Modeling Transformation

BOOK:

1.COMPUTER GRAPHICS (SCHAUMS SERIES)-2ND EDITION.[CHAPTER 4 & 6]

2. BOOK: COMPUTER GRAPHICS: PRINCIPLES AND PRACTICE- 2ND EDITION, FOLEY, VAN DAM, FEINER, HUGHE. [CHAPTER 5]

PREPARED BY ANIKA BINTE ISLAM

What is Transformation?

- An operation that changes one configuration into another.
- For images, shapes, etc.
  - A geometric transformation maps positions that define the object to other positions
  - Linear transformation means the transformation is defined by a linear function... which is what matrices are good
- A function that maps points x to points x':

Modeling Transformation

- When we want to place the object into a scene, we need to transform the object coordinates that we used to define the object into the world coordinate system that we are using for the scene. The transformation that we need is called a modeling transformation.
- Two types:
  - Geometric Transformation: The object itself is transformed relative to stationary coordinate system or background.
  - Coordinate Transformation: The object is held stationary while the coordinate system is transformed relative to the object.

Homogeneous Coordinate

- Translation, scaling and rotation are expressed (non-homogeneously) as:

  Translation: $P' = P + T$

  Scale: $P' = S \cdot P$

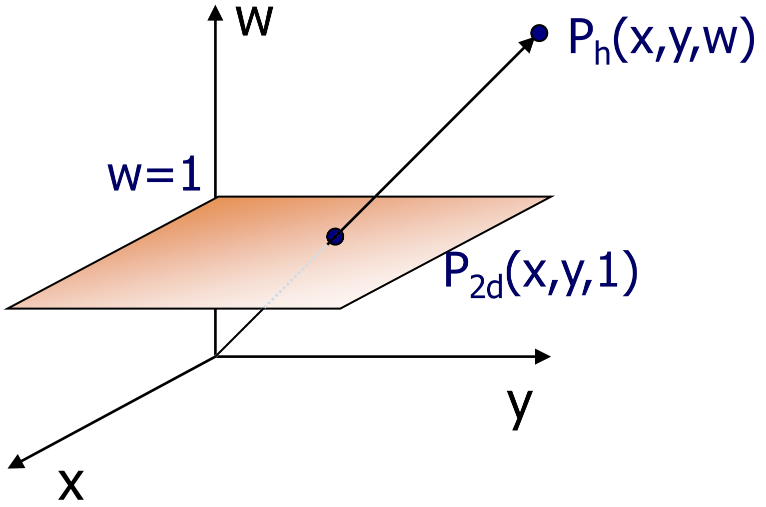
  Rotate: $P' = R \cdot P$

- Composition is difficult to express, since translation not expressed as a matrix multiplication
- Homogeneous coordinates allow all three to be expressed homogeneously, using multiplication by 3 ´ 3 matrices
- W is 1 for affine transformations in graphics

Homogeneous Coordinate

- Add an extra dimension
  - in 2D, we use 3 x 3 matrices
  - in 3D, we use 4 x 4 matrices
- Each point has an extra value, w. Most of the time w=1, and we can ignore it.

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & \mathrm{b} & \mathrm{c} & d \\ \mathrm{e} & \mathrm{f} & \mathrm{g} & \mathrm{h} \\ \mathrm{i} & \mathrm{j} & \mathrm{k} & l \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Geometric Transformation

2D Translation

$x' = x + t_x$

$y' = y + t_y$

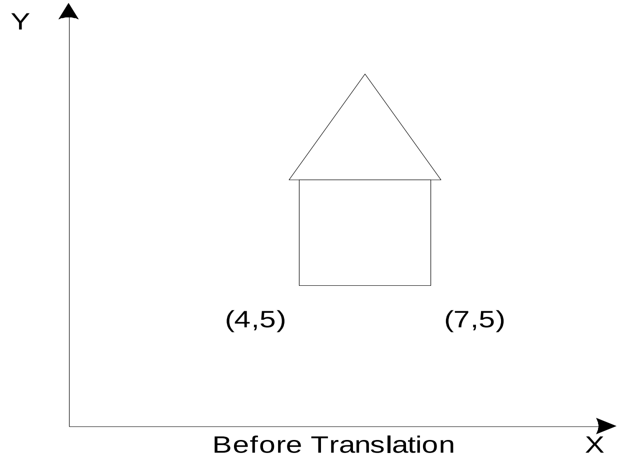
$P = \begin{bmatrix} x \\ y \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

