

# Coordinate Descent and Coordinate Gradient Descent

It is generally believed that coordinate descent should perform better than first-order methods. We have our minimization problem  $\min_{\mathbf{x}} f(\mathbf{x})$  where  $f$  is convex.

We can use coordinate descent. This algorithm is also called 'coordinatewise minimization'.

let the initial value be  $\mathbf{x}^{(0)} \in \mathbb{R}^n$ , and repeat:  $x_i^{(k)} = \operatorname{argmin}_{x_i} f(x_1^{(k)}, \dots, x_{i-1}^{(k)}, x_i, x_{i+1}^{(k-1)}, \dots, x_n^{(k-1)})$ , for  $i = 1, \dots, n$  and  $k = 1, 2, 3, \dots$ . Essentially, we minimize  $f$  with respect to one element  $x_i$ , plug it back in  $f$ , and move to the next index.

Some guidelines for coordinate descent:

- The order of cycle through coordinates is arbitrary, we can use any permutation of  $\{1, \dots, n\}$ ;
- we can replace everywhere individual coordinates with blocks of coordinates (Just remember that this is possible if you see some such step in the future.)
- The "one-at-a-time" update scheme is critical, and "all-at-once" scheme does not necessarily converge;

Why is it used?

- Very simple and easy to implement.
- Careful implementations can achieve state-of-the-art solution.
- Scalable, e.g., don't need to keep full data in memory

## 1 Example: linear regression

Given  $\mathbf{y} \in \mathbb{R}^n$  and  $\mathbf{X}$  is  $n \times p$  with columns  $X_1, \dots, X_p$ , consider the linear regression problem:  $\min_{\boldsymbol{\beta}} \frac{1}{2} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2$ .

Minimizing over  $\beta_i$  with all  $\beta_j, j \neq i$  fixed:  $0 = \frac{\partial f(\boldsymbol{\beta})}{\partial \beta_i} = X_i^T (\mathbf{X}\boldsymbol{\beta} - \mathbf{y}) = X_i^T (X_i\beta_i + \mathbf{X}_{-i}\boldsymbol{\beta}_{-i} - \mathbf{y})$  where  $\mathbf{X}_{-i}$  and  $\boldsymbol{\beta}_{-i}$  are original matrix or vector with  $i$ -th column or element removed, respectively.

Thus

$$\beta_i = \frac{X_i^T (\mathbf{y} - \mathbf{X}_{-i}\boldsymbol{\beta}_{-i})}{X_i^T X_i}. \quad (1)$$

Like gradient descent, we will update  $\beta$  sequentially. At  $k + 1$ -th iteration, the update for  $\beta$  be  $\beta^{(k+1)}$ . Then according to the gradient descent algorithm  $\beta^{(k+1)} = \beta^{(k)} - L \frac{2}{n} \mathbf{X}^T (\mathbf{y} - \mathbf{X}\beta^{(k)}) = \beta^{(k)} - L \frac{2}{n} \mathbf{X}^T (\mathbf{r}^k)$ , where  $\mathbf{r}^k = \mathbf{y} - \mathbf{X}\beta^{(k)}$  is the residual error vector.

In case of coordinate descent,  $\beta_i^{k+1} = \frac{X_i^T (\mathbf{y} - \mathbf{X}_{-i} \beta_{-i}^{(k)} + \mathbf{X}_i \beta_i^{(k)})}{X_i^T X_i} = \beta_i^{(k)} + \frac{X_i^T \mathbf{r}^{(k)}}{X_i^T X_i}$ . Thus, the two updates look very similar. The only difference is in the step size. For gradient-descent, it is fixed at  $L/n$ . However, here, it is coordinate specific,  $1/X_i^T X_i$ .

Coordinate descent repeats this update for  $i = 1, 2, \dots, p$ , Note that the computational cost for one cycle of coordinate descent is  $O(np)$  where  $O(n)$  to compute  $X_i^T (\mathbf{y} - \mathbf{X}_{-i} \beta_{-i})$  for each update in a cycle (it is  $O(n)$  because we can precompute  $X_i^T X_i \beta_i$ ), which is the same as gradient descent.

```
x <- rnorm(1000)
z <- rnorm(1000)
w <- rnorm(1000)

y <- rnorm(1000, 2*x+3*z)

#We fit the model y=beta1*x+beta2*z+beta3*w + error
```

