In [1]:

```
import numpy as np
from scipy.linalg import solve
from pandas import DataFrame
import matplotlib.pyplot as plt
```

1. Runge 效应

(a) Polynomial Interpolation with Same Intervals ¶

First, we construct the Neville algorithm as a function, which receives coordinates of n+1 nodes as input and return an interpolation polynomial of n-th degree. I use a simple nodes to test its correctness.

Here we use a numpy class named "poly1d" to construct a polynomial from a list of coefficients. More information in
https://docs.scipy.org/doc/numpy/reference/generated/numpy.poly1d.h
(https://docs.scipy.org/doc/numpy/reference/generated/numpy.poly1d.h

In [2]:

```
def Neville(nodes):
    n = len(nodes) - 1
    T = [[None for j in range(n+1)] for i in range(n
+1)] # T[i][j] stands for T_j,i
    T[0] = [np.poly1d([nodes[i][1]])  for i in range
(n+1)]
    for k in range(1, n+1):
        for j in range(k, n+1):
            T[k][j] = ([1, -nodes[j-k][0]] * T[k-1][
j] - [1, -nodes[j][0]] * T[k-1][j-1]) /\
            (nodes[j][0] - nodes[j-k][0])
    return T[n][-1] # T[n][n] is the final result

print(Neville([[1,1],[2,4],[3,9],[4,16], [5,25]]))
```

```
   2
1 x
```

Construct Runge function. Compute 21 uniform nodes of it on [-1,1] and 41 points (including midpoints) equally dividing the interval.

In [3]:

```
def Runge(x):
    return 1 / (1 + 25 * x * x)

order_of_interpolation = 20
interval = 2 / order_of_interpolation

uniform_nodes_of_Runge = [[x, Runge(x)] for x in np.
arange(-1, 1+interval, interval)]
uniform_points_of_Runge = [x for x in np.arange(-1,
1+interval/2, interval/2)]
```

Calculate $f(x),\ P_{20}(x),\ |P_{20}(x)-f(x)|$ for each dividing points. The result is showed in the table below.

In [4]:

```
P20 = Neville(uniform_nodes_of_Runge)
P20_values = P20(np.array(uniform_points_of_Runge))
Runge_values = Runge(np.array(uniform_points_of_Rung
e))

df = DataFrame({'x':uniform_points_of_Runge, 'P20
(x)':P20_values,
                'f(x)':Runge_values, '|P20(x)-f(x)|'
:abs(P20_values - Runge_values)},
               columns=['x', 'f(x)', 'P20(x)', '|P20
(x)-f(x)|'])
df.round({'x': 2})
```

Out[4]:

	x	f(x)	P20(x)	|P20(x)-f(x)|
0	-1.00	0.038462	0.038462	2.030601e-11
1	-0.95	0.042440	-39.952449	3.999489e+01
2	-0.90	0.047059	0.047059	8.011099e-12
3	-0.85	0.052459	3.454958	3.402499e+00
4	-0.80	0.058824	0.058824	3.293685e-12
5	-0.75	0.066390	-0.447052	5.134420e-01
6	-0.70	0.075472	0.075472	6.617346e-13
7	-0.65	0.086486	0.202423	1.159361e-01
8	-0.60	0.100000	0.100000	1.325468e-13
9	-0.55	0.116788	0.080660	3.612833e-02
10	-0.50	0.137931	0.137931	5.925815e-14
11	-0.45	0.164948	0.179763	1.481418e-02
12	-0.40	0.200000	0.200000	1.637579e-15
13	-0.35	0.246154	0.238446	7.707912e-03
14	-0.30	0.307692	0.307692	2.220446e-16
15	-0.25	0.390244	0.395093	4.849151e-03
16	-0.20	0.500000	0.500000	4.440892e-16
17	-0.15	0.640000	0.636755	3.244664e-03
18	-0.10	0.800000	0.800000	1.110223e-16
19	-0.05	0.941176	0.942490	1.313909e-03
20	0.00	1.000000	1.000000	0.000000e+00

	x	f(x)	P20(x)	|P20(x)-f(x)|
21	0.05	0.941176	0.942490	1.313909e-03
22	0.10	0.800000	0.800000	0.000000e+00
23	0.15	0.640000	0.636755	3.244664e-03
24	0.20	0.500000	0.500000	1.110223e-16
25	0.25	0.390244	0.395093	4.849151e-03
26	0.30	0.307692	0.307692	7.771561e-16
27	0.35	0.246154	0.238446	7.707912e-03
28	0.40	0.200000	0.200000	4.996004e-16
29	0.45	0.164948	0.179763	1.481418e-02
30	0.50	0.137931	0.137931	1.162959e-14
31	0.55	0.116788	0.080660	3.612833e-02
32	0.60	0.100000	0.100000	2.559758e-13
33	0.65	0.086486	0.202423	1.159361e-01
34	0.70	0.075472	0.075472	2.276693e-12
35	0.75	0.066390	-0.447052	5.134420e-01
36	0.80	0.058824	0.058824	1.651181e-11
37	0.85	0.052459	3.454958	3.402499e+00
38	0.90	0.047059	0.047059	7.119161e-11
39	0.95	0.042440	-39.952449	3.999489e+01
40	1.00	0.038462	0.038462	6.018995e-10

We see $P_{20}(x)$ fits $f(x)$ near the origin, but the differences between them become enormous when x deviates from the origin. When $x = \pm 0.95$, interpolation polynomial has a deviation of approximately 40!

Draw the following diagram to illustrate the difference, which shows the values of $f(x)$ and $P_{20}(x)$ on the 41 dividing points together.

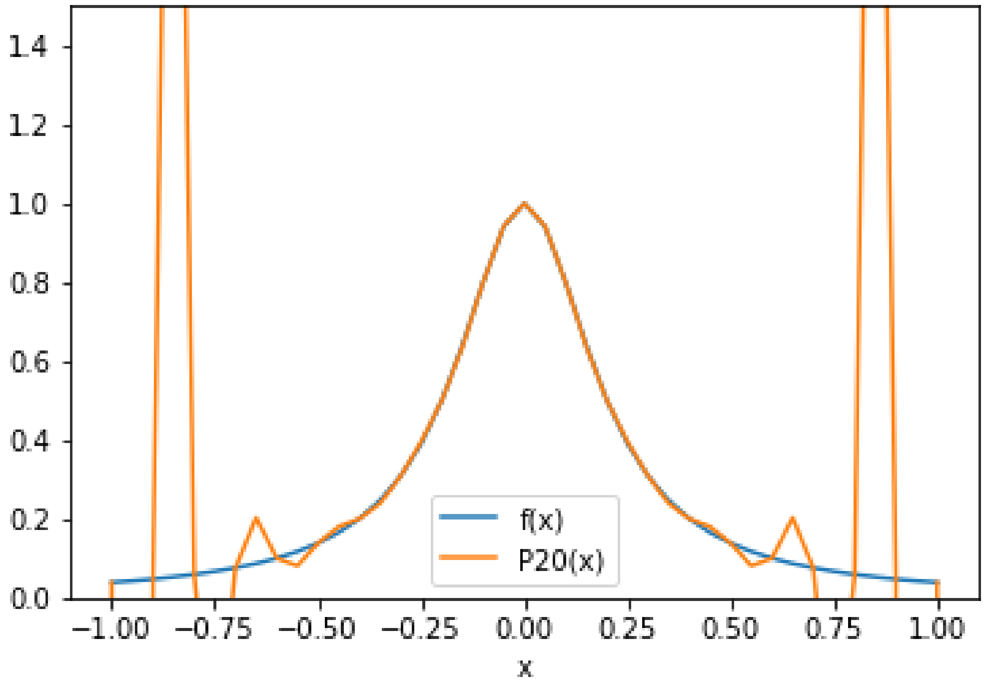
In [5]:

```
df.plot.line(x='x', y=['f(x)','P20(x)'], ylim=[0,1.5
])
```

```
/Library/Frameworks/Python.framework/Ver
sions/3.6/lib/python3.6/site-packages/pa
ndas/plotting/_core.py:1716: UserWarnin
g: Pandas doesn't allow columns to be cr
eated via a new attribute name - see htt
ps://pandas.pydata.org/pandas-docs/stabl
e/indexing.html#attribute-access
  series.name = label
```