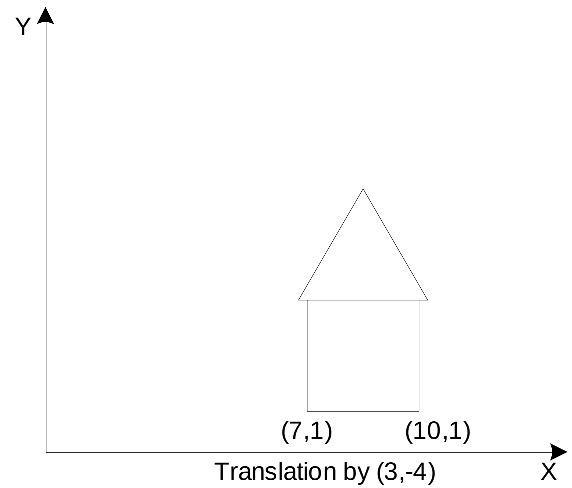
$P' = P + T$

Homogeneous Form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translation Matrix, $T_{(tx,\, ty)}$





3D Translation

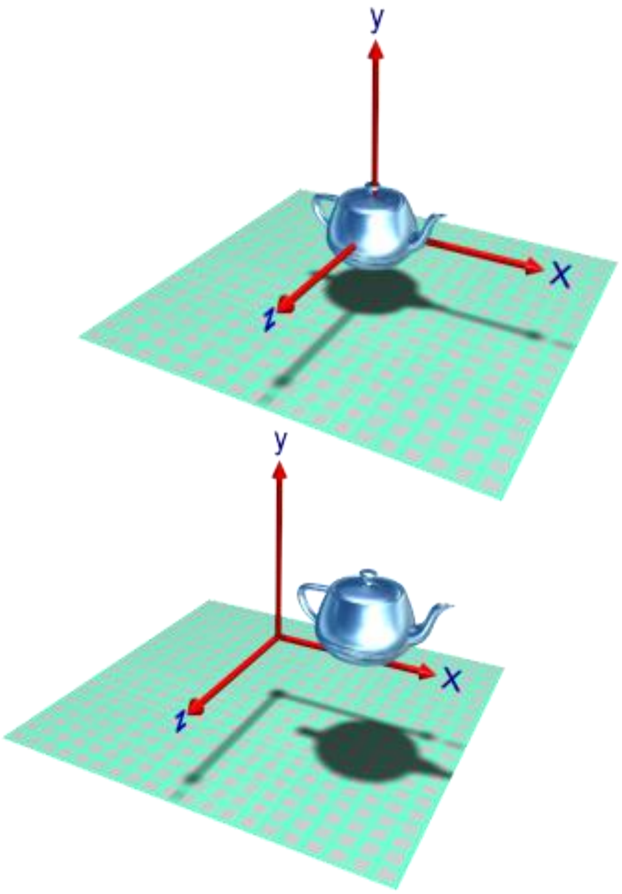
$x' = x + t_x \qquad y' = y + t_y \qquad z' = z + t_z$

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad T = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Homogeneous Form:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Translation Matrix, $T_{(tx,\, ty,\, tz)}$



2D Scaling w.r.t origin

$x' = x * s_x \qquad y' = y * s_y$

$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad S = \begin{bmatrix} s_x \\ s_y \end{bmatrix}$

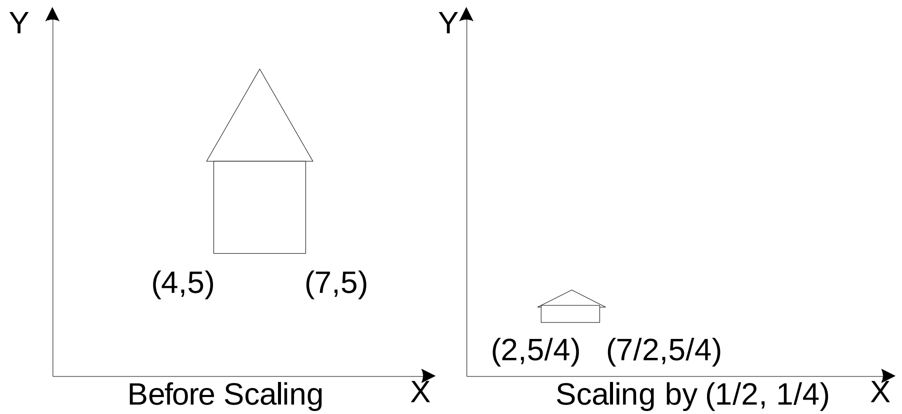
$P' = P * S$

Homogeneous Form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$\downarrow$

