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Flips in planar graphs

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A B S T R A C T

We review results concerning edge flips in planar graphs concentrating mainly on various aspects of the following problem: Given two different planar graphs of the same size, how many edge flips are necessary and sufficient to transform one graph into another? We overview both the combinatorial perspective (where only a combinatorial embedding of the graph is specified) and the geometric perspective (where the graph is embedded in the plane, vertices are points and edges are straight-line segments). We highlight the similarities and differences of the two settings, describe many extensions and generalizations, highlight algorithmic issues, outline several applications and mention open problems.

1. Introduction

An *edge flip* in a graph is the operation of removing one edge and inserting a different edge such that the resulting graph remains in the same graph class. The edge flip operation has been studied for many different graph classes and has many applications in various settings. In particular, for two given graphs with an equal number of vertices and edges, the number of edge flips required to transform one into the other gives a notion of distance. This notion acts as a measure of similarity that can be varied by constraining the family of allowable flips [49]. Moreover, flips have played a fundamental role in the enumeration of different types of planar graphs as well as in the computation of planar graphs where some criterion of the graph is optimized.

Our goal in this paper is to review the results on edge flips pertaining mainly to planar graphs both in the combinatorial and the geometric setting. We briefly mention results in both settings for flips in graphs embedded in 2-dimensional surfaces. We also outline different applications and generalizations of the flip operation. As one can imagine, there are an enormous number of generalizations of the flip operation and reviewing all of them is beyond the scope of this article. Thus, we concentrate on the generalizations we feel are most closely related to the edge flip operation as defined above. In addition to surveying results on edge flips and their applications, we also mention several open problems and sketch some of the proofs of the results so that the reader has some flavor of techniques used in this area.

Once the graph class $\mathcal{G}$ has been specified and a precise flip operation f has been described, one can define the *flip graph* in the following way. Each distinct n-vertex graph of the given class $\mathcal{G}$ is a vertex of the flip graph and there is an edge between two vertices in the flip graph provided that their representative graphs differ by exactly one flip. In this way many properties of the transformation f operating in $\mathcal{G}$ become graph theoretic questions on the flip graph. For example,

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if we denote the flip graph by G_f, determining how many flips are required to transform one graph in $\mathcal{G}$ into another is precisely the graph distance between the two corresponding nodes in G_f. The diameter of G_f is the maximum number of flips that may be required for any such transformation. The property that the graph G_f is connected means that moving between any two objects in $\mathcal{G}$ via the iterated use of the operation f is always possible. Finally, when the flip graph G_f admits a Hamiltonian path/cycle, we can extract a *Gray code* for the objects in the class $\mathcal{G}$, i.e., a method for generating or listing all the objects in the class, without repetition so that successive objects differ in a pre-specified way, ideally as small as possible [92].

The class of graphs we are most interested in is *triangulations*, i.e., maximal planar simple graphs. The edge flip operation generates many interesting questions about triangulations. For example, what is the maximum number of edges that can be flipped in any triangulation? Is the class of triangulations closed under the flip operation, i.e., given a triangulation T_1 and a different triangulation T_2 of equal size, does there always exist a finite sequence of edge flips that transforms T_1 into a triangulation isomorphic to T_2? Wagner [103] answered this question in the affirmative. This affirmative answer led to other intriguing questions such as given two triangulations, what is the shortest sequence of edge flips that transforms one triangulation into the other? How quickly can such a sequence be computed? What is the pair of triangulations that requires the longest sequence of edge flips, i.e., what is the diameter of the flip graph? If edges can be flipped simultaneously, are there shorter sequences? What is the maximum number of edges that can be flipped simultaneously in a triangulation? All of these questions and many other variants have been addressed in the literature. In Section 2, we present a review of some of the main results in this area followed by a discussion of some open issues that still need to be addressed.

