Verification of the Weil conjectures for cell complexes related to n-dimmensional projective space

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1 Introduction

In this paper we do an explicit verification of the Weil conjectures for various cell complexes related to n-dimensional projective space defined over F_q. Here F_q is a finite field with q elements, where $q = p^l$, p a rational prime.To give motivation, we define the Riemann Zeta function, discussing its various properties such as convergence, analytic continuation, functional equation, and the Riemann Hypothesis.The Weil conjectures are then stated, namely, rationality of Hasse-Weil zeta function, the functional equation, Riemann hypothsis, and Betti numbers. We then verify the conjectures explicitly for n-dimensional projective space, explicitely calculating the Hasse-Weil zeta function as predicted by the conjectures and verifying the functional equation. We also outline proofs for elliptic curves and curves in general.We then consider the Hasse-Weil zeta function of n-dimensional projective space and vary it, changing factors in a way so it corresponds to a new cell structure, while keeping the dimension and Euler characteristic the same.The Hasse-Weil zeta function of this space is shown to satisfy the Weil conjectures. These variations are then generalized and their associated Hasse-Weil zeta functions are also shown to satisfy the Weil conjectures.

2 Riemann Zeta function

The Riemann zeta function is defined as

$$\zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s} \tag{1}$$

where here s is a complex variable. One sees that $\zeta(s)$ diverges at $s = 1$. Since by unique factorization one can rewrite $\zeta(s)$ as a product given by

$$\zeta(s) = \prod_{p}^{\infty} \frac{1}{1 - p^{-s}} \tag{2}$$

we see that the divergence of $\zeta(s)$ at $s = 1$ gives a proof that there are an infinite number of primes.

More precisely, $\zeta(s)$ has a simple pole at $s = 1$ and is only defined for the Re $s > 1$. With the exception of this pole, it can be analytically continued to the whole complex plane.Furthermore, it satisfies a functional equation.For background material relating to the Riemann Zeta function one can look at [Apo76] or [Edw74].

It is conjectured that all the non-trivial zeroes of $\zeta(s)$ lie on the line Re $s = \frac{1}{2}$. (Zeroes at $s = -2, -4, ...$ are called trivial zeroes.) This conjecture is known as the Riemann Hypothesis. This is still unknown. What is known,due to Deligne, is that the analogous statement for the zeta function of a nonsingular projective variety over the finite field F_q is true.Here $q = p^n$, p a rational prime. What we will do here is verify these conjectures for the varieties $P^1_{F_q}$ and $P^n_{F_q}$. We will also outline proofs for elliptic curves and curves in general. Finally, we will show that if we vary the cell structure of projective space in a specific way, the new space will have a Hasse-Weil zeta function that will also satisfy the conjectures.Before doing this, we must first define the zeta function for a nonsingular projective variety.

3 Zeta function for a nonsingular projective variety

We now define the zeta function associated to a nonsingular projective variety over F_q where $q = p^n$, for p a rational prime. The definition comes from a straight forward generalization of the Riemann Zeta function given by

$$Z(X, s) = \sum_D \frac{1}{N(D)^s} \tag{3}$$

D is a positive divisor of X. For a curve,D is just a finite sum of points that lie on the curve.

Now the product formula for the Riemann Zeta function is given by

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \tag{4}$$

.

There is an analogous product formula for $Z(X, s)$ involving divisors. After a certain amount of manipulation, i.e., expansion of the individual terms in the product, interchanging of summations, we obtain the standard formula for the $Z(X, s)$ given by

$$Z(X, s) = exp(\sum_1^\infty \frac{X(F_{q^m})}{m} q^{-ms}) \tag{5}$$

.

Here $X(F_{q^m})$ are the number of points on X with coordinates in F_{q^m}, the degree n extension of the finite field F_q.

4 Weil conjectures

The Weil conjectures are a prediction of the behavior of the Hasse-Weil zeta function for a nonsingular projective variety.There are four predictions, one of them being a corresponding Riemann Hypothisis for this setting.Background for the conjectures, along with proofs, can be found in [Sza19].

