Course: Numerical Solution of Ordinary Differential Equations

Module1: Numerical Solution of Ordinary Differential Equations

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1	Solution of first order ordinary differential equations Approximate Solution: Picard Iteration Method, Taylor Series method	1
2	Numerical Solution: Euler method; Algorithm; Example; analysis	1
3	Modified Euler Method: Algorithm; Example; analysis	1
4	Runge Kutta methods, Second Order methods	1
5	Fourth Order Runge Kutta methods	1
6	Higher Order Runge Kutta methods	1

References:

Bradie B A **Friendly Introduction to Numerical Anaysis** Pearson Education,2007

Burden RL, Faires J D **Numerical Analysis** Cengage Learning, 2007

Chapra SC, Canale, R P **Numerical Methods for Engineers** Tata McGraw Hill, 2003

Gerald C.F., Wheatley P O **Applied Numerical analysis, Addison Wesley,** 1998

Module1: Numerical Solution of Ordinary Differential Equations

Lecture 1

Numerical solution of first order ordinary differential equations

Keywords: Initial Value Problem, Approximate solution, Picard method, Taylor series

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Solution of first order ordinary differential equations

Consider y(t) to be a function of a variable t. A **first order Ordinary differential equation** is an equation relating y, t and its first order derivatives. The most general form is :

$$F(t, y(t), y'(t)) = 0$$

The variable y is known as a dependent variable and t is independent variable. The equation is of first order as it is the order of highest derivative present in the equation. Sometimes it is possible to rewrite the equation in the form

$$y'(t) = f(t, y(t)) \qquad (1.1)$$

$y = g(t)$ is a **solution** of the first order differential equation (1.1) means

i) y(t) is differentiable

ii) Substitution of y(t) and y'(t) in (1.1) satisfies the differential equation identically

The differential equations are commonly obtained as mathematical representations of many real world problems. Then the solution of the underlying problem lies in the solution of differential equation. Finding solution of the differential equation is then critical to that real world problem.

Examples of first order differential equations are:

$$y' + 2y = 0$$
$$y' + \sin y = \exp(t)$$

The first of these equations represents the exponential decay of radioactive material where y represent the amount of material at any given time and k=2 is the rate of decay.

It may be noted that $y(t) = c\exp(-2t)$ is the solution of differential equation as it identically satisfies the given differential equation for arbitrarily chosen constant c. This

means that the differential equation has infinitely many solutions for different choices of c. In other words, the real world problem has infinitely many solutions which we know is not true. In fact, an initial condition should be specified for finding the unique solution of the problem:

$$y(0) = A$$

That is, the amount of radioactive material present at time t=0 is A. When this initial condition is imposed on the solution, the constant c is evaluated as A and the solution $y(t) = A\exp(-2t)$ is now unique. The expression can now be used for computing the amount of material at any given time.

The solution with arbitrary constant is known as the **general solution** of the differential equation. The solution obtained using the initial condition is a **particular solution**.

A first order **Initial Value Problem** (IVP) is defined as a first order differential equation together with specified initial condition at $t=t_0$:

$$y' = f(t,y); \quad t_0 \le t \le b \quad \text{with} \quad y(t_0) = y_0 \tag{1.2}$$

There exist several methods for finding solutions of differential equations. However, all differential equations are not solvable. The following well known theorem from theory of differential equations establishes the **existence and uniqueness** of solution of the IVP:

Let f(t,y(t)) be continuous in a domain $D=\{ (t,y(t)): t_0 \le t \le b,\ c \le y \le d \} \subset R^2$. *If f satisfies Lipschitz condition on the Variable y and* (t_0,y_0) *in D, then IVP has a unique solution* $y=y(t)$ *on the some interval* $t_0 \le t \le t_0+\delta$.

{The function f satisfies Lipschitz condition means that there exists a positive constant L such that $|f(t,y)-f(t,w)| < L|y-w|$}

The theorem gives conditions on function f(t, y) for existence and uniqueness of the solution. But the solution has to be obtained by available methods. It may not be possible to obtain analytical solution (in closed form) of a given first order differential equation by known methods even when the above theorem guarantees its existence.

Sometimes it is very difficult to obtain the solution. In such cases, the approximate solution of given differential equation can be obtained.

Approximate Solution

The classical methods for approximate solution of an IVP are:

i) Picard Iteration method

ii) Taylor Series method

Picard Iteration Method:

Picard method is an iterative method. An iterative method gives a sequence of approximations $y_1(x)$, $y_2(x)$, ...,$y_k(x)$,...to the solution of differential equations such that the nth approximation is obtained from one or more prevoius approximations.

