Sphere packings

Introduction

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Sphere packing original problem

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What's the densest packing of spheres into Euclidean space?

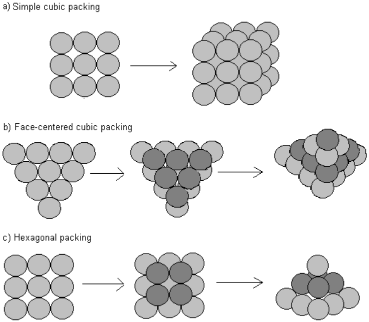

Figure: Packaging examples Arnold [2000]

Density function

Density function of spheres packaging in n-dimensional Euclidean space

Foremost, by definition it will be given that $x \in \mathbb{R}^d$, $r \in \mathbb{R}_{>0}$ and $B_d(x,r)$ will be the open ball in $\mathbb{R}$ with center x and radius r. Besides, $X \subset \mathbb{R}^d$ will be all the discrete points such that $\forall x, y \in X : x \neq y : ||x-y|| \geq 2$. Then:

$$\mathcal{P} = \bigcup_{x \in X} B_d(x,1) \tag{1}$$

will be defined to be a sphere packing, because we assume X is not necessarily a $\mathbb{R}^d$-lattice. Subsequently, the density function $\Delta_{\mathcal{P}}(r)$ will be introduced.

Definition (Density function)

The density function $\Delta_{\mathcal{P}}(r)$ will be defined as:

$$\Delta_{\mathcal{P}} := \frac{Vol(\mathcal{P} \cap B_d(0,r))}{Vol(B_d(0,r))}, r > 0. \tag{2}$$

Density function of spheres packaging in n-dimensional Euclidean space

Besides, the density of a packing $\mathcal{P}$ will be defined as the limit superior of $\Delta_{\mathcal{P}}(r)$, subsequently:

$$\Delta_{\mathcal{P}} := \limsup_{r\to\infty} \Delta_{\mathcal{P}}(r). \tag{3}$$

The number we are interested is therefore the supremum of all the given possible packing densities Viazovska [2017], i.e:

$$\Delta_d := \sup_{\mathcal{P}\subset\mathbb{R}^d} \Delta_{\mathcal{P}} \tag{4}$$

Sphere packing in $\mathbb{R}$

Sphere packing in 1-dimensional Euclidean space

Given a 1-sphere in a 1-dimensional Euclidean space, it is easy to see that $\Delta_d = 1$

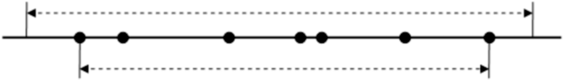

Figure: The density of 1-spheres

Sphere packing in $\mathbb{R}^2$

Sphere packing in 2-dimensional Euclidean space

The case of a 2-sphere is a little bit more complex than in the 1-dimension case. Given a 2-sphere in a 2-dimensional Euclidean space, it was proved that $\Delta_d = \frac{1}{6}\pi\sqrt{3} \approx 0.91.$ by Thue [1911]—and it is the one referenced in Viazovska [2017]—, however, the proof is considerably more complex than the one given in by Chang and Wang [2010] and therefore this last one will be used instead.

Lemma 1

Let θ be the largest internal angle of a given triangle ΔABC in a Delaunay triangulation for a saturated circle configuration C. Then

$$\frac{\pi}{3} \leq \theta < \frac{2\pi}{3}. \tag{5}$$

Proof Lemma 1

The largest internal angle is always equal or bigger than $\frac{\pi}{3}$. So, without loss of generality if assumed $\theta \geq \frac{2\pi}{3} > \frac{\pi}{3}$, then let A the smallest internal angle. We will have that $\sin(A)$ and $\overline{BC} \geq 2$.

Sphere packing in 2-dimensional Euclidean space

We will assume R is the circumradius of $\overline{ABC}$. Then, using the sine law we will get:

$$2R = \frac{\overline{BC}}{\sin A} \geq \frac{2}{\sin A} \geq 4. \tag{6}$$

Consequently the circumradius of ΔABC can be added to the circle configuration $\mathcal{C}$, therefore we get:

$$\theta < \frac{2\pi}{3}.\ \text{Q.E.D.} \tag{7}$$

Therefore, the density of a triangle ΔABC in a Delaunnay triangulation for a satured circle configuration $\mathcal{C}$ is given by:

$$\frac{\frac{1}{2}A + \frac{1}{2} + B\frac{1}{2}C}{\text{area of } \Delta ABC} = \frac{\frac{\pi}{2}}{\text{area of } \Delta ABC}. \tag{8}$$

Sphere packing in 2-dimensional Euclidean space

Lemma 2

The density of a triangle ΔABC in a Delaunay triangulation for a satred circle configuration C, holds that $C \leq \pi/\sqrt{12}$. The equality holds only for the regular triangle with side-length 2.

Proof Lemma 2

Let's assume, without loss of generality, that B is the largest internal angle of ΔABC. Then, by the first lemma,

$$\text{area of } \Delta ABC = \frac{1}{2}\overline{AB} \cdot \overline{BC} \cdot \sin B \geq \frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \tag{9}$$

$$\implies \text{density of } \Delta ABC = \frac{\pi/2}{\text{area of } \Delta ABC} \leq \frac{\pi}{\sqrt{12}}.$$

If ΔABC is a regular triangle with side-length $= 2$ then it becomes a equality

Simulations of sphere packing in $\mathbb{R}^2$

Simulations

Simulations in Python for $\mathbb{R}^2$.

