Adv QM

第一次作业

1.2 $X = a_0 + \vec{\sigma}\cdot\vec{a}$

a. $\mathrm{Tr}\,X = 2a_0$

$\mathrm{Tr}\,\sigma_i X = 2a_i \quad (\sigma_{1,2,3} \text{ traceless})$

b. $a_0 = \frac{1}{2}\mathrm{Tr}\,X = \frac{1}{2}(X_{11} + X_{22})$

$a_1 = \frac{1}{2}\mathrm{Tr}\,\sigma_1 X = \frac{1}{2}(X_{12} + X_{21})$

$a_2 = \frac{1}{2}\mathrm{Tr}\,\sigma_2 X = \frac{1}{2}(-iX_{12} + iX_{21})$

$a_3 = \frac{1}{2}\mathrm{Tr}\,\sigma_3 X = \frac{1}{2}(X_{11} - X_{22})$

1.3 Show that the determinant of a 2×2 matrix $\boldsymbol{\sigma}\cdot\mathbf{a}$ is invariant under

$$\boldsymbol{\sigma}\cdot\mathbf{a} \to \boldsymbol{\sigma}\cdot\mathbf{a}' \equiv \exp\left(\frac{i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right)\boldsymbol{\sigma}\cdot\mathbf{a}\exp\left(\frac{-i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right).$$

Find a_k' in terms of a_k when $\hat{\mathbf{n}}$ is in the positive z-direction and interpret your result.

$\hat{n} = \hat{z} \Rightarrow \vec{\sigma}\cdot\hat{n} = \sigma_3$

$$\vec{\sigma}\cdot\vec{a}' = \vec{\sigma}\cdot\vec{a} + \frac{i\phi}{2}\cdot h_{\sigma_3}\vec{\sigma}\cdot\vec{a} + \frac{1}{2}\left(\frac{i\phi}{2}\right)^2 h_{\sigma_3}^2\,\vec{\sigma}\cdot\vec{a} + \cdots$$

(B-H formula)

Here, $h_A B =: [A, B]$

$$h_{\sigma_3} \vec{\sigma}\cdot\vec{a} = i a_1 \sigma_2 - i a_2 \sigma_1$$

$$h^2_{\sigma_3} \vec{\sigma}\cdot\vec{a} = -i^2 a_1 \sigma_1 - i^2 \sigma_2 \sigma_2$$

$$\cdots\cdots$$

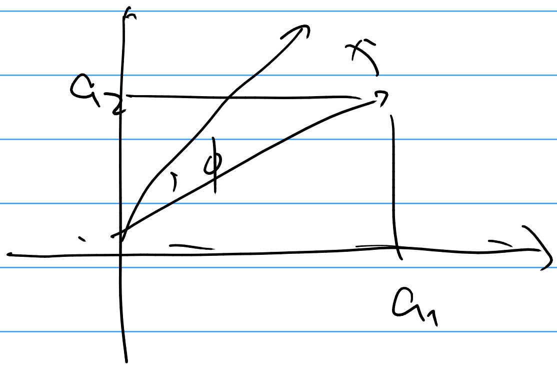
$$a_1' = a_1 + \frac{i\phi}{2}(-i a_2) + \frac{1}{2}\left(\frac{i\phi}{2}\right)^2 (-i^2)\, a_1 + \cdots$$

$$= a_1 \cos\phi + a_2 \sin\phi$$

$$a_2' = a_2 + \frac{i\phi}{2}\, i a_1 + \frac{1}{2}\left(\frac{i\phi}{2}\right)^2 (-i^2)\, a_2 + \cdots$$

$$= -a_1 \sin\phi + a_2 \cos\phi$$

Interpretation: it is like a rotation w/ angle, ϕ.



1.4 $$X = \sum_x x\, |x\rangle\langle x| \qquad Y = \sum_y y\, |y\rangle\langle y|$$

a. $$\mathrm{Tr}\, XY = \mathrm{Tr} \sum xy\, |x\rangle\langle x|y\rangle\langle y|$$

$$= Tr \sum xy |y\rangle\langle y|x\rangle\langle x| = Tr\, Y X$$

b. $$(XY)^{+} = \left(\sum xy\ |x\rangle\langle x|y\rangle\langle y|\right)^{+}$$

$$= \sum x^{*} y^{*}\ |y\rangle\langle y|x\rangle\langle x|$$

$$= Y^{+} X^{+}$$

c. $$A = \sum_{a} a\ |a\rangle\langle a|$$

$$\exp(if(A)) = \sum_{a} e^{if(a)}\ |a\rangle\langle a|$$

d. $$\sum_{a'} \psi^{*}_{a'}(x')\ \psi_{a'}(x'')$$

$$= \sum_{a'} \langle a'|x'\rangle\langle x''|a'\rangle = Tr\ |x'\rangle\langle x''|$$

$$= \delta(x - x'')$$

1.5

a. $$|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle a|\alpha\rangle\langle\beta|a\rangle & \langle a|\alpha\rangle\langle\beta|a'\rangle \\ \langle a'|\alpha\rangle\langle\beta|a\rangle & \langle a'|\alpha\rangle\langle\beta|a'\rangle \\ \vdots & \ddots \end{pmatrix}$$

b. $|\alpha\rangle = |S_z = \frac{\hbar}{2}\rangle =: |0\rangle$

$|\beta\rangle = |S_x = \frac{\hbar}{2}\rangle =: |+\rangle$

$|a\rangle = |0\rangle$, $|a'\rangle = |1\rangle$

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle 0|0\rangle\langle +|0\rangle & \langle 0|0\rangle\langle +|1\rangle \\ \langle 1|0\rangle\langle +|0\rangle & \langle 1|0\rangle\langle +|1\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

第二次作业

1.8 $S_x S_y = (\frac{\hbar}{2})^2 i \left(|+\rangle\langle +| - |-\rangle\langle -| \right)$

$= i\frac{\hbar}{2} S_z$

$S_y S_x = (\frac{\hbar}{2})^2 i \left(-|+\rangle\langle +| + |-\rangle\langle -| \right)$

$= -i\frac{\hbar}{2} S_z$

$[S_x, S_y] = i\hbar S_z$

$[S_y, S_x] = -i\hbar S_z$

$\{S_x, S_y\} = 0$

$$S_x S_x = \left(\frac{\hbar}{2}\right)^2 \mathbb{1} \qquad [S_x, S_x] = 0$$

$$\{S_x, S_x\} = \frac{\hbar^2}{2}$$

Similiarly, we have other com. and anti-comm.

1.9

$$\vec{S}\cdot\hat{n}\,|\vec{S}\cdot\hat{n};+\rangle = \frac{\hbar}{2}\,|\vec{S}\cdot\hat{n};+\rangle$$

$$\hat{n} = (\cos\alpha\sin\beta,\ \sin\alpha\sin\beta,\ \cos\beta)$$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\begin{pmatrix} \cos\beta & \cos\alpha\sin\beta - i\sin\alpha\sin\beta \\ \cos\alpha\sin\beta + i\sin\alpha\sin\beta & -\cos\beta \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\beta & e^{-i\alpha}\sin\beta \\ e^{i\alpha}\sin\beta & -\cos\beta \end{pmatrix}$$

Let $|\phi\rangle = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\begin{pmatrix} \cos\beta\,\phi_0 + e^{-i\alpha}\sin\beta\,\phi_1 \\ e^{i\alpha}\sin\beta\,\phi_0 - \cos\beta\,\phi_1 \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$$

$$\Rightarrow \cos\beta\, \phi_0 + e^{-i\alpha} \sin\beta\, \phi_1 = \phi_0$$

$$\Rightarrow \frac{\phi_1}{\phi_0} = \frac{1-\cos\beta}{e^{-i\alpha}\sin\beta} = \frac{e^{i\alpha}\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}}$$

$$\Rightarrow |\vec{S}\cdot\hat{n};+\rangle = \begin{pmatrix} \cos\frac{\beta}{2} \\ e^{i\alpha}\sin\frac{\beta}{2} \end{pmatrix}$$