Out[5]:

```
<matplotlib.axes._subplots.AxesSubplot a
t 0x1124a9860>
```



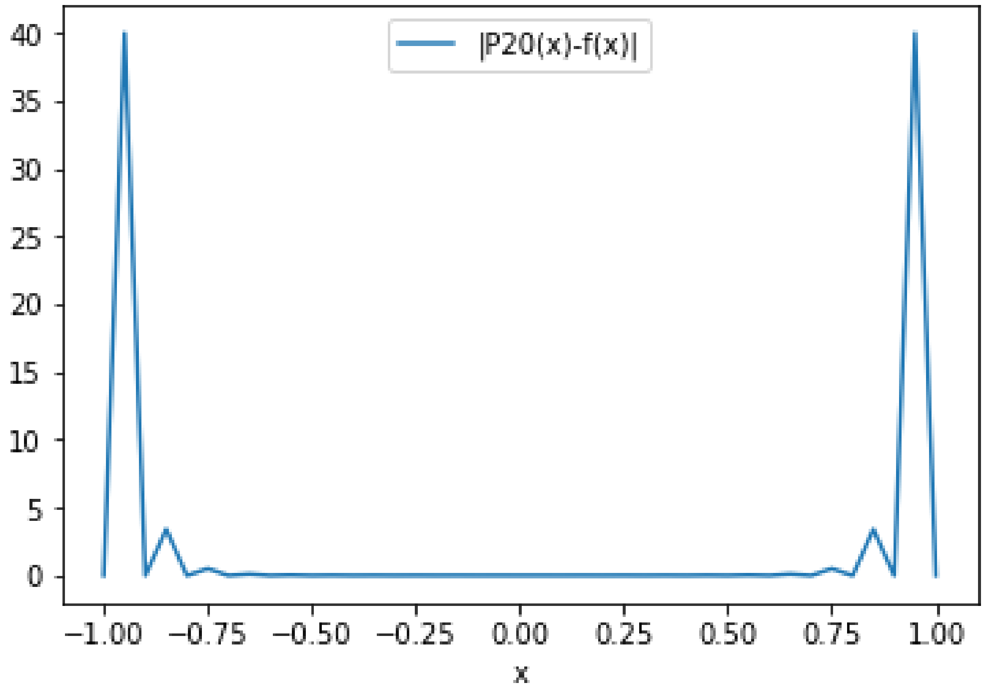
And draw this diagram to show the values of $|P_{20}(x) - f(x)|$.

In [6]:

```
df.plot.line(x='x', y='|P20(x)-f(x)|')
```

Out[6]:

```
<matplotlib.axes._subplots.AxesSubplot a
t 0x11257a6d8>
```



(b) Chebyshev Interpolation

First, construct 20 standard Chebyshev nodes.

In [7]:

```
number_of_Chebyshev_nodes = 20 # N
Chebyshev_nodes = [np.cos(np.pi * (k + 1/2) / number
_of_Chebyshev_nodes) for k in range(20)]
```

Then, compute the coefficient of each Chebyshev polynomial c_m. Given that Runge function is even, c_m should be 0 when m is odd. In our result, they turn out to be very small instead.

In [8]:

```
coefficients_of_Chebyshev = []
for m in range(number_of_Chebyshev_nodes):
    coefficient = 0
    for k in range(number_of_Chebyshev_nodes):
        coefficient = coefficient +\
        np.cos(m * np.pi *(k + 1/2) / number_of_Cheb
yshev_nodes) *\
        Runge(np.cos(np.pi * (k + 1/2) / number_of_C
hebyshev_nodes))
    coefficient = coefficient * (2 - (m==0)) / numbe
r_of_Chebyshev_nodes
    coefficients_of_Chebyshev.append(coefficient)

for i in range(len(coefficients_of_Chebyshev)):
    print('c', i ,'=', coefficients_of_Chebyshev[i])
```

```
c 0 = 0.19597752688801637
c 1 = -1.1796119636642288e-17
c 2 = -0.2633114580781154
c 3 = 6.245004513516506e-18
c 4 = 0.176797695672696 6
c 5 = -8.326672684688674e-17
c 6 = -0.118571564574909
c 7 = 1.464106613724425 3e-16
c 8 = 0.07931688380910713
c 9 = 1.97758476261356e-17
c 10 = -0.05275290445276218 5
c 11 = -6.21031004399696 9e-17
c 12 = 0.0346293898088590 3
c 13 = -1.457167719820518e-17
c 14 = -0.0220465775343737 8
c 15 = 2.114627917215728e-16
c 16 = 0.0129912176653884 14
c 17 = -3.157196726277789e-17
c 18 = -0.006014452622864767
c 19 = -3.690624195140657e-17
```

Use Clenshaw's algorithm to get the expansion under Chebyshev polynomial $S(x)$. Again, odd-order entries should have cofficients of zero. But very small coefficients are given instead.

In [9]:

```
def Clenshaw(coefficients, alpha, beta, F_0, F_1):
    N = len(coefficients) - 1
    b = [None] * (N+1) + [np.poly1d([0])] * 2
    for k in range(N, 0, -1):
        b[k] = np.poly1d([coefficients[k]]) + alpha[
k] * b[k+1] + beta[k+1] * b[k+2]
    return np.poly1d(coefficients[0]) * F_0 + b[1] *
 F_1 + beta[1] * F_0 * b[2]

alpha_for_Chebyshev = [None] + [np.poly1d([2, 0])] *
 (number_of_Chebyshev_nodes + 1)
beta_for_Chebyshev = [None] + [np.poly1d([-1])] * (n
umber_of_Chebyshev_nodes + 1)
Chebyshev_expansion = Clenshaw(coefficients_of_Cheby
shev, alpha_for_Chebyshev,
                               beta_for_Chebyshev, F
_0=np.poly1d([1]), F_1=np.poly1d([1,0]))
print(np.poly1d(Chebyshev_expansion))
```

```
            19          18               17
        16               15
-9.675e-12 x  - 788.3 x  + 4.389e-11 x
+ 3973 x  - 7.965e-11 x
         14               13              12
             11          10
 - 8535 x  + 7.209e-11 x  + 1.02e+04 x
- 3.143e-11 x  - 7413 x
              9         8              7
      6             5
 + 3.164e-12 x + 3379 x + 2.402e-12 x -
960.8 x - 8.742e-13 x
          4               3         2
 + 165.5 x + 1.013e-13 x - 16.54 x - 3.8
08e-15 x + 0.9624
```

Calculate $f(x),\ S(x),\ |S(x)-f(x)|$ for each dividing points. The result is showed in the table below.

In [10]:

```
Chebyshev_values = Chebyshev_expansion(np.array(unif
orm_points_of_Runge))

df = DataFrame({'x':uniform_points_of_Runge, 'S(x)':
Chebyshev_values,
                'f(x)':Runge_values, '|S(x)-f(x)|':a
bs(Chebyshev_values - Runge_values)},
               columns=['x', 'f(x)', 'S(x)', '|S(x)-
f(x)|'])
df.round({'x': 2})
```

