A Short Lecture on Classical Optimization

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Outline

Linear Programming

Consider the problem P.

$$
\begin{array}{llrcl}
\text{P:} & \text{maximize} & x_1 + x_2 & & \\
& \text{subject to} & x_1 + 2x_2 & \leq & 6 \\
& & x_1 - x_2 & \leq & 3 \\
& & x_1 \ , \ x_2 & \geq & 0
\end{array}
$$

- objective function
- constraints

Linear Programming

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Matrix/ Vector Notation

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$$\begin{array}{lllcl} \text{P:} & \text{maximize} & \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & & \\ & \text{subject to} & \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \leq & \begin{bmatrix} 6 \\ 3 \end{bmatrix} \\ & & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \geq & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}$$

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$$\begin{array}{llrcl} \text{P:} & \text{maximize} & f(x) & & \\ & \text{subject to} & g(x) & = & b \\ & & x & \in & X \end{array}$$

- $Ax \leq b$ will be the same as $Ax + z = b$
- z is the **slack variable**

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General Optimization

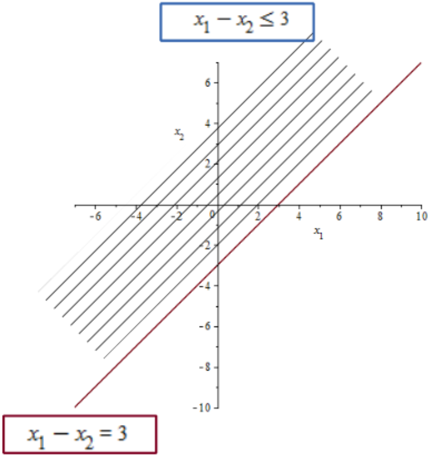
$$\begin{array}{llrcl} \text{P:} & \text{maximize} & f(x) & & \\ & \text{subject to} & g(x) & = & b \\ & & x & \in & X \end{array}$$

where

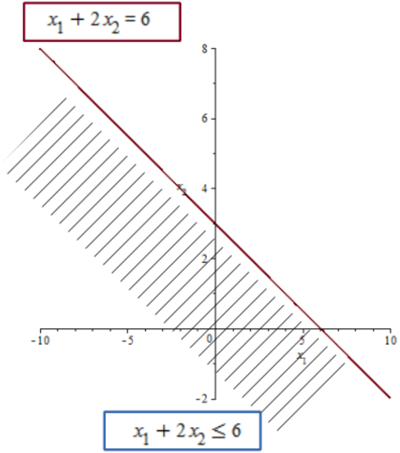
$$\begin{array}{rcll} x & \in & \mathbb{R}^n & (n \text{ decision variables}) \\ f:\mathbb{R}^n & \to & \mathbb{R} & (\text{objective function}) \\ X & \subseteq & \mathbb{R}^n & (\text{regional constraints}) \\ g:\mathbb{R}^n & \to & \mathbb{R}^m & (m \text{ functional constraints}) \\ b & \subseteq & \mathbb{R}^m & \end{array}$$

Note that maximizing $f(x)$ is the same as minimizing $-f(x)$

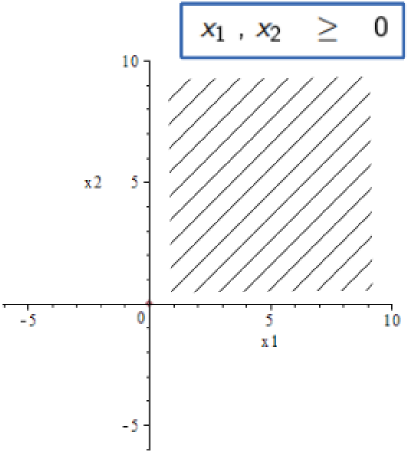
1st Functional Constraint



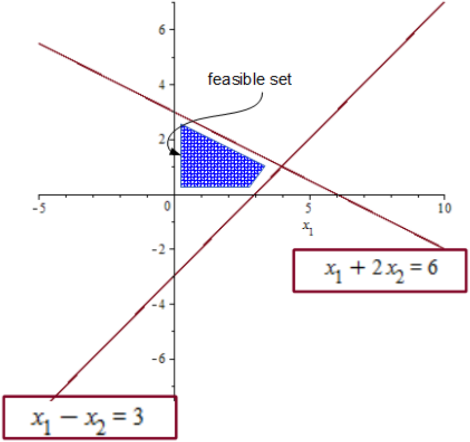
2nd Functional Constraint



Regional Constraint



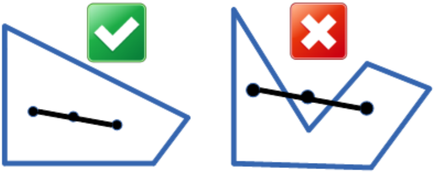
Feasible Set



Convex Set

Definition

1.1 For a convex set S, any line segment connecting two points in S must also lie in S.
Equivalently, a set $S \subseteq \mathbb{R}^n$ is a **convex set** if $\forall x, y \in S$ and $0 \leq \lambda \leq 1 \implies \lambda x + (1-\lambda) y \in S$



LP Feasible Set

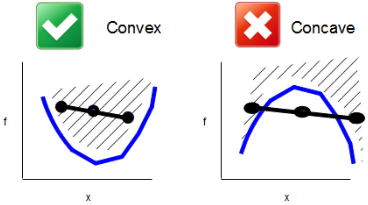
Theorem

1.1 The feasible set of an LP problem is convex

Convex Function

Definition

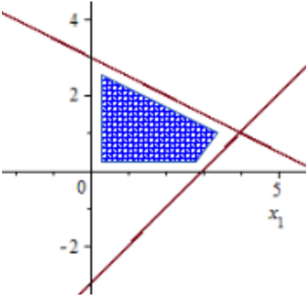
1.2 A function $f : S \to \mathbb{R}$ is a convex function if the set above its graph is a convex set.
Equivalently, $\forall 0 \leq \lambda \leq 1$, $\lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$



Convexity

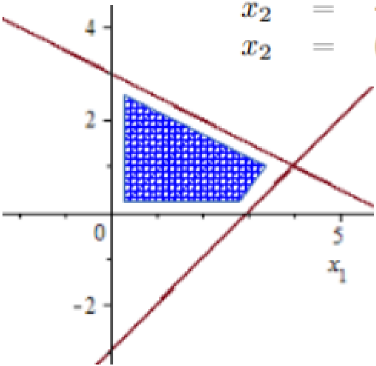
- In a general problem of minimizing f over a general S , there may be a lot of local minima
- It may be difficult to find the global minimum
- However, **if S is a convex set and f is a convex function in S, then any local minima found is the global minimum**
- **if S is convex and f is a concave function in S, then any local maxima found is the global maximum**
- Linear functions (e.g. those in LP) are both concave and convex at the same time. Any local optima found are also global.