Suppose X is a smooth projective variety of dinension n over F_q. Then $Z(X,s)$ has the following properties:

1) It is rational function in $\mathbf{Q}(t)$, where $t=q^{-s}$.

2)$Z(X.s)$ obeys a functional equation given by

$$Z(X,n-s)=\pm q^{E(\frac{n}{2}-s)}Z(X,s) \tag{6}$$

where E is the Euler characteristic of X.

3) Riemann Hypothsis,i.e., $Z(X,s)$ is of the form

$$Z(X,s)=\frac{P_1(t)P_3(t)...P_{2n-1}(t)}{P_0(t)P_2(t)...P_{2n}(t)} \tag{7}$$

where $P_0(t)=1-t$ and $P_{2n}(t)=1-q^n t$ and the other $P_i(t)$ are of the form

$$P_j(t)=\prod_{1}^{degP_j(t)}(1-\alpha_{ij}t) \tag{8}$$

where $|\alpha_{ij}|=q^{\frac{j}{n}}$.One should note that in the case of a nonsingular projective variety of dimension n the zeroes are coming from the polnomials $P_1(t),P_3(t)...P_{2n-1}(t)$ and are along the vertical lines $Res=\frac{1}{2},\frac{3}{2},..\frac{2n-1}{2}$ in the complex plane.

4) The degree of the polynomial $P_j(t)$ are equal to the Betti number b_j associated with the j_{th} cohomology group of the nonsingular projective variety $\overline{X}(\mathbf{C})$, assuming the variety X_{F_q} is the good reduction of $\overline{X}(\mathbf{C})$.

5 Verification of the Weil conjectures for $P^1_{F_q}$

As was calculated previously, the zeta function for $P^1_{F_q}$ is given by

$$Z(P^1_{F_q},s)=\frac{1}{(1-q^{1-s})(1-q^{-s})} \tag{9}$$

Since the zeta function has the above form, we see the only thing that needs to be verified is the functional equation.In this case this is given by

$$Z(P^1_{F_q},n-s)=\pm q^{E(\frac{n}{2}-s)}Z(P^1_{F_q},s) \tag{10}$$

Where here E denotes the Euler characteristic of $P^1_{F_q}$ and n denotes the dimension of the variety.

Since $n=1$ and $E=2$ for $P^1_{F_q}$, after dropping the $\pm$, the functional equation becomes

$$\frac{1}{(1-q^s)(1-q^{s-1})}=q^{2(\frac{1}{2}-s)}\frac{1}{(1-q^{1-s})(1-q^{-s})} \tag{11}$$

Now $q^{2(\frac{1}{2}-s)}=q^{1-2s}$.Getting rid of denominators, the functional equation can be written as

$$(1-q^{1-s})(1-q^{-s}) = q^{1-2s}(1-q^{s})(1-q^{s-1}) \tag{12}$$

.

The right hand side can be rewritten as

$$q^{-s}(1-q^{s})q^{1-s}(1-q^{s-1}) = (q^{-s}-1)(q^{1-s}-1) = (1-q^{-s})(1-q^{1-s}) \tag{13}$$

So, in the case of $P^1_{F_q}$ we get equality.

6 Verification of the Weil conjectures for $P^n_{F_q}$

The calculation here is similar. First one observes that the number of points that $P^n_{F_q}$ has with coordinates in F_{q^m} is given by

$$X(F_{q^m}) = \frac{(q^m)^{n+1}-1}{q^m-1} = 1 + q^m + q^{2m} + q^{3m} ... + (q^m)^n. \tag{14}$$

Plugging this into the expression for the zeta function we obtain

$$Z(P^n_{F_q}, s) = \frac{1}{(1-q^{-s})(1-q^{1-s})...(1-q^{n-s})} \tag{15}$$

As before, because of the form of the zeta function we only need to verify the functional equation. As before, we will verify

$$Z(P^n_{F_q}, n-s) = \pm q^{E(\frac{n}{2}-s)} Z(P^n_{F_q}, s). \tag{16}$$

Here the dimension of $P^n_{F_q}$ is n and its Euler characteristic is $n+1$. As before, we drop the $\pm$ and the above expression becomes

$$\frac{1}{(1-q^{s-n})(1-q^{s-(n-1)})...(1-q^{-s})} = q^{\frac{n(n+1)}{2}-(n+1)s} \frac{1}{(1-q^{-s})(1-q^{1-s})...(1-q^{n-s})} \tag{17}$$

Taking the reciprocal of the above equation,we get

$$(1-q^{s-n})(1-q^{s-(n-1)})...(1-q^{-s}) = q^{(n+1)s-\frac{n(n+1)}{2}}(1-q^{-s})(1-q^{1-s})...(1-q^{n-s}) \tag{18}$$

.

