Adv QM
第一次作业

1.2 $X = a_0 + \vec{\sigma} \cdot \vec{a}$

a. $\mathrm{Tr}\, X = 2a_0$

$\mathrm{Tr}\, \sigma_i X = 2a_i$ ($\sigma_{1,2,3}$ traceless)

b. $a_0 = \frac{1}{2} \mathrm{Tr}\, X = \frac{1}{2}(X_{11} + X_{22})$

$a_1 = \frac{1}{2} \mathrm{Tr}\, \sigma_1 X = \frac{1}{2}(X_{12} + X_{21})$

$a_2 = \frac{1}{2} \mathrm{Tr}\, \sigma_2 X = \frac{1}{2}(-iX_{12} + iX_{21})$

$a_3 = \frac{1}{2} \mathrm{Tr}\, \sigma_3 X = \frac{1}{2}(X_{11} - X_{22})$

1.3 Show that the determinant of a 2×2 matrix $\boldsymbol{\sigma}\cdot\mathbf{a}$ is invariant under

$$\boldsymbol{\sigma}\cdot\mathbf{a} \to \boldsymbol{\sigma}\cdot\mathbf{a}' \equiv \exp\left(\frac{i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right)\boldsymbol{\sigma}\cdot\mathbf{a}\exp\left(\frac{-i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right).$$

Find a_k' in terms of a_k when $\hat{\mathbf{n}}$ is in the positive z-direction and interpret your result.

$\hat{n} = \hat{z} \Rightarrow \vec{\sigma} \cdot \hat{n} = \sigma_3$

$$\vec{\sigma} \cdot \vec{a}' = \vec{\sigma} \cdot \vec{a} + \frac{i\phi}{2} \cdot h_{\sigma_3} \vec{\sigma} \cdot \vec{a}$$

$$+ \frac{1}{2}\left(\frac{i\phi}{2}\right)^2 h_{\sigma_3}^2 \vec{\sigma} \cdot \vec{a} + \cdots$$

(B-H formula)

Here, $h_A B =: [A, B]$

$$h_{\sigma_3} \vec{\sigma}\cdot\vec{a} = i a_1 \sigma_2 - i a_2 \sigma_1$$

$$h^2_{\sigma_3} \vec{\sigma}\cdot\vec{a} = -i^2 a_1 \sigma_1 - i^2 \sigma_2 \sigma_2$$

$$\cdots \cdots$$

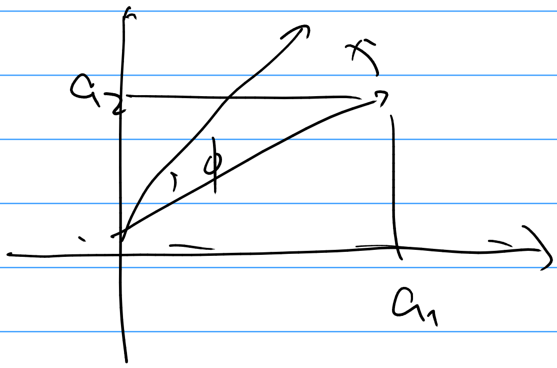
$$a_1' = a_1 + \frac{i\phi}{2}(-i a_2) + \frac{1}{2}\left(\frac{i\phi}{2}\right)^2 (-i^2)\, a_1 + \cdots$$

$$= a_1 \cos\phi + a_2 \sin\phi$$

$$a_2' = a_2 + \frac{i\phi}{2}\, i a_1 + \frac{1}{2}\left(\frac{i\phi}{2}\right)^2 (-i^2)\, a_2 + \cdots$$

$$= -a_1 \sin\phi + a_2 \cos\phi$$

Interpretation: it is like a rotation w/ angle, ϕ.



1.4 $X = \sum_x x\, |x\rangle\langle x|$ $\quad Y = \sum_y y\, |y\rangle\langle y|$

a. $\mathrm{Tr}\, XY = \mathrm{Tr} \sum xy\, |x\rangle\langle x|y\rangle\langle y|$

$$= \mathrm{Tr} \sum xy |y\rangle\langle y|x\rangle\langle x| = \mathrm{Tr}\, Y X$$

b. $$(XY)^{+} = \left(\sum xy\ |x\rangle\langle x|y\rangle\langle y|\right)^{+}$$

$$= \sum x^{*} y^{*}\ |y\rangle\langle y|x\rangle\langle x|$$

$$= Y^{+} X^{+}$$

c. $$A = \sum_{a} a\ |a\rangle\langle a|$$

$$\exp(if(A)) = \sum_{a} e^{if(a)}\ |a\rangle\langle a|$$

d. $$\sum_{a'} \psi_{a'}^{*}(x')\ \psi_{a'}(x'')$$

$$= \sum_{a'} \langle a'|x'\rangle\langle x''|a'\rangle = \mathrm{Tr}\ |x'\rangle\langle x''|$$

$$= \delta(x - x'')$$

1.5

a. $$|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle a|\alpha\rangle\langle\beta|a\rangle & \langle a|\alpha\rangle\langle\beta|a'\rangle \\ \langle a'|\alpha\rangle\langle\beta|a\rangle & \langle a'|\alpha\rangle\langle\beta|a'\rangle \\ \vdots & \ddots \end{pmatrix}$$

b. $|\alpha\rangle = |S_z = \frac{\hbar}{2}\rangle =: |0\rangle$

$|\beta\rangle = |S_x = \frac{\hbar}{2}\rangle =: |+\rangle$

$|a\rangle = |0\rangle, \quad |a'\rangle = |1\rangle$

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle 0|0\rangle\langle +|0\rangle & \langle 0|0\rangle\langle +|1\rangle \\ \langle 1|0\rangle\langle +|0\rangle & \langle 1|0\rangle\langle +|1\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