1.10

$$H = a\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|H-\lambda| = \begin{vmatrix} a-\lambda & a \\ a & -a-\lambda \end{vmatrix}$$

$$= \lambda^2 - 2a^2 = 0$$

$$\lambda = \pm\sqrt{2}\, a \Rightarrow \text{eig. vals.}$$

$$(H \mp \sqrt{2} a)\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$$

$$= a\begin{pmatrix} 1\pm\sqrt{2} & 1 \\ 1 & -1\pm\sqrt{2} \end{pmatrix}\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = 0$$

$$\Rightarrow (1\pm\sqrt{2})\phi_0 + \phi_1 = 0$$

eig. states

$\Downarrow$

$$\phi_0 = \frac{-1}{\sqrt{(1\pm\sqrt{2})^2+1}}, \quad \phi_1 = \frac{1\pm\sqrt{2}}{\sqrt{(1\pm\sqrt{2})^2+1}}$$

第三次作业 第一章第17，18，19，20题、PPT第二页习题

1.17 $[A_1, H] = 0 \quad [A_2, H] = 0 \quad [A_1, A_2] \neq 0$

Suppose $H|\psi\rangle = \lambda|\psi\rangle$, i.e. $|\psi\rangle$ is eig state

Then $HA_1|\psi\rangle = A_1 H|\psi\rangle = \lambda A_1|\psi\rangle$

$HA_2|\psi\rangle = A_2 H|\psi\rangle = \lambda A_2|\psi\rangle$

To keep non-deg., $A_1|\psi\rangle \propto A_2|\psi\rangle \propto |\psi\rangle$.

And for $[A_1, A_2] \neq 0$, this is generally impossible.

Unless $A_1|\psi\rangle = 0$

$A_2|\psi\rangle = 0$

$k = m\omega^2$

Example: $H = \frac{1}{2m}\vec{P}^2 + \frac{1}{2}k\vec{r}^2 = \hbar\omega(a^\dagger a + \frac{1}{2})$

$L_x = yP_z - zP_y = i\hbar(a_z^\dagger a_y - a_y^\dagger a_z)$

$L_z = xP_y - yP_x = i\hbar(a_y^\dagger a_x - a_x^\dagger a_y)$

Note that
$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$
$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a).$$

eig state of H: $|n_x, n_y, n_z\rangle$

g.s. $|n_x = 0, n_y = 0, n_y = 0\rangle$ is non-deg.

Because $L_x|0,0,0\rangle = 0$

$L_z|0,0,0\rangle = 0$

1.18

a. $$\left(\langle\alpha| + \lambda^* \langle\beta|\right)\left(|\alpha\rangle + \lambda|\beta\rangle\right)$$

$$= \left\| |\alpha\rangle + \lambda|\beta\rangle \right\|^2 \geq 0$$

Schwarz ineq: $\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle$

$$\leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$$

Let $\lambda = -\dfrac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$

then $0 \leq \langle\alpha|\alpha\rangle + \lambda\lambda^* \langle\beta|\beta\rangle$

$$+ \lambda^* \langle\beta|\alpha\rangle + \lambda \langle\alpha|\beta\rangle$$

$$= \langle\alpha|\alpha\rangle + \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$$

$$- \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} - \frac{\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}$$

$$= \frac{1}{\langle\beta|\beta\rangle}\left(\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle - \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle\right)$$

$\Rightarrow$ Schwarz ineq. ✓

b. Generalised uncertainty relation:

$$\langle \Delta A^2\rangle\langle\Delta B^2\rangle \leq \frac{1}{4}\left|\langle[\Delta A, \Delta B]\rangle\right|^2$$

Suppose $\Delta A|\alpha\rangle = \lambda\,\Delta B|\alpha\rangle$ λ: pure img

First, equality in (1.4.53) holds, which is a Schwarz ineq.

Second, in (1.4.62), the 2nd term on l.h.s

$$\frac{1}{2}\langle\alpha|\{\Delta A, \Delta B\}|\alpha\rangle$$

$$= \frac{1}{2}\left(\langle\alpha|\Delta A\,\Delta B|\alpha\rangle + \langle\alpha|\Delta B\,\Delta A|\alpha\rangle\right)$$

$$= \frac{1}{2}\left(\langle\alpha|\lambda^*\Delta B^2|\alpha\rangle + \langle\alpha|\Delta B^2\lambda|\alpha\rangle\right)$$

$$= \frac{1}{2}(\lambda+\lambda^*)\langle\alpha|\Delta B^2|\alpha\rangle = 0$$

Which is for λ is pure img.

Thus, the whole equality holds.

c.

$$\langle x'|\alpha\rangle = (2\pi d^2)^{-1/4}\exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}\right]$$

Formula of Gaussian integral:

$$\int_{-\infty}^{\infty} x^{2n}e^{-\alpha x^2}\,dx = (-1)^n\int_{-\infty}^{\infty}\frac{\partial^n}{\partial\alpha^n}e^{-\alpha x^2}\,dx$$
$$= (-1)^n\frac{\partial^n}{\partial\alpha^n}\int_{-\infty}^{\infty}e^{-\alpha x^2}\,dx$$
$$= \sqrt{\pi}(-1)^n\frac{\partial^n}{\partial\alpha^n}\alpha^{-\frac{1}{2}}$$
$$= \sqrt{\frac{\pi}{\alpha}}\frac{(2n-1)!!}{(2\alpha)^n}$$

$$\langle\alpha|\Delta x^2|\alpha\rangle = (2\pi d^2)^{-\frac{1}{2}}\int(x'-\langle x\rangle)^2\exp\left(-\frac{(x'-\langle x\rangle)^2}{2d^2}\right)dx'$$
$$= (2\pi d^2)^{-\frac{1}{2}}(2\pi d^2)^{\frac{1}{2}}\frac{1}{\left(2\frac{1}{2d^2}\right)} = d^2.$$

$$\langle x'|\Delta p|\alpha\rangle = (2\pi d^2)^{-\frac{1}{4}}\left(-i\hbar\,\partial_{x'} - \langle p\rangle\right)\exp\left(\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}\right)$$
$$= (2\pi d^2)^{-\frac{1}{4}}\left(-i\hbar\left(\cancel{\frac{i\langle p\rangle}{\hbar}} - \frac{2(x'-\langle x\rangle)}{4d^2}\right) - \cancel{\langle p\rangle}\right)$$
$$\exp\left(\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}\right)$$

$$= (2\pi d^2)^{-\frac{1}{4}}\frac{i\hbar(x'-\langle x\rangle)}{2d^2}\exp(\cdot\text{——})$$

Note that $\langle \alpha | \Delta p | x' \rangle = \langle x' | \Delta p | \alpha \rangle^*$

$$\langle \alpha | \Delta p^2 | \alpha \rangle = (2\pi d^2)^{-\frac{1}{2}} \frac{\hbar^2}{(2d^2)^2} \int (x' - \langle x \rangle)^2 \exp\left(-\frac{(x' - \langle x \rangle)^2}{2d^2} \right) dx'$$

$$= (2\pi d^2)^{-\frac{1}{2}} \frac{\hbar^2}{(2d^2)^2} (2\pi d^2)^{\frac{1}{2}} \frac{1}{\left(2 \cdot \frac{1}{2d^2}\right)} = \frac{\hbar^2}{4d^2}.$$

$$\sqrt{\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle} = \sqrt{\frac{1}{4}\hbar^2} = \frac{\hbar}{2}.$$

$$\langle x' | \Delta x | \alpha \rangle = (2\pi d^2)^{-\frac{1}{4}} (x' - \langle x \rangle) \exp(\text{———})$$

$$= \left(\frac{i\hbar}{2d^2} \right)^{-1} \langle x' | \Delta p | \alpha \rangle$$

where $\left(\frac{i\hbar}{2d^2} \right)^{-1}$ is pure imag.