Feasible Set



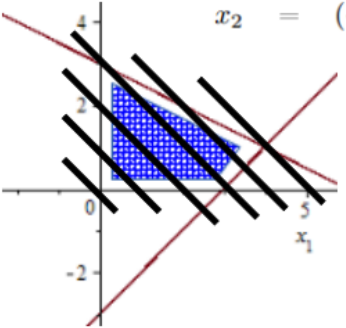
Objective Function

$$\begin{aligned} &x_1 + x_2 \\ x_1 + x_2 &= b \\ x_2 &= -x_1 + b \\ x_2 &= (-1)x_1 + b \end{aligned}$$



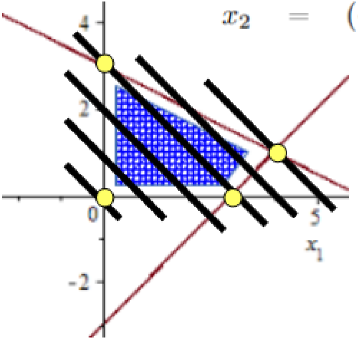
Family of Curves

$$\begin{aligned} & x_1 + x_2 \\ x_1 + x_2 &= b \\ x_2 &= -x_1 + b \\ x_2 &= (-1)x_1 + b \end{aligned}$$



Extreme Points

$$\begin{aligned} x_1 + x_2 & \\ x_1 + x_2 &= b \\ x_2 &= -x_1 + b \\ x_2 &= (-1)x_1 + b \end{aligned}$$



Writing out the Lagrangian

From our previous example,

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& & x_1 - x_2 & \leq & 3 \\
& & x_1 \;,\; x_2 & \geq & 0
\end{array}
$$

The Lagrangian will be ,

$$L(x, \lambda) = x_1 + x_2 - \lambda_1 \left(x_1 + 2x_2 + z_1 - 6\right) - \lambda_2 \left(x_1 - x_2 + z_2 - 3\right)$$

With the constraint that $x, z \geq 0$
where z is the slack variable and λ is the Lagrange multiplier

Other Representations

$$L(x,\lambda) = x_1 + x_2 - \lambda_1 (x_1 + 2x_2 + z_1 - 6) - \lambda_2 (x_1 - x_2 + z_2 - 3)$$

$$L(x,\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right)$$

$$L(x,\lambda) = c^T x - \lambda^T (Ax + z - b)$$

$$L(x,\lambda) = f(x) - \lambda^T (g(x) + z - b)$$

where $x, z \geq 0$

The Lagrangian

For the general optimization problem
P: minimize $f(x)$ s.t. $g(x) = b$, $x \in X$
The Lagrangian is

$$L(x, \lambda) = f(x) - \lambda^T (g(x) - b)$$

with $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^m$, meaning there are n decision variables, m constraints and m Lagrange multipliers.

Lagrangian Sufficiency Theorem

Theorem

2.1 If x^* and λ^* exist such that x^* is feasible for P and

$$L(x^*, \lambda^*) \leq L(x, \lambda^*) \qquad \forall x \in X$$

then x^* is optimal for P.

Proof

Lagrangian Sufficiency Theorem

Proof.

Define

$$X_b = \{x : x \in X \, and \, g(x) = b\}$$

Note that $X_b \subseteq X$ and that for any $x \in X_b$ and any $\lambda \in \mathbb{R}^m$

$$L(x, \lambda) = f(x) - \lambda^T (g(x) - b) = f(x)$$

$$L(x, \lambda^*) = f(x) - \lambda^T (g(x) - b) = f(x)$$

Note also that for $x \in X_b$,

$$L(x^*, \lambda^*) = f(x^*)$$

Proof.

Since,

$$f(x^*) \leq f(x)$$

So,

$$f(x^*) = L(x^*, \lambda^*) \leq L(x, \lambda^*) = f(x)$$

Thus $x^*$is optimal for P

□

- This is useful as long as we can find λ^* , which is usually the case
- This helps us find x^* by limiting the values of x to consider

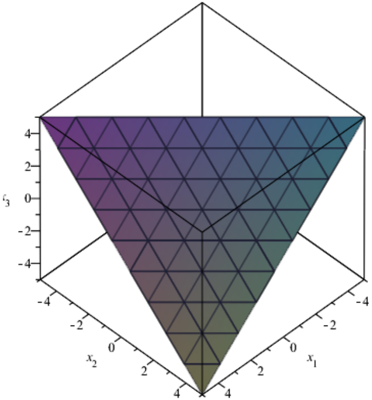
Example 1

Example

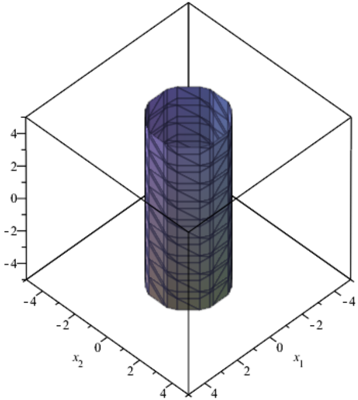
$$\begin{array}{rrcl} \text{minimize} & x_1 - x_2 - 2x_3 & & \\ \text{s.t.} & x_1 + x_2 + x_3 & = & 5 \\ & x_1^2 + x_2^2 & = & 4 \\ & x \in X & = & \mathbb{R}^3 \end{array}$$

A Look at the Constraints

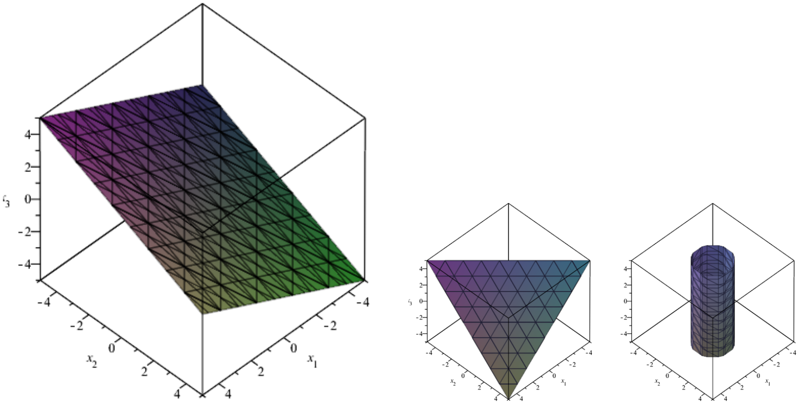
$x_1 + x_2 + x_3 = 5$



$x_1^2 + x_2^2 = 4$



The Objective Function



$$x_1 - x_2 - 2x_3$$

Step 1

Solution to Example 1

Setup $L(x,\lambda)$ and for each λ , minimize $L(x,\lambda)$ subject to $x \in X$

$$
\begin{aligned}
L(x,\lambda) &= f(x) - \lambda^T (g(x) - b) \\
&= (x_1 - x_2 - 2x_3) - \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \cdot \left(\begin{bmatrix} (x_1 + x_2 + x_3) \\ (x_1^2 + x_2^2) \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \end{bmatrix} \right) \\
&= (x_1 - x_2 - 2x_3) - \lambda_1 (x_1 + x_2 + x_3 - 5) - \lambda_2 (x_1^2 + x_2^2 - 4) \\
&= \left[x_1 (1 - \lambda_1) - \lambda_2 x_1^2 \right] + \left[x_2 (-1 - \lambda_1) - \lambda_2 x_2^2 \right] \\
&+ \left[-x_3 (2 + \lambda_1) \right] + 5\lambda_1 + 4\lambda_2
\end{aligned}
$$