第二次作业

1.8 $S_x S_y = (\frac{\hbar}{2})^2 i \left(|+\rangle\langle +| - |-\rangle\langle -|\right)$

$= i\frac{\hbar}{2} S_z$

$S_y S_x = (\frac{\hbar}{2})^2 i \left(-|+\rangle\langle +| + |-\rangle\langle -|\right)$

$= -i\frac{\hbar}{2} S_z$

$[S_x, S_y] = i\hbar S_z$

$[S_y, S_x] = -i\hbar S_z$

$\{S_x, S_y\} = 0$

$$S_x S_x = \left(\frac{\hbar}{2}\right)^2 \mathbb{1} \qquad [S_x, S_x] = 0$$

$$\{S_x, S_x\} = \frac{\hbar^2}{2}$$

Similiarly, we have other comm. and anti-comm.

1.9

$$\vec{S}\cdot\hat{n}\,|\vec{S}\cdot\hat{n};+\rangle = \frac{\hbar}{2}\,|\vec{S}\cdot\hat{n};+\rangle$$

$$\hat{n} = (\cos\alpha\sin\beta,\ \sin\alpha\sin\beta,\ \cos\beta)$$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\begin{pmatrix} \cos\beta & \cos\alpha\sin\beta - i\sin\alpha\sin\beta \\ \cos\alpha\sin\beta + i\sin\alpha\sin\beta & -\cos\beta \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\beta & e^{-i\alpha}\sin\beta \\ e^{i\alpha}\sin\beta & -\cos\beta \end{pmatrix}$$

Let $|\phi\rangle = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\begin{pmatrix} \cos\beta\,\phi_0 + e^{-i\alpha}\sin\beta\,\phi_1 \\ e^{i\alpha}\sin\beta\,\phi_0 - \cos\beta\,\phi_1 \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$$

$$\Rightarrow \cos\beta\, \phi_0 + e^{-i\alpha} \sin\beta\, \phi_1 = \phi_0$$

$$\Rightarrow \frac{\phi_1}{\phi_0} = \frac{1-\cos\beta}{e^{-i\alpha}\sin\beta} = \frac{e^{i\alpha}\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}}$$

$$\Rightarrow |\vec{S}\cdot\hat{n};+\rangle = \begin{pmatrix} \cos\frac{\beta}{2} \\ e^{i\alpha}\sin\frac{\beta}{2} \end{pmatrix}$$

1.10

$$H = a\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|H-\lambda| = \begin{vmatrix} a-\lambda & a \\ a & -a-\lambda \end{vmatrix}$$

$$= \lambda^2 - 2a^2 = 0$$

$$\lambda = \pm\sqrt{2}\,a \Rightarrow \text{eig. vals.}$$

$$(H \mp \sqrt{2}a)\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$$

$$= a\begin{pmatrix} 1\pm\sqrt{2} & 1 \\ 1 & -1\pm\sqrt{2} \end{pmatrix}\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = 0$$

eig. states

$$\Rightarrow (1\pm\sqrt{2})\phi_0 + \phi_1 = 0 \quad \Downarrow$$

$$\phi_0 = \frac{-1}{\sqrt{(1\pm\sqrt{2})^2+1}}, \quad \phi_1 = \frac{1\pm\sqrt{2}}{\sqrt{(1\pm\sqrt{2})^2+1}}$$

第三次作业 第一章第17，18，19，20题、PPT第二页习题

1.17 $[A_1, H] = 0 \quad [A_2, H] = 0 \quad [A_1, A_2] \neq 0$

Suppose $H|\psi\rangle = \lambda|\psi\rangle$, i.e. $|\psi\rangle$ is eig state

Then $H A_1 |\psi\rangle = A_1 H |\psi\rangle = \lambda A_1 |\psi\rangle$

$H A_2 |\psi\rangle = A_2 H |\psi\rangle = \lambda A_2 |\psi\rangle$

To keep non-deg., $A_1|\psi\rangle \propto A_2|\psi\rangle \propto |\psi\rangle$.

And for $[A_1, A_2] \neq 0$, this is generally impossible.

Unless $A_1|\psi\rangle = 0$

$A_2|\psi\rangle = 0$

$k = m\omega^2$

Example: $H = \frac{1}{2m}\vec{p}^2 + \frac{1}{2}k\vec{r}^2 = \hbar\omega(a^\dagger a + \frac{1}{2})$

$L_x = y p_z - z p_y = i\hbar(a_z^\dagger a_y - a_y^\dagger a_z)$

$L_z = x p_y - y p_x = i\hbar(a_y^\dagger a_x - a_x^\dagger a_y)$

Note that
$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$
$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)\,.$$

eig state of H: $|n_x, n_y, n_z\rangle$

g.s. $|n_x = 0, n_y = 0, n_y = 0\rangle$ is non-deg.