Since

$$\frac{n(n+1)}{2} = 1 + 2... + n \tag{19}$$

We can rewrite the above equation as

$$(1-q^{s-n})(1-q^{s-(n-1)})...(1-q^{s}) = q^{s}(1-q^{-s})q^{s-1}(1-q^{1-s})...q^{s-n}(1-q^{n-s}) \tag{20}$$

.

But then the right hand side is just

$$(q^{s}-1)(q^{s-1}-1)...(q^{s-n}-1) \tag{21}$$

which is just $\pm$ the left hand side, where we get '+' in the case of n odd and '$-$' in the case of n even.

7 Weil conjectures for elliptic curves

For elliptic curves, the conjectures take a particular form. Here we will focus on rationality and the Riemann hypothesis, where the $Z(E_{F_q}, s) \in \mathbf{Q}(t)$ (here $t = q^{-s}$) will take the form

$$Z(E_{F_q}, s) = \frac{1-at+qt^2}{(1-qt)(1-t)} \tag{22}$$

where here

$$1-at+qt^2 = (1-\alpha t)(1-\beta t) \in \mathbf{C}(t) \tag{23}$$

with $|\alpha| = |\beta| = \sqrt{q}$ and $a = q+1-E(F_q)$.

Here we will follow [Kor20] which addresses the conjectures for elliptic curves. For a more general study of elliptic curves [Sil09] is a good reference.In order to prove this result, we need an expression for $E(F_{q^m})$ in order to calculate the zeta function for $Z(E_{F_q}, s)$.The following can be found in [Kor20]:

Let E_{F_q} be an elliptic curve,$\rho : E \to E$ be the q^{th} power Frobenius endomorphism and $a = q+1-E(F_q)$. Assume $\alpha, \beta \in \mathbf{C}(t)$ are the roots of the polynomial $t^2 - at + q$. Then α and β are conjugates satisfying $|\alpha| = |\beta| = \sqrt{q}$ and for every $m \geq 1$ we have

$$X(F_{q^m}) = q^m + 1 - \alpha^m - \beta^m. \tag{24}$$

Setting $t = q^{-s}$ in [5] and using the above result, we see that

$$ln(Z(X,t)) = (\sum_{1}^{\infty} \frac{X(F_{q^m})}{m} t^m) \tag{25}$$

$$= (\sum_{1}^{\infty} \frac{q^m + 1 - \alpha^m - \beta^m}{m} t^m) \tag{26}$$

$$= -ln(1-t) - ln(1-qt) + ln(1+\beta t) + ln(1+\alpha t). \tag{27}$$

Exponentiating we get the expression for $Z(E_{F_q}, s)$.

First note that rationality is clear. In terms of the Riemann Hypothesis, if we rewrite $Z(E_{F_q}, s)$ in terms of q^{-s} we get

$$Z(E_{F_q}, s) = \frac{(1-\alpha q^{-s})(1-\beta q^{-s})}{(1-q^{1-s})(1-q^{-s})}. \tag{28}$$

Note that the zeroes of $Z(E_{F_q}, s)$ occur when the factors in the numerator vanish: since $|\alpha| = |\beta| = q^{\frac{1}{2}}$, we see that the zeroes occur for $Re(s) = \frac{1}{2}$.

8 Weil conjectures for curves

In order to deal with curves in general, we must use some basic algebraic geometry as it applies to curves.Elliptic curves had a multiplication, from which one can construct the Tate module and Weil pairing. Via the Weil pairing one can calculate $X(F_{q^n})$, obtain an expression for the Hasse Weil Zeta function and ultimately verify the conjectures. For a general curve we don't have this structure, so our starting point will be the Riemann Roch theorem for curves.