The integration of differential equation (1.2) yields

$$y(t) = y_0 + \int_{t_0}^{t} f(x, y(x))dx$$

Note that the exact solution of IVP is obtained for $t=t_0$

For approximate solution, the exact solution y(x) is approximated by y_0 in the integrand to get

$$y(t) \cong y_1(x) = y_0 + \int_{t_0}^{t} f(x, y_0)dx$$

The approximation can be improved as

$$y_2(x) = y_0 + \int_{t_0}^{t} f(x, y_1)dx$$

A sequence of approximations $y_1(x)$, $y_2(x)$, ...,$y_k(x)$,...can be obtained as

$$y_{k+1}(x) = y_0 + \int_{t_0}^{t} f(x, y_k)dx \; ; k = 0,1,2,... \qquad (1.3)$$

From the theory of differential equations, it can be proved that the above sequence of approximations converges to the exact solution of IVP.

Example 1.1: Obtain the approximate solution of IVP using Picard method. Obtain its exact solution also

$$y' = 1 + ty; \quad y(0) = 1$$

Solution: Given that y_0=1. Using (1.3) gives

$$y_{k+1}(t) = 1 + \int_0^t f(x, y_k)dx \quad f(x, y_k) = 1 + xy_k$$

$$y_{k+1}(t) = 1 + \int_0^t (1 + xy_k)dx = 1 + t + \int_0^t (xy_k)dx$$

Simplification yields the sequence of approximations as

$$y_1(t) = 1 + t + t^2/2$$

$$y_2(t) = 1 + t + \int_0^t x(1 + x + x^2/2)dx = 1 + t + t^2/2 + t^3/3 + t^4/8$$

$$y_3(t) = 1 + t + \int_0^t x(1 + x + x^2/2 + x^3/3 + x^4/8)dx \quad \text{and so on.}$$

$$= 1 + t + t^2/2 + t^3/3 + t^4/8 + t^5/15 + t^6/48$$

The differential equation in example 1.1 is a linear first order equation. Its exact solution can be obtained as

$$y(t) = \exp(x^2/2)[1 + \int_0^t \exp(-x^2)dx]$$

The closed form solution of differential equation in this case is possible. But the expression involving an integral is difficult to analyze. The sequence of polynomials as obtained by Picard method gives only approximate solution, but for many practical problems this form of solution is preferred.

Taylor Series method:

The IVP gives the solution y_0 at initial point $t=t_0$. for given step size h, the solution at $t=t_0+h$ can be computed from Taylor series as

$$y(t_1) = y(t_0 + h) = y(t_0) + h y'(t_0) + \frac{h^2}{2} y''(t_0) + \frac{h^3}{6} y''(t_0) + \ldots \qquad (1.4)$$

From the differential equation, it is observed that

$$y'(t_0) = f(t_0, y(t_0)) = f(t_0, y_0)$$

Repeated differentiation gives $y''(t_0), y'''(t_0), \ldots$ as

$$y''(t_0) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} y'\right]_{t=t_0}$$

$$y'''(t_0) = \left[\frac{\partial^2 f}{\partial t^2} + 2\frac{\partial^2 f}{\partial t \partial y} y' + (\frac{\partial f}{\partial y} y')^2\right]_{t=t_0} \text{ and so on}$$

Substituting these derivatives and truncating the series (1.4) gives the approximate solution at t_1.

Example 1.2: Obtain the approximate solution y(t) of IVP using Taylor series method. Obtain approximate solution at t= 0.1 correct to 4 places of decimal.

$$y' = 1 + ty; \quad y(0) = 1$$

Solution: Given that $y' = 1 + ty = f(t, y)$

Repeated differentiations yield

$$y'' = y + ty'$$
$$y''' = 2y' + ty''$$
$$y^{iv} = (2y' + 1)y'' + xy'y''' \text{ and so on}$$

Or $\quad y(0) = 1,\ y'(0) = 1,\ y''(0) = 1,\ y'''(0) = 2,\ y^{iv}(0) = 3, \ldots$

Substitution in (1.4) with $t_0=0$ and h=t gives

$$y(t) = y(0 + t) = 1 + t + t^2/2 + t^3/3 + t^4/8 + \ldots$$

Taking t=0.1 and substitution in the above series gives

$$y(0.1) = 1 + 0.1 + 0.01/2 + 0.001/3 + 0.0001/8 + \ldots.$$

Or $\quad y(0.1) = 1 + 0.1 + 0.005 + 0.00033 + 0.0000125 + ..$

It may be noted that fifth term and subsequent terms are smaller than the accuracy requirement, the Taylor series can be truncated beyond fourth term. Accordingly y(0.1)=1.1.53.

Observe that the Picard method involves integration while Taylor series method involves differentiation of the function f. Depending on the ease of operation, one can select the appropriate method for finding the approximate solution. The number of iterations in Picard method depends upon the accuracy requirement. The step size h can be chosen sufficiently small to meet the accuracy requirement in case of Taylor series method. For fixed h, more terms have to be included in the solution when more accuracy is desired.

In the category of methods that include Picard method and Taylor series method, the approximate solution is given in the form of a mathematical expression.