APMA 0160 Lecture 12

Ordinary Differential Equations

Brown University, Summer 2017

Monday, July 24

Announcements

- Grades for HW 6 and HW 7 are online
- HW 7 due Wednesday
- HW 8 out tomorrow
- Recitaiton Wednesday–bring your computer

Review

What function, when differentiated, gives the function $2x$?

What function, when differentiated, does not change?

What function, when differentiated, doubles in magnitude?

What function, whe differentiated, keeps the same magnitude but changes sign?

What function, when differentiated twice, keeps the same magnitude but changes sign?

What is a differential equation?

Much of middle/high school math is dedicated to solving algebraic equations. Examples include

- linear equations: $12x + 13 = 25$
- quadratic equations: $x^2 - 3x - 2 = 0$
- trigonometric equations: $\cos(2\theta) = 1$

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Ordinary differential equations (ODEs) involve derivatives

- $y'(x) = y(x)$
- $y''(x) - 3y'(x) - 2y(x) = 0$
- $y'(x) = \frac{y(x)}{2} \cos x$

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Solutions to ODEs are functions.

Examples of ODEs

Differential equations occur naturally in the modeling of all kind of phenomena:

- Newton's law: $F = ma$. However, a is short hand for $x''(t)$.
- Population models: $y' = ky$ or $y' = ky(y - M)$
- Mathematical finance: Black-Scholes equation
- Fluids/solids: Navier-Stokes equations/Navier-Lame equations
- Chemistry/quantum physics: Schrodinger's equation
- Electromagnetism: Maxwell's equations

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Any time you have an equation with a quantity that is changing, you have a differential equation.

A basic ODE

Let y be a function of x. Consider the differential equation

$$y' = y$$

What is the answer?

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$y(x) = e^x$ works. Is it the only answer?

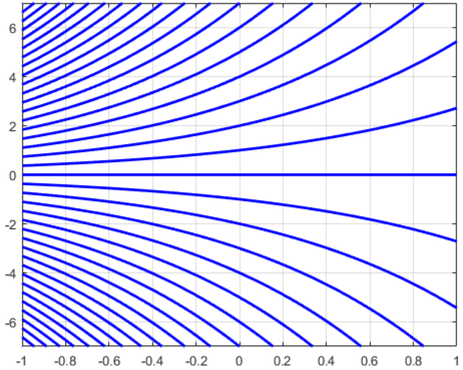
There are infinitely many solutions, all of the form $y(x) = Ce^x$, where C is any constant. We call this the general solution.

A basic ODE

Let y be a function of x. Consider the differential equation

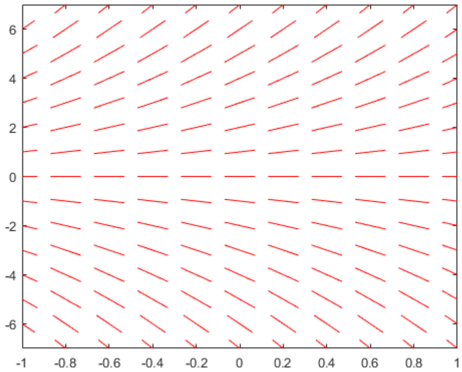
$$y' = y \implies \boxed{y = Ce^x}$$

A plot of the infinitely many solutions.



A basic ODE

In this case, we obtained the solution, but we could have guessed the behavior of the solutions by examining the direction field. Recall that y' represents the slope of a curve.



At any point (x_0, y_0), we can draw a vector with slope $y' = y_0$.

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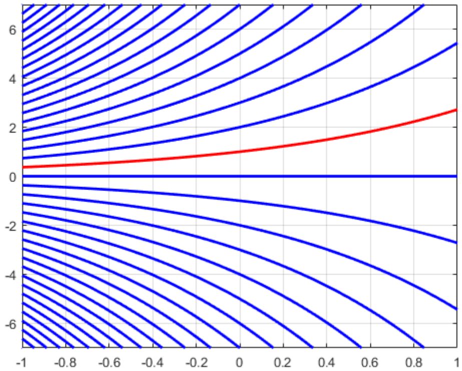
This allows us to select out THE solution to the ODE that satisfies this condition.

A basic ODE

Let y be a function of x. Consider the differential equation

$$y' = y \implies \boxed{y = Ce^x}$$

Only ONE of the infinitely many solutions satisfies the given initial condition.



We call an ordinary differential equation together with an initial condition an initial value problem.