The above setting of the problem is often referred to as the *combinatorial setting* of the problem since only a combinatorial embedding of the triangulation is specified; in other words, we are given for each vertex of the graph the clockwise order of the edges adjacent to the vertex. Although many other settings of the problem have been studied in the literature, we continue with a review of the results in the geometric version of the problem. In this setting, the graphs are geometric graphs, i.e., the vertices are points in the plane and edges are straight-line segments joining the points. There are a number of similarities as well as differences with the combinatorial setting. One of the differences is that the class of graphs studied is usually *near-triangulations* as opposed to triangulations. A near-triangulation is a triangulation with the property that one particular face (called the outerface) need not be a triangle. In the geometric setting, the near-triangulation is embedded in the plane such that the vertices are points in the plane and the edges are straight-line segments with the property that two edges not sharing a common vertex do not intersect. Such an embedding is often referred to as a *planar straight-line embedding*. An edge flip is still a valid operation in the geometric setting (see Fig. 1). Thus, similar questions have been studied. For example, Lawson [69] showed that given any two near-triangulations N_1 and N_2 embedded on the same n points in the plane, there always exists a finite sequence of edge flips that transforms the edge set of N_1 to the edge set of N_2. In Section 3, we present a review of some of the main results in the geometric setting of the problem as well as a discussion of some open problems.

Note that there is quite a disparity between the combinatorial setting of the problem and the geometric one. The disparity arises because not all combinatorially valid edge flips are geometrically valid (see Fig. 2). In the combinatorial setting, Wagner [103] showed that *every* triangulation on n vertices can be transformed to every other triangulation via edge flips. On the other hand, in the geometric setting, Lawson [69] showed that only the near-triangulations that are defined on a specified point set can be attained via edge flips. For example, in the geometric setting, given a set of points in convex position, the only plane graphs that can be drawn without crossing are outerplanar graphs. In fact, the number of edges in a near-triangulation on n vertices depends on the number of vertices on the outer-face whereas the number of vertices in a combinatorial triangulation on n vertices is always the same. This disparity has initiated a new line of investigation. Namely, does there exist a set of local operations in addition to edge flips that permits the enumeration of all n-vertex triangulations in the geometric setting. In Section 4, we present a review of some of these results and describe many extensions and generalizations, both in the combinatorial and geometric setting, with specific attention to pseudotriangulations. We also discuss edge flips for 2-dimensional surfaces other than the plane and outline many open problems along the way.

In Section 5 we review some of the main applications of edge flips, particularly exhaustive enumeration, random generation and geometric optimization. We end with some concluding remarks.

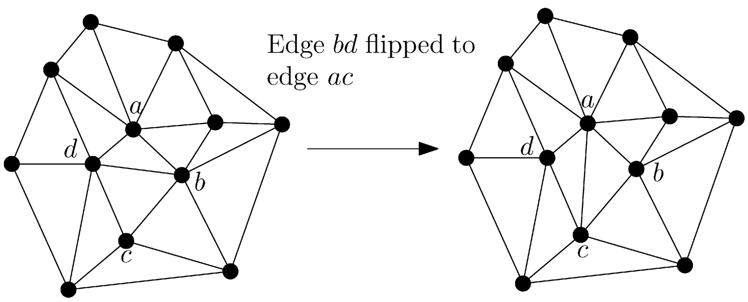


Fig. 1. Example of an edge flip.