1.19 a.

$$\langle S_x \rangle = 0$$

$$\langle S_x^2 \rangle = \langle 0 | S_x^2 | 0 \rangle = \frac{1}{4}\hbar^2$$

$$\langle \Delta S_x^2 \rangle = \frac{1}{4}\hbar^2$$

Similarly, $\langle \Delta S_y^2 \rangle = \frac{1}{4}\hbar^2$

$$[S_x, S_y] = 2i S_z \qquad \langle \Delta S_z^2 \rangle = 0$$

$$\text{l.h.s} = \left(\frac{1}{4}\hbar^2\right)^2 \geq 0 = \text{r.h.s}.$$

b. $\langle \Delta S_x^2 \rangle = 0$

$\langle \Delta S_y^2 \rangle = \frac{1}{4}\hbar^2$ $\qquad$ l.h.s. $= 0$

$\langle [S_x, S_y] \rangle = \langle 2i S_z \rangle = 0$ $\qquad$ r.h.s. $= 0$

$\Rightarrow$ l.h.s. $\geq$ r.h.s.

1.20. Note that $|\pm\rangle$ are eig states of z (not x)

$|\psi\rangle = \left(\cos\frac{\theta}{2} , \sin\frac{\theta}{2} e^{i\phi} \right)^T$ $\qquad$ $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$

$\langle S_x \rangle_\psi = \frac{\hbar}{2} \sin\theta \cos\phi$ $\qquad$ $\langle \Delta S_x^2 \rangle_\psi = \frac{\hbar^2}{4}(1 - \sin^2\theta \cos^2\phi)$

$\langle S_x^2 \rangle_\psi = \frac{\hbar^2}{4} \langle I \rangle_\psi = \frac{\hbar^2}{4}$

$\langle S_y \rangle_\psi = \frac{\hbar}{2} \sin\theta \sin\phi$

$\langle S_y^2 \rangle_\psi = \frac{\hbar^2}{4}$ $\qquad$ $\langle \Delta S_y^2 \rangle_\psi = \frac{\hbar^2}{4}(1 - \sin^2\theta \sin^2\phi)$

$$\langle \Delta S_x^2 \rangle \langle \Delta S_y^2 \rangle = \left(\frac{\hbar^2}{4}\right)^2 (1 - \sin^2\theta \cos^2\phi)(1 - \sin^2\theta \sin^2\phi)$$

$$\leq \left(\frac{\hbar^2}{4}\right)^2$$

So max is obtained i.f.f.

$\sin^2\theta \cos^2\phi = 0$ and $\sin^2\theta \sin^2\phi = 0$

$\Leftrightarrow \theta = 0$

$$|\psi\rangle \propto |-\rangle$$

$$|\langle [S_x, S_y] \rangle|^2 = |\langle i S_z \rangle|^2 = \frac{\hbar^2}{4}$$

Uncertainty relation holds.

Note.

逆定理：若$\hat{A},\hat{B}$对易，则$\hat{A},\hat{B}$一定有完备的共同本征态。

证明：设$|n\rangle$是$\hat{A}$的本征态，

$$\hat{A}|n\rangle = a_n|n\rangle,$$

由$[\hat{A},\hat{B}]=0$，有

$$\hat{A}\hat{B}|n\rangle = \hat{B}\hat{A}|n\rangle = a_n\hat{B}|n\rangle$$

说明$\hat{B}|n\rangle$也是$\hat{A}$的对应本征值a_n的本征态。若$\hat{A}$无简并，则$\hat{B}|n\rangle$与$|n\rangle$是同一个态，只能相差一个常数：

$$\hat{B}|n\rangle = b_n|n\rangle$$

故$|n\rangle$也是$\hat{B}$的本征态，即$\hat{A},\hat{B}$有共同的完备本征态。

若$\hat{A}$有简并，则可以用施密特方法来证明有同样的结果（作为习题）。

Suppose m-fold degeneracy on eigvalue a_n,

those eigstates are $|n,1\rangle \ldots |n,m\rangle$

Our goal is to construct $\{|n,i'\rangle\}_{i=1}^{m}$ s.t. $\langle n,i'|B|n,j'\rangle \propto \delta_{ij}$

This can be done by Schmidt orth.

(Here, normalization is ignored)

$$|n,1'\rangle = |n,1\rangle$$

$$|n,2'\rangle = |n,2\rangle - \frac{\langle n,1'|B|n,2\rangle}{\langle n,1'|B|n,1'\rangle}|n,1'\rangle$$

$$|n,3'\rangle = |n,3\rangle - \frac{\langle n,1'|B|n,3\rangle}{\langle n,1'|B|n,1'\rangle}|n,1'\rangle - \frac{\langle n,2'|B|n,3\rangle}{\langle n,2'|B|n,2'\rangle}|n,2'\rangle$$

$\sim \ldots \ldots -$

第四次作业　　第2章第1，2，3，4题

2.1

$$H = -\left(\frac{eB}{mc}\right) S_z = \omega S_z,$$

where $h_A B =: [A, B]$

$$\hat{O}(t) = e^{\frac{iHt}{\hbar}} \hat{O} e^{-\frac{iHt}{\hbar}} = \sum_{n=0}^{\infty} \frac{i^n}{n!} h_H^n \hat{O} = \sum_{n=0}^{\infty} \frac{(i\omega t)^n}{n!} h_{S_z/\hbar}^n \hat{O}$$

$$h_{S_z/\hbar} S_x = i S_y, \quad h_{S_z} S_y = -i S_x, \quad h_{S_z/\hbar} S_z = 0.$$

① $$S_x(t) = \sum_{n:\text{even}} \frac{(i\omega t)^n}{n!} S_x + \sum_{n:\text{odd}} \frac{(i\omega t)^n}{n!} i S_y$$

$$= \cos \omega t \, S_x - \sin \omega t \, S_y.$$

② $$S_y(t) = \sum_{n:\text{even}} \frac{(i\omega t)^n}{n!} S_y + \sum_{n:\text{odd}} \frac{(i\omega t)^n}{n!} (-i) \cdot S_x$$

$$= \cos \omega t \, S_y + \sin \omega t \, S_x.$$

③ $$S_z(t) = S_z.$$

2.2.

H is not Hermitian, so prob. conservation may be broken.

Suppose $H = h |2\rangle\langle 1|$

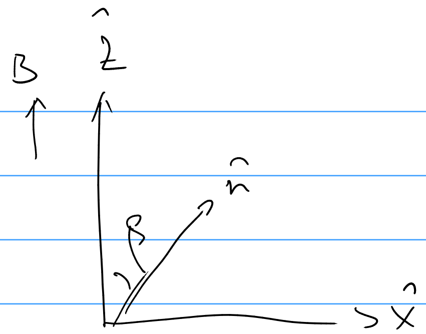
$$U = e^{-iHt} = \sum_{n=0}^{\infty} \frac{(-iHt)^n}{n!} = 1 - iHt = 1 - ih|2\rangle\langle 1|$$

Note that $H^n = 0, \ n \geq 2.$

$U|1\rangle = |1\rangle - ih|2\rangle$, whose norm is large that 1.

That breaks prob. conservation.