- Minimize $L(x,\lambda)$ for a fixed λ in $x \in \mathbb{R}^3$. We can minimize with respect to our decision variables,

$$
\begin{aligned}
\partial L/\partial x_1 &= 1 - \lambda_1 - 2\lambda_2 x_1 &= 0 \implies x_1 &= \frac{1-\lambda_1}{2\lambda_2} \\
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$$

Step 2
Solution to Example 1

Define the set $Y = \left\{ \lambda : \min_{x \in X} L(x, \lambda) > -\infty \right\}$, i.e. λ for which the Lagrangian will have a finite minimum.

$$L(x,\lambda) = \left[x_1(1-\lambda_1) - \lambda_2 x_1^2\right] + \left[x_2(-1-\lambda_1) - \lambda_2 x_2^2\right] + \left[-x_3(2+\lambda_1)\right] + 5\lambda$$

- notice that $[-x_3(2+\lambda_1)]$ has a minimum of $-\infty$ unless $\lambda_1 = -2$. So we only consider $\lambda_1 = -2$.
- also, observe that $\left[x_1(1-\lambda_1) - \lambda_2 x_1^2\right] + \left[x_2(-1-\lambda_1) - \lambda_2 x_2^2\right]$ will only have a finite minimum if λ_2 is strictly < 0

$$Y = \{\lambda : \lambda_1 = -2,\ \lambda_2 < 0\}$$

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$$Y = \{\lambda : \lambda_1 = -2,\ \lambda_2 < 0\}$$

Step 3

Solution to Example 1

The minimum will be found at some $x(\lambda)$, for $\lambda \in Y$. We find $x(\lambda)$ that minimizes $L(x, \lambda)$.

$$Y = \{\lambda : \lambda_1 = -2,\ \lambda_2 < 0\}$$

$$\begin{aligned} x_1 &= \frac{1-\lambda_1}{2\lambda_2} &\implies x_1 &= 3/2\lambda_2 \\ x_2 &= \frac{-1-\lambda_1}{2\lambda_2} &\implies x_2 &= 1/2\lambda_2 \end{aligned}$$

$$x(\lambda) = (3/2\lambda_2, 1/2\lambda_2, x_3)^T$$

Step 3
Solution to Example 1

The minimum will be found at some $x(\lambda)$, for $\lambda \in Y$. We find $x(\lambda)$ that minimizes $L(x, \lambda)$.

$$Y = \{\lambda : \lambda_1 = -2,\ \lambda_2 < 0\}$$

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$$x(\lambda) = (3/2\lambda_2, 1/2\lambda_2, x_3)^T$$

Step 4

Solution to Example 1

Adjust or find λ values so that $x(\lambda)$ is feasible.

If $\lambda^* \in Y$ exists such that $x^* = x(\lambda^*)$is feasible, then x^* is optimal for P by the Lagrangian sufficiency theorem

$$Y = \{\lambda : \lambda_1 = -2, \lambda_2 < 0\}$$

$$x(\lambda) = \left(\frac{3}{2\lambda_2}, \frac{1}{2\lambda_2}, x_3\right)^T$$

To find a feasible $x(\lambda)$, we can use the second constraint.

$$x_1^2 + x_2^2 = 4 \implies \frac{9}{4\lambda_2^2} + \frac{1}{4\lambda_2^2} = 4 \implies \lambda_2 = \pm\sqrt{5/8}$$

Since $\lambda_2 < 0$, $\lambda_2 = -\sqrt{5/8}$

Step 4
Solution to Example 1

Adjust or find λ values so that $x(\lambda)$ is feasible.
If $\lambda^* \in Y$ exists such that $x^* = x(\lambda^*)$is feasible, then x^* is optimal for P by the Lagrangian sufficiency theorem

$$Y = \{\lambda : \lambda_1 = -2,\ \lambda_2 < 0\}$$

$$x(\lambda) = \left(\frac{3}{2\lambda_2}, \frac{1}{2\lambda_2}, x_3 \right)^T$$

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Since $\lambda_2 < 0$, $\lambda_2 = -\sqrt{5/8}$

Solution to Example 1

We can now solve for the values of x_1 and x_2

$$\begin{aligned} x_1 &= \frac{3}{2\lambda_2} &= -3\sqrt{\frac{2}{5}} &\approx \text{-1.90} \\ x_2 &= \frac{1}{2\lambda_2} &= -\sqrt{\frac{2}{5}} &\approx \text{-0.632} \end{aligned}$$

To find the value of x_3 we can use the first constraint

$$x_3 = 5 - x_1 - x_2 = 5 + 4\sqrt{2/5} \approx 7.53$$

$$x^* = \left(-3\sqrt{2/5},\ -\sqrt{2/5},\ 5 + 4\sqrt{2/5}\right)^T$$

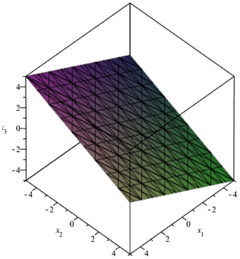
$$\lambda^* = \left(-2,\ -\sqrt{5/8}\right)^T$$

The conditions of the theorem are satisfied, so x^* is optimal. :)

A Review of the Plots

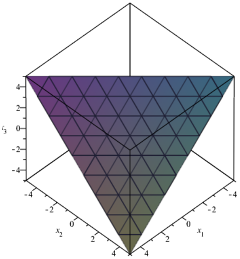
Objective Function

$$x_1 - x_2 - 2x_3$$



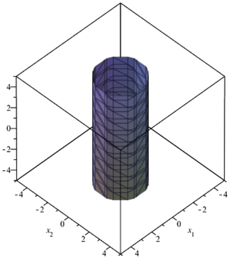
Constraint 1

$$x_1 + x_2 + x_3 = 5$$



Constraint 2

$$x_1^2 + x_2^2 = 4$$



Summary of Steps

1. Setup $L(x,\lambda)$ and for each λ , minimize $L(x,\lambda)$ subject to $x \in X$
2. Define the set $Y = \left\{\lambda : \min_{x \in X} L(x,\lambda) > -\infty\right\}$, i.e. λ for which the Lagrangian will have a finite minimum.
3. The minimum will be found at some $x(\lambda)$, for $\lambda \in Y$. We find $x(\lambda)$ that minimizes $L(x,\lambda)$.
4. Adjust or find λ values so that $x(\lambda)$ is feasible.