Theorem

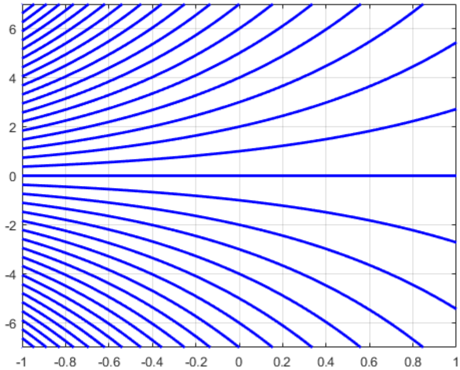
The initial value problem

$$y' = f(t, y)$$
$$y(0) = y_0$$

always has a unique solution if f and $\partial f/\partial y$ are continuous.

Existence and Uniqueness

In other words, if you pick any point in the xy-plane:



Existence says that there is a solution passing through that point. Uniqueness says that there is only one solution passing through that point, so two different curves cannot cross.

Autonomous ODEs

Suppose $y = y(t)$. Autonomous ODEs are first order ODEs (only first derivatives and no higher derivatives) that do not depend explicitly on the independent variable t.

- $y' = y$
- $y' = 2y^2 - y + \sin y + 14$
- $y' = ry(1 - y/K)$ where r and K are constants

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Examples of non-autonomus ODEs:

- $y' = t$
- $y' = 2y^2 - y + \sin y + 14t$
- $y'' = \sin y' - y \tan y$

Autonomous ODEs

We are interested in analyzing ODEs without explicitly solving them. Note that autonomous ODEs can always be solved explicitly in principle:

$$\frac{dy}{dt} = f(y) \implies \frac{1}{f(y)}dy = dt \implies t = \int \frac{dy}{f(y)}$$

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However, we are not interested in using this approach for two reasons:

- We want to build intuition, not memorize a formula.
- Often the integral is difficult to do or cannot be done analytically.

Autonomous ODEs

Consider the logistic equation

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The roots of $f(y)$ are called equilibrium solutions. Why?

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Furthermore, since $f(y) > 0$ on the interval from $[0, K]$, any solution in that interval must always be increasing.

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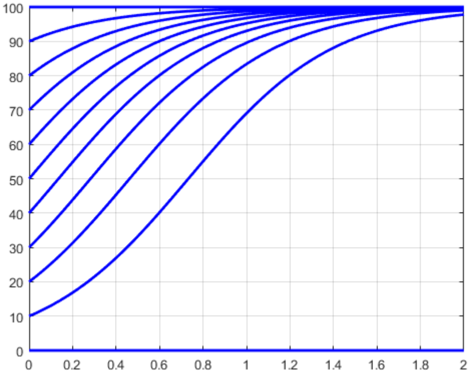
Furthermore, since $f(y) > 0$ on the interval from $[0, K]$, any solution in that interval must always be increasing.

Any solution starting in this interval is bounded above by $y = K$ but always increasing towards $y = K$. The result is that all solutions tend asymptotically to K as $t \to \infty$.

Autonomous ODEs

In the language of dynamical systems, we say $y = 0$ is an unstable equilibrium and $y = K$ is an asymptotically stable equilibrium.



What does this mean in the context of population modeling?

Stability criteria

A different (faster?) way of reaching the same conclusion:

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right) = f(y)$$

$$\implies f'(y) = r\left(1 - \frac{2y}{K}\right)$$

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When $y = 0$, $f'(0) = r > 0 \implies$ unstable
When $y = K$, $f'(K) = -r < 0 \implies$ asymptotically stable

We can summarize this in a theorem:

Theorem

Let $y' = f(y)$ be an autonomous ODE with equilibrium solution $y = y^*$. If $f'(y^*) > 0$, then the solution $y = y^*$ is unstable. On the other hand, if $f'(y^*) < 0$, $y = y^*$ is asymptotically stable.

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Proof

Taylor expand $f(y)$ around the fixed point $y = y^*$:

$$f(y) = f(y^*) + f'(y^*)(y - y^*) + \frac{f''(\xi)}{2}(y - y^*)^2$$

$$\implies y' \simeq f'(y)(y - y^*)$$

$$\implies y = Ce^{f'(y^*)t} + y^*$$

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If $f(y^*) < 0$, the solution flows toward $y = y^*$.

Note that one case is missing...

Another example

A second order chemical reaction involves the collision of one molecule of substance P with one molcule of substance Q to create one molecule of a new substance X. This is denoted by the equation $P + Q \to X$.

Let p_0 and q_0 be the initial concentrations of chemicals P and Q. Let $x(t)$ be the concentration of X as a function of time.