Because $L_x|0,0,0\rangle = 0$

$L_z|0,0,0\rangle = 0$

1.18

a. $$\left(\langle\alpha| + \lambda^* \langle\beta|\right)\left(|\alpha\rangle + \lambda|\beta\rangle\right)$$

$$= \left\| |\alpha\rangle + \lambda|\beta\rangle \right\|^2 \geq 0$$

Schwarz ineq: $\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle \leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$

Let $\lambda = -\dfrac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$

then $$0 \leq \langle\alpha|\alpha\rangle + \lambda\lambda^*\langle\beta|\beta\rangle + \lambda^*\langle\beta|\alpha\rangle + \lambda\langle\alpha|\beta\rangle$$

$$= \langle\alpha|\alpha\rangle + \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} - \frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} - \frac{\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle}$$

$$= \frac{1}{\langle\beta|\beta\rangle}\left(\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle - \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle\right)$$

$\Rightarrow$ Schwarz ineq. ✓

b. Generalised uncertainty relation:

$$\langle \Delta A^2\rangle\langle\Delta B^2\rangle \leq \frac{1}{4}\left|\langle[\Delta A, \Delta B]\rangle\right|^2$$

Suppose $\Delta A|\alpha\rangle = \lambda\,\Delta B|\alpha\rangle$ $\quad\lambda$: pure img

First, equality in (1.4.5?) holds, which is a Schwarz ineq.

Second, in (1.4.62), the 2nd term on l.h.s

$$\frac{1}{2}\langle\alpha|\{\Delta A, \Delta B\}|\alpha\rangle$$

$$= \frac{1}{2}\left(\langle\alpha|\Delta A\,\Delta B|\alpha\rangle + \langle\alpha|\Delta B\,\Delta A|\alpha\rangle\right)$$

$$= \frac{1}{2}\left(\langle\alpha|\lambda^{*}\Delta B^{2}|\alpha\rangle + \langle\alpha|\Delta B^{2}\lambda|\alpha\rangle\right)$$

$$= \frac{1}{2}(\lambda+\lambda^{*})\langle\alpha|\Delta B^{2}|\alpha\rangle = 0$$

which is for λ is pure img.

Thus, the whole equality holds.

c.

$$\langle x'|\alpha\rangle = (2\pi d^{2})^{-1/4}\exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^{2}}{4d^{2}}\right]$$

Formula of Gaussian integral:

$$\int_{-\infty}^{\infty} x^{2n}e^{-\alpha x^{2}}\,dx = (-1)^{n}\int_{-\infty}^{\infty}\frac{\partial^{n}}{\partial\alpha^{n}}e^{-\alpha x^{2}}\,dx$$
$$= (-1)^{n}\frac{\partial^{n}}{\partial\alpha^{n}}\int_{-\infty}^{\infty}e^{-\alpha x^{2}}\,dx$$
$$= \sqrt{\pi}(-1)^{n}\frac{\partial^{n}}{\partial\alpha^{n}}\alpha^{-\frac{1}{2}}$$
$$= \sqrt{\frac{\pi}{\alpha}}\frac{(2n-1)!!}{(2\alpha)^{n}}$$

$$\langle\alpha|\Delta x^{2}|\alpha\rangle = (2\pi d^{2})^{-\frac{1}{2}}\int(x'-\langle x\rangle)^{2}\exp\left(-\frac{(x'-\langle x\rangle)^{2}}{2d^{2}}\right)dx'$$

$$= (2\pi d^{2})^{-\frac{1}{2}}(2\pi d^{2})^{\frac{1}{2}}\frac{1}{\left(2\frac{1}{2d^{2}}\right)} = d^{2}.$$

$$\langle x'|\Delta p|\alpha\rangle = (2\pi d^{2})^{-\frac{1}{4}}\left(-i\hbar\,\partial_{x'} - \langle p\rangle\right)\exp\left(\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^{2}}{4d^{2}}\right)$$

$$= (2\pi d^{2})^{-\frac{1}{4}}\left(-i\hbar\left(\cancel{\frac{i\langle p\rangle}{\hbar}} - \frac{2(x'-\langle x\rangle)}{4d^{2}}\right) - \cancel{\langle p\rangle}\right)\exp\left(\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^{2}}{4d^{2}}\right)$$

$$= (2\pi d^{2})^{-\frac{1}{4}}\frac{i\hbar\,(x'-\langle x\rangle)}{2d^{2}}\exp(\;\cdot\;\text{———}\;)$$

Note that $\langle \alpha | \Delta p | x' \rangle = \langle x' | \Delta p | \alpha \rangle^*$

$$\langle \alpha | \Delta p^2 | \alpha \rangle = (2\pi d^2)^{-\frac{1}{2}} \frac{\hbar^2}{(2d^2)^2} \int (x' - \langle x \rangle)^2 \exp\left(-\frac{(x' - \langle x \rangle)^2}{2d^2} \right) dx'$$

$$= (2\pi d^2)^{-\frac{1}{2}} \frac{\hbar^2}{(2d^2)^2} (2\pi d^2)^{\frac{1}{2}} \frac{1}{\left(2 \cdot \frac{1}{2d^2}\right)} = \frac{\hbar^2}{4d^2}.$$

$$\sqrt{\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle} = \sqrt{\frac{1}{4} \hbar^2} = \frac{\hbar}{2}.$$

$$\langle x' | \Delta x | \alpha \rangle = (2\pi d^2)^{-\frac{1}{4}} (x' - \langle x \rangle) \exp(\text{———})$$

$$= \left(\frac{i\hbar}{2d^2} \right)^{-1} \langle x' | \Delta p | \alpha \rangle$$

where $\left(\frac{i\hbar}{2d^2} \right)^{-1}$ is pure imag.