If $\lambda^* \in Y$ exists such that $x^* = x(\lambda^*)$is feasible, then x^* is optimal for P by the Lagrangian sufficiency theorem

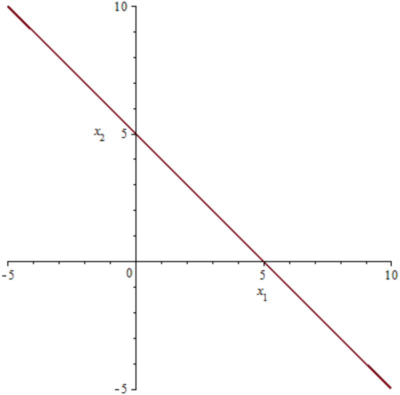
Example 2

Example

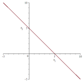
minimize $\frac{1}{1+x_1} + \frac{1}{2+x_2}$ s.t. $x_1 + x_2 = b$, $x_1, x_2 \geq 0$

The Functional Constraint

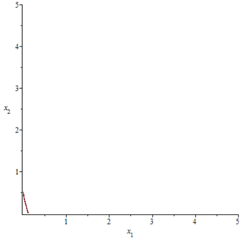
$$x_1 + x_2 = b$$



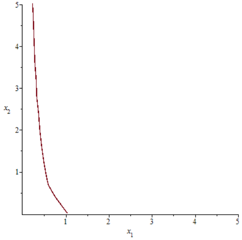
The Objective Function to Minimize



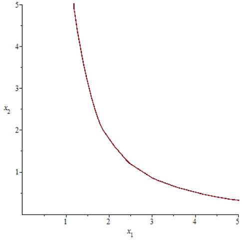
$$\frac{1}{1+x_1}+\frac{1}{2+x_2}=1.4$$



$$\frac{1}{1+x_1}+\frac{1}{2+x_2}=1.0$$



$$\frac{1}{1+x_1}+\frac{1}{2+x_2}=0.6$$



Step 1
Solution to Example 2

Setup $L(x,\lambda)$ and for each λ , minimize $L(x,\lambda)$ subject to $x \in X$

Definition

$X = \{x : x \geq 0\}$,
Lagrangian $L(x,\lambda) = f(x) - \lambda(g(x) - b)$

$$
\begin{aligned}
L(x,\lambda) &= \frac{1}{1+x_1} + \frac{1}{2+x_2} - \lambda(x_1 + x_2 - b) \\
&= \left(\frac{1}{1+x_1} - \lambda x_1\right) + \left(\frac{1}{2+x_2} - \lambda x_2\right) + \lambda b
\end{aligned}
$$

$$
\begin{aligned}
\partial L/\partial x_1 &= -\frac{1}{(1+x_1)^2} - \lambda = 0 \implies x_1 = -1 + \sqrt{\frac{-1}{\lambda}} \\
\partial L/\partial x_2 &= -\frac{1}{(2+x_2)^2} - \lambda = 0 \implies x_2 = -2 + \sqrt{\frac{-1}{\lambda}}
\end{aligned}
$$

Step 1
Solution to Example 2

Setup $L(x,\lambda)$ and for each λ , minimize $L(x,\lambda)$ subject to $x \in X$

Definition

$X = \{x : x \geq 0\}$,
Lagrangian $L(x,\lambda) = f(x) - \lambda(g(x) - b)$

$$\begin{aligned} L(x,\lambda) &= \tfrac{1}{1+x_1} + \tfrac{1}{2+x_2} - \lambda(x_1 + x_2 - b) \\ &= \left(\tfrac{1}{1+x_1} - \lambda x_1\right) + \left(\tfrac{1}{2+x_2} - \lambda x_2\right) + \lambda b \end{aligned}$$

$$\begin{aligned} \partial L/\partial x_1 &= -\frac{1}{(1+x_1)^2} - \lambda &= 0 &\implies & x_1 &= -1 + \sqrt{\frac{-1}{\lambda}} \\ \partial L/\partial x_2 &= -\frac{1}{(2+x_2)^2} - \lambda &= 0 &\implies & x_2 &= -2 + \sqrt{\frac{-1}{\lambda}} \end{aligned}$$

Step 2

Solution to Example 2

Define the set $Y = \left\{\lambda : \min_{x \in X} L(x,\lambda) > -\infty\right\}$, i.e. λ for which the Lagrangian will have a finite minimum.

$$L(x,\lambda) = \left(\frac{1}{1+x_1} - \lambda x_1\right) + \left(\frac{1}{2+x_2} - \lambda x_2\right) + \lambda b$$

- given that $x \geq 0$, the first 2 terms of the rhs will only have a finite minimum if $\lambda \leq 0$

$$Y = \{\lambda : \lambda \leq 0\}$$

Step 2

Solution to Example 2

Define the set $Y = \left\{ \lambda : \min_{x \in X} L(x, \lambda) > -\infty \right\}$, i.e. λ for which the Lagrangian will have a finite minimum.

$$L(x, \lambda) = \left(\frac{1}{1 + x_1} - \lambda x_1 \right) + \left(\frac{1}{2 + x_2} - \lambda x_2 \right) + \lambda b$$

- given that $x \geq 0$, the first 2 terms of the rhs will only have a finite minimum if $\lambda \leq 0$

$$Y = \{ \lambda : \lambda \leq 0 \}$$

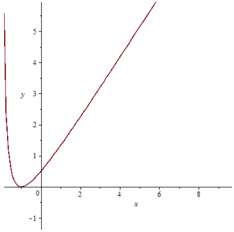
Step 3

Solution to Example 2

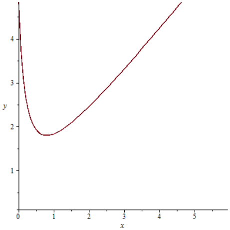
The minimum will be found at some $x(\lambda)$, for $\lambda \in Y$. We find $x(\lambda)$ that minimizes $L(x,\lambda)$.

- for $x \geq 0$ and $\lambda \leq 0$, we examine functions of the form

$$\left(\frac{1}{a+x} - \lambda x\right)$$



$(2+x)^{-1}+x=y$



$(0.2+x)^{-1}+x=y$

Step 3
Solution to Example 2

The minimum will be found at some $x(\lambda)$, for $\lambda \in Y$. We find $x(\lambda)$ that minimizes $L(x, \lambda)$.