Scaling Matrix, $S_{(s_x\,,\,s_x)}$



s_x & s_y <1 means reduction

s_x & sy >1 means magnification

$s_x = s_y$ means uniform scaling

s_x=! s_y means differential scaling

s_x and s_y is the scaling factor along x and y axis respectively

3D Scaling w.r.t origin

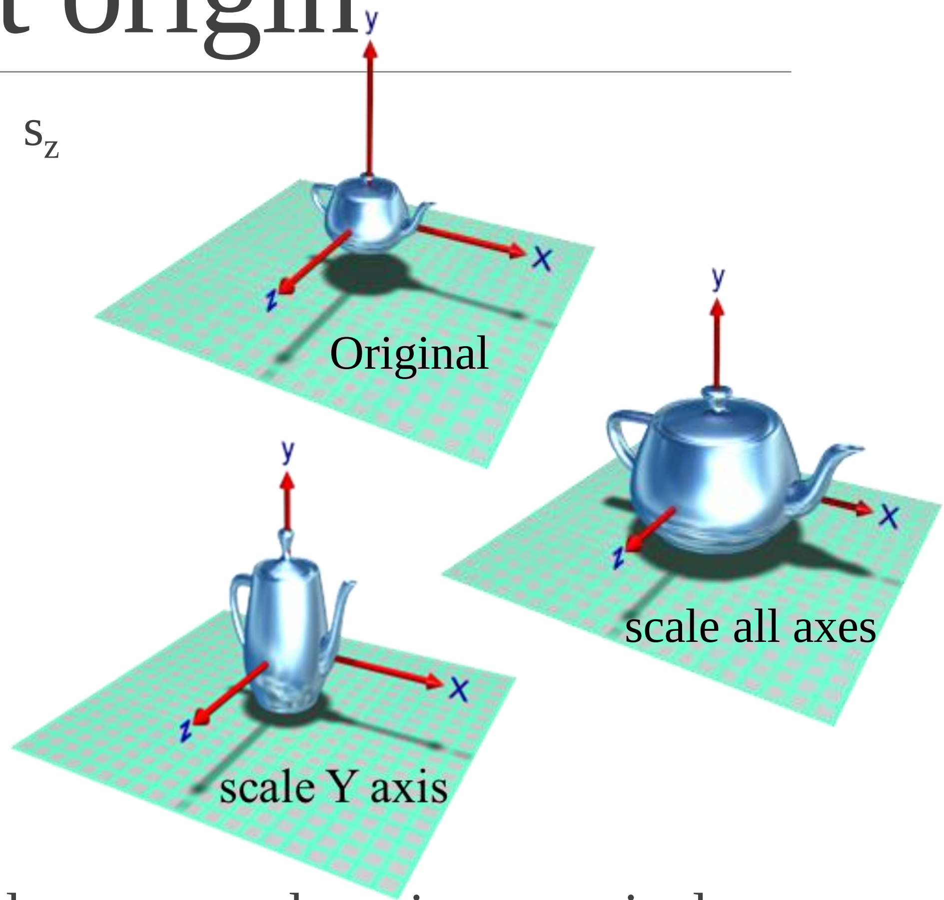
$x' = x * s_x \qquad y' = y * \ s_y \qquad z' = z * \ s_z$

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \qquad S = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}$$

Homogeneous Form:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ \sigma & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Scaling Matrix, $S_{(\ s_x\ ,\ s_x\ ,\ s_z)}$



s_x , s_y and s_z is the scaling factor along x, y and z axis respectively

Rotation about an origin (2D)

x= r cosα y= r sinα

x′=r cos (α+ θ) y′=r sin (α+ θ)