Out[10]:

	x	f(x)	S(x)	|S(x)-f(x)|
0	-1.00	0.038462	0.037016	0.001446
1	-0.95	0.042440	0.040849	0.001592
2	-0.90	0.047059	0.048685	0.001626
3	-0.85	0.052459	0.052261	0.000198
4	-0.80	0.058824	0.056713	0.002110
5	-0.75	0.066390	0.067169	0.000779
6	-0.70	0.075472	0.078252	0.002780
7	-0.65	0.086486	0.086534	0.000047
8	-0.60	0.100000	0.096413	0.003587
9	-0.55	0.116788	0.114126	0.002663
10	-0.50	0.137931	0.140523	0.002592
11	-0.45	0.164948	0.171124	0.006176
12	-0.40	0.200000	0.202763	0.002763
13	-0.35	0.246154	0.240175	0.005979
14	-0.30	0.307692	0.296333	0.011360
15	-0.25	0.390244	0.385335	0.004909
16	-0.20	0.500000	0.511895	0.011895
17	-0.15	0.640000	0.663854	0.023854
18	-0.10	0.800000	0.812606	0.012606
19	-0.05	0.941176	0.922073	0.019103
20	0.00	1.000000	0.962410	0.037590

	x	f(x)	S(x)	|S(x)-f(x)|
21	0.05	0.941176	0.922073	0.019103
22	0.10	0.800000	0.812606	0.012606
23	0.15	0.640000	0.663854	0.023854
24	0.20	0.500000	0.511895	0.011895
25	0.25	0.390244	0.385335	0.004909
26	0.30	0.307692	0.296333	0.011360
27	0.35	0.246154	0.240175	0.005979
28	0.40	0.200000	0.202763	0.002763
29	0.45	0.164948	0.171124	0.006176
30	0.50	0.137931	0.140523	0.002592
31	0.55	0.116788	0.114126	0.002663
32	0.60	0.100000	0.096413	0.003587
33	0.65	0.086486	0.086534	0.000047
34	0.70	0.075472	0.078252	0.002780
35	0.75	0.066390	0.067169	0.000779
36	0.80	0.058824	0.056713	0.002110
37	0.85	0.052459	0.052261	0.000198
38	0.90	0.047059	0.048685	0.001626
39	0.95	0.042440	0.040849	0.001592
40	1.00	0.038462	0.037016	0.001446

Plot $f(x)$ and $S(x)$ together. From the diagram, we see $S(x)$ generally fits $f(x)$ well; but not so well close to origin, which is opposed to the result in (a). And there are no singular points like $x = \pm 0.95$ in (a).

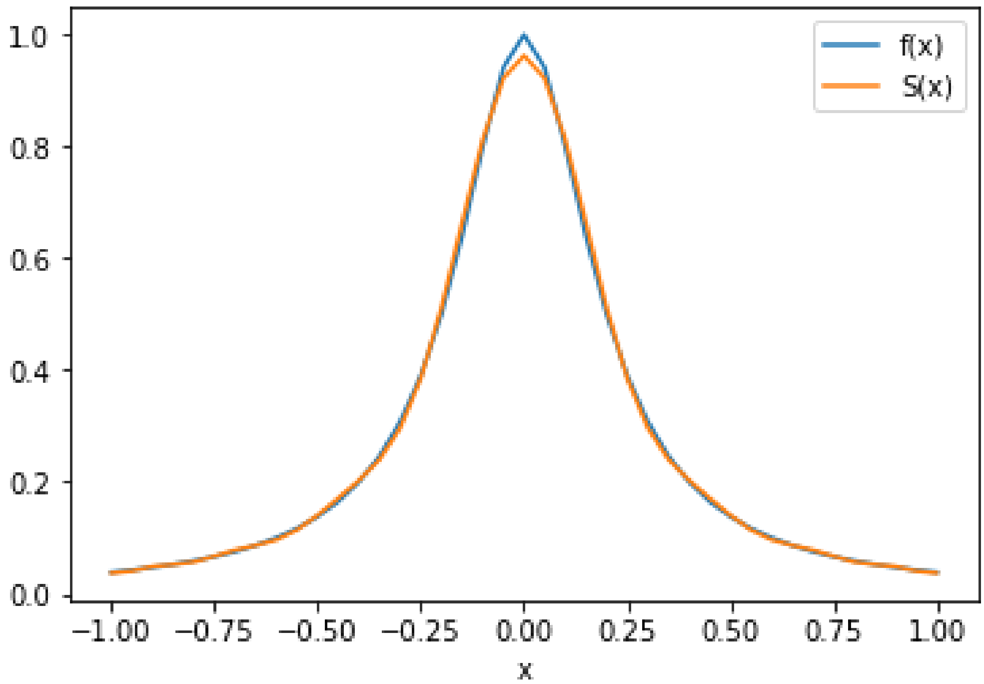
In [11]:

```
df.plot.line(x='x', y=['f(x)','S(x)'])
```

```
/Library/Frameworks/Python.framework/Versions/3.6/lib/python3.6/site-packages/pandas/plotting/_core.py:1716: UserWarning: Pandas doesn't allow columns to be created via a new attribute name - see https://pandas.pydata.org/pandas-docs/stable/indexing.html#attribute-access
  series.name = label
```

Out[11]:

```
<matplotlib.axes._subplots.AxesSubplot at 0x112689588>
```



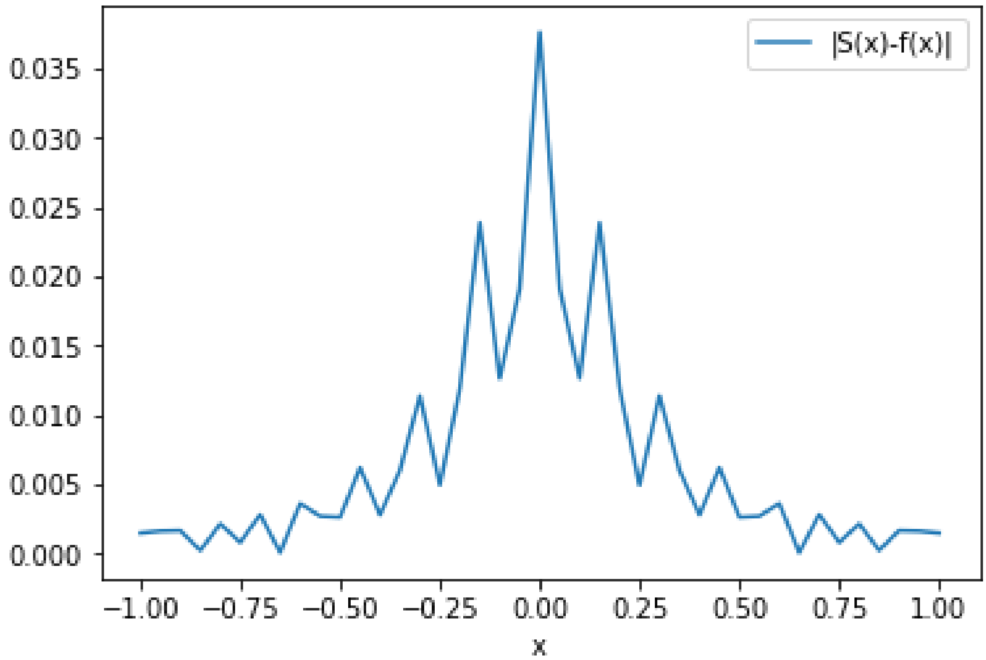
Plot $|S(x) - f(x)|$ together. We can see more clearly that the difference generally increases when approaching the origin.

In [12]:

```
df.plot.line(x='x', y='|S(x)-f(x)|')
```

Out[12]:

```
<matplotlib.axes._subplots.AxesSubplot a
t 0x112698208>
```



(c) Cubic Spline Function

To get the cubic spline function, we firstly construct an algorithms to compute the cubic polynomial on each small interval. The result is a list of polynomials, whose j-th component is the polynomial on each interval $[x_j, x_{j+1}], \quad j = 0, \cdots, n$. (Showed in output)

We add two constrains to the spline function. That is $S''(x)\Big|_{x=\pm 1} = f''(x) \approx 0.2105$, i.e. $M_0 = M_n = 0.2105$. According to equation(36), we have $\lambda_0 = \mu_n = 0, \quad d_0 = d_n = 2M_0 = 0.421$