2.3



$$H = \frac{eB}{mc} S_z =: \omega S_z$$

$$|\psi(t=0)\rangle = \left(\cos\frac{\beta}{2}, \sin\frac{\beta}{2}\right)^T$$

a. As calculation in problem 2.2 suggests;

$$S_x(t) = \cos(\omega t) S_x - \sin\omega t\, S_y.$$

$$\langle S_x(t)\rangle = \frac{\hbar}{2}\cos(\omega t)\sin\beta$$

$$P_r\left(S_x = \frac{\hbar}{2}\right) = \frac{1}{2}\left(\cos(\omega t)\sin\beta + 1\right).$$

b. $\langle S_x(t)\rangle = \frac{\hbar}{2}\cos(\omega t)\sin\beta$ is given in a.

c. (i) $\beta \to 0$ $\quad P_r\left(S_x = +\frac{\hbar}{2}\right) = \frac{1}{2}$ $\quad \langle S_x(t)\rangle = 0$

(ii) $\beta \to \frac{\pi}{2}$ $\quad P_r\left(S_x = +\frac{\hbar}{2}\right) = \frac{1}{2}(\cos\omega t + 1)$ $\quad \langle S_x(t)\rangle = \frac{\hbar}{2}\cos(\omega t)$

2.4

$$H = \frac{1}{2m}p^2$$

$$X(t) = e^{iHt/\hbar}\, X\, e^{-iHt/\hbar} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{it}{\hbar}\right)^n \mathcal{L}_H^n X$$

$[x,p] = i\hbar$

$$\mathcal{L}_H X = \frac{1}{2m}[p^2, x] = -\frac{i\hbar}{m}p$$

$$\mathcal{L}_H^n X = 0, \quad n \geq 2$$

$$X(t) = X + \left(\frac{it}{\hbar}\right)\cdot\left(-\frac{i\hbar}{m}p\right) = x + \frac{p}{m}t.$$

$$[X(t), X(0)] = \left[\frac{p}{m}t, x\right] = -\frac{i\hbar}{m}t.$$

第五次作业

第2章第9，10，11题。

2.9 $H = \Delta\left(|L\rangle\langle R| + |R\rangle\langle L|\right) = \Delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ under $\{|L\rangle, |R\rangle\}$

a. Eigstate: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^T$ Eigval: $+\Delta$

$\quad\cdots\quad$: $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)^T$ $\quad\cdots\quad$: $-\Delta$

b.
$$|\alpha\rangle = \frac{1}{\sqrt{2}}\left(\langle R|\alpha\rangle + \langle L|\alpha\rangle\right)\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} + \frac{1}{\sqrt{2}}\left(\langle R|\alpha\rangle - \langle L|\alpha\rangle\right)\begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$|\alpha, t_0 = 0; t\rangle = e^{-\frac{iHt}{\hbar}}|\alpha\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\langle R|\alpha\rangle + \langle L|\alpha\rangle\right)\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} e^{-\frac{i\Delta t}{\hbar}} + \frac{1}{\sqrt{2}}\left(\langle R|\alpha\rangle - \langle L|\alpha\rangle\right)\begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} e^{\frac{i\Delta t}{\hbar}}$$

$$= \begin{pmatrix} \frac{1}{2}\left(\langle R|\alpha\rangle + \langle L|\alpha\rangle\right)e^{-\frac{i\Delta t}{\hbar}} + \frac{1}{2}\left(\langle R|\alpha\rangle - \langle L|\alpha\rangle\right)e^{\frac{i\Delta t}{\hbar}} \\ \frac{1}{2}\left(\langle R|\alpha\rangle + \langle L|\alpha\rangle\right)e^{-\frac{i\Delta t}{\hbar}} - \frac{1}{2}\left(\langle R|\alpha\rangle - \langle L|\alpha\rangle\right)e^{\frac{i\Delta t}{\hbar}} \end{pmatrix}$$

$$= \begin{pmatrix} \langle R|\alpha\rangle \cos\frac{\Delta t}{\hbar} - \langle L|\alpha\rangle\, i\sin\frac{\Delta t}{\hbar} \\ \langle L|\alpha\rangle \cos\frac{\Delta t}{\hbar} + \langle R|\alpha\rangle\, i\sin\frac{\Delta t}{\hbar} \end{pmatrix}$$

c. Substitute $\langle R|\alpha\rangle = 0$, $\langle L|\alpha\rangle = 1$ to the result of b.

$$|\alpha, t_0=0; t\rangle = \begin{pmatrix} -i\sin\frac{\Delta t}{\hbar} \\ \cos\frac{\Delta t}{\hbar} \end{pmatrix}$$

$$\left|\langle L|\alpha, t_0=0; t\rangle\right|^2 = \sin^2\frac{\Delta t}{\hbar}$$

d. Denote $|\alpha, t_0=0; t\rangle =: |\alpha(t)\rangle$

$$\frac{\partial}{\partial t}|\alpha(t)\rangle = -\frac{i}{\hbar}H|\alpha(t)\rangle$$

$$\begin{pmatrix} \langle L|\alpha(t)\rangle \\ \langle R|\alpha(t)\rangle \end{pmatrix} = -\frac{i\Delta}{\hbar}\begin{pmatrix} \langle R|\alpha(t)\rangle \\ \langle L|\alpha(t)\rangle \end{pmatrix},$$

which is satisfied by result of b.

e. $$e^{-\frac{iHt}{\hbar}} = \sum_n \frac{1}{n!}\left(\frac{-it}{\hbar}\right)^n H^n$$

$$H = \Delta\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \qquad H^n = 0, \quad n \geq 2$$

$$|\alpha(t)\rangle = |\alpha\rangle - \frac{i\Delta t}{\hbar}\langle R|\alpha\rangle|L\rangle$$

$$= \begin{pmatrix} \langle L|\alpha\rangle - \frac{i\Delta t}{\hbar}\langle R|\alpha\rangle \\ \langle R|\alpha\rangle \end{pmatrix}$$

$$|\langle\alpha(t)|\alpha(t)\rangle|^2 = 1 - \left(\frac{\Delta t}{\hbar}\right)^2 \langle R|\alpha\rangle^2,$$

which will fall from unity in general.

2.10. $H = \frac{1}{2}m\omega^2 x^2 + \frac{1}{2m}p^2$

(a). x, p are not changed in Schrödinger picture.

In Heisenberg picture,

$$x(t) = e^{\frac{iHt}{\hbar}} x \, e^{-\frac{iHt}{\hbar}} = \sum \frac{1}{n!}\left(\frac{it}{\hbar}\right)^n h_H^n x$$

$$\frac{it}{\hbar} h_H x = [\frac{1}{2m}p^2, x] = \frac{pt}{m}$$

$$\left(\frac{it}{\hbar}\right)^2 h_H^2 x = \frac{it^2}{\hbar m}[H, p] = -\frac{it^2}{\hbar m}[\frac{1}{2}m\omega^2 x^2, p]$$

$$= -\frac{it^2}{\hbar m} m\omega^2 \cdot i\hbar x = \omega^2 t^2 x$$

$$\Rightarrow x(t) = \cos(\omega t)\, x + \sin(\omega t)\frac{p}{m\omega}$$

Similarly, $p(t) = \cos(\omega t)\, p - \sin(\omega t)\, \omega m x$

(b) Denote the most general state vec. as $|\psi\rangle$.

In Heisenberg picture, $|\psi\rangle$ is not changed.

In Schrödinger picture, $|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\psi\rangle$.

2-11. As calculated in 2.10,

$$x(t) = \cos(\omega t)\, x + \sin(\omega t)\frac{p}{m\omega}$$

$$\langle x\rangle|_t = \langle 0| e^{\frac{ipa}{\hbar}} \left(\cos(\omega t) x + \sin(\omega t)\frac{p}{m\omega}\right) e^{-\frac{ipa}{\hbar}} |0\rangle$$

$$= \cos(\omega t) \langle 0| e^{\frac{ipa}{\hbar}} x\, e^{-\frac{ipa}{\hbar}} |0\rangle$$

$$= \cos(\omega t) \langle 0| (x + a) |0\rangle$$

$$= a \cos(\omega t).$$

Note that $e^{\frac{-ipa}{\hbar}}$ is a spatial transition operator. So the result fits w/ its classical case.

