A Dynamical Analysis of Espresso Flow

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We present a theoretical model of the flow of an ideal espresso shot through a portafilter, accounting for its non-Newtonian behavior, time-dependent viscosity, and anisotropic puck property. The dynamics are described using the incompressible Navier-Stokes equation simplified under laminar and low-Reynolds number conditions, and Darcy's Law and the Brinkman equation for porous media, and its time-evolving viscosity modeled as a function of temperature and time. Anisotropy in the coffee puck is modeled via a permeability tensor, allowing directional variations in flow to be considered. We consider thermal energy relations through a heat equation with viscous dissipation, ensuring consistency with the first and second law of thermodynamics. We discuss analytical solutions for idealized isotropic pucks and numerical approaches for anisotropic cases, with a normalized flux variance introduced as a quantitative factor to measure uniformity. This framework suggests how anisotropy affect flow distribution and highlights methods for predicting channeling in espresso extraction.

I. INTRODUCTION

The extraction of espresso is governed by interactions between fluid mechanics, thermal effects, and the porous structure of the coffee puck. Unlike simple Newtonian fluids, espresso exhibits highly non-Newtonian behavior due to the colloidal suspension of oils, solids, and gases. The viscosity evolves over time as crema forms, and the flow is strongly influenced by the geometry and packing of the coffee puck. Understanding these dynamics is crucial for optimizing extraction quality and minimizing undesirable effects such as channeling. Previous studies [1][2] have largely focused on either experimental characterization of espresso flow or simplified one-dimensional Darcy models. However, these approaches do not fully capture the coupled effects of non-Newtonian fluids, anisotropic permeability, and thermal dissipation. In this work, we develop a comprehensive mathematical model of espresso flow through a portafilter, incorporating non-newtonian, time-evolving viscosity, anisotropic permeability, and its relation to thermodynamics and energy dissipation. The equations are analyzed first in isotropic conditions and later in general configurations both analytically and numerically. By integrating fluid dynamics, continuum mechanics, and thermodynamics, this work provides a framework for predicting extraction behavior, identifying potential sources of flow nonuniformity, and exploring a practical application for producing a better cup of coffee in the morning.

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II. TOPOLOGY OF THE PORTAFILTER

Let the fluid domain inside the portafilter be

$$\Omega_P = D^2 \times [0,d] \setminus \bigcup_{i=1}^{N} B_\epsilon(p_i),$$

where $D^2 \subset \mathbb{R}^2$ is the circular cross-section with a radius R and $B_\epsilon(p_i)$ represent small circular holes at the base. To account for the geometry of the portafilter, we introduce a smooth diffeomorphism

$$\phi : D^2 \times [0,d] \setminus \bigcup_{i=1}^{N} B_\epsilon(\tilde{p}_i) \longrightarrow \Omega_P, \quad \phi \in C^\infty,$$

mapping the ideal punctured cylinder to the physical portafilter. This mapping enables us to consider an outflow condition

$$\mathbf{v}(\mathbf{r}) \cdot \mathbf{n} = \begin{cases} 0, & p(\mathbf{r}) < P_c \\ g(\mathbf{r}), & p(\mathbf{r}) \geq P_c \end{cases}, \quad \mathbf{r} \in \partial_B \Omega_P,$$

where we can now study the dynamics of fluids inside the ideal cylinder.

III. DYNAMICS OF VISCOUS FLUIDS

We begin by recalling the incompressible Navier-Stokes equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} \tag{1}$$

where $\boldsymbol{\sigma}$ is the viscous stress tensor, and $\mathbf{f}$ is the body force. To model the particular flow, we consider a modified power-law viscosity with time-evolving and temperature dependent factors

$$\mu(t, \dot{\gamma}, T) = \dot{\gamma}^{n-1}(\mu_0 e^{-Kt} + \mu_M)e^{-b(T-T_0)}, \qquad (2)$$

where $\dot{\gamma}$ is the shear rate [3][4], T_0 is the reference temperature, μ_M, μ_0, b, and K are coefficients. Clearly,