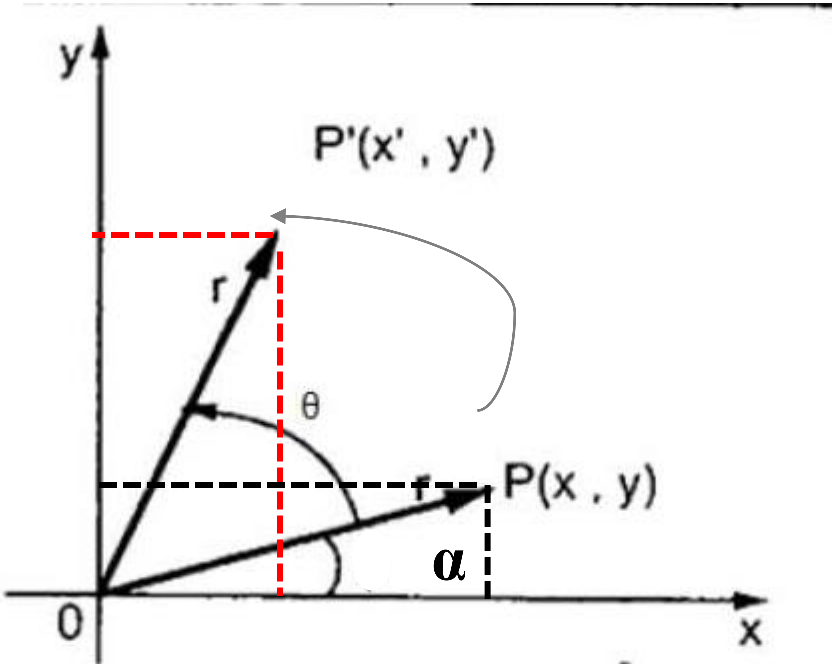
After expanding cos (α+ θ) and

sin (α+ θ), we get,

$x' = x \cos\theta - y \sin\theta \quad y' = x \sin\theta + y \cos\theta$

Homogeneous Form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Rotation matrix, } R_\theta} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Note: θ positive->counter clockwise
θ negative-> clockwise

3D Rotation

For 3D rotation 2 parameters are needed:

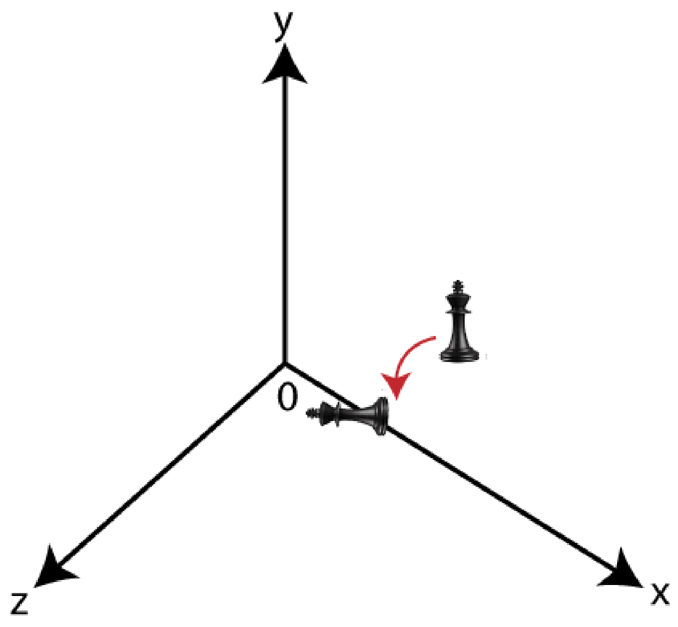
- Angle of rotation
- Axis of rotation

Rotation about X-axis

$$R\begin{cases} y' = ycos\theta - zsin\theta \\ z' = ysin\theta + zcos\theta \\ x' = x \end{cases}$$

$P' = R_{\theta,i} * P$

$$\backslash \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

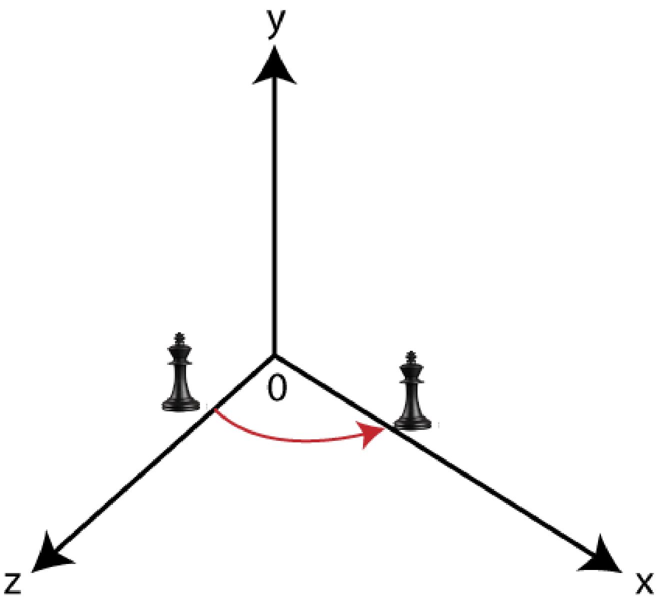


Rotation about Y-axis

$$\mathrm{R}\begin{cases} z' = zcos\theta - xsin\theta \\ x' = zsin\theta + xcos\theta \\ \quad\quad y' = y \end{cases}$$

