Chapter 4: Continuous Random Variables

A r.v. X is <u>continuous</u> if its possible values are in an interval or union of intervals on the number line.

ex 4.1) X = depth of a lake at a randomly chosen location on the lake's surface

A = minimum depth
B = maximum depth

Possible values: $A \le X \le B$

$$X \in [A, B]$$

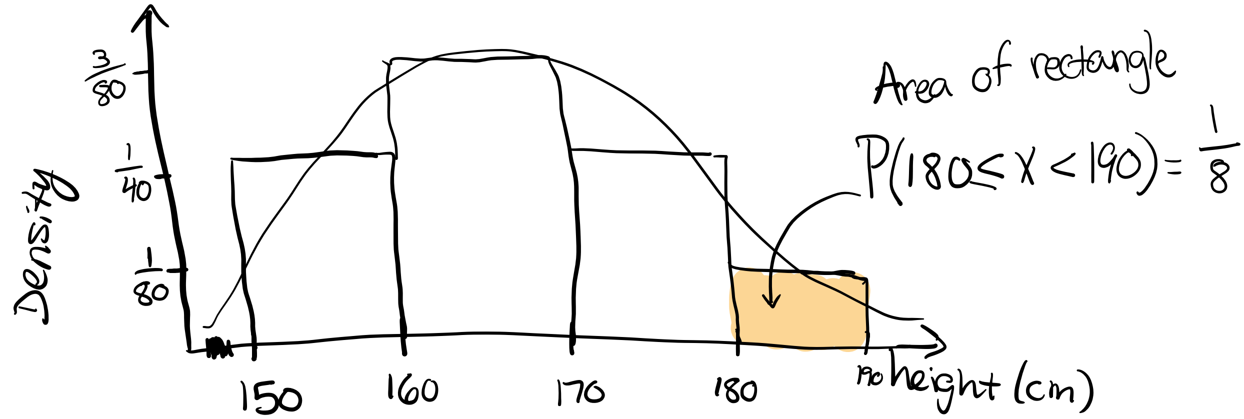
ex 4.2) X = the pH value of a randomly chosen chemical compound

$$0 \le X \le 14$$

Ex) X = height of students (in cm)

X	frequency	relative frequency	Density
$150 \le X < 160$	10	$1/4$	$1/40$
$160 \le X < 170$	15	$3/8$	$3/80$
$170 \le X < 180$	10	$1/4$	$1/40$
$180 \le X < 190$	5	$1/8$	$1/80$

Density histogram of heights



Area of a rectangle in the histogram
= P(X is in that interval)

Sum of areas of rectangles = 1

We can change the width of the rectangles

As the rectangle widths get smaller, the shape of the histogram approaches the <u>density curve</u>

Area under the density curve = 1

The density curve for a r.v. X is the graph of the <u>probability density function</u>, $f(x)$ (pdf)

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

$$P(X = c) = 0 \quad \text{for any } c \in \mathbb{R}$$

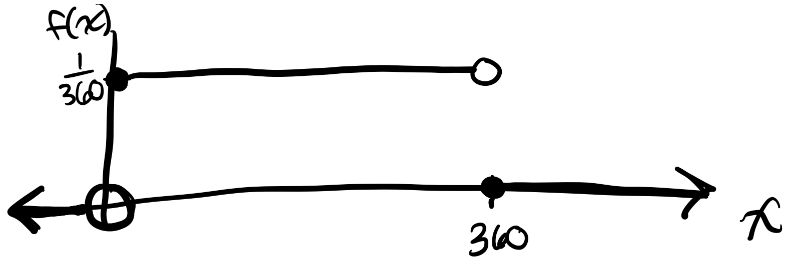
Properties of pdfs

① $f(x) \geq 0$ for all x

② $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Ex 4.4 Consider a reference line connecting the valve stem on a tire to the center point. Let X = the angle (in degrees) measured clockwise from the reference line to the location of an imperfection. The pdf of X is

$$f(x) = \begin{cases} \frac{1}{360} & 0 \leq x < 360 \\ 0 & \text{otherwise} \end{cases}$$



The area under the density curve is $\left(\frac{1}{360}\right)(360) = 1$

What is the probability that the angle is between $90°$ and $180°$?

$$P(90 \le X \le 180) = \int_{90}^{180} \frac{1}{360}\, dx = \frac{x}{360}\Big|_{90}^{180}$$

$$= \frac{180-90}{360} = \frac{90}{360} = \frac{1}{4} = \boxed{0.25}$$

For this pdf, whenever $0 \le a \le b \le 360$,
$P(a \le X \le b)$ depends <u>only</u> on $b-a$
(the width of the interval)

This is called <u>uniform distribution</u>

$X \sim \text{Unif}([A,B])$

$X \sim \text{Unif}(A,B)$

X is uniform on the interval $[A,B]$

pdf:
$$f(x) = \begin{cases} \frac{1}{B-A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

Note: for continuous r.v.'s, since $P(X=c)=0$ for any $c \in \mathbb{R}$,

$$P(a \le X \le b) = P(a < X < b)$$

$$= P(a \le X < b) = P(a < X \le b)$$

$$= \int_a^b f(x)\,dx$$

Ex 4.5 "Time headway" between 2 consecutive cars measures traffic flow = elapsed time between one car passing a fixed point and the next.