Apply coordinate descent in the above problem to estimate beta1, beta2 and beta3. Use (1) to get the updates of components of beta.

```
#Initialize
beta <- rep(0, 3)

#Design matrix
X = cbind(x, z, w)

for(itr in 1:10000){
  beta[1] <- t(X[,1])%*(y-X[,-1])%(t(X[,-1])%*X[,-1])
  beta[2] <- t(X[,2])%*(y-X[,-2])%(t(X[,-2])%*X[,-2])
  beta[3] <- t(X[,3])%*(y-X[,-3])%(t(X[,-3])%*X[,-3])
}

#Least square estimate
```

```

betals <- solve(crossprod(X))%*%crossprod(X, y)

#Compare with least square estimate
sum((beta-betals)^2)

```

Here is the quicker code along with a stopping rule.

```

#Quicker code
x <- rnorm(1000)
z <- rnorm(1000)
w <- rnorm(1000)

y <- rnorm(1000, 2*x+3*z)

beta <- rep(0, 3)
#Design matrix

X = cbind(x, z, w)
res <- y-X%*%beta
for(itr in 1:10000){
  beta0itr <- beta0
  beta0 <- beta
  beta[1] <- beta[1] + sum(X[,1]*res)/sum(X[,1]*X[,1])
  res <- res + beta0[1] * X[,1] - beta[1] * X[,1]

  beta0 <- beta
  beta[2] <- beta[2] + sum(X[,2]*res)/sum(X[,2]*X[,2])
  res <- res + beta0[2] * X[,2] - beta[2] * X[,2]

  beta0 <- beta
  beta[3] <- beta[3] + sum(X[,3]*res)/sum(X[,3]*X[,3])
  res <- res + beta0[3] * X[,3] - beta[3] * X[,3]

  #stopping rule
  if(sum(((beta0itr-beta)/beta0itr)^2)<1e-3){
    break;
  }
}

```

}

beta

## 2 Pathwise Coordinate Descent for Lasso

The objective function is given by,

$$\hat{\beta} = \arg \min_{\beta} \left( \frac{1}{n} \|\mathbf{y} - \mathbf{X}\beta\|_2^2 + \lambda \|\beta\|_1 \right),$$

where the part  $\lambda \|\beta\|_1$  is called the LASSO penalty [2]. Here  $\|\beta\|_1 = \sum_{j=1}^p |\beta_j|$  where  $\beta = (\beta_1, \dots, \beta_p)$ .

If we minimize above expression with respect to  $\beta_i$ , keeping everything else fixed, the minimizer would be  $\hat{\beta}_i$ ?

Start with an initial solution of  $\hat{\beta}$ . Define,  $\beta_i^* = \frac{X_i^T(\mathbf{y} - \mathbf{X}_{-i}\hat{\beta}_{-i})}{X_i^T X_i}$ , the coordinate descent update ‘like’ in the least square case.

$$\hat{\beta}_i = \begin{cases} \beta_i^* - \frac{n}{2}\lambda & \text{if } \beta_i^* > 0, \frac{n}{2}\lambda \leq |\beta_i^*| \\ \beta_i^* + \frac{n}{2}\lambda & \text{if } \beta_i^* < 0, \frac{n}{2}\lambda \leq |\beta_i^*| \\ 0 & \text{if } \frac{n}{2}\lambda > |\beta_i^*|, \end{cases}$$

This above estimate help us to get the following way to get ‘solution-path’ of LASSO. Specifically, [1] proposed this the pathwise-coordinate descent method for Lasso problem. The algorithm runs over two loops. We start with the maximum value of  $\lambda_{\max}$  for which the solution is zero. Outer Loop (pathwise strategy): • Compute the solution over a sequence  $\lambda_{\max} = \lambda_1 > \lambda_2 > \dots > \lambda_r$  of tuning parameter values • For tuning parameter value  $\lambda_k$ , initialize coordinate descent algorithm at the computed solution for  $\lambda_{k-1}$  for warm start.

## 3 Coordinate Gradient Descent

For a smooth function  $f$ , the iterations  $x_i^{(k)} = x_i^{(k-1)} - t_{ki} \nabla f(x_1^{(k)}, \dots, x_{i-1}^{(k)}, x_i, x_{i+1}^{(k-1)}, \dots, x_n^{(k-1)})$ ,  $i = 1, \dots, n$  for  $k = 1, 2, 3 \dots$  are called coordinate gradient descent.

## References

- [1] Jerome Friedman, Trevor Hastie, and Rob Tibshirani. Regularization paths for generalized linear models via coordinate descent. *Journal of statistical software*, 33(1):1, 2010.
- [2] R. Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society B*, 58:267–288, 1996.