In [13]:

```
def cubic_splines(nodes, lam_0, mu_n, d_0, d_n):
    n = len(nodes) - 1
    x = [node[0] for node in nodes]
    y = [node[1] for node in nodes]
    h = [x[j+1] - x[j] for j in range(0, n)]
    mu = [None] + [h[j-1] / (h[j-1] + h[j]) for j in
 range(1,n)] + [mu_n]
    lam = [lam_0] + [h[j] / (h[j-1] + h[j]) for j in
 range(1,n)] + [None]
    d = [6 * (y[j-1] / h[j-1] / (h[j-1]+h[j]) +\
             y[j+1] / h[j] / (h[j-1]+h[j]) -\
             y[j] / h[j-1] / h[j]) for j in range(1,
 n)]
    d = [d_0] + d + [d_n]

    coefficient_matrix =  np.zeros((n+1,n+1))
    for i in range(n+1):
        coefficient_matrix[i][i] = 2
    for i in range(n):
        coefficient_matrix[i][i+1] = lam[i]
        coefficient_matrix[i+1][i] = mu[i+1]

    M = solve(coefficient_matrix, d)

    A = [(y[j+1] - y[j]) / h[j] - h[j] / 6 * (M[j+1]
 - M[j])  for j in range(n)]
    B = [y[j] - M[j] * h[j] ** 2 / 6  for j in range
(n)]

    S = [np.poly1d([-M[j]]) / 6 / h[j] * np.poly1d([
1, -x[j+1]]) ** 3 +\
        np.poly1d([M[j+1]]) / 6 / h[j] * np.poly1d([
1, -x[j]]) ** 3 + A[j] * np.poly1d([1, -x[j]]) + B[j
]
        for j in range(n)]
    # Bug if M[] is not converted into 'poly1d' obje
ct!
    return S
```

```
splines = cubic_splines(uniform_nodes_of_Runge, lam_
0=0, mu_n=0, d_0=0.421, d_n=0.421)

for i in range(len(splines)):
    print('[', '%.1f' % (uniform_nodes_of_Runge[i][0
]), ',',
          '%.1f' % (uniform_nodes_of_Runge[i+1][0]),
  ']:')
    print(np.poly1d(splines[i]))
```

```
[ -1.0 , -0.9 ]:
        3          2
0.1577 x + 0.5785 x + 0.7576 x + 0.3753
[ -0.9 , -0.8 ]:
        3          2
0.2738 x + 0.8917 x + 1.04 x + 0.4599
[ -0.8 , -0.7 ]:
        3         2
0.4633 x + 1.347 x + 1.403 x + 0.5569
[ -0.7 , -0.6 ]:
        3       2
0.8697 x + 2.2 x + 2.001 x + 0.6964
[ -0.6 , -0.5 ]:
      3         2
1.58 x + 3.479 x + 2.768 x + 0.8498
[ -0.5 , -0.4 ]:
       3         2
3.544 x + 6.425 x + 4.241 x + 1.095
[ -0.4 , -0.3 ]:
       3         2
5.728 x + 9.046 x + 5.29 x + 1.235
[ -0.3 , -0.2 ]:
       3         2
12.53 x + 15.17 x + 7.127 x + 1.419
[ -0.2 , -0.1 ]:
        3         2
-32.79 x - 12.02 x + 1.688 x + 1.056
[ -0.1 , -0.0 ]:
        3         2
-89.07 x - 28.91 x - 3.109e-15 x + 1
[ -0.0 , 0.1 ]:
       3         2
89.07 x - 28.91 x - 3.997e-15 x + 1
[ 0.1 , 0.2 ]:
       3         2
32.79 x - 12.02 x - 1.688 x + 1.056
[ 0.2 , 0.3 ]:
        3         2
-12.53 x + 15.17 x - 7.127 x + 1.419
[ 0.3 , 0.4 ]:
```

```
         3         2
-5.728 x + 9.046 x - 5.29 x + 1.235
[ 0.4 , 0.5 ]:
         3         2
-3.544 x + 6.425 x - 4.241 x + 1.095
[ 0.5 , 0.6 ]:
        3         2
-1.58 x + 3.479 x - 2.768 x + 0.8498
[ 0.6 , 0.7 ]:
          3       2
-0.8697 x + 2.2 x - 2.001 x + 0.6964
[ 0.7 , 0.8 ]:
          3         2
-0.4633 x + 1.347 x - 1.403 x + 0.5569
[ 0.8 , 0.9 ]:
          3          2
-0.2738 x + 0.8917 x - 1.04 x + 0.4599
[ 0.9 , 1.0 ]:
          3          2
-0.1577 x + 0.5785 x - 0.7576 x + 0.3753
```

Using the polynomial expressions above, we can compute the function value $S(x)$ of each of the 41 points.

Show $f(x),\ S(x),\ |S(x)-f(x)|$ in the following table. Compared to polynomial interpolation in (a), (b), the cubic spline function is more accurate in fitting Runge function. And no sigular points as well.

In [14]:

```
spline_function_values = []
for j in range(len(uniform_points_of_Runge)-1):
    spline_function_values.append(splines[j//2](unif
orm_points_of_Runge[j]))
spline_function_values.append(splines[-1](uniform_po
ints_of_Runge[-1]))

df = DataFrame({'x':uniform_points_of_Runge, 'S(x)':
spline_function_values,
                'f(x)':Runge_values, '|S(x)-f(x)|':a
bs(spline_function_values - Runge_values)},
               columns=['x', 'f(x)', 'S(x)', '|S(x)-
f(x)|'])
df.round({'x': 2})
```

Out[14]:

	x	f(x)	S(x)	|S(x)-f(x)|
0	-1.00	0.038462	0.038462	8.326673e-17
1	-0.95	0.042440	0.042438	2.412308e-06
2	-0.90	0.047059	0.047059	2.844947e-16
3	-0.85	0.052459	0.052457	1.922724e-06
4	-0.80	0.058824	0.058824	9.714451e-17
5	-0.75	0.066390	0.066387	2.902889e-06
6	-0.70	0.075472	0.075472	4.163336e-17
7	-0.65	0.086486	0.086475	1.100250e-05
8	-0.60	0.100000	0.100000	2.359224e-16
9	-0.55	0.116788	0.116786	1.943161e-06
10	-0.50	0.137931	0.137931	3.053113e-16
11	-0.45	0.164948	0.164865	8.376322e-05
12	-0.40	0.200000	0.200000	6.383782e-16
13	-0.35	0.246154	0.246268	1.142769e-04
14	-0.30	0.307692	0.307692	4.440892e-16
15	-0.25	0.390244	0.389420	8.243211e-04
16	-0.20	0.500000	0.500000	1.110223e-16
17	-0.15	0.640000	0.643169	3.168936e-03
18	-0.10	0.800000	0.800000	1.110223e-16
19	-0.05	0.941176	0.938866	2.310258e-03
20	0.00	1.000000	1.000000	1.110223e-16

	x	f(x)	S(x)	|S(x)-f(x)|
21	0.05	0.941176	0.938866	2.310258e-03
22	0.10	0.800000	0.800000	1.110223e-16
23	0.15	0.640000	0.643169	3.168936e-03
24	0.20	0.500000	0.500000	1.110223e-16
25	0.25	0.390244	0.389420	8.243211e-04
26	0.30	0.307692	0.307692	2.220446e-16
27	0.35	0.246154	0.246268	1.142769e-04
28	0.40	0.200000	0.200000	5.551115e-17
29	0.45	0.164948	0.164865	8.376322e-05
30	0.50	0.137931	0.137931	1.387779e-16
31	0.55	0.116788	0.116786	1.943161e-06
32	0.60	0.100000	0.100000	1.526557e-16
33	0.65	0.086486	0.086475	1.100250e-05
34	0.70	0.075472	0.075472	3.747003e-16
35	0.75	0.066390	0.066387	2.902889e-06
36	0.80	0.058824	0.058824	1.249001e-16
37	0.85	0.052459	0.052457	1.922724e-06
38	0.90	0.047059	0.047059	1.110223e-16
39	0.95	0.042440	0.042438	2.412308e-06
40	1.00	0.038462	0.038462	1.179612e-16