(Riemann-Roch). Let X be a smooth projective curve over k of genus g with canonical divisor K. Then for any divisor D we have

$$l(D) - l(K-D) = dimD - g + 1 \tag{29}$$

Eventually we get and expression for the Hasse Weil Zeta function of X given by

$$Z(X,t) = \frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-qt)} \tag{30}$$

where here the α_i come in conjugate pairs and $\prod_{i=1}^{2g} \alpha_i = q^g$. Note that this reduces to the elliptic curve case for $g = 1$.

9 Verification of the functional equation for curves

Here we will outline set up of the proof. The interested reader can go to [Ji21] for details.First note that the Hasse-Weil Zeta function can be written as the series

$$Z(X,t) = \sum_{r=0}^{\infty} A_r t^r \tag{31}$$

where here A_r are the number of effective divisors of degree r. We break $Z(X,t) = Z_1(X,t) + Z_2(X,t)$ where

$$Z_1(X,t) = \sum_{0 \leq degD \leq 2-2g} A_r t^r \tag{32}$$

and

$$Z_2(X,t) = \sum_{degD>2-2g} A_r t^r \tag{33}$$

Observe that the coefficient A_r represents the number of effective divisors of degree r. The number of effective divisors of degree r is given by Prop2.2 in [Ji21] as $\frac{q^{l(D)}-1}{q-1}$ where here $l(D)$ is the vector space of rational functions whose poles are bounded by D. The explicit expressions for these are given by

$$Z_1(X,t) = \sum_{0\leq degD\leq 2g-2} \frac{q^{l(D)}-1}{q-1} t^{deg(D)} \tag{34}$$

and

$$Z_2(X,t) = \sum_{degD>2g-2} \frac{q^{deg(D)+1-g}-1}{q-1} t^{deg(D)} \tag{35}$$

The difference in the exponents for q in the above formulas is due to the fact for divisors D where $deg(D) > 2g-2$ implies $l(K-D) = 0$. So via Riemann-Roch for curves

$$l(D) - l(K-D) = deg(D) + 1 - g \tag{36}$$

we have an explicit expression for $l(D)$.

In order to prove the functional equation, we "reshuffle" terms among $Z_1(X)$ and $Z_2(X)$ to get

$$Z_3(X,t) = \sum_{0\leq degD<g} \frac{q^{l(D)}-1}{q-1} + \sum_{g\leq degD\leq 2g-2} \frac{q^{l(D)}-q^{deg(D)+1-g}}{q-1} t^{deg(D)} \tag{37}$$

and

$$Z_4(X,t) = \sum_{degD>g} \frac{q^{deg(D)+1-g}-1}{q-1} t^{deg(D)} \tag{38}$$

.

We can see this if we think of the numerator in the summand of $Z_3(X)$ as $q^{h^0(D)}-1-(q^{deg(D)+1-g}-1)$ as opposed to $q^{l(D)} - q^{deg(D)+1-g}$, i.e., we've subtracted "$q^{deg(D)+1-g} - 1$" terms from $Z_1(X)$ (giving $Z_3(X)$) and thus must add them back into $Z_2(X)$ (giving $Z_4(X)$).

To prove the functional equation holds, we must show

$$Z(X, \frac{1}{qt}) = q^{1-g}t^{2-2g} Z(X,t) \tag{39}$$

.

Finally this is verified individually for $Z_3(X,t)$ and $Z_4(X,t)$.The verification for $Z_4(X,t)$ is somewhat straight forward, where as the verification for $Z_3(X,t)$ is a bit more delicate since it uses the involution $D \to K - D$ to rearrange divisors. Again, details can be found in [Ji21].

10 Verification of the Riemann hypothesis for curves

Now we verify the Riemann Hypothesis.In order to do this, we will need the Hasse-Weil bound, proven in [Ji] which states

$$|N_r - q^r - 1| \leq 2g\sqrt{q^r} \tag{40}$$

.

Since the term in absolute values will come up in the proof, for convenience we define $a_r = |N_r - q^r - 1|$.