P′= $R_{\theta,j}$ * P

$$\backslash\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} cos\,\theta & 0 & sin\,\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

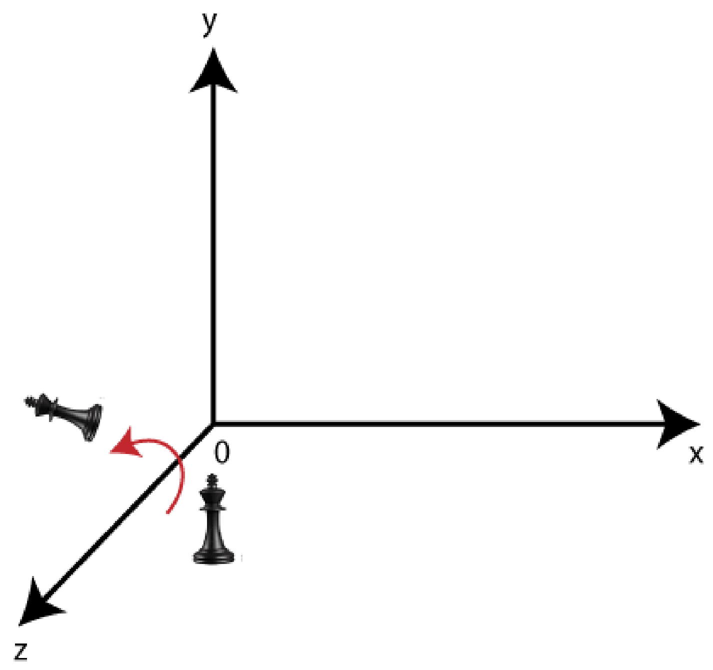


Rotation about Z-axis

$$R\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \\ z' = z \end{cases}$$

$P' = R_{\theta,k} * P$

$$\backslash \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

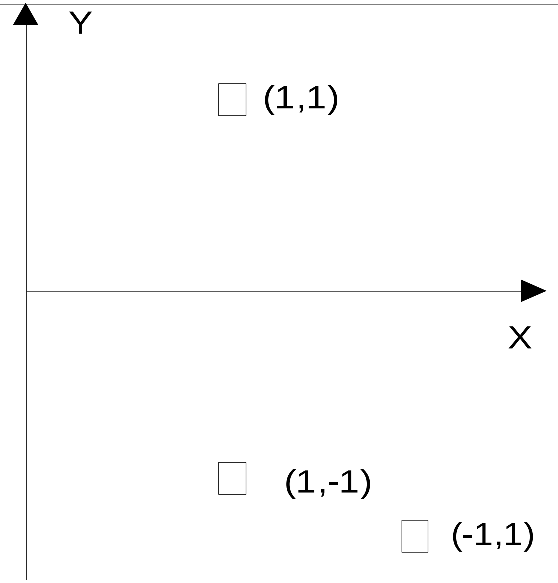


Mirror Reflection

Reflection about X-axis:

$x' = x \quad y' = -y$

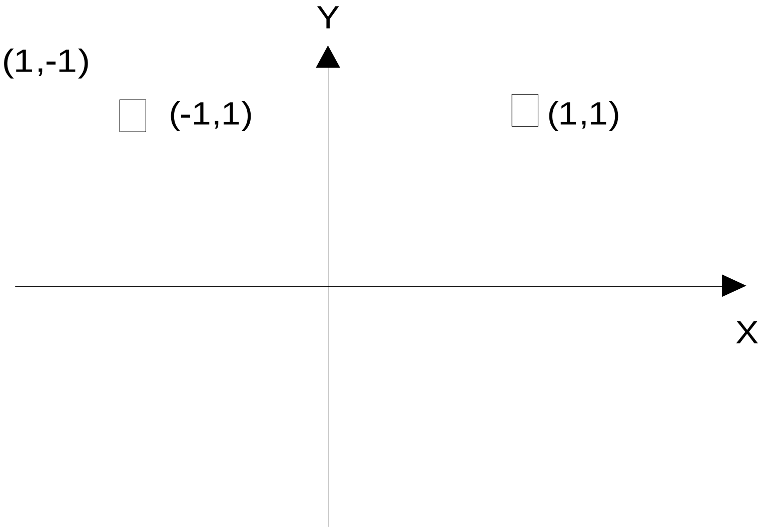
$$M_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Reflection about Y-axis:

$x' = -x \quad y' = -y$

$$M_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Shearing Transformation

$$Sh_x = \begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Sh_y = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Sh_{xy} = \begin{bmatrix} 1 & sh_x & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Shx>0 :diagram moves to right side

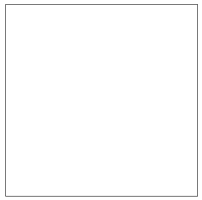
Shx<0 :diagram moves to left side

Shx=0 :No change

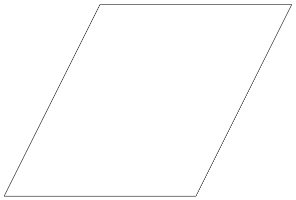
Shy>0 :diagram moves to above

Shy<0 :diagram moves to low

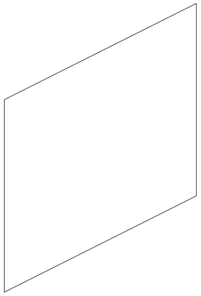
Shy=0 :No change



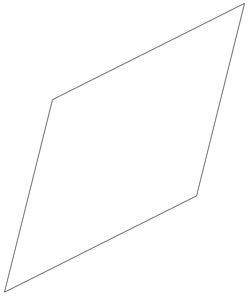
unit cube



Sheared in X direction



Sheared in Y direction



Sheared in both X and Y direction

Inverse Transformation

- Translation: Negate translation factor that is tx and ty.
- Rotation: Transpose the rotation matrix.
- Scale: Invert diagonal
- Mirror Reflection:

Translation : $T^{-1}_{(tx,ty)} = T_{(-tx,-ty)}$

Rotation : $R^{-1}_{(\theta)} = R_{(-\theta)} = R^{T}_{(\theta)}$

Sclaing : $S^{-1}_{(sx,sy)} = S_{(1/sx,\,1/sy)}$

Mirror Ref : $M^{-1}_{x} = M_{x}$

$$M^{-1}_{y} = M_{y}$$

Coordinate Transformation

Translation

Point fixed. Only translate the axis.

$x' = x - t_x$

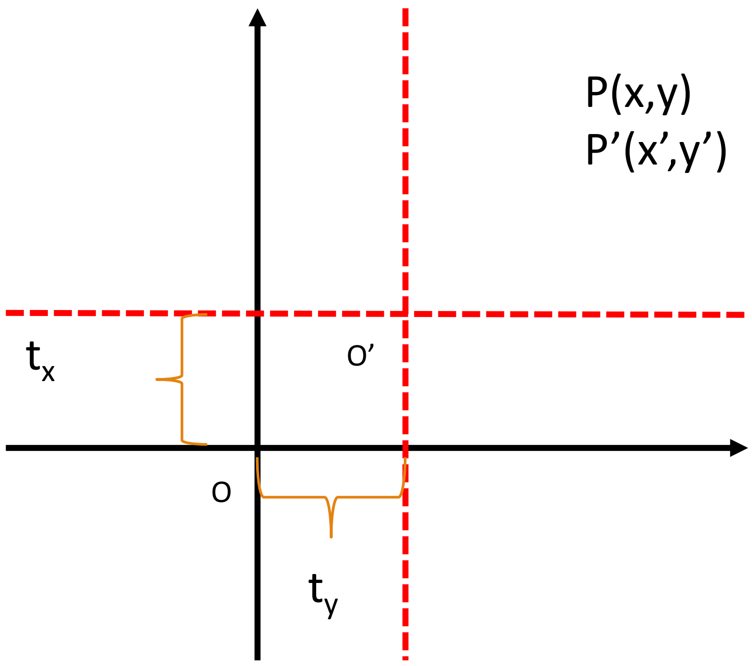
$y' = y - t_y$

$P = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ $\quad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