1.19 a.

$$\langle S_x \rangle = 0$$

$$\langle S_x^2 \rangle = \langle 0 | S_x^2 | 0 \rangle = \frac{1}{4} \hbar^2$$

$$\langle \Delta S_x^2 \rangle = \frac{1}{4} \hbar^2$$

Similarly, $\langle \Delta S_y^2 \rangle = \frac{1}{4} \hbar^2$

$$[S_x, S_y] = 2i S_z \qquad \langle \Delta S_z^2 \rangle = 0$$

$$\text{l.h.s} = \left(\frac{1}{4} \hbar^2 \right)^2 \geq 0 = \text{r.h.s}.$$

b. $\langle \Delta S_x^2 \rangle = 0$

$\langle \Delta S_y^2 \rangle = \frac{1}{4}\hbar^2$ $\qquad$ l.h.s. $= 0$

$\langle [S_x, S_y] \rangle = \langle 2i S_z \rangle = 0$ $\qquad$ r.h.s. $= 0$

$\Rightarrow$ l.h.s. $\geq$ r.h.s.

1.20. Note that $|\pm\rangle$ are eig states of z (not x)

$$|\psi\rangle = \left(\cos\frac{\theta}{2},\ \sin\frac{\theta}{2}e^{i\phi}\right)^T \qquad \theta \in [0, \pi),\ \phi \in [0, 2\pi)$$

$\langle S_x \rangle_\psi = \frac{\hbar}{2}\sin\theta\cos\phi$ $\qquad$ $\langle \Delta S_x^2 \rangle_\psi = \frac{\hbar^2}{4}(1 - \sin^2\theta\cos^2\phi)$

$\langle S_x^2 \rangle_\psi = \frac{\hbar^2}{4}\langle I \rangle_\psi = \frac{\hbar^2}{4}$

$\langle S_y \rangle_\psi = \frac{\hbar}{2}\sin\theta\sin\phi$

$\langle S_y^2 \rangle_\psi = \frac{\hbar^2}{4}$ $\qquad$ $\langle \Delta S_y^2 \rangle_\psi = \frac{\hbar^2}{4}(1 - \sin^2\theta\sin^2\phi)$

$$\langle \Delta S_x^2 \rangle \langle \Delta S_y^2 \rangle = \left(\frac{\hbar^2}{4}\right)^2 (1 - \sin^2\theta\cos^2\phi)(1 - \sin^2\theta\sin^2\phi)$$

$$\leq \left(\frac{\hbar^2}{4}\right)^2$$

So max is obtained i.f.f.

$\sin^2\theta\cos^2\phi = 0$ and $\sin^2\theta\sin^2\phi = 0$

$\Leftrightarrow \theta = 0$

$|\psi\rangle \propto |-\rangle$

$$|\langle [S_x, S_y] \rangle|^2 = |\langle i S_z \rangle|^2 = \frac{\hbar^2}{4}$$

Uncertainty relation holds.

Note.

逆定理：若$\hat{A},\hat{B}$对易，则$\hat{A},\hat{B}$一定有完备的共同本征态。

证明：设$|n\rangle$是$\hat{A}$的本征态，

$$\hat{A}|n\rangle = a_n|n\rangle,$$

由$[\hat{A},\hat{B}]=0$，有

$$\hat{A}\hat{B}|n\rangle = \hat{B}\hat{A}|n\rangle = a_n\hat{B}|n\rangle$$

说明$\hat{B}|n\rangle$也是$\hat{A}$的对应本征值a_n的本征态。若$\hat{A}$无简并，则$\hat{B}|n\rangle$与$|n\rangle$是同一个态，只能相差一个常数：

$$\hat{B}|n\rangle = b_n|n\rangle$$

故$|n\rangle$也是$\hat{B}$的本征态，即$\hat{A},\hat{B}$有共同的完备本征态。

若$\hat{A}$有简并，则可以用施密特方法来证明有同样的结果（作为习题）。

Suppose m-fold degeneracy on eigvalue a_n,

those eig states are $|n,1\rangle \ldots |n,m\rangle$

Our goal is to construct $\{|n,i'\rangle\}_{i=1}^{m}$ s.t. $\langle n,i'|B|n,j'\rangle \propto \delta_{ij}$

This can be done by Schmidt orth.