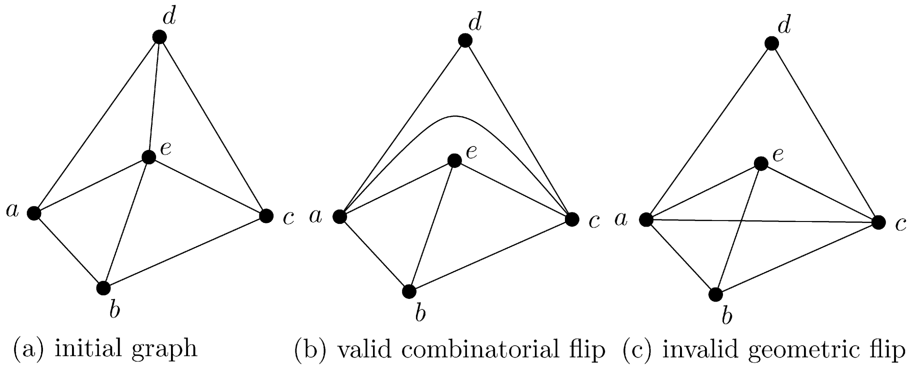


Fig. 2. Edge *de* can be flipped to *ac* combinatorially but not geometrically.

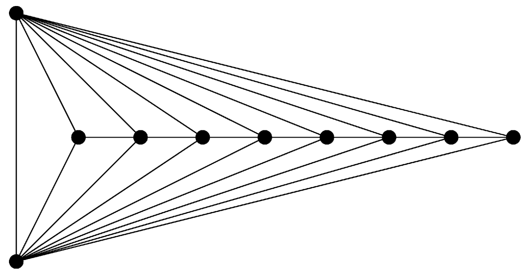


Fig. 3. Wagner [103]'s canonical triangulation.

2. Combinatorial setting

The result that initiated the research on edge flips in triangulations is due to Wagner [103]. He proved that given a triangulation with n vertices, with a finite sequence of edge flips, one can transform this graph to any other triangulation on n vertices. The main idea behind Wagner [103]'s proof is that a finite sequence of edge flips allow one to transform a given triangulation to a canonical one. The canonical triangulation defined is one where there are two vertices, called *dominant* vertices, in the triangulation that are adjacent to every other vertex of the triangulation (see Fig. 3). The graph induced by these other vertices is a path and is referred to as the *spine* of the canonical triangulation. With this tool in hand, to transform an n-vertex triangulation T_1 to a triangulation T_2, one first transforms T_1 into canonical form, then applies the flips to transform T_2 to canonical form in reverse order.

Given this seminal result, several natural questions about edge flips in triangulations leap to mind. Indeed, this result incited a flurry of activity in many different directions. We restrict our attention to results directly related to edge flips in the combinatorial setting. A careful analysis of Wagner [103]'s result reveals that the length of the edge flip sequence is at most $O(n^2)$ where n is the size of the triangulation. Essentially, he shows that a linear number of flips are sufficient to increase the degree of a vertex by 1. In this way, to create two dominant vertices, a quadratic number of flips suffice. It is easy to see that there exist pairs of triangulations that require $\Omega(n)$ edge flips. Consider a triangulation having a vertex of linear degree and one where every vertex has constant degree. Since an edge flip only reduces the degree of a vertex by one, a linear number of edge flips is required to reduce the degree of a vertex from linear to constant. Komuro [66] proved that this bound is tight by showing that $O(n)$ edge flips suffice to transform any n-vertex triangulation to any other. Komuro's argument was similar to Wagner's where two adjacent vertices are selected to be transformed into dominant vertices with flips. However, he used a clever amortization argument to show that essentially 2 flips were sufficient to increase the degree of one of these two vertices by 1. This lead to a total of $4(n-1)$ flips to create the two dominant vertices and $8(n-1)$ total for the transformation. Mori et al. [79] currently have the best bound where they show that at most $6n-30$ edge flips are sufficient. One can view this from a different perspective via the triangulation flip graph. The triangulation flip graph is a graph whose vertices are combinatorially distinct n-vertex triangulations and two vertices in the flip graph are adjacent provided that the two corresponding triangulations differ by one flip. Viewed from this perspective, Wagner [103] showed that the triangulation flip graph is connected and its diameter is $O(n^2)$. Komuro [66] showed that in fact the diameter is $O(n)$, and Mori et al. [79] reduced the constants to show that the diameter is at most $6n-30$. On the way to proving their result, Mori et al. [79] showed that given any n-vertex triangulation, at most $n-4$ edge flips are sufficient to convert this to a 4-connected triangulation (which by a result of Tutte [101] is Hamiltonian), and $4n-22$ edge flips are sufficient to convert any 4-connected triangulation to any other 4-connected triangulation. Several interesting questions remain open.