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We can write down an equation for the reaction in terms of $x(t)$ by assuming that the rate at which the reaction occurs is proportional to the concentration chemical P and Q.

This leads to the equation $x' = \alpha(p_0 - x)(q_0 - x)$ for some constant α.

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Does this make sense?

One more example

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Since $f(y)$ is increasing for both $y < 1$ and $y > 1$, the solutions flow toward $y = 1$ for $y < 1$ and away from $y = 1$ for $y > 1$. This is an example of a semistable equilibrium.

Higher order equations

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You can rearrange it into a first order system of ODEs by defining $u_1 = y$, $u_2 = y'$.

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Futhermore, if the system is linear, it may be represented as a matrix

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Numerical ODEs

Now let's return to computations. If we have a first-order initial value problem

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how can we solve it approximately?

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how can we solve it approximately?

One approach is to integrate both sides of the equation over a short time interval (time-stepping):

$$\int_{t_i}^{t_{i+1}} y' dt = \int_{t_i}^{t_{i+1}} f(t, y) dt \implies y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t, y) dt$$

Then, we can use any of the rules we have learned to approximate the integral on the RHS. Suppose we use the left rectangle rule:

$$\int_{t_i}^{t_{i+1}} f(t, y) dt \simeq hf(t_i, y_i)$$

Euler methods

If we rearrange, we get

$$\boxed{y_{i+1} = y_i + hf(t_i, y_i)}$$

where we introduced the notation $y_i = y(t_i)$. This is known as the forward Euler method.

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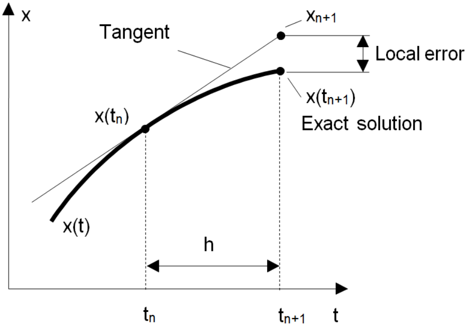

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What are the inputs?

What are the outputs?

Euler methods

In the code, what did you notice happened after a long period of time?

We will come back to the error shortly.

Euler methods

What would have happened if we had used a backward finite difference to approximate y'?

$$y' \simeq \frac{y(t_i) - y(t_i - h)}{h} = f(t_i, y_i) \implies \boxed{y_{i+1} = y_i + hf(t_{i+1}, y_{i+1})}$$

This is called the backward Euler method. Do you see a potential issue here?

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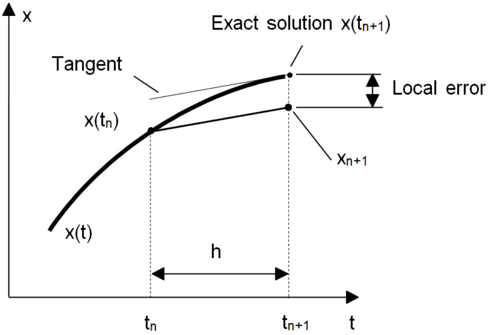
Forward Euler is an example of an explicit method. Backward Euler is an example of an implicit method.

Although we are again approximating by a tangent line, we're using a different location and then translating the line to our current location.



The Crank-Nicolson method

This method is obtained by averaging the results of the forward and backward Euler methods. Alternatively, this is obtained by integrating the equation and approximating the integral using the midpoint rule.

$$\boxed{y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, y_{i+1})]}$$

Is this method implicit or explicit?

Error analysis

As usual, we want to know which method is the most accurate. To this end, we let Y_i be the true solution to the ODE and define two types of error:

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Local truncation error: $\tau_i(h) = Y_i - y_i$

Global error: $\tau(h) = \max_i |\tau_i(h)|$

Error analysis

Theorem

The local truncation error of the forward Euler method is $O(h^2)$*.*
The global error of the forward Euler method is $O(h)$*.*

Proof

Taylor expand: $Y(x_{i+1}) = Y(x_i) + hY'(x_i) + \frac{h^2}{2}Y''(x_i) + \frac{h^3}{6}Y'''(\xi_i)$.
Euler method: $y_{i+1} = y_i + hf(t_i, y_i)$
Taylor expand $f(t_i, y_i)$ *about the true solution:*
$f(t_i, y_i) = f(t_i, Y_i) + hf'(t_i, Y_i) + \dots$
All terms $O(h)$ *and lower cancel out, so the dominant remaining term is* $O(h^2)$*.*

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All terms $O(h)$ *and lower cancel out, so the dominant remaining term is* $O(h^2)$.
Global error: At each step, we accumulate $O(h^2)$ *error. So, at after* N *steps, the error is* $O(Nh^2) = O(h)$.