(Here, normalization is ignored)

$$|n,1'\rangle = |n,1\rangle$$

$$|n,2'\rangle = |n,2\rangle - \frac{\langle n,1'|B|n,2\rangle}{\langle n,1'|B|n,1'\rangle}|n,1'\rangle$$

$$|n,3'\rangle = |n,3\rangle - \frac{\langle n,1'|B|n,3\rangle}{\langle n,1'|B|n,1'\rangle}|n,1'\rangle - \frac{\langle n,2'|B|n,3\rangle}{\langle n,2'|B|n,2'\rangle}|n,2'\rangle$$

$\sim \cdots \cdots -$

第四次作业　　第2章第1，2，3，4题

2.1 $$H = -\left(\frac{eB}{mc}\right)S_z = \omega S_z,$$

where $h_A B =: [A, B]$

$$\hat{O}(t) = e^{\frac{iHt}{\hbar}} \hat{O} e^{-\frac{iHt}{\hbar}} = \sum_{n=0}^{\infty} \frac{i^n}{n!} h_H^n \hat{O} = \sum_{n=0}^{\infty} \frac{(i\omega t)^n}{n!} h_{S_z/\hbar}^n \hat{O}$$

$$h_{S_z/\hbar} S_x = i S_y, \quad h_{S_z} S_y = -i S_x, \quad h_{S_z/\hbar} S_z = 0.$$

① $$S_x(t) = \sum_{n:\text{even}} \frac{(i\omega t)^n}{n!} S_x + \sum_{n:\text{odd}} \frac{(i\omega t)^n}{n!} i S_y$$

$$= \cos\omega t \, S_x - \sin\omega t \, S_y.$$

② $$S_y(t) = \sum_{n:\text{even}} \frac{(i\omega t)^n}{n!} S_y + \sum_{n:\text{odd}} \frac{(i\omega t)^n}{n!} (-i) \cdot S_x$$

$$= \cos\omega t \, S_y + \sin\omega t \, S_x.$$

③ $$S_z(t) = S_z.$$

2.2. H is not Hermitian, so prob. conservation may be broken.

Suppose $H = h|2\rangle\langle 1|$

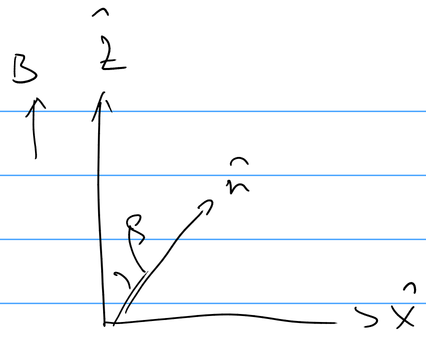
$$U = e^{-iHt} = \sum_{n=0}^{\infty} \frac{(-iHt)^n}{n!} = 1 - iHt = 1 - ih|2\rangle\langle 1|$$

Note that $H^n = 0$, $n \geq 2$.

$U|1\rangle = |1\rangle - ih|2\rangle$, whose norm is large that 1.

That breaks prob. conservation.

2.3



$$H = \frac{eB}{mc} S_z =: \omega S_z$$

$$|\psi(t=0)\rangle = \left(\cos\frac{\beta}{2}, \sin\frac{\beta}{2}\right)^T$$

a. As calculation in problem 2.2 suggests;

$$S_x(t) = \cos(\omega t) S_x - \sin\omega t \, S_y.$$

$$\langle S_x(t)\rangle = \frac{\hbar}{2}\cos(\omega t)\sin\beta$$

$$P_r\left(S_x = \frac{\hbar}{2}\right) = \frac{1}{2}\left(\cos(\omega t)\sin\beta + 1\right).$$

b. $\langle S_x(t)\rangle = \frac{\hbar}{2}\cos(\omega t)\sin\beta$ is given in a.

c. (i) $\beta \to 0$ $\quad P_r\left(S_x = +\frac{\hbar}{2}\right) = \frac{1}{2}$ $\quad \langle S_x(t)\rangle = 0$

(ii) $\beta \to \frac{\pi}{2}$ $\quad P_r\left(S_x = +\frac{\hbar}{2}\right) = \frac{1}{2}(\cos\omega t + 1)$ $\quad \langle S_x(t)\rangle = \frac{\hbar}{2}\cos(\omega t)$

2.4

$$H = \frac{1}{2m} p^2$$

$$X(t) = e^{iHt/\hbar} \, X \, e^{-iHt/\hbar} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{it}{\hbar}\right)^n h_H^n \, X$$

$[x, p] = i\hbar$

$$h_H X = \frac{1}{2m}[p^2, x] = -\frac{i\hbar}{m} p$$

$$h_H^n X = 0 \quad , \quad n \geq 2$$

$$X(t) = X + \left(\frac{it}{\hbar}\right)\cdot\left(-\frac{i\hbar}{m}p\right) = x + \frac{p}{m}t.$$

$$[X(t), X(0)] = \left[\frac{p}{m}t \, x\right] = -\frac{i\hbar}{m}t.$$

第五次作业　　第2章第9，10，11题。

2.9　$H = \Delta\left(|L\rangle\langle R| + |R\rangle\langle L|\right) = \Delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ under $\{|L\rangle, |R\rangle\}$

a. Eigstate: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^T$　Eigval: $+\Delta$