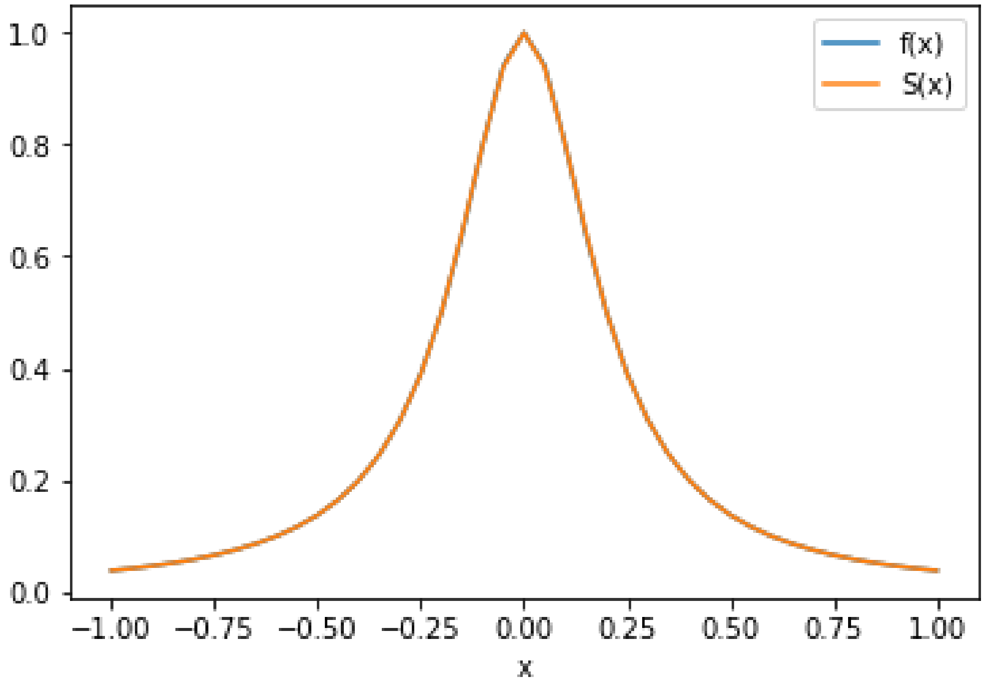
Plot $f(x)$ and $S(x)$ together. Compared to polynomial interpolation, cubic spline function fits much better in general.

In [15]:

```
df.plot.line(x='x', y=['f(x)', 'S(x)'])
```

Out[15]:

```
<matplotlib.axes._subplots.AxesSubplot a
t 0x112f7af28>
```



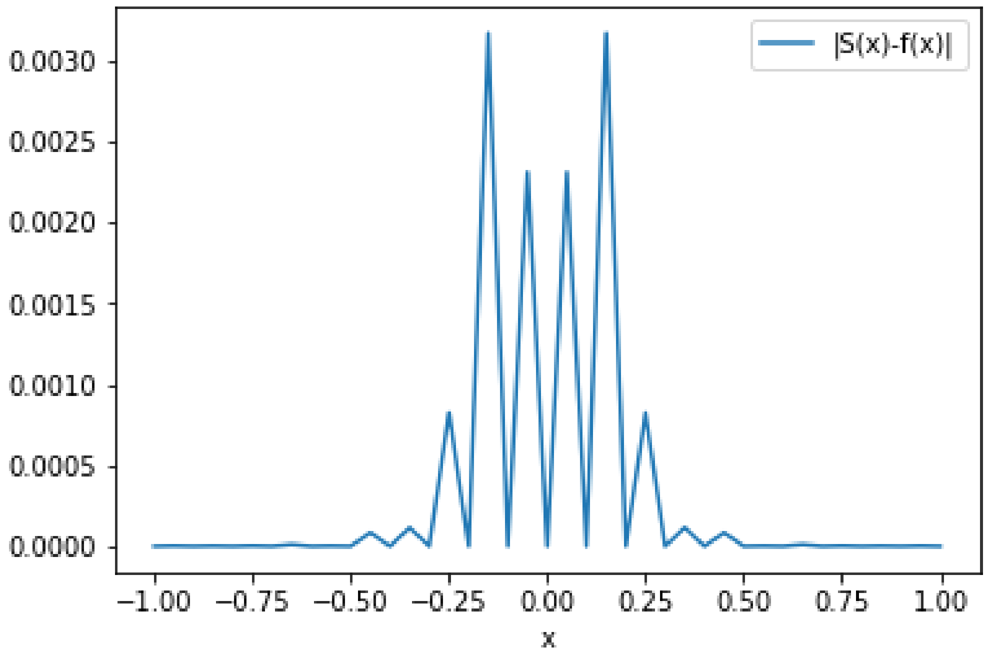
Plot $|S(x) - f(x)|$. It's very interesting that the error is larger around the origin, just opposed to polynomial interpolation in (a).

In [16]:

```
df.plot.line(x='x', y='|S(x)-f(x)|')
```

Out[16]:

```
<matplotlib.axes._subplots.AxesSubplot a
t 0x1125b64a8>
```



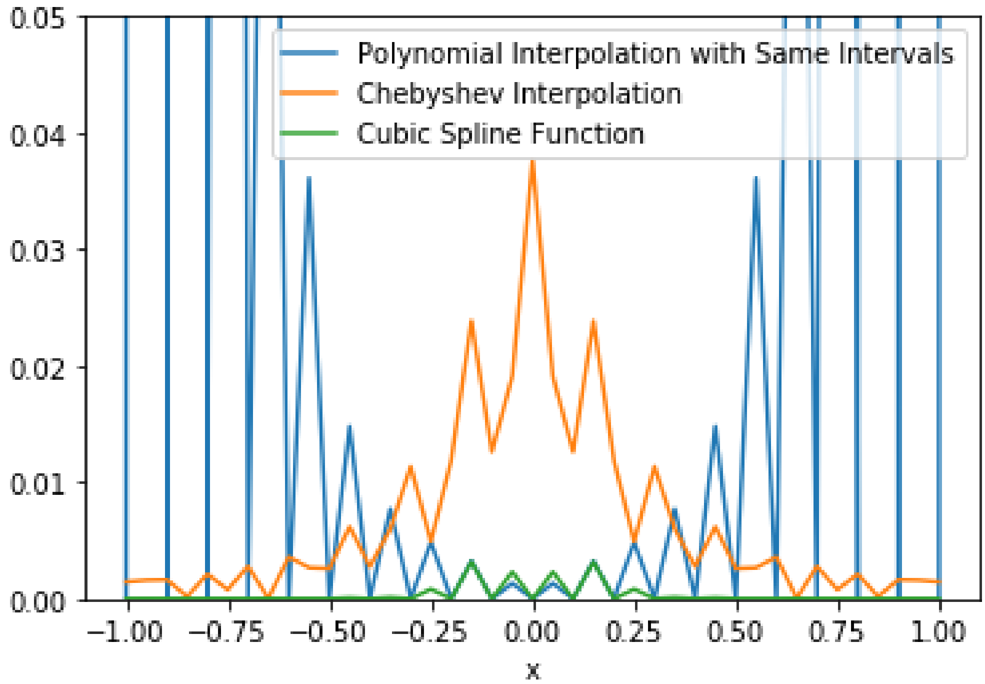
Lastly, let's put the deviation of the fitting function from $f(x)$ in three method together in the diagram below.

In [17]:

```
df = DataFrame({'x':uniform_points_of_Runge,
                'Polynomial Interpolation with Same
 Intervals':abs(P20_values - Runge_values),
                'Chebyshev Interpolation':abs(Chebys
hev_values - Runge_values),
                'Cubic Spline Function':abs(spline_f
unction_values - Runge_values)},
               columns=['x', 'Polynomial Interpolati
on with Same Intervals',
                        'Chebyshev Interpolation',
'Cubic Spline Function'])
df.plot.line(x='x', y=['Polynomial Interpolation wit
h Same Intervals',
                        'Chebyshev Interpolation',
'Cubic Spline Function'], ylim=[0,.05])
```

Out[17]:

<matplotlib.axes._subplots.AxesSubplot at 0x112634c50>



Summary:

1. Cubic spline function is the most accurate method in this problem, followed by Chebyshev interpolation. Polynomial interpolation with same intervals does not work when x deviates from the origin.
2. Cubic spline function and Chebyshev interpolation have a larger deviation near the origin, which is opposed to polynomial interpolation with same intervals.

2. 样条函数在计算机绘图中的应用

(a)

Consruct the polar equation of cardioid $r(\phi)$. Calculate $\phi,\ x_t,\ y_t$, showed in the table below.

