Operator-Theoretic Regularization of Logarithmic Divergences

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Abstract

One of the challenges in modern theoretical physics is dealing with infinities that arise in quantum field theory, quantum electrodynamics, and statistical mechanics to name a few. These divergences must be controlled in order to extract a physically meaningful quantity, and one approach to do so is by regularization. While analytic regularization is a widely used method in mathematical physics, few attempts have been made to regularize series with a $n^\alpha \log n$ term in the denominator — more generally, series in the form of

$$\sum_{n}^{\infty} n^{-p(n)},$$

structurally similar to the Riemann zeta function. This individual study investigates the analytic methods as means of continuation, and proposes an operator theoretic approach of borderline divergent series.

Introduction

A function of sum is first defined

$$S(n) = \sum_{n=2}^{\infty} \frac{1}{n \log n},$$

where the sum can be expressed as

$$S(n) = \sum_{n=2}^{\infty} n^{-p(n)}, \qquad p(n) = 1 + \frac{\log(\log n)}{\log n}.$$

One can define

$$p(n) = 1 + f(n)$$

and expand the summand

$$n^{-1-f(n)} = \frac{1}{n}\left[1 - \log(\log n) + O(f(n)^2)\right].$$

Assume that as $n \to \infty$, the $O(f(n)^2)$ term becomes negligible. The summand simplifies to

$$n^{-p(n)} = \frac{1}{n} - \frac{\log(\log n)}{n}.$$

To study the behavior of the $\log(\log n)/n$ term, a function of sum can be defined

$$F(s,b) = \sum_{n=2}^{\infty} \frac{\log^{-b} n}{n^s}$$

for $\Re(s) > 1$, $b \in \mathbb{C}$. Taking a partial derivative with respect to b gives

$$-\frac{\partial}{\partial b} F(s,0) = Z(s) = \sum_{n=2}^{\infty} \frac{\log(\log n)}{n^s}.$$

By letting

$$f(x) = \frac{\log(\log x)}{x^{1+\epsilon}}, \qquad \epsilon > 0,$$

it can be found that

$$Z(1+\epsilon) = \int_2^N f(x)\,dx + \frac{f(N)+f(2)}{2} + \sum_{k=1}^{\lfloor p/2 \rfloor} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(N) - f^{(2k-1)}(2)\right) + R_p$$

using Euler-Maclaurin expansion, and further simplified to

$$Z(1+\epsilon) = \frac{\gamma - \log\epsilon}{\epsilon} + C + o(1), \qquad C = \lim_{\epsilon\to 0}\left[Z(1+\epsilon) + \frac{\log\epsilon - \gamma}{\epsilon}\right]$$

from a well known result of asymptotic expansion. In the future, I aim to extend this regularization by investigating an alternative method to expand $n^{-p(n)}$ that produces a discrete imaginary term,

$$p(k) = \frac{\pi i(2k-1)}{n}, \quad k \in \mathbb{Z},$$

similar to that of the Matsubara frequency modes often seen in finite-temperature field theory. This allows the divergent logarithmic sum to produce a regularized trace of a complex operator, where it can be represented as a Hermitian operator with real spectrum after analytic continuation. Future work will investigate the physical interpretation of the $p(k)$ term as spectral mode of an operator $\hat{P}$, such that

$$\mathrm{Tr}\left(e^{-\log(n)\hat{P}}\right) = \sum_k e^{-p(k)\log n}$$

where its eigenvalues correspond to frequency modes, analogous to energy levels in partition functions of a quantum system. By defining

$$\hat{H} = \hbar\hat{P}, \quad t = \log n,$$

the given expression can be written as a unitary time evolution trace,

$$\mathrm{Tr}\left(e^{-i\hat{H}t/\hbar}\right) = \sum_k e^{-ip(k)t}.$$

Let $t = -i\hbar\tau$ such that $\tau = \beta \in \mathbb{R}_*^+$, then

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta\hat{H}}\right)$$

from the standard Wick Rotation relation, $t \mapsto -i\hbar\tau$. Thus with $\beta = \log n$ the logarithmic regulator can be viewed as an inverse temperature, and the analytic continuation $\log n \mapsto i\beta$ relates the Euclidean partition function to the Lorentzian time-evolution trace.