Sorting networks

The 0-1-principle

An indispensable tool for the proof of correctness of sorting networks is the 0-1-principle [Knu 73]. The 0-1-principle states the following:

Theorem: (0-1-principle)

If a sorting network sorts every sequence of 0's and 1's, then it sorts every arbitrary sequence of values.

The proof of the 0-1-principle is not very difficult. However, it is quite helpful to have some definitions and lemmas ready.

Preliminaries

Definition: Let A and B be ordered sets. A mapping $f : A \rightarrow B$ is called monotonic if for all $a_1, a_2 \in A$

$$a_1 \leq a_2 \quad \Rightarrow \quad f(a_1) \leq f(a_2)$$

Lemma: Let $f : A \rightarrow B$ be a monotonic mapping. Then the following holds for all $a_1, a_2 \in A$:

$$f(\min(a_1, a_2)) = \min(f(a_1), f(a_2))$$

Proof: Let $a_1 \leq a_2$ and thus $f(a_1) \leq f(a_2)$. Then

$$\min(a_1, a_2) = a_1 \quad \text{and} \quad \min(f(a_1), f(a_2)) = f(a_1)$$

This implies

$$f(\min(a_1, a_2)) = f(a_1) = \min(f(a_1), f(a_2))$$

Similarly, if $a_2 \leq a_1$ and therefore $f(a_2) \leq f(a_1)$, we have

$$f(\min(a_1, a_2)) = f(a_2) = \min(f(a_1), f(a_2))$$

An analogous property holds for the max-function.

Definition: Let $f : A \rightarrow B$ be a mapping. The extension of f to finite sequences $a = a_0, ..., a_{n-1}, \; a_i \in A$ is defined as follows:

$$f(a_0, ..., a_{n-1}) = f(a_0), ..., f(a_{n-1}) \;, \quad \text{i.e.}$$

$$f(a)_i = f(a_i)$$

Lemma: Let f be a monotonic mapping and N a comparator network. Then N and f commutate, i.e. for every finite sequence $a = a_0, ..., a_{n-1}$ the following holds:

$$N(\,f(a)\,) = f(\,N(a)\,)$$

In other words: a monotonic mapping f can be applied to the input sequence of comparator network N or to the output sequence, the result is the same.

Proof: For a single comparator $[i:j]$ the following holds (see definition of comparator):

$$[i:j](\,f(a)\,)_i = [i:j](\,f(a_0), ..., f(a_{n-1})\,)_i = \min(\,f(a_i), f(a_j)\,)$$

$$= f(\,\min(a_i, a_j)\,) = f(\,[i:j](a)_i\,) = f(\,[i:j](a)\,)_i$$

This means that the ith element is the same regardless of the order of application of f and $[i:j]$. The same can be shown for the jth element and for all other elements. Therefore

$$[i:j](\,f(a)\,) = f(\,[i:j](a)\,)$$

For an arbitrary comparator network N (which is a composition of comparators) and a monotonic mapping f we have therefore

$$N(\ f(a)\ ) \ = \ f(\ N(a)\ )$$

Proof of the 0-1-principle

Theorem: (0-1-principle)

Let N be a comparator network. If every 0-1-sequence is sorted by N, then every arbitrary sequence is sorted by N.

Proof: Suppose a with $a_i \in A$ is an arbitrary sequence which is not sorted by N. This means $N(a) = b$ is unsorted, i.e. there is a position k such that $b_k > b_{k+1}$.

Now define a mapping $f : A \longrightarrow \{0, 1\}$ as follows. For all $c \in A$ let

$$f(c) \ = \ \begin{cases} 0 & \text{if } \ c < b_k \\ 1 & \text{if } \ c \geq b_k \end{cases}$$

Obviously, f is monotonic. Moreover we have:

$f(b_k) = 1$ and $f(b_{k+1}) = 0$

i.e. $f(b) = f(N(a))$ is unsorted.

This means that $N(f(a))$ is unsorted or, in other words, that the 0-1-sequence $f(a)$ is not sorted by the comparator network N.

We have shown that, if there is an arbitrary sequence a that is not sorted by N, then there is a 0-1-sequence $f(a)$ that is not sorted by N.

Equivalently, if there is no 0-1-sequence that is not sorted by N, then there can be no sequence a whatsoever that is not sorted by N.

Equivalently again, if all 0-1-sequences are sorted by N, then all arbitrary sequences are sorted by N.

References

[Knu 73] D.E. KNUTH: The Art of Computer Programming, Vol. 3 - Sorting and Searching. Addison-Wesley (1973)

H.W. Lang Hochschule Flensburg lang@hs-flensburg.de Impressum