$P' = P + T$

Homogeneous Form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Rotation

x= r cosφ y= r sinφ

x′=r cos (φ-θ)

y′=r sin (φ-θ)

After expanding cos (φ-θ) and

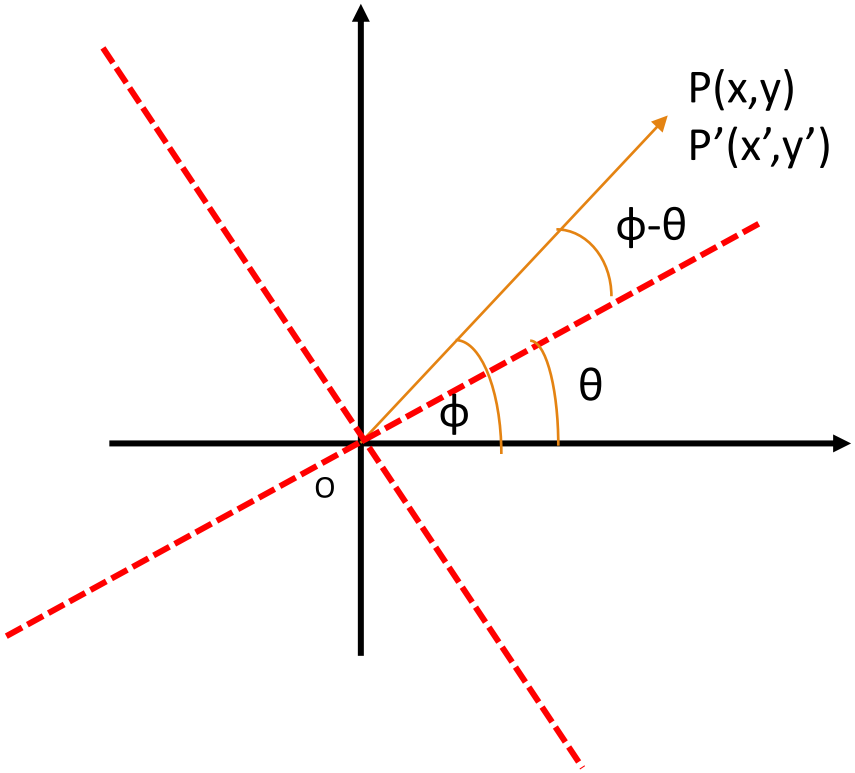
sin (φ-θ). we get,

x′ = xcosθ + y sinθ

y′ = -x sinθ + y cosθ

Homogeneous Form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



Scaling w.r.t origin

Coordinate will be unchanged. Only unit measurement will be changed along x and y axis.

$x' = x * 1/s_x$ $y' = y * 1/s_y$

$P' = P * S$

Homogeneous Form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1/sx & 0 & 0 \\ 0 & 1/sy & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Composite Transformation

Composite Transformation

- A number of transformations or sequence of transformations can be combined into single one called as composition. The resulting matrix is called as composite matrix. The process of combining is called as concatenation.

- Suppose we want to perform rotation about an arbitrary point, then we can perform it by the sequence of three transformations
  - Translation
  - Rotation
  - Inverse Translation