...... : $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)^T$　...... : $-\Delta$

b. $$|\alpha\rangle = \frac{1}{\sqrt{2}}\left(\langle R|\alpha\rangle + \langle L|\alpha\rangle\right)\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} + \frac{1}{\sqrt{2}}\left(\langle R|\alpha\rangle - \langle L|\alpha\rangle\right)\begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$|\alpha, t_0 = 0; t\rangle = e^{-\frac{iHt}{\hbar}}|\alpha\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\langle R|\alpha\rangle + \langle L|\alpha\rangle\right)\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} e^{-\frac{i\Delta t}{\hbar}} + \frac{1}{\sqrt{2}}\left(\langle R|\alpha\rangle - \langle L|\alpha\rangle\right)\begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} e^{\frac{i\Delta t}{\hbar}}$$

$$= \begin{pmatrix} \frac{1}{2}\left(\langle R|\alpha\rangle + \langle L|\alpha\rangle\right) e^{-\frac{i\Delta t}{\hbar}} + \frac{1}{2}\left(\langle R|\alpha\rangle - \langle L|\alpha\rangle\right) e^{\frac{i\Delta t}{\hbar}} \\ \frac{1}{2}\left(\langle R|\alpha\rangle + \langle L|\alpha\rangle\right) e^{-\frac{i\Delta t}{\hbar}} - \frac{1}{2}\left(\langle R|\alpha\rangle - \langle L|\alpha\rangle\right) e^{\frac{i\Delta t}{\hbar}} \end{pmatrix}$$

$$= \begin{pmatrix} \langle R|\alpha\rangle \cos\frac{\Delta t}{\hbar} - \langle L|\alpha\rangle i\sin\frac{\Delta t}{\hbar} \\ \langle L|\alpha\rangle \cos\frac{\Delta t}{\hbar} + \langle R|\alpha\rangle i\sin\frac{\Delta t}{\hbar} \end{pmatrix}$$

c. Substitute $\langle R|\alpha\rangle = 0$, $\langle L|\alpha\rangle = 1$ to the result of b.

$$|\alpha, t_0=0; t\rangle = \begin{pmatrix} -i\sin\frac{\Delta t}{\hbar} \\ \cos\frac{\Delta t}{\hbar} \end{pmatrix}$$

$$|\langle L|\alpha, t_0=0; t\rangle|^2 = \sin^2\frac{\Delta t}{\hbar}$$

d. Denote $|\alpha, t_0=0; t\rangle =: |\alpha(t)\rangle$

$$\frac{\partial}{\partial t}|\alpha(t)\rangle = -\frac{i}{\hbar} H |\alpha(t)\rangle$$

$$\begin{pmatrix} \langle L|\alpha(t)\rangle \\ \langle R|\alpha(t)\rangle \end{pmatrix} = -\frac{i\Delta}{\hbar}\begin{pmatrix} \langle R|\alpha(t)\rangle \\ \langle L|\alpha(t)\rangle \end{pmatrix},$$

which is satisfied by result of b.

e. $$e^{-\frac{iHt}{\hbar}} = \sum_n \frac{1}{n!}\left(\frac{-it}{\hbar}\right)^n H^n$$

$$H = \Delta\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \qquad H^n = 0, \quad n \geq 2$$

$$|\alpha(t)\rangle = |\alpha\rangle - \frac{i\Delta t}{\hbar}\langle R|\alpha\rangle|L\rangle$$

$$= \begin{pmatrix} \langle L|\alpha\rangle - \frac{i\Delta t}{\hbar}\langle R|\alpha\rangle \\ \langle R|\alpha\rangle \end{pmatrix}$$

$$|\langle\alpha(t)|\alpha(t)\rangle|^2 = 1 - \left(\frac{\Delta t}{\hbar}\right)^2 \langle R|\alpha\rangle^2,$$

which will fall from unity in general.

2.10. $H = \frac{1}{2}m\omega^2 x^2 + \frac{1}{2m}p^2$

(a). x, p are not changed in Schrödinger picture.

In Heisenberg picture,

$$x(t) = e^{\frac{iHt}{\hbar}} x\, e^{-\frac{iHt}{\hbar}} = \sum \frac{1}{n!}\left(\frac{it}{\hbar}\right)^n h_H^n x$$

$$\frac{it}{\hbar} h_H x = [\tfrac{1}{2m}p^2, x] = \frac{pt}{m}$$

$$\left(\frac{it}{\hbar}\right)^2 h_H^2 x = \frac{it^2}{\hbar m}[H, p] = -\frac{it^2}{\hbar m}[\tfrac{1}{2}m\omega^2 x^2, p]$$

$$= -\frac{it^2}{\hbar m} m\omega^2 \, i\hbar\, x = \omega^2 t^2 x$$

$$\Rightarrow x(t) = \cos(\omega t)\, x + \sin(\omega t)\frac{p}{m\omega}$$

Similarly, $p(t) = \cos(\omega t)\, p - \sin(\omega t)\, \omega m x$

(b) Denote the most general state vec. as $|\psi\rangle$.

In Heisenberg picture, $|\psi\rangle$ is not changed.

In Schrödinger picture, $|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\psi\rangle$.