第六次作业

第2章第15，16，17，18题

2.15 Recapture 2.10,

$$x(t) = \cos(\omega t)\, x + \sin(\omega t) \frac{p}{m\omega}.$$

$$\langle x(t)\, x(0) \rangle = \langle 0 | \cos(\omega t)\, x^2 + \sin(\omega t) \frac{px}{m\omega} | 0 \rangle$$

$$= \cos(\omega t) \langle 0 | x^2 | 0 \rangle$$

$$= \cos(\omega t) \frac{\hbar}{2m\omega}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a).$$

2.16

a. $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\alpha^* \langle 0| + \beta^* \langle 1|\right)(a^\dagger + a)\left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\alpha^* \langle 0| + \beta^* \langle 1|\right)\left(\alpha|1\rangle + \beta|0\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\alpha^*\beta + \beta^*\alpha\right) = 2\sqrt{\frac{\hbar}{2m\omega}} \cdot \mathrm{Re}(\alpha^*\beta)$$

$$\mathrm{Re}(\alpha^*\beta) \le |\alpha^*\beta| = |\alpha||\beta| \le \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) = 1.$$

Eq. holds when $|\alpha| = |\beta|$ and $\alpha^*\beta$ is real.

Up to a global phase, $\begin{cases} \alpha = \frac{\sqrt{2}}{2} \\ \beta = \frac{\sqrt{2}}{2} \end{cases}$

$$|\psi\rangle = \frac{\sqrt{2}}{2}\left(|0\rangle + |1\rangle\right)$$

b.

$$|\psi(t)\rangle = \frac{\sqrt{2}}{2}\left(e^{-i\frac{\omega t}{2}}|0\rangle + e^{-i\frac{3\omega t}{2}}|1\rangle\right),$$

where ω is from $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2$

(i) $$\langle\psi(t)|x|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\cdot\frac{1}{2}\cdot\left(e^{\frac{i\omega t}{2}}\langle 0| + e^{i\frac{3\omega t}{2}}\langle 1|\right)(a + a^{\dagger})\left(e^{-i\frac{\omega}{2}t}|0\rangle + e^{-i\frac{3\omega}{2}t}|1\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cdot\frac{1}{2}\cdot\left(e^{\frac{i\omega t}{2}}\langle 0| + e^{i\frac{3\omega t}{2}}\langle 1|\right)\left(e^{-i\frac{\omega}{2}t}|1\rangle + e^{-i\frac{3\omega t}{2}}|0\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cdot\frac{1}{2}\cdot\left(e^{\frac{i\omega t}{2}}\cdot e^{-\frac{i3\omega t}{2}} + e^{\frac{i3\omega t}{2}}e^{-\frac{i\omega t}{2}}\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)$$

(ii) $$x(t) = \cos(\omega t)\,x + \sin(\omega t)\frac{p}{m}.$$

$$= \cos(\omega t)\sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{\dagger})$$
$$+ \sin(\omega t)\,\frac{1}{m}\,i\sqrt{\frac{\hbar m\omega}{2}}\,(a^{\dagger} - a)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(e^{i\omega t}a^{\dagger} + e^{-i\omega t}a\right)$$

$$\langle\psi|x(t)|\psi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\frac{1}{2}\left(\langle 0| + \langle 1|\right)\left(e^{i\omega t}a^{\dagger} + e^{-i\omega t}a\right)\left(|0\rangle + |1\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cdot\frac{1}{2}\cdot\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t).$$

c. Use Heisenberg.

$$\langle\psi|x(t)^2|\psi\rangle = \frac{\hbar}{2m\omega}\frac{1}{2}\left(\langle 0|+\langle 1|\right)\left(e^{i\omega t}a^{\dagger}+e^{-i\omega t}a\right)^2\left(|0\rangle+|1\rangle\right)$$

$$= \frac{\hbar}{2m\omega}\frac{1}{2}\left(\langle 0|+\langle 1|\right)\left(a^{\dagger}a+aa^{\dagger}\right)\left(|0\rangle+|1\rangle\right)$$

$$= \frac{\hbar}{2m\omega}\cdot\frac{1}{2}\cdot 2 = \frac{\hbar}{2m\omega}.$$

$$\langle \Delta X\rangle^2 = \frac{\hbar}{2m\omega} - \left(\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)\right)^2$$

$$= \frac{\hbar}{2m\omega}\left(1-\cos^2(\omega t)\right).$$

2.17 $$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\langle 0|e^{ikx}|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\int e^{ikx}e^{-\frac{m\omega x^2}{\hbar}}dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{k^2\hbar}{4m\omega}}\int e^{-\frac{m\omega}{\hbar}\left(x-\frac{ik\hbar}{2m\omega}\right)^2}dx$$

$$= e^{-\frac{k^2\hbar}{4m\omega}}$$

$$\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}$$

So, $\langle 0|e^{ikx}|0\rangle = \exp\left(-k^2\frac{\langle 0|x^2|0\rangle}{2}\right)$

2.18

a.

$$|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}} e^{\lambda a^+} |0\rangle$$

$$a|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}} a\, e^{\lambda a^+} |0\rangle$$

$$= e^{-\frac{|\lambda|^2}{2}} \sum_n \frac{1}{n!} \lambda^n a (a^+)^n |0\rangle$$

$$= e^{-\frac{|\lambda|^2}{2}} \sum_n \frac{\lambda^n}{n!} a \sqrt{n!}\, |n\rangle \quad - - - - \quad (*)$$

$$= e^{-\frac{|\lambda|^2}{2}} \sum_n \frac{\lambda^n}{n!} \cdot n \cdot \sqrt{(n-1)!}\, |n-1\rangle$$

$$= e^{-\frac{|\lambda|^2}{2}} \lambda \sum_n \frac{\lambda^{n-1}}{(n-1)!} (a^+)^{n-1} |n-1\rangle$$

$$= \lambda |\lambda\rangle.$$

b.

$$\langle\lambda|x|\lambda\rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle\lambda|(a^+ + a)|\lambda\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} (\lambda + \lambda^*)$$

$$\langle\lambda|x^2|\lambda\rangle = \frac{\hbar}{2m\omega} \langle\lambda|(a^+ + a)^2|\lambda\rangle$$

$$= \frac{\hbar}{2m\omega} \langle\lambda| \left(a^{+2} + a^2 + 2a^+a + 1\right) |\lambda\rangle$$

$$= \frac{\hbar}{2m\omega} \left(1 + (\lambda + \lambda^*)^2\right)$$

$$\langle \Delta x^2 \rangle = \frac{\hbar}{2m\omega}$$

Similarly, $\langle \Delta p^2 \rangle = \frac{\hbar m \omega}{2}$, $\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle = \frac{\hbar^2}{4}$,

which is the minimal uncertainty relation.

c. $f(n) = \langle n | \lambda \rangle = e^{-\frac{|\lambda|^2}{2}} \frac{\lambda^n}{\sqrt{n!}}$ from $(*)$ in a.

$$|f(n)|^2 = e^{-|\lambda|^2} \frac{|\lambda|^{2n}}{n!} = \frac{1}{n!} \Theta^n e^{-\Theta},$$

where $\Theta =: |\lambda|^2$.

$$|f(n)|^2 - |f(n-1)|^2$$

$$= \frac{e^{-\Theta}}{(n-1)!} \left(\frac{\Theta}{n} - 1 \right) \begin{cases} \geq 0 & , n \leq \Theta \\ < 0 & , n > \Theta \end{cases}$$