In [18]:

```
def cardioid(phi, a=0.5):
    return 2 * a * (1 - np.cos(phi))

def phi(t):
    return t * np.pi /4

def cardioid_coord(t):
    return { 'x': cardioid(phi(t)) * np.cos(phi(t)),
            'y': cardioid(phi(t)) * np.sin(phi(t))}

t_nodes = list(range(9))
phi_nodes = [phi(t)  for t in t_nodes]

x_nodes = [cardioid_coord(t)['x'] for t in t_nodes]
y_nodes = [cardioid_coord(t)['y'] for t in t_nodes]

DataFrame({'t':t_nodes, 'phi':phi_nodes, 'x_t':x_nod
es, 'y_t':y_nodes},
               columns=['t', 'phi', 'x_t', 'y_t'])
```

Out[18]:

	t	phi	x_t	y_t
0	0	0.000000	0.000000e+00	0.000000e+00
1	1	0.785398	2.071068e-01	2.071068e-01
2	2	1.570796	6.123234e-17	1.000000e+00
3	3	2.356194	-1.207107e+00	1.207107e+00
4	4	3.141593	-2.000000e+00	2.449294e-16
5	5	3.926991	-1.207107e+00	-1.207107e+00
6	6	4.712389	-1.836970e-16	-1.000000e+00
7	7	5.497787	2.071068e-01	-2.071068e-01
8	8	6.283185	0.000000e+00	-0.000000e+00

(b)

Use cubic spline function coded earlier to construct $S_\Delta(x;t),\ S_\Delta(y;t)$.

In this case, it's natural to choose periodic constrains on the spline functions, for the 1st and 2nd derivative of spline functions at $\phi = 0$ must equal to that at $\phi = 2\pi$. Under this constrains, however, the earlier code no longer works, because the coefficient matrix is not a three diagonal one. Therefore, I still choose to make constrains on 2nd derivatives.

For $S_\Delta(x;t),\ S''_\Delta(x;t)\Big|_{t=0\text{ or }8} = x''_t\Big|_{t=0\text{or}8} \approx 1.2337. \quad M_0 = M_{n=8} = 1.2$

For $S_\Delta(y;t),\ S''_\Delta(y;t)\Big|_{t=0\text{ or }8} = y''_t\Big|_{t=0\text{or}8} \approx 0. \quad \lambda_0 = \mu_n = 0,\ d_0 = d_n =$

In [19]:

```
splines_x = cubic_splines(nodes=[[t, x] for t, x in zip(t_nodes, x_nodes)],
                          lam_0=0, mu_n=0, d_0=2.4674, d_n=2.4674)

print('cubic spline function of x:')
for i in range(len(splines_x)):
    print('[', str(t_nodes[i]), ',', str(t_nodes[i+1]),  ']:')
    print(np.poly1d(splines_x[i], variable='t'))
```

```
cubic spline function of x:
[ 0 , 1 ]:
         3          2
-0.2992 t + 0.6169 t - 0.1105 t
[ 1 , 2 ]:
         3          2
-0.1518 t + 0.1745 t + 0.3318 t - 0.1474
[ 2 , 3 ]:
        3        2
0.3206 t - 2.66 t + 6 t - 3.926
[ 3 , 4 ]:
        3         2
0.2837 t - 2.327 t + 5.003 t - 2.93
[ 4 , 5 ]:
         3         2
-0.2837 t + 4.481 t - 22.23 t + 33.38
[ 5 , 6 ]:
         3         2
-0.3206 t + 5.034 t - 25 t + 37.99
[ 6 , 7 ]:
        3         2
0.1518 t - 3.468 t + 26.02 t - 64.04
[ 7 , 8 ]:
        3         2
0.2992 t - 6.565 t + 47.69 t - 114.6
```

In [20]:

```
splines_y = cubic_splines(nodes=[[t, y] for t, y in zip(t_nodes, y_nodes)],
                          lam_0=0, mu_n=0, d_0=0, d_n=0)

print('cubic spline function of y:')
for i in range(len(splines_y)):
    print('[', str(t_nodes[i]), ',', str(t_nodes[i+1]),  ']:')
    print(np.poly1d(splines_y[i], variable='t'))
```

```
cubic spline function of y:
[ 0 , 1 ]:
       3
0.1735 t + 0.03361 t
[ 1 , 2 ]:
        3         2
-0.2817 t + 1.366 t - 1.332 t + 0.4552
[ 2 , 3 ]:
        3          2
-0.2183 t + 0.9853 t - 0.5714 t - 0.0518
5
[ 3 , 4 ]:
       3         2
0.3265 t - 3.918 t + 14.14 t - 14.76
[ 4 , 5 ]:
       3         2
0.3265 t - 3.918 t + 14.14 t - 14.76
[ 5 , 6 ]:
        3         2
-0.2183 t + 4.254 t - 26.72 t + 53.34
[ 6 , 7 ]:
        3         2
-0.2817 t + 5.395 t - 33.57 t + 67.03
[ 7 , 8 ]:
       3         2
0.1735 t - 4.164 t + 33.34 t - 89.1
```

(c)

Plot cubic spline function (x_t, y_t) and original Cardioid curve (x, y) together with nodes pointed. Given that the number of nodes is limited, the result turned out to be satisifing!

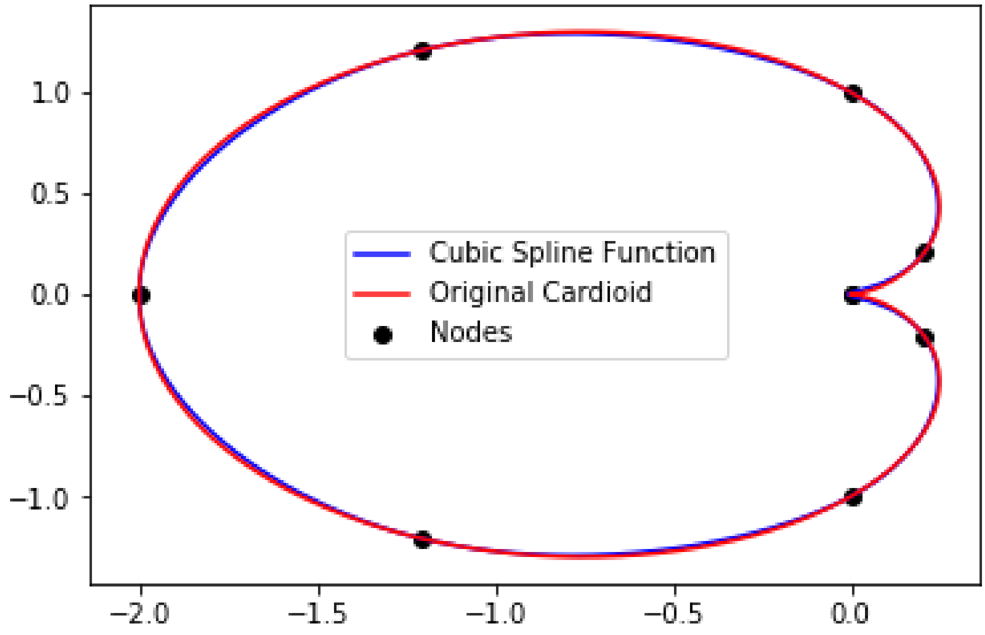
In [21]:

```
def cubic_spline_function(t, t_nodes, splines):
    return sum([splines[i](t) * ((t >= t_nodes[i]) a
nd (t < t_nodes[i+1]))
                for i in range(len(t_nodes)-1)])

step_number = 100000
# this step number may cost much time, you can dimin
ish it if you don't mind getting worse quanlity
t_values = np.linspace(0, 8, step_number)

fit_x_values = np.array([])
fit_y_values = np.array([])
for t in t_values:
    x = cubic_spline_function(t, t_nodes, splines_x)
    fit_x_values = np.append(fit_x_values, np.linspa
ce(x, x, 1))
    y = cubic_spline_function(t, t_nodes, splines_y)
    fit_y_values = np.append(fit_y_values, np.linspa
ce(y, y, 1))

plt.plot(fit_x_values, fit_y_values,'b',label='Cubic
 Spline Function')

real_x_values = np.array([])
real_y_values = np.array([])
for t in t_values:
    x = cardioid_coord(t)['x']
    real_x_values = np.append(real_x_values, np.lins
pace(x, x, 1))
    y = cardioid_coord(t)['y']
    real_y_values = np.append(real_y_values, np.lins
pace(y, y, 1))

plt.plot(real_x_values, real_y_values,'r',label='Ori
ginal Cardioid')



plt.scatter(x_nodes, y_nodes, color='black', label=
```

```
'Nodes')

plt.legend()
```

Out[21]:

```
<matplotlib.legend.Legend at 0x10e8e5710
>
```



(c)

The point is that the 1st and 2nd derivatives of x and y with respect to t are continuous along the fitting curve, including the nodes. Therefore, the gradient and curvature are continuous, which gives the fitting curve a smooth appearance.