$$\mu(0, T) = C_0\mu_i\dot{\gamma}^{n-1}e^{-bT}, \quad \lim_{t\to\infty} \mu(t, T) = C_0\mu_M\dot{\gamma}^{n-1}e^{-bT},$$

$$\mu(t, 0) = C_0\dot{\gamma}^{n-1}(\mu_0 e^{-Kt} + \mu_M), \quad \lim_{T\to\infty} \mu(t, T) = 0,$$

where we define $\mu_i = \mu_0 + \mu_M$ and $C_0 = e^{bT_0}$ to simplify. The volumetric flow rate Q through the puck is

$$Q(z, t) = \frac{kA}{\mu(t, \dot{\gamma}, T)}\frac{\Delta p}{\Delta h} = \frac{\pi R^2 k}{\mu(t, \dot{\gamma}, T)}\frac{\partial p}{\partial z}, \qquad (3)$$

where k is the permeability of the medium, R is the radius of the cylinder, p is the pressure, and $\mu(t, \dot{\gamma}, T)$ is the dynamic viscosity, and the Darcy velocity v_z is

$$v_z(z, t) = \frac{Q}{\pi R^2} = \frac{k}{\mu}\frac{\partial p}{\partial z}(r, \theta, z) \qquad (4)$$

which for both Q and v_z we assume that the flow is approximately vertical, $\mathbf{v} \approx v_z\hat{z}$. A shot of espresso is highly non-Newtonian, so to generalize the Darcy velocity given above, we also assume the shear rate to be

$$\dot{\gamma} = \alpha\frac{v_z}{h}$$

where α is a coefficient and h is the thickness of the puck. Using Eq. 2 and Eq. 4, we obtain

$$v_z = \frac{ke^{bT}\,\partial_z p}{C_0(\alpha\frac{v_z}{h})^{n-1}(\mu_0 e^{-Kt} + \mu_M)},$$

which explicitly simplifies to

$$v_z = K_0\left(\frac{\partial_z p}{\mu_0 e^{-kt} + \mu_M}\right)^{\frac{1}{n}} e^{\frac{bT}{n}}, \qquad (5)$$

where $K_0 = \left(\frac{h}{\alpha}\right)^{\frac{n-1}{n}}\left(\frac{k}{C_0}\right)^{\frac{1}{n}}$, in analogy with power-law porous-media flow [5]. This yields

$$Q \propto \left(\frac{\Delta p}{\Delta h}\right)^{\frac{1}{n}},$$

consistent with the volumetric flow relation [6].

IV. FLOW THROUGH ANISOTROPIC POROUS MEDIA

Under ideal condition, the shot exhibits a laminar flow, $\mathrm{Re} \ll 1$. We may then ignore the inertial terms and the body force and obtain

$$\frac{\partial p}{\partial z} \approx \nabla\cdot\boldsymbol{\sigma}.$$

For a non-Newtonian fluid, the viscous stress tensor [7] is

$$\boldsymbol{\sigma} = 2\mu(t, \dot{\gamma}, T)\mathbf{D},$$

$$\mathbf{D} = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T),$$

$$D_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$$

Substituting Eq. 6 into the simplified equation, one can see

$$\nabla\cdot 2\mu(t, \dot{\gamma}, T)\mathbf{D} = 2(\partial_j\mu)D_{ij} + 2\mu\partial_j D_{ij} \approx 2\mu\nabla\cdot\mathbf{D}$$

for a spatially uniform viscosity. For incompressible flow,

$$\nabla\cdot\mathbf{v} = 0, \quad \nabla\cdot\mathbf{D} = \frac{1}{2}\nabla^2\mathbf{v},$$

and Eq. 4,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{v_z}{k} = 0$$

under the vertical flow assumption $\mathbf{v} \approx v_z\hat{z}$. It can be generalized to the vectorial form

$$\nabla^2\mathbf{v} + \frac{1}{k}\mathbf{v} = 0,$$

the Brinkman equation.