So, most probable value of n is $\lfloor \Theta \rfloor = \lfloor |\lambda|^2 \rfloor$

d. <u>lemma</u> : $e^{\frac{ip\ell}{\hbar}} a e^{-\frac{ip\ell}{\hbar}} = a - \ell \sqrt{\frac{m\omega}{2\hbar}}$

$$\text{l.h.s.} = \sum \frac{1}{n!} \left(\frac{i\ell}{\hbar} \right)^n h_p^n a$$

$$h_p a = i \sqrt{\frac{m\omega\hbar}{2}} [a^\dagger - a, a] = -i \sqrt{\frac{m\omega\hbar}{2}}$$

And higher commutators vanish.

$$\Rightarrow \text{l.h.s.} = a - \left(\frac{i\ell}{\hbar} \right) \cdot \left(-i \sqrt{\frac{m\omega\hbar}{2}} \right)$$

$$= a - \ell \sqrt{\frac{m\omega}{2\hbar}}$$

$$a e^{+\frac{ip\ell}{\hbar}} |\lambda\rangle = e^{+\frac{ip\ell}{\hbar}} e^{-\frac{ip\ell}{\hbar}} a e^{+\frac{ip\ell}{\hbar}} |\lambda\rangle$$

$$= e^{+\frac{ip\ell}{\hbar}} \left(a + \ell \sqrt{\frac{m\omega}{2\hbar}} \right) |\lambda\rangle$$

$$= e^{+\frac{ip\ell}{\hbar}} \left(\lambda + \ell \sqrt{\frac{m\omega}{2\hbar}} \right) |\lambda\rangle .$$

(Note that sgn is opposite to that in lemma).

Take $l = \lambda \sqrt{\frac{2\hbar}{m\omega}}$, $a\, e^{+\frac{i p l}{\hbar}} |\lambda\rangle = 0$

$$\Rightarrow \quad e^{+\frac{i p l}{\hbar}} |\lambda\rangle = |0\rangle$$

$$\Rightarrow \quad |\lambda\rangle = e^{-\frac{i p l}{\hbar}} |0\rangle$$

Reference of this solution:

https://homepage.univie.ac.at/reinhold.bertlmann/pdfs/T2_Skript_Ch_5.pdf

第七次作业

第2章第29，30，31题

2.29

$$Z = \int d\vec{x}' \, K(\vec{x}', t; \vec{x}'; 0) \Big|_{\beta = \frac{it}{\hbar}}$$

$$= \sum_{a'} \exp\left(\frac{-iE_{a'}t}{\hbar}\right) \Big|_{\beta = \frac{it}{\hbar}}$$

$$= \sum_{a'} \exp(-\beta E_{a'})$$

$$\frac{\partial Z}{\partial \beta} = \sum_{a'} -E_{a'} \exp(-\beta E_{a'})$$

$$-\frac{1}{Z}\frac{\partial Z}{\partial \beta}\Big|_{\beta\to\infty} = \frac{\sum_{a'} E_{a'} \exp(-\beta E_{a'})}{\sum_{a'} \exp(-\beta E_{a'})}\Big|_{\beta\to\infty}$$

$= E_0$, which is the ground energy.

Note that $\frac{E_{a'}}{E_0} \to 0$, if $a' \neq 0$.

For a part. in 1-D box (Length of l), $E_n = \frac{\hbar^2 \pi^2}{2m} n^2$.

$E_0 = 0$

$$Z(\beta) = \sum_n e^{-\beta \frac{\hbar^2 \pi^2}{2m} n^2}$$

$$\frac{\partial Z}{\partial \beta} = \sum_n \frac{\hbar^2 \pi^2}{2m} n^2 e^{-\beta \frac{\hbar^2 \pi^2}{2m} n^2} \overset{\beta\to\infty}{=} 0 = E_0.$$

2.30. $\langle \vec{p}'', t \mid \vec{p}', t_0 \rangle$

$$= \langle \vec{p}'' \mid e^{\frac{-iH(t-t_0)}{\hbar}} \mid \vec{p}' \rangle$$

$$= e^{-\frac{ip'^2(t-t_0)}{2m\hbar}} \delta^3(\vec{p}'' - \vec{p}')$$

2.31

a. $$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

$$S = \int \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right) dt$$

b. Feymann's side:

Letting $x_n =: x$, $x_{n-1} =: x - \xi$, $t_n - t_{n-1} =: \Delta t$:

$$\langle x_n, t_n \mid x_{n-1}, t_{n-1} \rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \exp\left(\frac{im\xi^2}{2\hbar\Delta t} - \frac{im\omega^2 x^2}{2\hbar}\Delta t\right)$$

Propagator given by (2.5.26)

$$\langle x_n t_n \mid x_{n-1}, t_{n-1} \rangle = \sum_{a'} \langle x_n | a' \rangle \langle a' | x_{n-1} \rangle \exp\left(\frac{-iE_{a'}\Delta t}{\hbar}\right)$$

$$= \sum_{a'} \psi_{a'}(x_n) \psi_{a'}(x_{n-1}) \exp\left(-i(a'+\tfrac{1}{2})\omega\Delta t\right)$$

$$= \sum_{a'} \frac{1}{2^{a'} a'!} \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega^2}{2\hbar}(x_n^2 + x_{n-1}^2)} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x_n\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}} x_{n-1}\right) \cdot \exp\left(-i(a'+\tfrac{1}{2})\omega\Delta t\right)$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega^2}{2\hbar}(x_n^2 + x_{n-1}^2)} e^{-\frac{i}{2}\omega\Delta t} \sum_{a'} \frac{1}{a'!} \left(\frac{e^{-i\omega\Delta t}}{2}\right)^{a'} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x_n\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}} x_{n-1}\right)$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\frac{i}{2}\omega\Delta t} \frac{1}{\sqrt{1-\rho^2}} \exp\left(\frac{\frac{m\omega}{\hbar}\left(4x_n x_{n-1}\rho - (1+\rho^2)(x_n^2+x_{n-1}^2)\right)}{2(1-\rho^2)}\right),$$

where $\rho = e^{-i\omega\Delta t}$. The last step is by using Mehler's kernel.

Consider that $\Delta t \to 0$, $\rho \to 1 - i\omega\Delta t \underbrace{- \frac{1}{2}\omega^2\Delta t^2}$, potential contribution

$\rho^2 \to 1 - 2i\omega\Delta t \underbrace{-2\omega^2\Delta t^2}$

$\langle x_n, t_n | x_{n-1}, t_{n-1}\rangle$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\frac{i\omega\Delta t}{2}} \sqrt{\frac{1}{2i\omega\Delta t}} \exp\left(\frac{m\omega}{2\hbar}\cdot\frac{1}{2i\omega\Delta t}\cdot\left(4(1-i\omega\Delta t - \tfrac{1}{2}\omega^2\Delta t^2)x_n x_{n-1} - (2-2i\omega\Delta t - 2\omega^2\Delta t^2)(x_n^2+x_{n-1}^2)\right)\right)$$

$$= \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\, e^{-\frac{i\omega\Delta t}{2}} \exp\left(\frac{-m}{4i\hbar\Delta t}\left(2(1-i\omega\Delta t)\xi^2 + 2\omega^2\Delta t^2 x^2\right)\right)$$

$$= \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\, \underbrace{e^{-\frac{i\omega\Delta t}{2}}} \exp\left(\frac{im\xi^2}{2\hbar\Delta t} - \frac{im\omega^2x^2}{2\hbar}\Delta t\right)$$

In comparison, two results agree up to a phase factor $e^{-\frac{i\omega\Delta t}{2}}$, which can be gauged out by adding $\frac{1}{2}\hbar\omega$ to the classical Lagrangian (or $-\frac{1}{2}\hbar\omega$ to the Ham.).