2-11. As calculated in 2.10,

$$x(t) = \cos(\omega t)\, x + \sin(\omega t)\frac{p}{m\omega}$$

$$\langle x\rangle|_t = \langle 0| e^{\frac{ipa}{\hbar}} \left(\cos(\omega t) x + \sin(\omega t)\frac{p}{m\omega}\right) e^{-\frac{ipa}{\hbar}} |0\rangle$$

$$= \cos(\omega t) \langle 0| e^{\frac{ipa}{\hbar}}\, x\, e^{-\frac{ipa}{\hbar}} |0\rangle$$

$$= \cos(\omega t) \langle 0| (x + a) |0\rangle$$

$$= a \cos(\omega t).$$

Note that $e^{\frac{-ipa}{\hbar}}$ is a spatial transition operator. So the result fits w/ its classical case.

第六次作业　　第2章第15，16，17，18题

2.15 Recapture 2.10,

$$x(t) = \cos(\omega t)\, x + \sin(\omega t) \frac{p}{m\omega}.$$

$$\langle x(t)\, x(0)\rangle = \langle 0|\, \cos(\omega t)\, x^2 + \sin(\omega t) \frac{px}{m\omega}\, |0\rangle$$

$$= \cos(\omega t) \langle 0|\, x^2\, |0\rangle$$

$$= \cos(\omega t) \frac{\hbar}{2m\omega}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a).$$

2.16

a. $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha^* \langle 0| + \beta^* \langle 1|\right)(a^\dagger + a)\left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha^* \langle 0| + \beta^* \langle 1|\right)\left(\alpha|1\rangle + \beta|0\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha^*\beta + \beta^*\alpha\right) = 2\sqrt{\frac{\hbar}{2m\omega}} \cdot \mathrm{Re}(\alpha^*\beta)$$

$$\mathrm{Re}(\alpha^*\beta) \le |\alpha^*\beta| = |\alpha||\beta| \le \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) = 1.$$

Eq. holds when $|\alpha| = |\beta|$ and $\alpha^*\beta$ is real.

Up to a global phase, $\begin{cases} \alpha = \frac{\sqrt{2}}{2} \\ \beta = \frac{\sqrt{2}}{2} \end{cases}$

$$|\psi\rangle = \frac{\sqrt{2}}{2}\left(|0\rangle + |1\rangle\right)$$

b.

$$|\psi(t)\rangle = \frac{\sqrt{2}}{2}\left(e^{-i\frac{\omega t}{2}}|0\rangle + e^{-i\frac{3\omega t}{2}}|1\rangle\right),$$

where ω is from $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2$

(i) $$\langle\psi(t)|x|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\cdot\frac{1}{2}\cdot\left(e^{\frac{i\omega t}{2}}\langle 0| + e^{\frac{i3\omega t}{2}}\langle 1|\right)(a+a^{\dagger})\left(e^{-i\frac{\omega}{2}t}|0\rangle + e^{-i\frac{3\omega}{2}t}|1\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cdot\frac{1}{2}\cdot\left(e^{\frac{i\omega t}{2}}\langle 0| + e^{\frac{i3\omega t}{2}}\langle 1|\right)\left(e^{-i\frac{\omega}{2}t}|1\rangle + e^{-\frac{i3\omega t}{2}}|0\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cdot\frac{1}{2}\cdot\left(e^{\frac{i\omega t}{2}}\cdot e^{-\frac{i3\omega t}{2}} + e^{\frac{i3\omega t}{2}}e^{-\frac{i\omega t}{2}}\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)$$

(ii) $$x(t) = \cos(\omega t)\, x + \sin(\omega t)\frac{p}{m}.$$

$$= \cos(\omega t)\sqrt{\frac{\hbar}{2m\omega}}\,(a+a^{\dagger}) + \sin(\omega t)\,\frac{1}{m}\, i\sqrt{\frac{\hbar m\omega}{2}}\,(a^{\dagger}-a)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(e^{i\omega t}a^{\dagger} + e^{-i\omega t}a\right)$$

$$\langle\psi|x(t)|\psi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\frac{1}{2}\left(\langle 0| + \langle 1|\right)\left(e^{i\omega t}a^{\dagger} + e^{-i\omega t}a\right)\left(|0\rangle + |1\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cdot\frac{1}{2}\cdot\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t).$$

c. Use Heisenberg.

$$\langle\psi|X(t)^2|\psi\rangle = \frac{\hbar}{2m\omega}\frac{1}{2}\left(\langle 0|+\langle 1|\right)\left(e^{i\omega t}a^{H}+e^{-i\omega t}a\right)^2 \left(|0\rangle+|1\rangle\right)$$

$$= \frac{\hbar}{2m\omega}\frac{1}{2}\left(\langle 0|+\langle 1|\right)\left(a^{H}a+aa^{H}\right)\left(|0\rangle+|1\rangle\right)$$

$$= \frac{\hbar}{2m\omega}\cdot\frac{1}{2}\cdot 2 = \frac{\hbar}{2m\omega}.$$

$$\langle \Delta X\rangle^2 = \frac{\hbar}{2m\omega} - \left(\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)\right)^2$$

$$= \frac{\hbar}{2m\omega}\left(1-\cos^2(\omega t)\right).$$

2.17 $$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\langle 0|e^{ikx}|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\int e^{ikx}e^{-\frac{m\omega x^2}{\hbar}}dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{k^2\hbar}{4m\omega}}\int e^{-\frac{m\omega}{\hbar}\left(x-\frac{ik\hbar}{2m\omega}\right)^2}dx$$

$$= e^{-\frac{k^2\hbar}{4m\omega}}$$

$$\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}$$

So, $\langle 0|e^{ikx}|0\rangle = \exp\left(-k^2\frac{\langle 0|x^2|0\rangle}{2}\right)$