In fact, one of the main factors that contribute to a poor shot of espresso comes from anisotropy of the puck. Permeability is then directional, where we must consider

$$\mathbf{v} = \frac{1}{\mu}\mathbf{K}\cdot\nabla p, \qquad (6)$$

[8] where $\mathbf{K}$ is the permeability tensor. Rewriting Eq. 4., we obtain

$$v_z(r, \theta, z, t) = \frac{1}{\mu}\left(K_{zr}\partial_r p + K_{z\theta}\frac{1}{r}\partial_\theta p + K_{zz}\partial_z p\right) \qquad (7)$$

and the total volumetric flow rate is found using

$$Q(t) = \int_{\partial_B\Omega_P} v_z(r, \theta, 0)\,dS. \qquad (8)$$

V. THERMAL COUPLING

The heat equation with viscous dissipation is given by

$$\rho c_p \frac{DT}{Dt} = \kappa \nabla^2 T + \Phi, \tag{9}$$

where ρ is the fluid density, c_p is the specific heat at constant pressure, κ is the thermal conductivity, and Φ is the viscous dissipation. For a vertical flow,

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + v_z \frac{\partial T}{\partial z},$$

$$\Phi = 2\mu D_{ij} D_{ij} = \mu \left[\left(\frac{\partial v_z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial v_z}{\partial \theta} \right)^2 + 2 \left(\frac{\partial v_z}{\partial z} \right)^2 \right].$$

Substituting the expressions into Eq. 9, we obtain

$$\begin{aligned} \rho c_p \big(\partial_t T + v_z \partial_z T \big) &= \kappa \Big[\frac{1}{r} \partial_r (r \partial_r T) + \frac{1}{r^2} \partial_\theta^2 T + \partial_z^2 T \Big] \\ &+ \mu \Big[\big(\partial_r v_z \big)^2 + \frac{1}{r^2} \big(\partial_\theta v_z \big)^2 + 2 \big(\partial_z v_z \big)^2 \Big], \end{aligned} \tag{10}$$

coupling thermal system and vertical flow together.

As a sanity check, integrating both sides of Eq. 9 gives

$$\begin{aligned} \frac{d}{dt} \int_V \rho c_p T \, dV &= \int_V \big[-\rho c_p \mathbf{v} \cdot \nabla T + \nabla \cdot (\kappa \nabla T) + \Phi \big] \, dV \\ &= -\int_V \big[\rho c_p \mathbf{v} \cdot \nabla T \big] \, dV + \oint_{\partial V} \kappa \nabla T \cdot \hat{n} \, dS + \int_V \Phi \, dV \end{aligned},$$

yielding the energy balance from convection, heat conduction, and viscous dissipation. Moreover, we obtain

$$\Phi = \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) - \mathbf{v} \cdot \nabla p$$

from Stokes-Brinkman equations, integrating both sides yields

$$\Phi = -\mathbf{v} \cdot \nabla p.$$

We can find

$$\mathbf{v} \cdot \nabla p = \frac{1}{\mu} (\nabla p)^T \mathbf{K} (\nabla p),$$

which is positive [9] since

$$-\mathbf{v} \cdot \nabla p \geq 0$$

and $\mathbf{K} = \mathbf{K}^T$, thus $\Phi \geq 0$, ensuring entropy production.

VI. SOLUTIONS, BOUNDARY / INITIAL CONDITIONS, AND APPLICATIONS

One instance where a simple analytical solution can be found is when the shear rate is given in a situation where $\mathbf{K} = k\mathbf{I}$. For a given time t_0, we consider a uniform viscosity so that the Brinkman equation can be linearized. Suppose the separable solution

$$v_z(r, \theta, z) = \frac{k \partial_z p(r, \theta, z)}{\mu(t, T)} = R(r) \Theta(\theta) Z(z) M(T).$$