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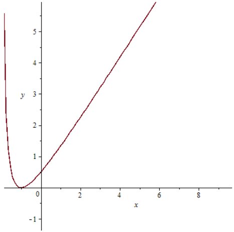
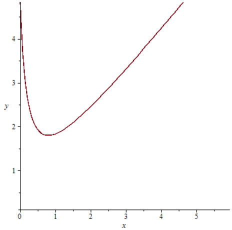
$$\left(\frac{1}{a+x} - \lambda x \right)$$



$(2+x)^{-1} + x = y$



$(0.2+x)^{-1} + x = y$

Step 3

Solution to Example 2 (slide 2)

- for a range of $x \geq 0$, a function of the form $\left(\frac{1}{a+x} - \lambda x\right)$ will have a minimum at either $x = 0$ or at the stationary point of the function depending on whether the function is increasing or decreasing at $x = 0$

$$\frac{d}{dt}\left(\frac{1}{a+x} - \lambda x\right) = -\frac{1}{(a+x)^2} - \lambda = 0 \implies x = -a + \sqrt{\frac{-1}{\lambda}}$$

- so the minimum will occur at

$$x = \begin{cases} 0 & , \quad \sqrt{-1/\lambda} \leq a \\ -a + \sqrt{-1/\lambda} & , \quad \sqrt{-1/\lambda} \geq a \end{cases}$$

Step 3

Solution to Example 2 (slide 3)

$$x = \begin{cases} 0 & , \quad \sqrt{-1/\lambda} \leq a \\ -a + \sqrt{-1/\lambda} & , \quad \sqrt{-1/\lambda} \geq a \end{cases}$$

- we look at the boundary values for λ
- we know that $\lambda \leq 0$
- we now find λ for $\sqrt{-1/\lambda} = a$ where $a = 1, 2$

$$\begin{aligned} \sqrt{-1/\lambda} &= a \\ -1/\lambda &= a^2 \\ -1/a^2 &= \lambda \end{aligned}$$

- There will be boundaries at $\lambda = -1$ and $\lambda = -1/4$

Step 4
Solution to Example 2

Adjust / Find λ values so that $x(\lambda)$ is feasible.

- we use our constraint and substitute our values for $x(\lambda)$ to it

$$x_1 + x_2 = b$$

$$x = \begin{cases} 0 & , \quad \sqrt{-1/\lambda} \leq a \\ -a + \sqrt{-1/\lambda} & , \quad \sqrt{-1/\lambda} \geq a \end{cases}$$

Definition

$$c^+ = \max(0, c)$$

$$x(\lambda) = \left(-a + \sqrt{-1/\lambda}\right)^+$$

Step 4
Solution to Example 2

Adjust / Find λ values so that $x(\lambda)$ is feasible.

- we use our constraint and substitute our values for $x(\lambda)$ to it

$$x_1 + x_2 = b$$

$$x = \begin{cases} 0 & , \quad \sqrt{-1/\lambda} \leq a \\ -a + \sqrt{-1/\lambda} & , \quad \sqrt{-1/\lambda} \geq a \end{cases}$$

Definition

$$c^+ = \max(0, c)$$

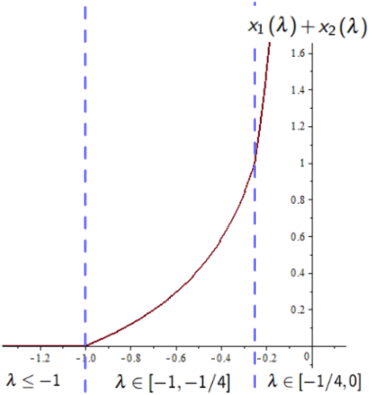
$$x(\lambda) = \left(-a + \sqrt{-1/\lambda}\right)^+$$

Step 4
Solution to Example 2 (slide 2)

- we use our constraint and substitute our values for $x(\lambda)$ to it

$$
\begin{aligned}
x_1 + x_2 &= b \\
x_1(\lambda) + x_2(\lambda) &= \left(-1 + \sqrt{-1/\lambda}\right)^+ + \left(-2 + \sqrt{-1/\lambda}\right)^+ \\
&= \begin{cases} 0, & \lambda \le -1 \\ -1 + 1/\sqrt{-\lambda}, & \lambda \in [-1, -1/4] \\ -3 + 2/\sqrt{-\lambda}, & \lambda \in [-1/4, 0] \end{cases}
\end{aligned}
$$

Step 4
Solution to Example 2 (slide 3)



- by the Intermediate Value Theorem , for any $b > 0$ there will be a unique value of λ,say λ^*, for which $x_1(\lambda^*) + x_2(\lambda^*) = b$
- x^* and λ^* will satisfy the conditions for the Lagrangian sufficiency theorem and thus, x^* will be optimal

Solution to Example 2

- if we had a value for b , we can find the optimum by substituting it below

$$\begin{aligned} x_1 + x_2 &= b \\ x_1(\lambda) + x_2(\lambda) &= \left(-1 + \sqrt{-1/\lambda}\right)^+ + \left(-2 + \sqrt{-1/\lambda}\right)^+ \end{aligned}$$

- Then we shall get λ^*, which we will substitute below

$$\begin{aligned} x_1 &= -1 + \sqrt{\frac{-1}{\lambda}} \\ x_2 &= -2 + \sqrt{\frac{-1}{\lambda}} \end{aligned}$$

- to get the optimal values for x_1 and x_2

Weak Duality Theorem

Some definitions

Definition

$$Y = \left\{ \lambda : \min_{\mathrm{x} \in \mathrm{X}} \mathrm{L}(\mathrm{x}, \lambda) > -\infty \right\}$$

For $\lambda \in Y$,

Definition

$$L(\lambda) = \min_{x \in X} L(x, \lambda)$$

Weak Duality Theorem

Theorem Proper (slide 2)

Theorem

3.1 Weak Duality Theorem
For any feasible $x \in X_b$ and any $\lambda \in Y$

$$L(\lambda) \leq f(x)$$

If the set Y is non-empty, we can pick any $\lambda \in Y$ and $L(\lambda)$ will be a lower bound for the minimum value of the objective function $f(x)$

Proof

Weak Duality Theorem

- The Lagrangian is $L(x,\lambda) = f(x) - \lambda^T(g(x) - b)$
- If x is within the feasible region, then the constraint $g(x) = b$ is satisfied and $g(x) - b = 0$
- in this case, the Lagrangian $L(x,\lambda)$ is equal to the objective function $f(x)$

Proof.

