Picard's Method for D.E.'s

The method of successive approximations uses the equivalent integral equation for (1) and an iterative method for constructing approximations to the solution. This is a traditional way to prove (1) and appears in most all differential equations textbooks. It is attributed to the French mathematician Charles Emile Picard (1856-1941).

Theorem 2 (Successive Approximations - Picard Iteration). The solution to the I.V.P in (1) is found by constructing recursively a sequence $\{Y_n(x)\}_{n=0}^{\infty}$ of functions

$Y_0(x) = y_0$, and

(2)

$$Y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, Y_n(t))\,dt \quad \text{for } n \geq 0.$$

Then the solution $y(x)$ to (1) is given by the limit:

(3) $$y(x) = \lim_{n\to\infty} Y_n(x).$$

Proof.

Begin by reformulating (1) as an equivalent integral equation. Integration of both sides of (1) yields

(4) $$\int_{x_0}^{x} y'(t)\,dt = \int_{x_0}^{x} f(t, y(t))\,dt.$$

Applying the Fundamental Theorem of Calculus to the left side of (4) yields $\int_{x_0}^{x} y'(t)\,dt = y(x) - y(x_0) = y(x) - y_0$ and we have $y(x) - y_0 = \int_{x_0}^{x} f(t, y(t))\,dt$, which can be rearranged to obtain

(5) $$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

Observe that $y(x)$ occurs on both the left and right hand sides of equation (5). We can use this formula and input $y(t)$ in the integrand $f(t, y(t))$ on the right and then output the next iteration for $y(x)$ on the left side. This is a type of fixed point iteration, the most familiar form of which is Newton's method for root finding.

Start the iteration with the initial function

$$Y_0(t) = y_0,$$

then define the next function $Y_1(x)$ as follows

$$Y_1(x) = y_0 + \int_{x_0}^{x} f(t, Y_0(t))\,dt.$$

Next $Y_1(x)$ is used to construct $Y_2(x)$ as follows

$$Y_2(x) = y_0 + \int_{x_0}^{x} f(t, Y_1(t))\,dt.$$

The process is repeated, and once $Y_n(t)$ has been obtained, the next function is given recursively by

(6) $$Y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, Y_n(t))\,dt.$$

We must take the limit as $n \to \infty$ in (6). Assume that the limit (3) exists, then $\lim_{n\to\infty} Y_{n+1}(x) = y(x)$ and we write

$$y(x) = \lim_{n\to\infty} Y_{n+1}(x) = \lim_{n\to\infty}\left(y_0 + \int_0^x f(t, Y_n(t))\,dt\right)$$

If there is "no problem" when taking limits on the right side then we might expect the following

$$y(x) = \lim_{n\to\infty}\left(y_0 + \int_0^x f(t, Y_n(t))\,dt\right)$$

$$y(x) = y_0 + \lim_{n\to\infty} \int_0^x f(t, Y_n(t))\,dt$$

$$y(x) = y_0 + \int_0^x \lim_{n\to\infty} f(t, Y_n(t))\,dt$$

$$y(x) = y_0 + \int_0^x f\left(t, \lim_{n\to\infty} Y_n(t)\right)dt$$

$$y(x) = y_0 + \int_0^x f(t, y(t))\,dt$$

This is the "intuitive proof" of equation (5). Q.E.D.

For More Proof. More details for the existence and uniqueness of $y(x)$ can be found in textbooks and the literature.

1. Stephen W. Goode, Differential Equations and Linear Algebra, 3rd edition, Prentice Hall, Upper Saddle River, NJ, 2005, Appendix 4.
2. R. Kent Nagle; Edward B. Saff; and Arthur David Snider, Fundamentals of Differential Equations and Boundary Value Problems, 4th edition, Addison-Wesley, Boston, MA, 2004, Chapter 13, Sections 1 and 2.
3. C. Henry Edwards; and David E. Penney, Differential Equations and Boundary Value Problems: Computing and Modeling, 3rd edition, Prentice Hall, Upper Saddle River, NJ, 2005, Appendix A.1.
4. William E. Boyce; and Richard C. DiPrima, Elementary Differential Equations and Boundary Value Problems, 7th Edition, John Wiley and Sons, New York, NY, 2002, Chapter 2, Section 8.
5. Garrett Birkhoff; and Gian-Carlo Rota, Ordinary Differential Equations, 4rd Ed., John Wiley and Sons, New York, NY, 1989, Chapter 1, p. 23.
6. Einar Hille, Lectures on Ordinary Differential Equations, Addison-Wesley Pub. Co., Reading, MA, 1969, pp. 32-41.
7. Arthur Wouk, On the Cauchy-Picard Method, The American Mathematical Monthly, Vol. 70, No. 2. (Feb., 1963), pp. 158-162, Jstor.

Picard Iterative Process

Indeed, often it is very hard to solve differential equations, but we do have a numerical process that can approximate the solution. This process is known as the **Picard iterative process**.
First, consider the IVP

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 .$$

It is not hard to see that the solution to this problem is also given as a solution to (called the integral associated equation)

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds .$$

The Picard iterative process consists of constructing a sequence (y_n) of functions which will get closer and closer to the desired solution. This is how the process works:

(1) $y_0(x) = y_0$ for every x;

(2) then the recurrent formula holds

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt,$$

for $n \geq 1$.

Example: Find the approximated sequence (y_n), for the IVP

$$y' = 2x(1+y) \quad y(0) = 0$$

Solution: First let us write the associated integral equation

$$y(x) = \int_0^x 2s(1+y(s))ds .$$

Set $y_0(x) = 0$. Then for any $n \geq 1$, we have the recurrent formula

$$y_{n+1}(x) = \int_0^x 2s(1+y_n(s))ds .$$

We have $y_1(x) = \int_0^x 2s\,ds = x^2$, and

$$y_2 = \int_0^x 2s(1+s^2)ds = x^2 + \frac{x^4}{2} .$$

We leave it to the reader to show that

$$y_n(x) = x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \dots + \frac{x^{2n}}{n!} .$$

We recognize the Taylor polynomials of (which also get closer and closer to) the function