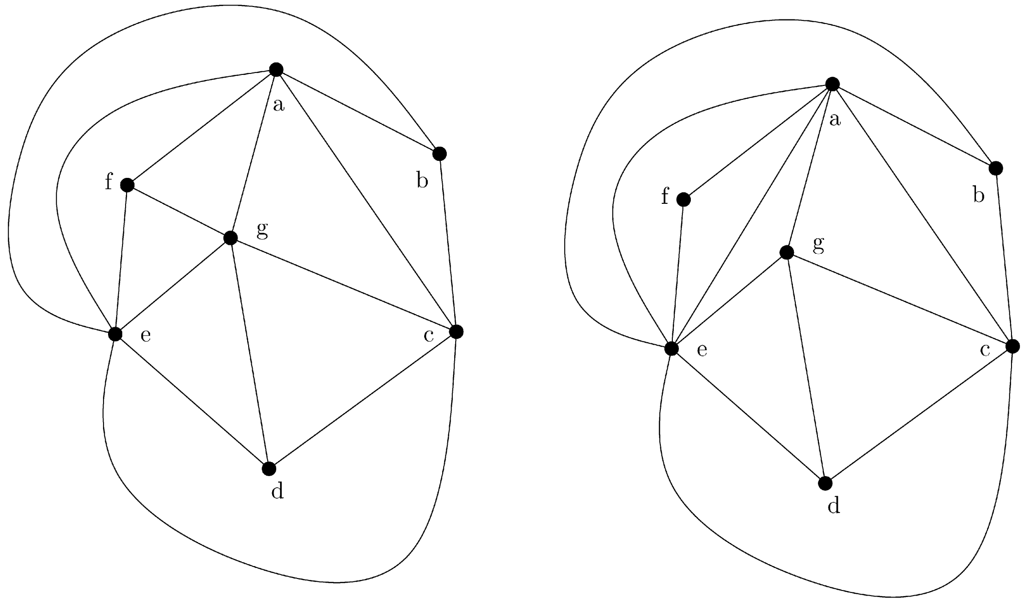


Fig. 4. Cannot flip edge fg as it results in a parallel edge.

Open Problems. Are there triangulations that require at least $n-4$ edge flips to be converted to Hamiltonian or 4-connected? Is $4n-22$ the best upper bound for converting one 4-connected triangulation to another? Is there a matching lower bound? Is $6n-30$ the best upper bound for converting one triangulation to another? Can one find matching upper and lower bounds?

To date, all of the bounds are proven by showing how to transform a given triangulation into a canonical one (of some form). Clearly, this is not necessarily the best way for transforming a triangulation T_1 into T_2. For example, it may be that a single edge flip is sufficient to transform T_1 into T_2 but by going via a canonical triangulation, $O(n)$ flips are performed. This gives rise to the following open questions.

Open Problems. Is it possible to efficiently compute the smallest number of flips sufficient to transform a given triangulation T_1 into T_2 (i.e., without constructing the whole flip graph)? Can a sequence of flips be found whose length is related to (i.e., bounded by a constant or a $(1+\epsilon)$-approximation) the length of the shortest sequence?

Notice that in terms of the flip graph, both of the above questions are asking for the shortest path or an approximation of the shortest path between two vertices of the flip graph.