For $x \in X_b$, $\lambda \in Y$,

$$f(x) = L(x,\lambda) \geq \min_{x \in X_b} L(x,\lambda) \geq \min_{x \in X} L(x,\lambda) = L(\lambda)$$

$\square$

Lagrangian Dual Problem

Consider the problem

$$\text{D: maximize } L(\lambda) \text{ subject to } \lambda \in Y$$

equivalently,

$$\text{D: } \underset{\lambda \in Y}{\text{maximize}} \left\{ \min_{x \in X} L(x, \lambda) \right\}$$

- this is the Lagrangian Dual Problem
- the original problem is called the Primal Problem
- The optimal value of the dual is $\leq$ to the optimal value of the primal
- If they are equal (e.g. in LP) , we call it a Strong Duality

Example 3

- We review Example 1

Example

$$
\begin{array}{rrrcl}
\text{minimize} & f(x) = & x_1 - x_2 - 2x_3 & & \\
\text{s.t.} & g_1(x) = & x_1 + x_2 + x_3 & = & 5 \\
 & g_2(x) = & x_1^2 + x_2^2 & = & 4 \\
 & & x \in X & = & \mathbb{R}^3
\end{array}
$$

- $Y = \{\lambda : \lambda_1 = -2,\ \lambda_2 < 0\}$
- The $\min_{x \in X} L(x, \lambda)$ occurred for $x(\lambda) = \left(\frac{3}{2\lambda_2}, \frac{1}{2\lambda_2}, x_3\right)$
- Solve for $L(\lambda) = L(x(\lambda), \lambda) = f(x(\lambda)) - \lambda^T (g(x(\lambda)) - b)$

Example 3
slide 2

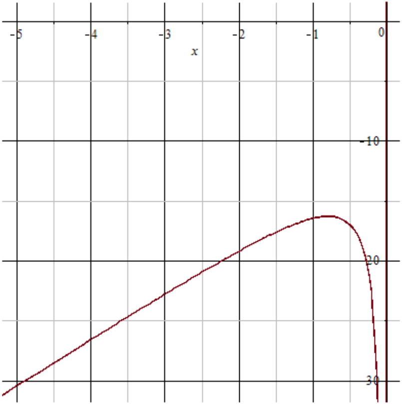
- The $\min_{x \in X} L(x, \lambda)$ occurred for $x(\lambda) = \left(\frac{3}{2\lambda_2}, \frac{1}{2\lambda_2}, x_3 \right)$
- $Y = \{\lambda : \lambda_1 = -2,\ \lambda_2 < 0\}$

$$
\begin{aligned}
L(x,\lambda) &= (x_1 - x_2 - 2x_3) - \lambda_1 (x_1 + x_2 + x_3 - 5) - \lambda_2 \left(x_1^2 + x_2^2 - 4\right) \\
L(\lambda) &= \left(\frac{3}{2\lambda_2} - \frac{1}{2\lambda_2} - 2x_3\right) - (-2)\left(\frac{3}{2\lambda_2} + \frac{1}{2\lambda_2} + x_3 - 5\right) \\
&- \lambda_2 \left(\left(\frac{3}{2\lambda_2}\right)^2 + \left(\frac{1}{2\lambda_2}\right)^2 - 4\right) \\
L(\lambda) &= \frac{3}{2\lambda_2} - \frac{1}{2\lambda_2} - 2x_3 + 2\left(\frac{3}{2\lambda_2} + \frac{1}{2\lambda_2} + x_3 - 5\right) \\
&- \left(\frac{9}{4\lambda_2} + \frac{1}{4\lambda_2} - 4\lambda_2\right) \\
L(\lambda) &= \frac{3}{2\lambda_2} - \frac{1}{2\lambda_2} - 2x_3 + \frac{3}{\lambda_2} + \frac{1}{\lambda_2} + 2x_3 - 10 - \frac{9}{4\lambda_2} - \frac{1}{4\lambda_2} + 4\lambda_2 \\
&= \frac{10}{4\lambda_2} - 10 + 4\lambda_2
\end{aligned}
$$

Example 3
slide 3

- We define the Dual Problem

$$\underset{\lambda_2<0}{\text{maximize}}\left\{\frac{10}{4\lambda_2}-10+4\lambda_2\right\}$$



Example 3
slide 4

- get the derivative and equate to 0

$$\begin{aligned}
\frac{10}{4}(-1)(\lambda_2)^{-2}+4 &= 0 \\
4 &= \frac{5}{2\lambda_2^2} \\
\lambda_2^2 &= \frac{5}{8} \\
\lambda_2 &= \pm\sqrt{5/8}
\end{aligned}$$

- Since a $\lambda_2 < 0$ is required to have a finite minimum, the max is at $\lambda_2 = \left(-\sqrt{5/8}\right) \approx -0.79$

Example 3
Dual Problem Optimum (slide 5)

- The optimum for the Dual Problem is at $L(\lambda_2)$

$$
\begin{aligned}
L(\lambda) &= \frac{10}{4\lambda_2} - 10 + 4\lambda_2 \\
&= \frac{10}{4\left(-\sqrt{5/8}\right)} - 10 + 4\left(-\sqrt{5/8}\right) \\
&= \frac{-10}{\sqrt{16 \cdot 5/8}} - 10 - \sqrt{16 \cdot 5/8} \\
&= \frac{-10}{\sqrt{10}} - 10 - \sqrt{10} \\
&= -10 - 2\sqrt{10} \approx -16.325
\end{aligned}
$$

Example 3

Primal Problem Optimum (slide 6)

- The primal optimum is given by $f(x^*)$ with $x^* = \left(-3\sqrt{2/5},\, -\sqrt{2/5},\, 5+4\sqrt{2/5}\right)^T$

$$
\begin{aligned}
f(x^*) &= x_1 - x_2 - 2x_3 \\
&= -3\sqrt{2/5} + \sqrt{2/5} - 2\cdot\left(5+4\sqrt{2/5}\right) \\
&= -2\sqrt{2/5} - 10 - 8\sqrt{2/5} \\
&= -10\sqrt{2/5} - 10 \\
&= -2\sqrt{25\cdot 2/5} - 10 \\
&= -2\sqrt{10} - 10
\end{aligned}
$$

Example 4

- We review example 2:

Example

minimize $\frac{1}{1+x_1} + \frac{1}{2+x_2}$ s.t. $x_1 + x_2 = b$, $x_1, x_2 \geq 0$

- $Y = \{\lambda : \lambda \leq 0\}$
- $x(\lambda) = \left(-a + \sqrt{-1/\lambda}\right)^+ = \left(-a + 1/\sqrt{-\lambda}\right)^+$
- This time we have a function for $x(\lambda)$ that changes depending on the value of λ
- The function for $L(\lambda)$ will be changing accordingly as well