Here N_r are the number of closed points of the n-dimensional variety X defined over F_{q^r}. Recall that for a curve X of genus g its Hasse -Weil zeta function $Z(X,t)$ will of the form

$$Z(X,t) = \frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-qt)} \tag{41}$$

.

Here the α_i are algebraic integers and $\prod_{i=1}^{2g} \alpha_i = q^g$. Here the Riemann hypothesis states that $\alpha_i = \sqrt{q}$. Since $\prod_{i=1}^{2g} \alpha_i = q^g$, we just have to show $\alpha_i \leq \sqrt{q}$ for each i.From the definition of $Z(X,t)$ we know

$$\sum_{r\geq 1} N_r t^{r-1} = \frac{d}{dt} Z(X,t) \tag{42}$$

.

Using the explicit form for $Z(X,t)$ we get after some computation

$$\sum_{r\geq 1} (q^r + 1 - \sum_{1\leq i\leq 2g} \alpha_i^r) t^{r-1} \tag{43}$$

.

Equating the coefficients for t^{r-1} we deduce

$$a_r = \sum_{1\leq i\leq 2g} \alpha_i^r \tag{44}$$

We will know show the result by contradiction. Form the expression for a_r above, consider the summation

$$\sum_{r\geq 1} a_r = \sum_{r\geq 1} \sum_{1\leq i\leq 2g} \alpha_i^r = \sum_{1\leq i\leq 2g} \frac{\alpha_i t}{1-\alpha_i t} \tag{45}$$

.

Assume that there is some $\alpha_i > \sqrt{q}$. For that particular i, take the limit as t goes to α_i^{-1}.By the Hasse-Weil bound $a_r \leq 2g\sqrt{q^r}$, we know the LHS summation converges.But the right hand side diverges due to the $\frac{\alpha_i t}{1-\alpha_i t}$ term. Thus $\alpha_i \leq \sqrt{q}$ for all i, and since $\prod_{i=1}^{2g} \alpha_i = q^g$ we have $\alpha_i = \sqrt{q}$ for each.

11 Weil conjectures for other complexes

One can now ask if the Weil conjectures hold for other spaces. One approach is to look at the Hasse-Weil Zeta function for a variety where the conjectures hold, One can then change the zeta function in a way such that the conjectures still hold but whose underlying space is different. We exemplify this with n-dimensional projective space.

We start by looking at the zeta function for $P^n_{F_q}$. It is given by

$$Z(P^n_{F_q}, s) = \frac{1}{(1-q^{-s})(1-q^{1-s})...(1-q^{n-s})} \tag{46}$$

.

Recall that by letting $t = q^{-s}$ we can rewrite the zeta function as

$$Z(P^n_{F_q}, t) = \frac{1}{(1-t)(1-qt)...(1-q^n t)} \tag{47}$$

.

In the functional equation, the relation between $Z(P^n_{F_q}, n-s)$ and $Z(P^n_{F_q}, s)$ is given by the factor $\pm q^{E(\frac{n}{2}-s)}$, where here E is the Euler characteristic of $P^n_{F_q}$ which is $n+1$.To show the functional equation holds, the $\frac{(n)(n+1)}{2}$ terms involving q are multiplied through along with $n+1$ terms involving t in order to get equality.

$$Z(P^n_{F_q}, n-s) = \pm q^{E(\frac{n}{2}-s)} Z(P^n_{F_q}, s). \tag{48}$$

The approach here is to change the polynomial in a way where both the functional equation and Riemann hypothesis still holds. As an example, consider the Hasse-Weil zeta function associated with $Z(P^2_{F_q})$ given by

$$Z(P^4_{F_q}, t) = \frac{1}{(1-t)(1-qt)(1-q^2t)(1-q^3t)(1-q^4t)} \tag{49}$$

.

Now consider the rational function

$$\frac{1}{(1-t)(1-q^2t)(1-q^2t)(1-q^2t)(1-q^4t)} \tag{50}$$

.