Substituting it into the generalized Brinkman equation, one can obtain

$$\frac{1}{R} \frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{1}{\Theta} \frac{1}{r^2} \frac{d^2 \Theta}{d\theta^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} + \frac{1}{k} = 0,$$

where the solutions are given by

$$\Theta(\theta) = A \cos(\lambda \theta) + B \sin(\lambda \theta), \quad \lambda \in \mathbb{Z},$$

$$Z(z) = C_1 e^{-\beta z} + C_2 e^{\beta z} \quad \text{or} \quad D_1 \cos(\beta z) + D_2 \sin(\beta z),$$

$$R(r) = \alpha J_\lambda(mr) + \beta Y_\lambda(mr).$$

Bessel functions of the second kind, $Y_\lambda(mr)$, diverges at $r = 0$ and are excluded, and we choose $Z(z) = C_1 e^{-\beta z} + C_2 e^{\beta z}$ for the axial solution. The particular solution for $v_z(r, \theta, z)$ is then

$$v_z(r, \theta, z) = \Bigg[\sum_{\lambda, n} \alpha_{\lambda, n} J_\lambda(m_{\lambda, n} r) \Theta_\lambda(\theta) Z_{\lambda, n}(z) \Bigg] e^{bT}. \tag{11}$$

Often times, however, this will not be the case, such as flows through anisotropic media discussed above. To find numerical solutions to model the flow, we could impose boundary conditions

$$v_z(R, \theta, z) = 0, \quad v_z(r, 0, z) = v_z(r, 2\pi N, z), \ \forall N \in \mathbb{Z},$$

$$v_z(r, \theta, d) = v_i, \ \forall r \in [0, R),$$

$$p(r, \theta, d) = p_i, \quad p(r, 0, z) = p(r, 2\pi N, z), \ \forall N \in \mathbb{Z},$$

and initial conditions

$$v_z(r, \theta, z, t = 0) = 0, \quad T(r, \theta, z, t = 0) = T_i.$$

An ideal shot of espresso exhibits an even flow through the bottom for all r and θ, whereas a poor shot tends to channel. As a quantitative factor to measure uniformity, we define the normalized flux variance

$$S(r,\theta) = \frac{\sqrt{\int (Q(r,\theta,0) - \langle Q\rangle)^2 \, dS}}{Q_t} \tag{12}$$

where $Q(r,\theta,0)$ is the volumetric flow rate at $z=0$, Q_t is the total volumetric flow, and $\langle Q\rangle$ is the average flux given by

$$\langle Q\rangle = \frac{1}{\pi R^2}\int_{\partial_B \Omega_P} Q(r,\theta)\, dS.$$

This normalized quantity measures the uniformity of the outgoing flux through the boundary $\partial_B \Omega_P$. A perfectly even flow in the entire domain has $Q = \langle Q\rangle$ for any given r and θ, yielding $S = 0$. On the other hand, for instance, an extreme case of channeling modeled by $Q(\mathbf{r}) = Q_t \delta^2(\mathbf{r} - \mathbf{r}_0)$ yields $S(r,\theta) = 1$ for all $\mathbf{r}$ except for the singularity.

In discrete numerical calculations, it can be realized that

$$\frac{\sqrt{\int (Q(r,\theta,0) - \langle Q\rangle)^2 \, dS}}{Q_t} = \frac{\sigma_Q}{Q_t},$$

and

$$Q_t = \sum_{n=1}^{N} Q_n, \quad \langle Q\rangle = \frac{1}{N}\sum_{n=1}^{N} Q_n,$$
$$\sigma_Q = \sqrt{\frac{1}{N}\sum_{n=1}^{N}(Q_n - \langle Q\rangle)^2},$$

where Q_n is the n-th volumetric flow through each partitioned area, thus

$$S = \frac{\sqrt{\frac{1}{N}\sum_{n=1}^{N}(Q_n - \langle Q\rangle)^2}}{\sum_{n=1}^{N} Q_n}.$$

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[8] Or $\mathbf{v} = -\frac{1}{\mu}\mathbf{K}\cdot\nabla p$. We use the positive equation in this paper.
[9] Once again, this comes from the sign convention of Eq. 6.