Another question of interest is the maximum number of edges that can be individually flipped (i.e., edges that are flippable) in a triangulation. Note that not all edges can be flipped because flipping an edge may result in parallel edges which are not allowed (see Fig. 4). Gao et al. [48] showed that every n-vertex triangulation has at least $n-2$ flippable edges and that there exist triangulations with at most $n-2$ flippable edges. The former result is proved by showing that every face has at least one edge that is flippable. The latter is through a simple construction where one starts with an initial triangulation T on m vertices and inserts a vertex inside each face of T, and completes the triangulation by joining the vertex to each of the three vertices of the face. The resulting n-vertex triangulation has only $n-2$ flippable edges. Note that separating triangles play a key role here. In a triangulation, every vertex of degree 3 is contained in a separating triangle. Therefore, triangulations with minimum degree at least 4 have more flippable edges. In fact, Gao et al. [48] show that every n-vertex triangulation with minimum degree at least 4 (for $n>8$) has at least $2n+3$ flippable edges. There exist triangulations that also achieve this bound, therefore, these bounds are tight. When viewed in terms of the flip graph, these questions are asking about the degree of a vertex. However, there is a subtle difference. Even if a triangulation has $n-2$ flippable edges, it does not necessarily mean that flipping each of those edges leads to $n-2$ combinatorially distinct triangulation. Therefore, it would be interesting to determine the maximum, minimum and average degree of a vertex in the flip graph.

In an n-vertex triangulation, since there are always a linear number of edges that can each be individually flipped, it seems natural to ask how many of these edges can be flipped simultaneously. This notion was introduced, albeit in the geometric setting, by Hurtado et al. [63]. A subset of edges of a triangulation is independent provided that no pair of edges in the set share a common face. Given an n-vertex triangulation T and a subset S of independent edges, the operation of a *simultaneous flip* consists of flipping all of the edges in E to produce a distinct triangulation T'. Such a set E of

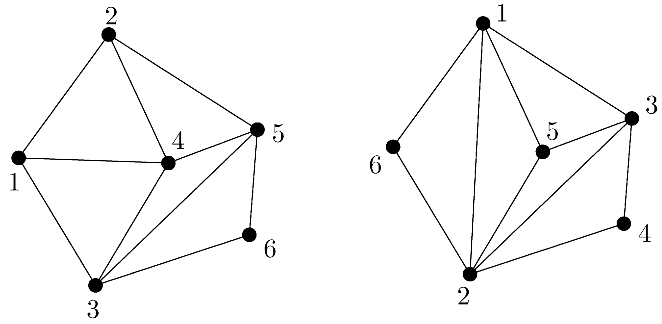


Fig. 5. One flip is not sufficient to convert the left graph into the right graph in the labelled setting.

edges is said to be *simultaneously flippable*. Although sets of simultaneously flippable edges have a strong connection to the notion of a flippable edge, they are quite different altogether. For example, it is possible for a set E of edges to be simultaneously flippable yet contain edges that are *not* individually flippable. It is also possible for every edge in a set E to be individually flippable but the set E itself *not* be simultaneously flippable. In this setting, the main question is how many simultaneous flips are sufficient to convert one n-vertex triangulation to another. The work on individually flippable edges trivially implies that $O(n)$ simultaneous flips are sufficient. The question is how much better can one do when one takes advantage of the ability to flip many edges at the same time. Bose et al. [25] showed that $O(\log n)$ simultaneous flips are sufficient to convert any n-vertex triangulation to any other. They showed that this bound is tight since there exist pairs of triangulations that require at least $\Omega(\log n)$ simultaneous flips to be converted to each other. The approach they take is to convert a triangulation into canonical form using simultaneous flips. As was shown by Mori et al. [79] for the case of single flips, Bose et al. [25] show that a few number of simultaneous flips are sufficient to convert a given triangulation into a 4-connected (Hamiltonian) one. In fact, they show that at most one simultaneous flip is sufficient. With respect to the maximum number of edges that can always be simultaneously flipped in an n-vertex triangulation, they show that at most $n-2$ edges can ever be flipped simultaneously, that every triangulation has at least $(n-2)/3$ edges that can be flipped simultaneously and that there exist triangulations where at most $6(n-2)/7$ edges can be flipped simultaneously. A number of open problems remain.