If there was a space Y that had this as its zeta function and identical dimension ($n = 4$) and Euler characteristic ($E = 5$)the functional equation would hold.Also observe we have three cells in the middle dimension, as opposed to one, as in the case for $P^4_{F_q}$ Note that

$$Z(Y_{F_q}, n-s) = \pm q^{E(\frac{n}{2}-s)} Z(Y_{F_q}, s). \tag{51}$$

Thinking in terms of $t = q^{-s}$, the functional equation becomes would give

$$\frac{1}{(1-\frac{1}{q^4t})(1-\frac{1}{q^2t})(1-\frac{1}{q^2t})(1-\frac{1}{q^2t})(1-\frac{1}{t})} = \pm q^{10}t^5 \frac{1}{(1-t)(1-q^2t)(1-q^2t)(1-q^2t)(1-q^4t)} \quad (52)$$

which is easily verified by dividing through by the $q^{10}t^5$ term on the left hand side.

We easily see that this works more generally. Namely, consider the Hasse-Weil zeta function for $P^n_{F_q}$ where n is even,$n \geq 4$.We define a polynomial to be a $q-sym$ variant of the polynomial $(1-t)(1-qt)(1-q^2t)...(1-q^nt)$ if it satisfies the following conditions:

1) It is of the same degree and each factor is of the form $(1-q^kt)$ where $k<n$.

2) The sum of the powers of q in the $q-sym$ variant add up to the sum of the powers of q in $(1-t)(1-qt)(1-q^2t)...(1-q^nt)$, namely $\frac{n(n+1)}{2}$.

3) The individual factors of the polynomial are symmetric with respect to the middle factor, i.e., the number of factors of $(1-q^kt)$ are equal to the number of factors of $(1-q^{n-k}t)$. As with the previous example, the corresponding cell complex has the same dimension and Euler characteristic as the $P^n_{F_q}$.

Theorem. A cell complexes which has a Hasse-Weil zeta functions with trivial numerator and denominator which is a $q-sym$ variation the polynomial $(1-t)(1-qt)(1-q^2t)...(1-q^nt)$ satisfies the Weil conjectures.

Proof. The only thing that must be verified is the functional equation.The proof is immediate due to the symmetry of the factors. As in the proof for $P^n_{F_q}$, the factor $(1-q^kt)$ goes to the factor $(1-q^{n-k}t)$.Since the $q-sym$ variation has the same degree and same number of powers of q as $(1-t)(1-qt)(1-q^2t)...(1-q^nt)$, the term $q^{E(\frac{n}{2}-s)} = q^{\frac{n(n+1)}{2}}t^{n+1}$ term rearranges the polynomial (and hence the zeta function) into its original form after the $t \rightarrow \frac{1}{q^nt}$ transformation.

Finally, let's consider the situation where the denominator of the Hasse-Weil zeta function of an n-dimensional variety is identical to n-dimensional projective space, i.e., we get only one cell for each even dimension, but in addition, we have factors in the numerator due to cells in the odd dimensions.(The most basic example here would be an elliptic curve.) More precisely, suppose we have an n-dimensional variety X whose Hasse-Weil zeta function is of the form

$$Z(X,s) = \frac{P_1(t)P_3(t)...P_{2n-1}(t)}{(1-t)(1-qt)(1-q^2t)...(1-q^nt)} \quad (53)$$

$$P_j(t) = \prod_{1}^{degP_j(t)} (1-\alpha_{ij}t) \quad (54)$$

Corollary.If the Hasse-Weil zeta function of X satisies the Weil conjectures, so does any cell complex whose Hasse-Weil zeta function has an identical numerator to $Z(X,s)$ and whose denominator is a $q-sym$ variant of $(1-t)(1-qt)(1-q^2t)...(1-q^nt)$.

Proof. The point here is that a $q-sym$ variant zeta function is associated to a space whose cell complex has the even cells shuffled around in a symmetric way about the middle dimension. This is still an n-dimensional projective variety and furthermore, its Euler characteristic stays the same as the contribution from the even dimensions stays the same. So we have the same number of factors of q and t in the functional equation as there were for original Hasse-Weil zeta function. The same number of them work for the denominator (as in the previous proof), and as the factors in the numerator are identical, what's left over will work for them. Note that since the numerator is identical, the roots are the same and so the Riemann hypothesis trivially holds, along with the Betti numbers.

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