第八次作业

第2章第32题

2-32 Schwinger:

$$\delta \langle x_2, t+dt | x_1, t \rangle = \frac{i}{\hbar} \langle x_2, t+dt | \delta [dt\, L_p] | x_1, t \rangle$$

Feymann:

$$\langle x_2, t_2 | x_1, t_1 \rangle = \int_{x_1}^{x_2} D[x(t)] \exp\left(\frac{i}{\hbar} \int_{t_1}^{t_2} L_c\, dt \right)$$

In classical limits, only the pathes very close to the one minimizing $\int_{t_1}^{t_2} L_c\, dt$ will survive.

第9次高量作业 第2章第35，36，37题

2.35
$$\frac{1}{2m}\left(\vec{p}-\frac{e\vec{A}}{c}\right)^2=\frac{1}{2m}\vec{p}^2-\frac{e}{mc}\vec{p}\cdot\vec{A}+\frac{e^2}{2mc^2}\vec{A}^2.$$

$\nabla\times\vec{A}=B\hat{z}$, so $\vec{A}=\frac{1}{2}B(y\hat{x}-x\hat{y})$

$$\frac{1}{2m}\left(\vec{p}-\frac{e\vec{A}}{c}\right)^2$$
$$=\frac{1}{2m}\vec{p}^2-\frac{e}{mc}\cdot\frac{B}{2}\cdot(yp_x-xp_y)+\frac{e^2}{2mc^2}\cdot\frac{1}{4}B^2(x^2+y^2)$$
$$=\frac{1}{2m}\vec{p}^2+\frac{e}{2mc}\vec{L}\cdot\vec{B}+\frac{e^2}{8mc^2}B^2(x^2+y^2).$$

The 2nd term is the correct interaction.

The 3rd term is $\propto B^2(x^2+y^2)$.

2.36

$$A_x=\frac{1}{2}By,\quad A_y=-\frac{1}{2}Bx$$

a.

$[\Pi_x,\Pi_y]$
$$=\left[p_x-\frac{eA_x}{c},\ p_y-\frac{eA_y}{c}\right]$$
$$=\left[p_x-\frac{eB}{2c}y,\ p_y+\frac{eB}{2c}x\right]$$
$$=\frac{eB}{2c}(-i\hbar)-\frac{eB}{2c}i\hbar=\frac{eB}{c}i\hbar.$$

b.

mag.	1D osc.
$H=\frac{1}{2m}\Pi_x^2+\frac{1}{2m}\Pi_y^2+\frac{p_z^2}{2m}$	$H=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2x^2$
$[\Pi_x,\Pi_y]=\frac{eB}{c}i\hbar$	$[x,p]=i\hbar$

So (2D) particle in mag. is equivalent to 1D osc, taking $\omega=\frac{|eB|}{mc}$.

Note that $[p_z,H]=0$. Denote eigvalue of p_z as $\hbar k$.

$$E_{k,n}=\frac{\hbar^2k^2}{2m}+\left(n+\frac{1}{2}\right)\frac{|eB|}{mc}.$$

2.37

Time duration: $\Delta t=\frac{l}{p/m_n}=\frac{m_nl\lambda}{h}$

Phase shift: $\frac{1}{\hbar}\Delta B\cdot\frac{g_n|e|\hbar}{2m_nc}\Delta t=\Delta B\frac{g_n|e|}{2c}\cdot\frac{l\lambda}{h}$

Let phase shift to be 2π.

$$\Delta B=\frac{4\pi\hbar c}{|e|g_n\lambda l}.$$

第10次高量作业 第3章第1，2，3题。

3.1

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

eigvals: $+1, -1$

eigstates: $|+i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$|-i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$|\psi\rangle =: \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\Pr[S_y = \tfrac{\hbar}{2}] = |\langle +i|\psi\rangle|^2 = \frac{1}{2}|\alpha + i\beta|^2.$$

3.2

$$A = a_0 + i\vec{a}\cdot\vec{\sigma} = \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix}$$

a.

$$\det A = a_0^2 + |\vec{a}|^2$$

$$AA^H = (a_0 + i\vec{a}\cdot\vec{\sigma})(a_0 - i\vec{a}\cdot\vec{\sigma}) = a_0^2 + |\vec{a}|^2 = \det A$$

$$U = \frac{A}{A^H} = \frac{A^2}{AA^H} = \frac{1}{\det A}A^2$$

$$UU^H = \frac{1}{(\det A)^2}A^2 A^{H2} = \mathbb{1} \qquad \text{(unitary)}$$

$$\det(U) = \frac{1}{(\det A)^2}\det(A^2) = 1 \qquad \text{(unimodular)}$$

b.

$$U = \frac{1}{\det A}A^2 = \frac{1}{\det A}\left(a_0^2 + 2ia_0\vec{a}\cdot\vec{\sigma} - |\vec{a}|^2\right)$$

$$= \frac{a_0^2 - |a|^2}{a_0^2 + |\vec{a}|^2} + \frac{2ia_0}{a_0^2 + |\vec{a}|^2}\vec{a}\cdot\vec{\sigma}$$

Write U into a "rotation" form:

$$U = \exp\left(\frac{-i\vec{\sigma}\cdot\hat{n}\phi}{2}\right) = \cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\,\hat{n}\cdot\vec{\sigma}$$

$$\begin{cases} \phi = 2\arccos\dfrac{a_0^2 - |a|^2}{a_0^2 + |\vec{a}|^2} \\ \hat{n} = -\dfrac{\vec{a}}{|\vec{a}|}. \end{cases}$$

3.3

a. Yes.

$$H \chi_+^{(e^-)} \chi_-^{(e^+)} \rightarrow \frac{eB}{mc}\left(S_z^{(e^-)} - S_z^{(e^+)}\right) \chi_+^{(e^-)} \chi_-^{(e^+)}$$

$$= \frac{eB}{mc}\left(\frac{\hbar}{2} - \left(-\frac{\hbar}{2}\right)\right) \chi_+^{(e^-)} \chi_-^{(e^+)}$$

$$= \frac{eB\hbar}{mc} \chi_+^{(e^-)} \chi_-^{(e^+)}$$

Eigval: $\frac{eB\hbar}{mc}$.

b. No. $H \rightarrow A\, \vec{S}^{(e^-)} \cdot \vec{S}^{(e^+)}$

$$= A\left(S_x^{(e^-)} S_x^{(e^+)} + S_y^{(e^-)} S_y^{(e^+)} + S_z^{(e^-)} S_z^{(e^+)}\right)$$

$$H \chi_+^{(e^-)} \chi_-^{(e^+)} = A\frac{\hbar^2}{4}\left(\chi_-^{(e^-)} \chi_+^{(e^+)} + \chi_-^{(e^-)} \chi_+^{(e^+)} - \chi_+^{(e^-)} \chi_-^{(e^+)}\right)$$

$$= A\frac{\hbar^2}{4}\left(2\chi_-^{(e^-)} \chi_+^{(e^+)} - \chi_+^{(e^-)} \chi_-^{(e^+)}\right)$$

$$\langle H \rangle = -A \cdot \frac{\hbar^2}{4}.$$

Hw11 第3章第4，5，6题

3.4

$$S_z = \hbar \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$S_z (S_z + \hbar)(S_z - \hbar) = \hbar^3 \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \begin{pmatrix} 2 & & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & & \\ & -1 & \\ & & -2 \end{pmatrix} = 0,$$