Open Problems. Can the gap between the lower bound of $(n-2)/3$ and upper bound of $6(n-2)/7$ be closed? Although asymptotically, the bounds on the number of simultaneous flips needed to convert any n-vertex triangulation to any other are tight, the constants are definitely not tight.

So far, all of the results that have been discussed pertain to the unlabelled setting, that is given an initial triangulation, we wish to convert it to a final triangulation but are satisfied if the edge flips terminate with a triangulation that is isomorphic to the final triangulation. In the labelled setting, the vertices of the graph are labelled. We are given an initial n-vertex labelled triangulation and a final n-vertex triangulation defined on the same labelled vertex set and we wish to bound the number of edge flips needed to convert the initial triangulation into the final one (see Fig. 5). For labelled triangulations, Sleator et al. [98] proved that $O(n\log n)$ flips are sufficient to transform one labelled triangulation with n vertices into any other, and $\Omega(n\log n)$ flips are sometimes necessary. Notice that if we transform both the initial and final triangulation into Wagner's canonical form without paying attention to vertex labels, then the problem in the labelled setting becomes one of sorting the vertices along the spine. This is essentially what Sleator et al. [98] do leading to the $O(n\log n)$ result. For the $\Omega(n\log n)$ lower bound, they show that there are at most $3^{2n-4}8^m$ distinct n-vertex labelled triangulations that are reachable from a given triangulation T via m flips. Since there are at least $(n-3)!$ different labelled triangulations on a set of n points,[3] the $\Omega(n\log n)$ lower bound follows. We note that this upper bound was independently rediscovered by Gao et al. [48]. Finally, we note that the $O(n)$ upper bound in the unlabelled setting [66,79] can also be obtained by a careful analysis of the proof by Sleator et al. [98].

3. Geometric setting

3.1. Generic point sets

In the geometric setting, the graphs studied are straight-line planar embeddings of near-triangulations where vertices are points in the plane and edges are straight-line segments. The seminal result by Lawson [69] initiated the study of flips in the geometric setting.[4] Lawson [69] showed that given any two near-triangulations N_1 and N_2 straight-line embedded

[3] This can be easily seen by looking at all the distinct labellings of an n-vertex wheel.

[4] In fact, Lawson [69] credits Weingarten [106] and independently Lawson [68] as having "considered" the result mentioned in Lawson [69] but he says that the proofs in Weingarten [106] and Lawson [68] were "obscure".

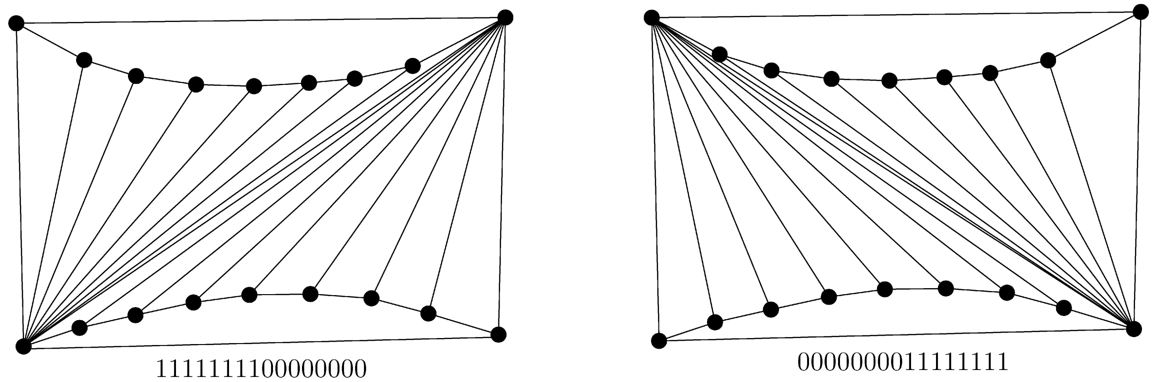


Fig. 6. The construction for the $\Omega(n^2)$ lower bound.