X = time headway between 2 randomly chosen consecutive cars

$$f(x) = \begin{cases} 0.15e^{-0.15(x-0.5)} & x \ge 0.5 \\ 0 & \text{otherwise} \end{cases}$$

① Check that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Check that $\int_{0.5}^{\infty} 0.15e^{-0.15(x-0.5)}\,dx = 1$

(verify as an exercise)

② Find the probability the headway time is at most 5 seconds

$$P(X \le 5) = \int_{-\infty}^{5} f(x)\,dx = \int_{0.5}^{5} 0.15e^{-0.15(x-0.5)}\,dx$$

$$= 0.15e^{0.075}\int_{0.5}^{5} e^{-0.15x}\,dx = \boxed{0.491}$$

On the TI-83/84: $\int_a^b f(x)\,dx = \text{fnInt}(f(x), X, a, b)$

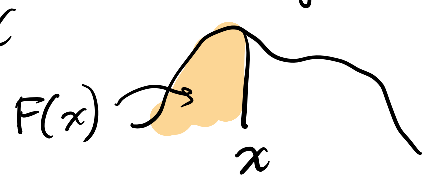
$\uparrow$ $\boxed{\text{math}}$, 9 $\qquad$ $\uparrow$ $\boxed{X,T,\theta,n}$

4.2: Cumulative Distribution Function (cdf)

$f(x)$ is the pdf of X, the cdf of X is

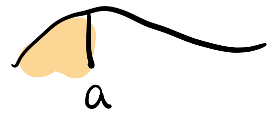
$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(y)\,dy$$

$F(x)$ = area under the density curve to left of x

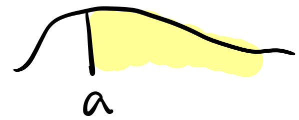


Using F(x) to compute Probabilities

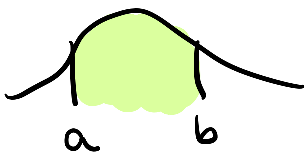
- $P(X<a) = F(a)$



- $P(X>a) = 1 - F(a)$



- $P(a \leq X \leq b) = F(b) - F(a)$



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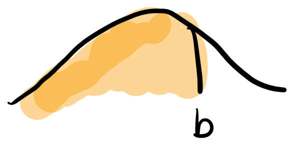

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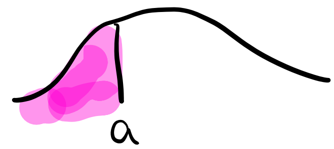

ex 4.7 Let

$$f(x) = \begin{cases} \frac{1}{8}(1+3x) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

For $0 \leq x \leq 2$

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_0^x \left(\frac{1}{8} + \frac{3}{8}y\right) dy$$

$$= \frac{x}{8} + \frac{3}{16}x^2$$

$$F(x) = \begin{cases} 0 & x<0 \\ \frac{x}{8} + \frac{3}{16}x^2 & 0 \leq x \leq 2 \\ 1 & x>2 \end{cases}$$

$$P(1 \leq X \leq 1.5) = F(1.5) - F(1) = \frac{19}{64}$$
$$= \boxed{0.297}$$

$$P(X > 1) = 1 - P(X \leq 1)$$
$$= 1 - F(1) = \frac{11}{16} = \boxed{0.688}$$

Obtaing $f(x)$ from $F(x)$

By the fundamental theorem of calculus, $F'(x) = f(x)$

ex 4.7)

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{8} + \frac{3}{16}x^2 & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$

$x<0$ $x>2$

$$\frac{d}{dx}(0) = 0 \qquad \frac{d}{dx}(1) = 0$$

For x, $0 \leq x \leq 2$,

$$\frac{d}{dx}\left[\frac{x}{8} + \frac{3}{16}x^2\right] = \frac{1}{8} + \frac{3}{8}x$$

$$f(x) = \begin{cases} \frac{1}{8} + \frac{3}{8}x & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Expected Value and Variance

For discrete r.v.'s : $E(x) = \sum_{\text{all } x} x \cdot p(x)$

Continuous r.v.'s:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx$$

$$E(X) = \mu_X$$

$$\mu_{h(x)} = E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f(x) \, dx$$

In particular

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) \, dx$$

Variance:

$$V(X) = E(X^2) - E(X)^2$$

<u>ex 4.10, 4.12</u> Find $E(X)$ and $V(X)$

X has the following pdf:

$$f(x) = \begin{cases} \frac{3}{2}(1-x^2) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \int_0^1 x \cdot \frac{3}{2}(1-x^2)\,dx$$

$$= \int_0^1 \frac{3}{2}(x - x^3)\,dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{8}$$

$$E(X^2) = \int_0^1 x^2 \cdot \frac{3}{2}(1-x^2)\,dx$$

$$= \int_0^1 \frac{3}{2}(x^2 - x^4)\,dx = \frac{3}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

$$V(X) = E(X^2) - E(X)^2$$

$$= \frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320} = \boxed{0.059}$$

$$\sigma_X = \sqrt{V(X)} = \sqrt{0.059} = \boxed{0.244}$$