3. 含有zeta函数的方程求解

(a)

Analyse the Problem

Define $\overline{\mathcal{Z}_{00}}(1;q^2)\Big|_l$ in the following way. For the second equality sign, we change $t \to t^2$ in the first integration to avoid the singularity at $t = 0$.

$$\overline{\mathcal{Z}_{00}}(1;q^2)\Big|_l = -\pi + \frac{\pi}{2}\int_0^1 dt\; t^{-3/2}\left(e^{tq^2} - 1\right) + \frac{1}{\sqrt{4\pi}}\sum_{\mathbf{n}\leq l}\frac{e^{q^2-\mathbf{n}^2}}{\mathbf{n}^2 - q^2}$$

$$= -\pi + \pi\int_0^1 dt\; t^{-2}\left(e^{t^2q^2} - 1\right) + \frac{1}{\sqrt{4\pi}}\sum_{\mathbf{n}\leq l}\frac{e^{q^2-\mathbf{n}^2}}{\mathbf{n}^2 - q^2} + \pi(1 - \delta_{\mathbf{n}0}$$

Here, the notation $\mathbf{n} = (n_1, n_2, n_3) \leq l$ means $n_i \leq l,\ i = 1, 2, 3$.

Naturally, we have:

$$\overline{\mathcal{Z}_{00}}(1;q^2)\Big|_{l\to\infty} = \mathcal{Z}_{00}(1;q^2)$$

To work out the problem, we need to compute the change of $\overline{\mathcal{Z}_{00}}(1;q^2)$ when you gradully increase the range of (each component of) $\mathbf{n}$.

Numerical Integration

To compute the integrals in the problem, we build algorithm using Simpson's rule. The parameter "peices*number" of it is the number of pieces you divide the whole interval into to later apply Simpson's rule. We use $\int*{0}^{\pi/2} sinx\ dx$ to test our algorithm, and find the result is accurate enough in our problem.

In [22]:

```
def integration_Simpson(integrand, bounds, pieces_nu
mber=1000):
    interval = (bounds[1] - bounds[0]) / (2 * pieces
_number)
    divide_points = np.array([integrand(x)
                              for x in np.arange(bounds[0], b
ounds[1] + interval, interval)])
    weights = np.array([1] + [4, 2] * (pieces_number
 - 1) + [4, 1])
    return interval / 3 * sum(divide_points * weight
s)

'%.15e' % integration_Simpson(lambda x: np.sin(x), (
0, np.pi / 2))
```

Out[22]:

```
'1.000000000000002e+00'
```

Computing Zeta Function

Construct an algorithm to compute $\overline{\mathcal{Z}_{00}}(1; q^2)\Big|_l$.

In [23]:

```
def zeta_00_sum_terms(q2, n2_max=150):
    sum_terms = {}
    for n2 in range(0, n2_max+1):
        term = np.exp(q2 - n2) / (n2 - q2 )
        if not n2 == 0:
            integrand = lambda t: 0 if t==0\
            else t ** (-3/2) * np.exp(t * q2) * np.e
xp(-np.pi ** 2 / t * n2)
            term = term + np.pi * integration_Simpso
n(integrand, bounds=(0, 1))
        term = term / np.sqrt(4 * np.pi)
        sum_terms[n2] = term
    return sum_terms

def zetabar_00(q2, l, s=1):
    result = -np.pi
    integrand = lambda t: q2 if t==0 else t ** (-2)
* (np.exp(t ** 2 * q2) - 1)
    result = result + np.pi * integration_Simpson(in
tegrand, bounds=(0, 1))
    #print(np.pi * integration_Simpson(integrand, bo
unds=(0, 1)))
    sum_terms = zeta_00_sum_terms(q2, n2_max=3 * l *
*2)
    result = result + sum([sum_terms[n1**2 + n2**2 +
 n3**2]
     for n1 in range(-l, l+1)
     for n2 in range(-l, l+1)
     for n3 in range(-l, l+1)])
    return result

zetabar_00(1.5, 5)
```

Out[23]:

```
1.3751096028849716
```

Compute $\overline{\mathcal{Z}_{00}}(1;q^2)\Big|_l$ for $q^2 = 2.9, 0.1$.

In [24]:

```
print('q^2=2.9')
for l in range(8):
    print('l=%d' % l, '%.11e' % zetabar_00(2.9, l))

print('q^2=0.1')
for l in range(8):
    print('l=%d' % l, '%.11e' % zetabar_00(0.1, l))
```

```
q^2=2.9
l=0 1.34324166161e+01
l=1 1.86456165446e+01
l=2 1.96575465200e+01
l=3 1.96592557848e+01
l=4 1.96592565599e+01
l=5 1.96592565600e+01
l=6 1.96592565600e+01
l=7 1.96592565600e+01
q^2=0.1
l=0 -5.93972038132e+00
l=1 -4.86578551506e+00
l=2 -4.84330440037e+00
l=3 -4.84323045469e+00
l=4 -4.84323041542e+00
l=5 -4.84323041541e+00
l=6 -4.84323041541e+00
l=7 -4.84323041541e+00
```

Two Extreme Cases

From the results above, we guess that $l = 7$ may be enough for 12-digit accuracy in most cases. But before trying on more values of q^2, it's necessary to consider two extreme cases.

- q^2 is very close to the square of an integer. $\lim_{q^2 \to n^2 \pm 0} \frac{e^{q^2 - \mathbf{n}^2}}{\mathbf{n}^2 - q^2} = \frac{1}{\mp 0} = \mp\infty$. Other terms ($\mathbf{n} \neq 0$) are negligible compared with it. So fewer terms of summation is needed in that case.
- $\mathcal{Z}_{00}(1; q^2) \approx 0$. Given that $\lim_{q^2 \to n^2 + 0} \frac{e^{q^2 - \mathbf{n}^2}}{\mathbf{n}^2 - q^2} = -\infty$ and $\lim_{q^2 \to (n+1)^2 - 0} \frac{e^{q^2 - (\mathbf{n+1})^2}}{(\mathbf{n+1})^2 - q^2} = +\infty$, there must be a zero point in $(n, n+1)$. And the relative error will go to infinity when approaching the zero points. We cannot guarantee any significance digit accuracy as a result.

Except the extrme cases, we compute $\overline{\mathcal{Z}_{00}}(1; q^2)\Big|_l$ for $l = 0, 2, \cdots, 7$ and for more "general" values of q^2 to get required l for 6-digit and 12-digit accuracy respectively.

In [25]:

```
def accuracy_check(q2):
    former, latter = 0, 0
    print('q^2=', q2)
    six_digit = False
    for l in range(8):
        latter = zetabar_00(q2, l)
        if '%.11e' % former == '%.11e' % latter:
            print('twelve digit:', l)
            break
        elif '%.5e' % former == '%.5e' % latter and
not six_digit:
            print('six digit:', l , end='\t')
            six_digit = True
        former = latter

for q2 in [1e-3] + list(np.arange(1-1e-3, 3, 0.2)):
    accuracy_check(q2)
```

```
q^2= 0.001
six digit: 3    twelve digit: 5
q^2= 0.999
six digit: 3    twelve digit: 5
q^2= 1.199
six digit: 4    twelve digit: 6
q^2= 1.399
six digit: 4    twelve digit: 6
q^2= 1.5990000000000002
six digit: 4    twelve digit: 6
q^2= 1.7990000000000004
six digit: 4    twelve digit: 5
q^2= 1.9990000000000006
six digit: 3    twelve digit: 5
q^2= 2.1990000000000003
six digit: 4    twelve digit: 5
q^2= 2.3990000000000005
six digit: 4    twelve digit: 6
q^2= 2.5990000000000006
six digit: 4    twelve digit: 6
q^2= 2.799000000000001
six digit: 4    twelve digit: 6
q^2= 2.999000000000001
six digit: 3    twelve digit: 5
```

From the results above, we see that 6-digit accuracy generally requires $l \geq 4$ ($9^3 = 729$ terms computed) and 12-digit accuracy $l \geq 6$ ($13^3 = 2197$ terms computed).