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\left(\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right) = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$S_x (S_x + \hbar)(S_x - \hbar) = \hbar^3 \begin{pmatrix} 0 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & 0 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 1 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & 1 \end{pmatrix} \begin{pmatrix} -1 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & -1 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -1 \end{pmatrix} = 0.$$

$$\begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

3.5

$$i\hbar \frac{dK_1}{dt} = [K_1, H] = \frac{1}{2I_2}[K_1, K_2^2] + \frac{1}{2I_3}[K_1, K_3^2]$$

$$= \frac{1}{2I_2}\left([K_1,K_2]K_2 + K_2[K_1,K_2]\right) + \frac{1}{2I_3}\left([K_1,K_3]K_3 + K_3[K_1,K_3]\right)$$

$$= \frac{i\hbar}{2I_2}\left(K_3K_2 + K_2K_3\right) + \frac{i\hbar}{2I_3}\left(-K_2K_3 - K_3K_2\right)$$

$$= \frac{i\hbar(I_3 - I_2)}{2I_2I_3}\left(K_2K_3 + K_3K_2\right).$$

Similarly, $i\hbar \dfrac{dK_2}{dt} = \dfrac{i\hbar(I_1 - I_3)}{2I_3I_1}(K_3K_1 + K_1K_3)$

$$i\hbar \frac{dK_3}{dt} = \frac{i\hbar(I_2 - I_1)}{2I_1I_2}(K_1K_2 + K_2K_1)$$

In classical limits, $[K_i, K_j] = 0$.

$$\frac{dK_1}{dt} = \frac{I_3 - I_2}{I_3I_2}K_2K_3, \quad \frac{dK_2}{dt} = \frac{I_1 - I_3}{I_1I_3}K_3K_1, \quad \frac{dK_3}{dt} = \frac{I_2 - I_1}{I_1I_2}K_1K_2.$$

3.6.

$$[G_i, G_j] = i\,\varepsilon_{ijk}G_k.$$

$$G_i =: \frac{J_i}{\hbar}$$

Hw12 第3章第13，14，15题。

3.13

$$G_1 = -i\hbar\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix} \qquad G_2 = -i\hbar\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix} \qquad G_3 = -i\hbar\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$$

$$[G_1, G_2] = (-i\hbar)^2\left(\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} - \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\right) = i\hbar\, G_3.$$

Similarly, $[G_2, G_3] = i\hbar G_1$, $[G_3, G_1] = i\hbar G_2$.

In (4), we have

$$J_1 = \hbar\begin{pmatrix}0&\sqrt{2}/2&0\\\sqrt{2}/2&0&\sqrt{2}/2\\0&\sqrt{2}/2&0\end{pmatrix}, \quad J_2 = \hbar\begin{pmatrix}0&\sqrt{2}/2\, i&0\\-\sqrt{2}/2\, i&0&\sqrt{2}/2\, i\\0&-\sqrt{2}/2\, i&0\end{pmatrix}, \quad J_3 = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}.$$

G_3: eigstates: $\frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle)$ (e.v. $= +\hbar$) $\longleftrightarrow$ $|1\rangle_J$ (e.v. $= \hbar$)

$\frac{1}{\sqrt{2}}(-|1\rangle + i|2\rangle)$ (e.v. $= -\hbar$) $\longleftrightarrow$ $|-1\rangle_J$ (e.v. $= -\hbar$)

$|3\rangle$ (e.v. $= 0$) $\longleftrightarrow$ $|0\rangle_J$ (e.v. $= 0$)

So the matrix is

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&0&-\frac{1}{\sqrt{2}}\\\frac{i}{\sqrt{2}}&0&\frac{i}{\sqrt{2}}\\0&1&0\end{pmatrix}$$

(I cannot find any "physical" meaning in this.)

And we can verify that the matrix works for $J_{1,2}$ ($G_{1,2}$):

```
>>> u = np.array([[1/2**0.5, 0, -1/2**0.5], [1j/2**0.5, 0, 1j/2**0.5], [0, 1, 0]])
>>> uu = np.transpose((np.conjugate(u)))
>>> u @ np.array([[0,0,0],[0,0,1],[0,-1,0]]) @ uu
array([[ 0.        +0.j        ,  0.        +0.j        ,
         0.70710678+0.j        ],
       [ 0.        +0.j        ,  0.        +0.j        ,
         0.        -0.70710678j],
       [-0.70710678+0.j        ,  0.        -0.70710678j,
         0.        +0.j        ]])
>>> uu @ np.array([[0,0,-1],[0,0,0],[1,0,0]]) @ u
array([[ 0.        +0.j, -0.70710678+0.j,  0.        +0.j],
       [ 0.70710678+0.j,  0.        +0.j, -0.70710678+0.j],
       [ 0.        +0.j,  0.70710678+0.j,  0.        +0.j]])
>>> uu @ np.array([[0,1,0],[-1,0,0],[0,0,0]]) @ u
array([[0.+1.j, 0.+0.j, 0.+0.j],
       [0.+0.j, 0.+0.j, 0.+0.j],
       [0.+0.j, 0.+0.j, 0.-1.j]])
```

$$\vec{V} \to \vec{V} + \hat{n}\,\delta\phi \times \vec{V} \iff V_i \to V_i + \delta\phi\, \varepsilon_{ijk} n_j V_k$$

$$= V_i - \frac{i\,\delta\phi (G_i)_{jk} n_j}{\hbar} V_k$$

$$\iff \vec{V} = \vec{V} - \frac{i}{\hbar}\left(\hat{n}\, G_i \vec{V}\right)\hat{e}_i$$

where G_i is related to J_i.

3.14 $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$

$J_+ - J_- = 2iJ_y$

$= J_z^2 + \frac{1}{4}(\cancel{J_+^2} + \cancel{J_-^2} + J_+J_- + J_-J_+) - \frac{1}{4}(\cancel{J_+^2} + \cancel{J_-^2} - J_+J_- - J_-J_+)$

$= J_z^2 + \frac{1}{2}J_+J_- + \frac{1}{2}J_-J_+$

Note that $[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y] = 2\hbar J_z$.

$\vec{J}^2 = J_z^2 + J_+J_- - \hbar J_z$.

b. $|c_-|^2 = \langle j,m|J_-^{H} J_-|j,m\rangle$

$= \langle j,m|J_+J_-|j,m\rangle$

$= \langle j,m|(\vec{J}^2 - J_z^2 + \hbar J_z)|j,m\rangle$

$= \hbar^2 j(j+1) - \hbar^2 m^2 + \hbar^2 m$

$c_- = \hbar\sqrt{j(j+1) - m^2 + m} = \hbar\sqrt{(j+m)(j-m+1)}$,

up to an arbitrary phase.

3.15 a. $\psi(\vec{x}) = r(\cos\phi\sin\theta + \sin\phi\sin\theta + 3\cos\theta)\, f(r)$

$= \sqrt{\frac{8\pi}{3}}\left(\frac{1}{2}Y_1^1(\theta,\phi) + \frac{1}{2}Y_1^{-1}(\theta,\phi) + \frac{1}{2i}Y_1^1(\theta,\phi) - \frac{1}{2i}Y_1^{-1}(\theta,\phi) + \frac{3}{\sqrt{2}}Y_1^0(\theta,\phi)\right) r\, f(r)$

$= \left(\sqrt{\frac{8\pi}{3}}\left(\frac{1}{2}+\frac{1}{2i}\right)Y_1^1(\theta,\phi) + \sqrt{\frac{8\pi}{3}}\left(\frac{1}{2}-\frac{1}{2i}\right)Y_1^{-1}(\theta,\phi) + \sqrt{12\pi}\, Y_1^0(\theta,\phi)\right) r\, f(r)$

So, $\psi(\vec{x})$ is an eigenstate w/ $l = 1$.

b. $m = \pm 1, 0$, proportion is $\left(\frac{8\pi}{3}\cdot\frac{1}{2}\right) : \left(\frac{8\pi}{3}\cdot\frac{1}{2}\right) : 12\pi = 1 : 1 : 9$.

So $\Pr(m=+1) = \Pr(m=-1) = \frac{1}{11}$, $\Pr(m=0) = \frac{9}{11}$.

c. $\left(\frac{1}{2m}\vec{p}^2 + V(r)\right)\psi(\vec{x}) = E\,\psi(\vec{x})$

$V(r) = E - \frac{\frac{1}{2m}\vec{p}^2\psi(\vec{x})}{\psi(\vec{x})}$.