2.18

a.

$$|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}} e^{\lambda a^\dagger} |0\rangle$$

$$a|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}} a\, e^{\lambda a^\dagger} |0\rangle$$

$$= e^{-\frac{|\lambda|^2}{2}} \sum_n \frac{1}{n!} \lambda^n a (a^\dagger)^n |0\rangle$$

$$= e^{-\frac{|\lambda|^2}{2}} \sum_n \frac{\lambda^n}{n!} a \sqrt{n!} |n\rangle \quad - - - - \quad (*)$$

$$= e^{-\frac{|\lambda|^2}{2}} \sum_n \frac{\lambda^n}{n!} \cdot n \cdot \sqrt{(n-1)!} |n-1\rangle$$

$$= e^{-\frac{|\lambda|^2}{2}} \lambda \sum_n \frac{\lambda^{n-1}}{(n-1)!} (a^\dagger)^{n-1} |n-1\rangle$$

$$= \lambda |\lambda\rangle .$$

b.

$$\langle\lambda| x |\lambda\rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle\lambda| (a^\dagger + a) |\lambda\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} (\lambda + \lambda^*)$$

$$\langle\lambda| x^2 |\lambda\rangle = \frac{\hbar}{2m\omega} \langle\lambda| (a^\dagger + a)^2 |\lambda\rangle$$

$$= \frac{\hbar}{2m\omega} \langle\lambda| \left(a^{\dagger 2} + a^2 + 2a^\dagger a + 1 \right) |\lambda\rangle$$

$$= \frac{\hbar}{2m\omega} \left(1 + (\lambda + \lambda^*)^2 \right)$$

$$\langle \Delta x^2 \rangle = \frac{\hbar}{2m\omega}$$

Similarly, $\langle \Delta p^2 \rangle = \frac{\hbar m\omega}{2}$, $\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle = \frac{\hbar^2}{4}$, which is the minimal uncertainty relation.

c. $f(n) = \langle n|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}} \frac{\lambda^n}{\sqrt{n!}}$ from (*) in a.

$$|f(n)|^2 = e^{-|\lambda|^2} \frac{|\lambda|^{2n}}{n!} = \frac{1}{n!} \theta^n e^{-\theta},$$

where $\theta =: |\lambda|^2$.

$$|f(n)|^2 - |f(n-1)|^2$$

$$= \frac{e^{-\theta}}{(n-1)!}\left(\frac{\theta}{n} - 1\right) \begin{cases} \geq 0 & , n \leq \theta \\ < 0 & , n > \theta \end{cases}$$

So, most probable value of n is $\lfloor \theta \rfloor = \lfloor |\lambda|^2 \rfloor$

d. <u>lemma</u>: $e^{\frac{ip\ell}{\hbar}} a e^{-\frac{ip\ell}{\hbar}} = a - \ell\sqrt{\frac{m\omega}{2\hbar}}$

$$\text{l.h.s.} = \sum \frac{1}{n!}\left(\frac{i\ell}{\hbar}\right)^n \mathcal{h}_p^n a$$

$$\mathcal{h}_p a = i\sqrt{\frac{m\omega\hbar}{2}}[a^\dagger - a, a] = -i\sqrt{\frac{m\omega\hbar}{2}}$$

And higher commutators vanish.

$$\Rightarrow \text{l.h.s.} = a - \left(\frac{i\ell}{\hbar}\right)\cdot\left(-i\sqrt{\frac{m\omega\hbar}{2}}\right)$$

$$= a - \ell\sqrt{\frac{m\omega}{2\hbar}}$$

$$a e^{+\frac{ip\ell}{\hbar}}|\lambda\rangle = e^{+\frac{ip\ell}{\hbar}} e^{-\frac{ip\ell}{\hbar}} a e^{+\frac{ip\ell}{\hbar}}|\lambda\rangle$$

$$= e^{+\frac{ip\ell}{\hbar}}\left(a + \ell\sqrt{\frac{m\omega}{2\hbar}}\right)|\lambda\rangle$$

$$= e^{+\frac{ip\ell}{\hbar}}\left(\lambda + \ell\sqrt{\frac{m\omega}{2\hbar}}\right)|\lambda\rangle.$$

(Note that sgn is opposite to that in lemma).

Take $\ell = \lambda \sqrt{\frac{2\hbar}{m\omega}}$, $a \, e^{+\frac{i p \ell}{\hbar}} |\lambda\rangle = 0$

$$\Rightarrow \quad e^{+\frac{i p \ell}{\hbar}} |\lambda\rangle = |0\rangle$$

$$\Rightarrow \quad |\lambda\rangle = e^{-\frac{i p \ell}{\hbar}} |0\rangle$$

Reference of this solution:

https://homepage.univie.ac.at/reinhold.bertlmann/pdfs/T2_Skript_Ch_5.pdf