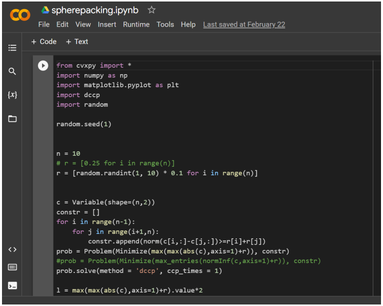

Figure: Collab platform

▸ SIMULATIONS

Sphere packing in $\mathbb{R}^3$

Sphere packing in $\mathbb{R}^3$

Lemma 3

The highest density amongst all possible 3-dimensional lattice packings of spheres in $\mathbb{R}^3$ corresponds to:

$$\Delta_3 := \frac{\pi}{\sqrt{18}} \approx 0.74. \tag{10}$$

The proof of Lemma 2 was originally solved in 1998, however it was 250 pages long, with 3 gigabytes of calculations and after 4 years, journal referees said that they were "99% certain" his proof was right. The journal published his paper, but Hales turned to giving a fully rigorous computerized proof of the Kepler Conjecture. He organized a team to do this named the "Flyspeck project" and they finished in 2014. The proof is 30 pages long using a "Proof assistant" but it is still quite complex. Hales et al. [2017]

Close-packing of equal spheres $\mathbb{R}^3$

Close-packing of equal spheres $\mathbb{R}^3$

Definition: Face-centred cubic

FCC is a face-centred cubic close packing structure of lattices. It is a space-efficient composition of crystal structures. The coordination number of this structure is 12, while the number of atoms per unit cell is 4. Here, the coordination number is the number of atoms the unit cell touches. Importantly, this structure efficiently occupies 74%. of the space; thus, the empty space is 26%.

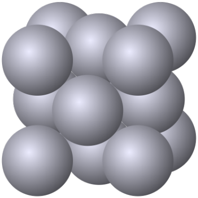

Figure: Collab platform

Close-packing of equal spheres $\mathbb{R}^3$

Definition: Hexagonal close packing

HCP is a hexagonal close packing structure of lattices. It is also a space-efficient composition of crystal structures. The coordination number of this structure is 2, while the number of atoms per unit cell is 6. The structure occupies 74% of total space; thus, the empty space is 26%. Here, HCP layers cycle among two layers. That means; the third layer of the structure is similar to the first layer.

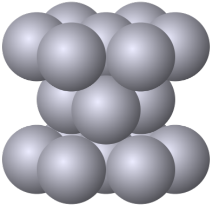

Figure: Collab platform

Contrast

Hexagonal close packing contrast: HCC and FCC

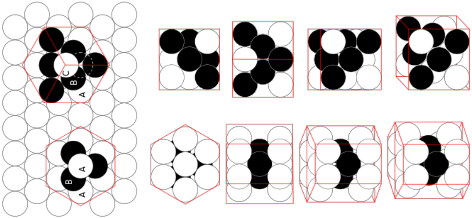

Figure: Contrast in contained space

Sphere packing in $\mathbb{R}^8$

Sphere packing in $\mathbb{R}^8$

Lemma 4

The highest density amongst all possible 8-dimensional lattice packings of spheres in $\mathbb{R}^3$ corresponds to:

$$\Delta_8 := \frac{\pi^4}{384} \approx 0.25. \tag{11}$$

Consequently, the E^8 lattice packing is the densest sphere packing in $\mathbb{R}^8$ Viazovska [2017].

Sphere packing in $\mathbb{R}^8$

Linear programming bounds for sphere packing

The idea is first to define the following Fourier transformation of f:

$$\tilde{f}(t) = \int_{\mathbb{R}^{\ltimes}} f(x) e^{2\pi\langle x,t\rangle} dx. \tag{12}$$

Then, using the theorem of Cohn-Elkies, if we suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a Schwartz function with the following properties:

1. 1. $f(0) = \tilde{f}(0) = 1.$
2. 2. $f(x) \leq 0$ for $|x| \geq r$
3. 3. $\tilde{f}(t) \forall t$

If the conditions are done then the density of any sphere packing in $\mathbb{R}^8$ is bounded avobe by:

$$vol(B_n)(\frac{r}{2})^n \tag{13}$$

Sphere packing in $\mathbb{R}^8$

Note then is a linear programming bound, by scaling f appropriately we can check if the conditions are fulfilled, s the density is bounded by $2^{-n}vol(B_n)f(0)$. Note that the constraints and objective function given are linear in f and therefore it is a linear convex program.

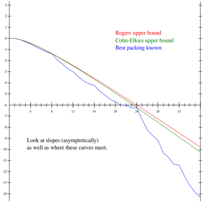

Figure: $\log$(density) vs. dimension

Desired functions

Let Λ be E_8 and $r_0, r_1, \ldots$ its nonzero vector lengths (square roots of the even natural numbers). To have a right upper bound that matches Λ, it is necessary to get a function f that looks like this:

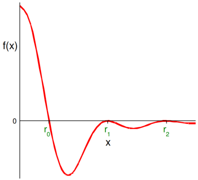

Figure: How f should look like

Desired functions

On the other hand, $\tilde{f}$ should look like this:

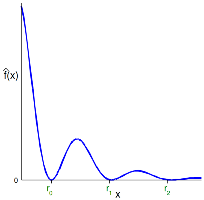

Figure: How $\tilde{f}$ should look like

In [Cohn-Kumar] they used a polynomial of degree 803 and 3000 digits of precision to find f and $\tilde{f}$ which looked like this with 200 forced double roots and r very close to 2. The key point in [Maryna] is to use a modular form with a lot of symmetries.

Sphere packing in $\mathbb{R}^8$ by Maryna S. Viazovska

Sphere packing in $\mathbb{R}^8$ by Maryna S. Viazovska

The sphere packing problem in dimension 8

Figure: Youtube video

VIDEO

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Thanks

Thanks for your attention :))