Example 4
slide 2

λ	$x(\lambda)$	$L(\lambda) = L(x(\lambda),\lambda)$
$\lambda \le -1$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\left(\frac{1}{1}+\frac{1}{2}\right) - \lambda(-b)$
$\lambda \in [-1,-1/4]$	$\begin{bmatrix} \left(-1+1/\sqrt{-\lambda}\right) \\ 0 \end{bmatrix}$	$\left(\frac{1}{1+\left(-1+1/\sqrt{-\lambda}\right)}+\frac{1}{2}\right)$ $-\lambda\left(\left(-1+1/\sqrt{-\lambda}\right)-b\right)$
$\lambda \in [-1/4,0]$	$\begin{bmatrix} \left(-1+1/\sqrt{-\lambda}\right) \\ \left(-2+\sqrt{-1/\lambda}\right) \end{bmatrix}$	$\frac{1}{1+\left(-1+1/\sqrt{-\lambda}\right)}$ $+\frac{1}{2+\left(-2+\sqrt{-1/\lambda}\right)}$ $-\lambda\left(-1+1/\sqrt{-\lambda}\right)$ $-\lambda\left(-2+1/\sqrt{-\lambda}\right)+\lambda b$

Example 4
$L(\lambda)$ for $\lambda \leq -1$ (slide 3)

- Solve for $L(\lambda)$ for $\lambda \leq -1$

$$\begin{aligned} L(\lambda) &= \left(\tfrac{1}{1} + \tfrac{1}{2}\right) - \lambda(-b) \\ &= 3/2 + \lambda b \end{aligned}$$

Example 4

$L(\lambda)$ for $\lambda \in [-1, -1/4]$ (slide 4)

- Solve for $L(\lambda)$ for $\lambda \in [-1, -1/4]$
- Note that $\lambda \leq -1$ and $(-\lambda)$ is either a positive number or zero

$$
\begin{aligned}
L(\lambda) &= \left(\frac{1}{1+\left(-1+1/\sqrt{-\lambda}\right)} + \frac{1}{2}\right) - \lambda\left(\left(-1+1/\sqrt{-\lambda}\right) - b\right) \\
&= \frac{1}{1/\sqrt{-\lambda}} + \frac{1}{2} - \lambda\left(-1+1/\sqrt{-\lambda}\right) + \lambda b \\
&= \sqrt{-\lambda} + \frac{1}{2} + \lambda + \frac{-\lambda}{\sqrt{-\lambda}} + \lambda b \\
&= \sqrt{-\lambda} + \frac{1}{2} + \sqrt{-\lambda} + \lambda b + \lambda \\
&= 2\sqrt{-\lambda} + \frac{1}{2} + \lambda(b+1)
\end{aligned}
$$

Example 4
$L(\lambda)$ for $\lambda \in [-1/4, 0]$ (slide 5)

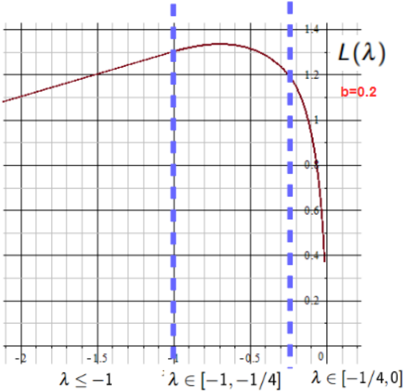
- Solve for $L(\lambda)$ for $\lambda \in [-1/4, 0]$
- Note that $\lambda \leq -1$ and $(-\lambda)$ is either a positive number or zero

$$
\begin{aligned}
L(\lambda) &= \frac{1}{1+\left(-1+1/\sqrt{-\lambda}\right)} + \frac{1}{2+\left(-2+1/\sqrt{-\lambda}\right)} \\
&- \quad \lambda\left(-1+1/\sqrt{-\lambda}\right) - \lambda\left(-2+1/\sqrt{-\lambda}\right) + \lambda b \\
L(\lambda) &= \frac{1}{1/\sqrt{-\lambda}} + \frac{1}{1/\sqrt{-\lambda}} + \lambda + \frac{(-\lambda)}{\sqrt{-\lambda}} + 2\lambda + \frac{(-\lambda)}{\sqrt{-\lambda}} + \lambda b \\
&= \sqrt{-\lambda} + \sqrt{-\lambda} + \lambda + \sqrt{-\lambda} + 2\lambda + \sqrt{-\lambda} + \lambda b \\
&= 4\sqrt{-\lambda} + (b+3)\lambda
\end{aligned}
$$

Ex4 Plotting $L(\lambda)$

b=0.2

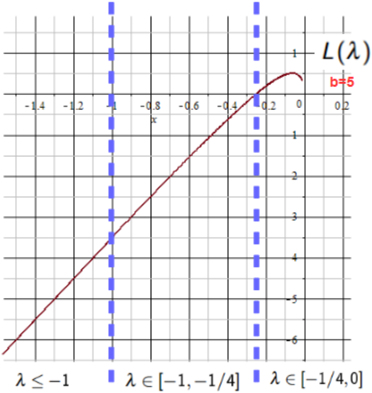
- The dual problem is to maximize $L(\lambda)$ s.t. $\lambda \leq 0$
- The solution lies in $-1 \leq \lambda \leq -1/4$ for $b \in [0, 1]$
- The solutions is in $-1/4 \leq \lambda \leq 0$ for $b \in [1, \infty]$



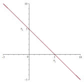
Ex4 Plotting $L(\lambda)$

b=5

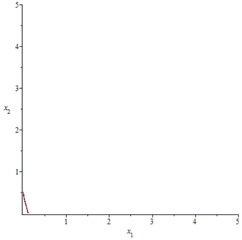
- The dual problem is to maximize $L(\lambda)$ s.t. $\lambda \leq 0$
- The solution lies in $-1 \leq \lambda \leq -1/4$ for $b \in [0, 1]$
- The solutions is in $-1/4 \leq \lambda \leq 0$ for $b \in [1, \infty]$

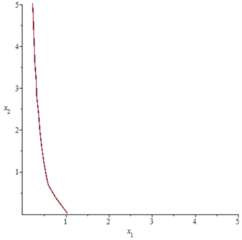


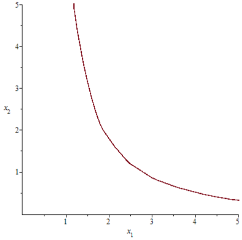
Ex4 Objective function f(x)



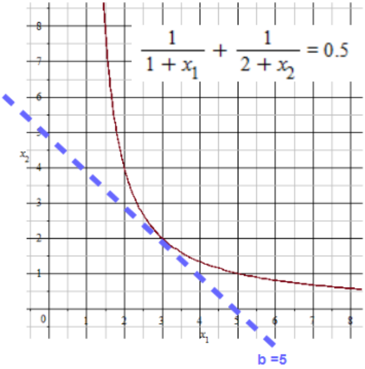
$$\frac{1}{1+x_1}+\frac{1}{2+x_2}=1.4 \qquad \frac{1}{1+x_1}+\frac{1}{2+x_2}=1.0 \qquad \frac{1}{1+x_1}+\frac{1}{2+x_2}=0.6$$







Ex4 f(x) with b=5



The primal and the dual have the same optimal values for this case

Construction of Dual Problem for LP

The Dual Problem for a given LP Problem can be constructed in the following manner

