three arcs. It appears that this method is using the circumcenter formula for the case where the three points are not collinear, but it might be helpful to handle the case where the points are collinear as well. In this case, the circumcenter is not defined, and you may want to handle this case appropriately (e.g., by returning `None` or raising an exception).

Finally, it might be helpful to add some error handling and sanity checks to the code to ensure that the input points and other values are valid and within reasonable bounds. This can help to prevent runtime errors and improve the reliability of the code.

I hope this helps! Let me know if you have any questions.

Confident descriptions



RO

Where should a new edge start during a site event?

During a site event, a new edge should start at the intersection of the parabola represented by the original arc and the vertical line passing through the new site event.

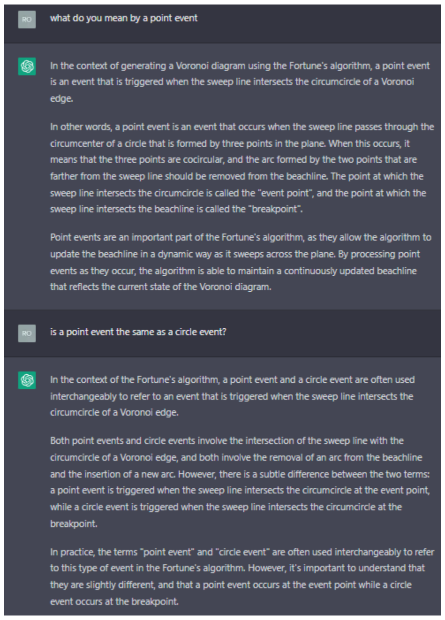
To calculate the start point of the new edge, you can use the following steps:

1. Calculate the x-coordinate of the start point:
   - The x-coordinate of the start point is simply the x-coordinate of the new site event.
2. Calculate the y-coordinate of the start point:
   - Find the y-coordinate of the point where the vertical line passing through the new site event intersects the parabola represented by the original arc.
   - To do this, you can use the `calcY` method of the original arc, passing in the x-coordinate of the new site event as the argument. This will return the y-coordinate of the intersection point.

Once you have calculated the x- and y-coordinates of the start point, you can create a new `Point` object with these coordinates and use it to create a new `Edge` object.

I hope this helps! Let me know if you have any further questions.

"Point Events"



RO

what do you mean by a point event

In the context of generating a Voronoi diagram using the Fortune's algorithm, a point event is an event that is triggered when the sweep line intersects the circumcircle of a Voronoi edge.

In other words, a point event is an event that occurs when the sweep line passes through the circumcenter of a circle that is formed by three points in the plane. When this occurs, it means that the three points are cocircular, and the arc formed by the two points that are farther from the sweep line should be removed from the beachline. The point at which the sweep line intersects the circumcircle is called the "event point", and the point at which the sweep line intersects the beachline is called the "breakpoint".

Point events are an important part of the Fortune's algorithm, as they allow the algorithm to update the beachline in a dynamic way as it sweeps across the plane. By processing point events as they occur, the algorithm is able to maintain a continuously updated beachline that reflects the current state of the Voronoi diagram.

RO

is a point event the same as a circle event?

In the context of the Fortune's algorithm, a point event and a circle event are often used interchangeably to refer to an event that is triggered when the sweep line intersects the circumcircle of a Voronoi edge.

Both point events and circle events involve the intersection of the sweep line with the circumcircle of a Voronoi edge, and both involve the removal of an arc from the beachline and the insertion of a new arc. However, there is a subtle difference between the two terms: a point event is triggered when the sweep line intersects the circumcircle at the event point, while a circle event is triggered when the sweep line intersects the circumcircle at the breakpoint.

In practice, the terms "point event" and "circle event" are often used interchangeably to refer to this type of event in the Fortune's algorithm. However, it's important to understand that they are slightly different, and that a point event occurs at the event point while a circle event occurs at the breakpoint.

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