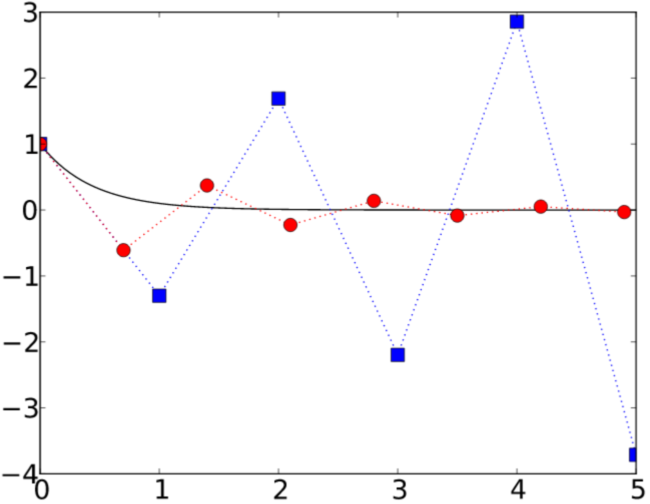
Error analysis

Backward Euler is also $O(h^2)$ locally and $O(h)$ globally.

Crank-Nicolson is $O(h^3)$ locally and $O(h^2)$ locally.

However, there are other things to worry about besides just order of accuracy.

Depending on the step size, Euler's method may fail to converge.



Stability

Consider the test equation $y' = \lambda y$ with $y(0) = y_0$. The forward Euler method reads

$$y_{k+1} = y_i + hf(t_j, y_k) = (1 + h\lambda)y_k$$

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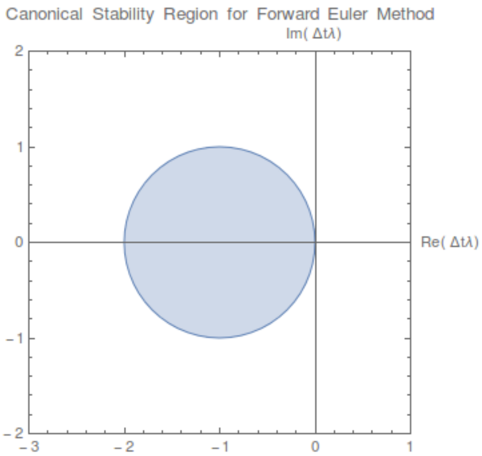
$$y_k = (1 + h\lambda)^k y_0$$

whereas the actual solution of the test equation is $Y(t) = y_0 e^{\lambda t}$ (can verify this).

Suppose that $Re\lambda < 0$. Then, the solution is an exponential decay, and we need $y_k \to 0$. This only happens if $\boxed{|1 + h\lambda| < 1}$.

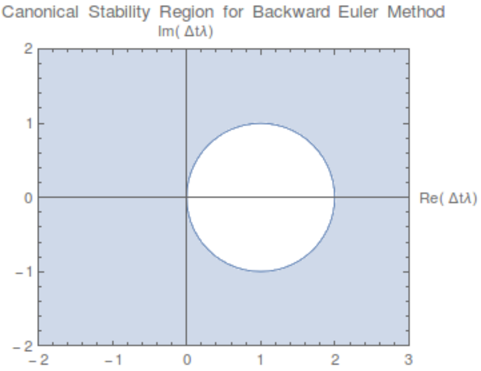
Stability

Note that if λ is real, then this simplifies to $0 < h < 2/|\lambda|$. So, when doing foward Euler, we need to be very careful about what step size we use.



Stability

Applying the same analysis to the backward euler method gives a much larger region of stability.



In general, implicit methods have better stability.

Stability

We will have more to say in the future but here is a summary of stabilities of various methods so far

	$\mathrm{Re}(\lambda) <= 0$	$\mathrm{Re}(\lambda) > 0$
Exact Soln.	Stable	Unstable
Forward Euler	"Conditionally Stable" (stable for $\|1 + h\lambda\| \leq 1$)	"Unconditionally Unstable" (unstable for $h > 0$)
Backward Euler	"Unconditionally Stable" (stable for $h \geq 0$)	"Conditionally Stable" (stable for $\|1 - h\lambda\| \geq 1$)
Trapezoid	"Unconditionally Stable" (stable for $h \geq 0$)	"Unconditionally Unstable" (unstable for $h > 0$)