1. Consider the Primal Problem
2. Write out the Lagrangian
3. Set Y such that $\lambda \in Y$ implies that the maximum of the Lagrangian is finite
4. For $\lambda \in Y$, compute the minimum of $L(\lambda)$

Step 1

Construction of Dual Problem for LP

Consider the primal problem P:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b,\ x \geq 0 \\ \text{equivalently} \quad & Ax + z = b,\ x, z \geq 0 \end{aligned}$$

Step 2
Construction of Dual Problem for LP

Write out the Lagrangian

$$\begin{aligned} L(x,z,\lambda) &= c^T x - \lambda^T (Ax + z - b) \\ &= c^T x - \lambda^T Ax - \lambda^T z - \lambda^T b \\ &= \left(c^T - \lambda^T A\right) x - \lambda^T z + \lambda^T b \end{aligned}$$

Step 3
Construction of Dual Problem for LP

Set Y such that $\lambda \in Y$ implies that the maximum of the Lagrangian is finite

$$L(x,z,\lambda) = \left(c^T - \lambda^T A\right) x - \lambda^T z + \lambda^T b$$

- for the linear term $-\lambda^T z$, if any $\lambda_i < 0$ we can make $\lambda_i z_i$ as large as we like by taking an arbitrarily large z_i
- there is only a finite maximum in $z \geq 0$ if $\lambda_i \geq 0$
- for the term $(c^T - \lambda^T A) x$, this can be made arbitrarily large for a positive $(c^T - \lambda^T A)$
- there is only a finite maximum if $(c^T - \lambda^T A) \leq 0$ or $\left(\lambda^T A - c^T\right) \geq 0$
- Therefore,

$$Y = \left\{\lambda : \lambda \geq 0, \lambda^T A - c^T \geq 0\right\}$$

Step 4
Construction of Dual Problem for LP

For $\lambda \in Y$, compute the minimum of $L(\lambda)$

- we pick a $\lambda \in Y$
- we then find $\max_{x,z \geq 0} -\lambda^T z = 0$
  - do this by choosing $z_i = 0$ if $\lambda_i > 0$ and any z_i if $\lambda_i = 0$
- we also find $\max_{x \geq 0} \left(c^T - \lambda^T A\right) = 0$
- Therefore, for $\lambda \in Y$,

$$L(\lambda) = \lambda^T b$$

The Dual Primal Pair

$$\begin{array}{lllll} \text{P:} & \text{maximize} & c^T x & \text{s.t.} & Ax \leq b, & x \geq 0 \\ \text{D:} & \text{minimize} & \lambda^T b & \text{s.t.} & \lambda^T A \geq c^T, & \lambda \geq 0 \end{array}$$

The Dual of an LP is also an LP

$$\begin{array}{llrcl}
\text{P:} & \text{maximize} & x_1 + x_2 & & \\
& \text{subject to} & x_1 + 2x_2 & \leq & 6 \\
& & x_1 - x_2 & \leq & 3 \\
& & x_1, x_2 & \geq & 0
\end{array}$$

$$\begin{array}{llrcl}
\text{D:} & \text{minimize} & 6\lambda_1 + 3\lambda_2 & & \\
& \text{subject to} & \lambda_1 + \lambda_2 & \geq & 1 \\
& & 2\lambda_1 - \lambda_2 & \geq & 1 \\
& & \lambda_1, \lambda_2 & \geq & 0
\end{array}$$

D can be written as

$$\text{D: maximize } (-b)^T \lambda \text{ s.t. } (-A)^T \lambda \leq (-c) \text{ , } \lambda \geq 0$$

Alternative Form of the Dual

D can be written as

$$\text{D: maximize } (-b)^T\lambda \text{ s.t. } (-A)^T\lambda \leq (-c) \text{ , } \lambda \geq 0$$

- D has the same form as P with the following changes in the variables

P	$\rightarrow$	D
c	$\rightarrow$	-b
b	$\rightarrow$	-c
A	$\rightarrow$	$-A^T$

- This also means that in Linear Programming, the Dual of the Dual is the Primal

The Weak Duality Theorem for LP

- recall

Theorem

3.1 Weak Duality Theorem
For any feasible $x \in X_b$ and any $\lambda \in Y$, then $L(\lambda) \leq f(x)$

- we apply this to Linear Programming

Theorem

3.3 Weak Duality Theorem for LP
if x is feasible for P (meaning $Ax \leq b$ for $x \geq 0$) and λ is feasible for D (meaning $\lambda \geq 0$, $A^T\lambda \geq c$) , then $c^T x \leq \lambda^T b$

Proof
The Weak Duality Theorem for LP

Theorem

3.3 Weak Duality Theorem for LP
if x is feasible for P (meaning $Ax \leq b$ for $x \geq 0$) and λ is feasible for D (meaning $\lambda \geq 0$, $A^T\lambda \geq c$) , then $c^T x \leq \lambda^T b$

Proof.

$$L(x,z,\lambda) = c^T x - \lambda^T (Ax + z - b)$$

where $Ax + z = b$, $z \geq 0$
Now for x and λsatisfying the conditions of the theorem,

$$c^T x = L(x,z,\lambda) = \left(c^T - \lambda^T A\right) x - \lambda^T z + \lambda^T b \leq \lambda^T b$$

$\square$

Sufficient Conditions for Optimality in LP

Theorem

3.4 Sufficient Conditions for Optimality in LP
If x^* , z^* is feasible for P, and λ^* is feasible for D and
$\left(c^T - \lambda^{*T} A\right) x^* = \lambda^{*T} z^* = 0$ *and*
Then x^* is optimal for P, λ^* is optimal for D , and $c^T x^* = \lambda^{*T} b$

The conditions $\left(c^T - \lambda^{*T} A\right) x^* = 0$ and $\lambda^{*T} z^* = 0$ are called *complimentary slackness conditions*

Proof

Sufficient Conditions for Optimality in LP

$$L(x^*, z^*, \lambda^*) = c^T x^* - \lambda^{*T}(Ax^* + z^* - b)$$

$$\text{Now,} \quad c^T x^* = L(x^*, z^*, \lambda^*) = \left(c^T - \lambda^{*T} A\right) x^* - \lambda^{*T} z^* + \lambda^{*T} b = \lambda^{*T} b$$

Proof.

For all x feasible for P, we have $c^T x \leq \lambda^{*T} b$ by the Weak Duality Theorem 3.3
and this implies that for all feasible x , $c^T x \leq c^T x^*$.
So x^* is optimal for P and similarly, λ^* is optimal for D □

The problems have the same optimums due to the strong duality of LP problems

Usefulness of the Primal-Dual Theory

1. One is sometimes easier to solve than the other
2. Sometimes, you want to solve both (e.g. two-person zero-sum games)
3. To assure optimality of the LP solution: primal feasibility, dual feasibility and complimentary slackness

R. Weber.
Optimization.
Easter Term, 2010.