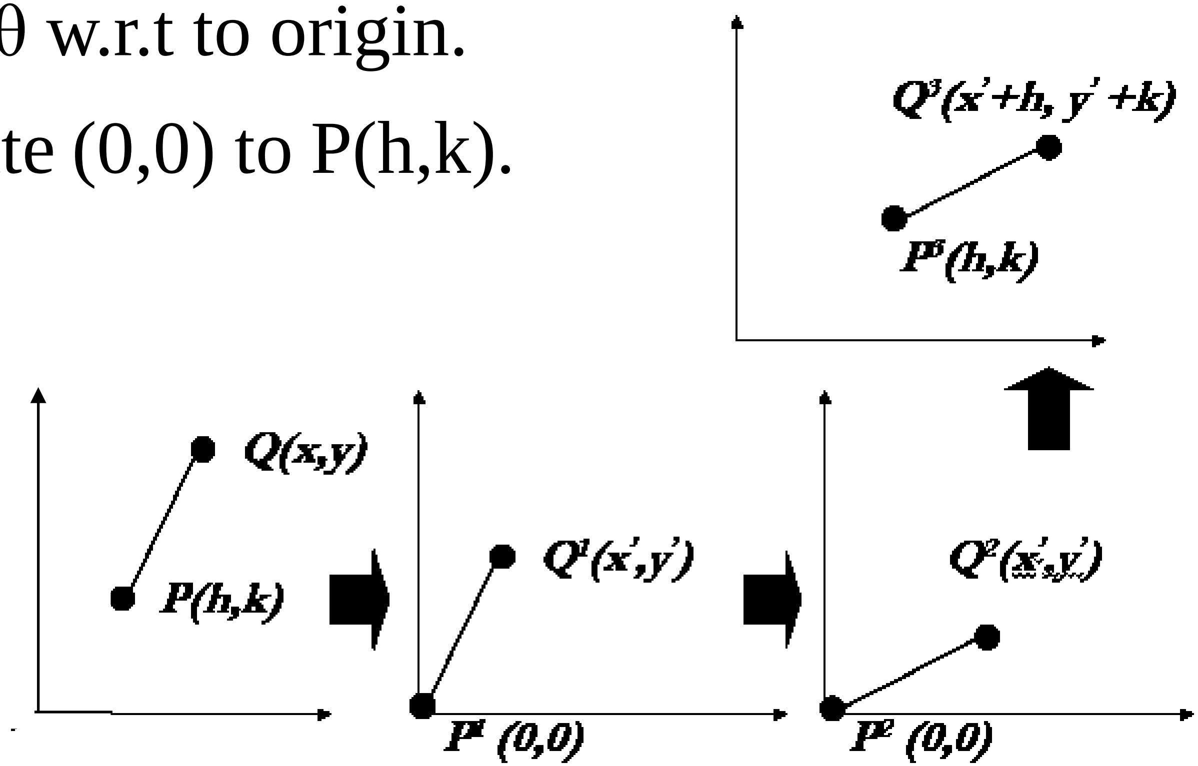
- The ordering sequence of these numbers of transformations must not be changed.

- If a matrix is represented in column form, then the composite transformation is performed by multiplying matrix in order from right to left side. Otherwise left to right.

Matrix multiplication is NOT commutative!

Rotation of θ about a fixed point P(h,k)

1. Translate P(h,k) to origin.
2. Rotate θ w.r.t to origin.
3. Translate (0,0) to P(h,k).



Rotation of θ about a fixed point P(h,k)

$\mathbf{R}_{\theta,P} = \mathbf{T(h,k) * R_{\theta} * T(-h,-k)}$

$$=\begin{bmatrix}1 & 0 & h\\0 & 1 & k\\0 & 0 & 1\end{bmatrix} * \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} * \begin{bmatrix}1 & 0 & -h\\0 & 1 & -k\\0 & 0 & 1\end{bmatrix}$$

……………………………………………………………

$$=\begin{bmatrix}\cos\theta & -\sin\theta & \mathrm{h}(1-\cos\theta)+\mathrm{k}\sin\theta\\ \sin\theta & \cos\theta & k(1-cos\theta)-hsin\theta\\ 0 & 0 & 1\end{bmatrix}$$

Scaling about a fixed point P(h,k)

1. Translate P(h,k) to origin.
2. Scaling w.r.t to origin.
3. Translate (0,0) to P(h,k).

$S_{sx,sy,P}$= T(h,k) * $S_{s,s}$ * T(-h,-k)

$$=\begin{bmatrix}1 & 0 & h\\0 & 1 & k\\0 & 0 & 1\end{bmatrix} * \begin{bmatrix}s_x & 0 & 0\\0 & s_y & 0\\0 & 0 & 1\end{bmatrix} * \begin{bmatrix}1 & 0 & -h\\0 & 1 & -k\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}s_x & 0 & \mathrm{h}(1-\mathrm{sx})\\0 & s_y & \mathrm{k}(1-\mathrm{sy})\\0 & 0 & 1\end{bmatrix}$$

Similar Problem:

An object has a point P (h,k). Magnify the object & keep P(h,k) at the same position.

Self Study

- 2D: Reflection about line L, M_L
- 3D: Rotation about a line parallel to an axis
- 3D: Rotation about an arbitrary axis

Practice Problem

- Book: Computer Graphics (Schaums Series)-2nd edition.

Solved Problem: 4.2,4.3,4.4,4.5, 4.6, 4.7 4.8, 4.9 4.10. 4.11-4.14, 6.1

- Book: Computer Graphics: Principles and Practice- 2nd Edition, Foley, van Dam, Feiner, Hughes

Almost at