In later computation, we will choose $l = 6$ for both accuracy (except for points near zero points) and economy.

(b)

In this section, we want to find the cross point of $S(q^2) = \pi^{3/2}(\frac{1}{A0} + \frac{R0}{2}q^2)$ and $\mathcal{Z}_{00}(1; q^2)$.

In [26]:

```
def scatter_expansion(q2):
    return (np.pi) ** (3/2) * (1 / 1.0 + 0.5 / 2 * q
2)

q2_values = np.linspace(0, 1, 50)
zeta_values = np.array([zetabar_00(q2, l=6) for q2 i
n q2_values])
expansion_values = np.array([scatter_expansion(q2) f
or q2 in q2_values])

df = DataFrame({'q^2':q2_values, 'Zeta Function Valu
es':zeta_values,
                'Scatter Expansion Values':expansion
_values},
               columns=['q^2', 'Zeta Function Value
s', 'Scatter Expansion Values'])
```

```
/Library/Frameworks/Python.framework/Ver
sions/3.6/lib/python3.6/site-packages/ip
ykernel_launcher.py:4: RuntimeWarning: d
ivide by zero encountered in double_scal
ars
  after removing the cwd from sys.path.
```

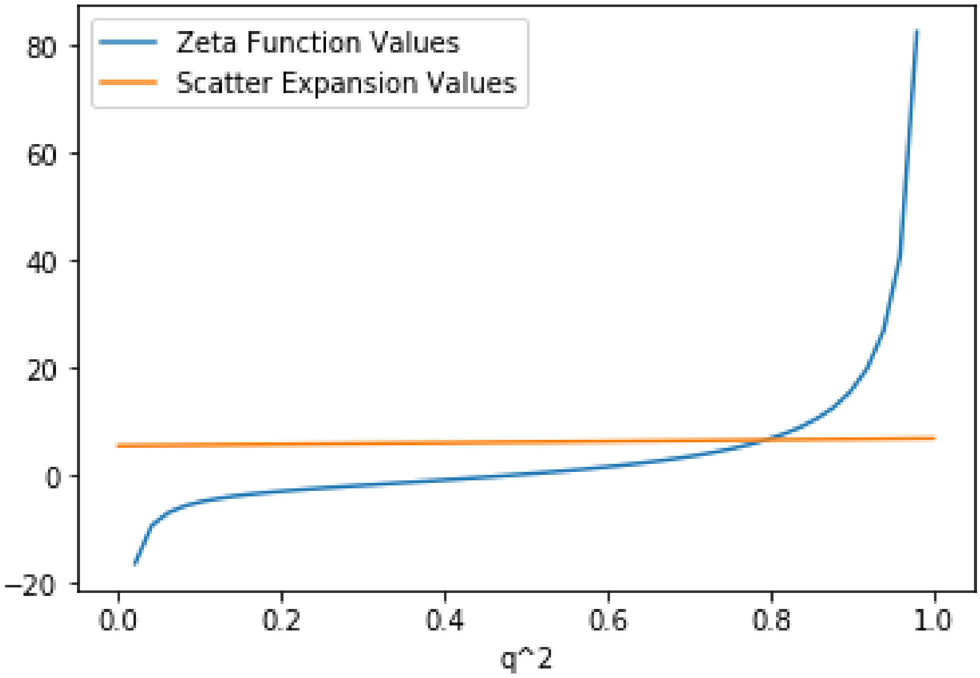
Plot two functions on $(0, 1)$. We get the crosspoint at q^2 slightly smaller that 1 as told.

In [27]:

```
df.plot.line(x='q^2', y=['Zeta Function Values', 'Sc
atter Expansion Values'])
```

Out[27]:

```
<matplotlib.axes._subplots.AxesSubplot a
t 0x1130f5748>
```



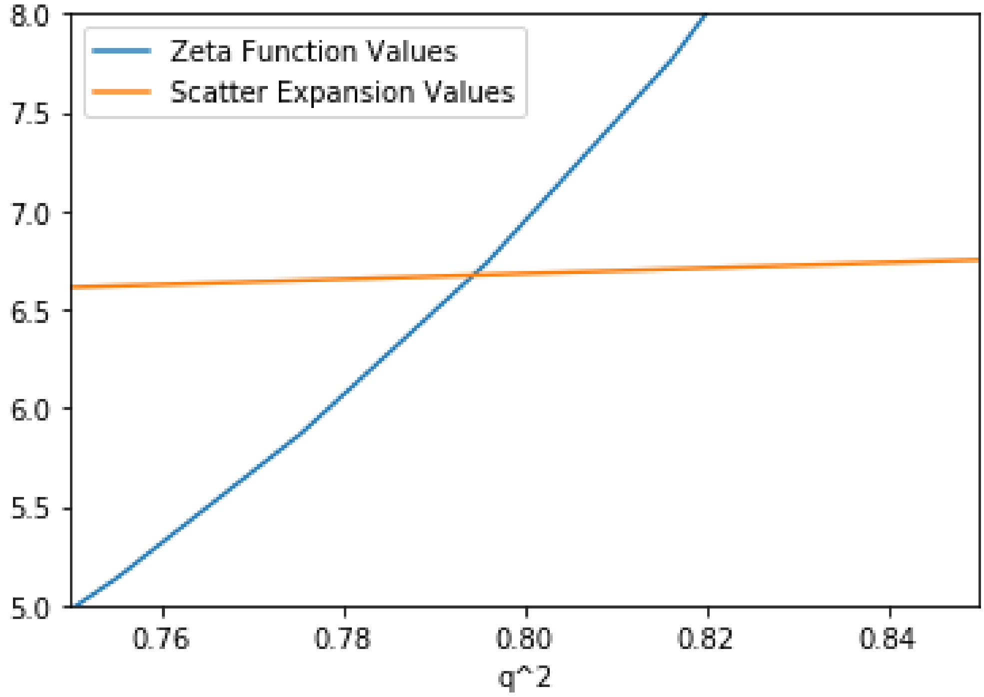
Zoom in to know the cross point exists near $q^2 = 0.8$.

In [28]:

```
df.plot.line(x='q^2', y=['Zeta Function Values', 'Scatter Expansion Values'],
             xlim=[0.75,0.85], ylim=[5,8])
```

Out[28]:

```
<matplotlib.axes._subplots.AxesSubplot at 0x113332978>
```



Search it on $[0.78, 0.82]$, naively using binary search algorithm. When the binary interval's length is smaller than 0.5×10^6, 6-digit accuracy is supposed to be satisfied.

In [29]:

```
q2_r, q2_l = 0.82, 0.78
while q2_r - q2_l > 5e-7:
    q2_m = (q2_r + q2_l) / 2
    if zetabar_00(q2_m, l=6) > scatter_expansion(q2_
m):
        q2_r = q2_m
    else:
        q2_l = q2_m

print('q^2 = %.6f' % q2_m)
print('Zeta function value =', zetabar_00(q2_m, l=6
))
```

```
q^2 = 0.794516
Zeta function value = 6.674360222784506
```

From the above result, we see the solution is not close to zero points. So the l we choose should be safe. And we check the accuray for more safety.

In [30]:

```
accuracy_check(q2_m)
```

```
q^2= 0.7945156860351562
six digit: 4    twelve digit: 5
```

In fact, the result is not that accurate, because the scattering expansion is merely 1st order accurate. But the first few digits is trustful at least.