on the same n points in the plane, there always exists a finite sequence of edge flips that transforms the edge set of N_1 to the edge set of N_2. The approach used by Lawson [69] is similar to that of Wagner [103] in that Lawson showed how to convert a given near-triangulation into canonical form using edge flips. The canonical form can be described as follows. Given a set of n points in the plane, label the points $p_1, p_2, \dots, p_n$ in sorted order by x-coordinate. Start with the triangle $\triangle(p_1p_2p_3)$. Add each remaining p_i to the current near-triangulation and connect it to all the vertices on the outerface that are visible to it. Note that among a set L of line segments in the plane, we say that two points are visible provided that the line segment between these two points does not properly intersect any of the line segments in L. Once all the points have been added, this represents the canonical near-triangulation. Lawson [69] proved that any near-triangulation T on a set of n points can be converted to a canonical triangulation with a finite number of flips. The proof is constructive and a simple analysis of the proof shows that $O(n^2)$ flips are sufficient.

If two triangles abd and bcd share the edge bd and their union is a convex quadrilateral, a flip is called a *Delaunay flip* if one replaces the diagonal bd by the diagonal ac provided that the circle through a, b and d contains c. In a follow-up paper, Lawson [70] showed that $O(n^2)$ flips are sufficient to convert any near-triangulation of a set of n points in the plane to the Delaunay triangulation[35,83] of the point set, using only Delaunay flips; this provides an alternate proof of the quadratic upper bound on the flip distance between triangulations with the Delaunay triangulation acting as the canonical triangulation.

Contrary to the situation in the combinatorial setting, Hurtado et al. [64] proved that the quadratic upper bound is tight by constructing a pair of n-vertex near-triangulations that require at least $\Omega(n^2)$ edge flips (of any kind) to convert one into the other. Their construction is as follows (Fig. 6): start with an axis-aligned rectangle and construct a convex curve with a total of n vertices connecting the two upper corners; construct also a concave curve with n vertices connecting the two lower corners; these curves are "very flat", in the sense that any segment connecting a vertex on the upper chain U and a vertex on the lower chain L doesn't cross either of them. In this way we have a point set S with $2n$ vertices, such that any near-triangulation of S will necessarily include both the lower and the upper chain entirely. We can ignore the triangulations inside the upper and lower convex regions, because the flip distance between triangulations of a convex n-gon is always $O(n)$. Nevertheless, any triangulation of the polygon between U and L consists exactly of $n-1$ triangles with two vertices in L and one in U (these we label with a "0") and $n-1$ triangles with two vertices in U and one in L (these we label with a "1"). If we read the numbers of the triangles from left to right, we obtain an ordered sequence of zeros and ones, and there is a bijection between the set of triangulations of the polygon and the set of binary sequences that consists of $n-1$ zeros and $n-1$ ones. As a flip is only possible for diagonals shared by triangles with different number, and the flip can only transpose those numbers, the triangulations encoded by $00\dots011\dots1$ and $11\dots100\dots0$ require $(n-1)^2$ edge flips to be transformed into each other.

Almost all approaches to transform a given near-triangulation to another make use of a canonical near-triangulation. This approach suffers from an inherent drawback that the number of flips used in the transformation may not be sensitive to the fewest number of flips required to achieve the transformation. This drawback also exists in the combinatorial setting as was highlighted in the previous section. As opposed to the combinatorial setting, in the geometric setting, an attempt has been made to address this problem by proposing an alternate proof method which is more sensitive [53]. Hanke et al. [53] show that given two near-triangulations defined on the same set of n points, the number of flips sufficient to convert one to the other is bounded by the number of intersections between the edges of the two near-triangulations. Since each near-triangulation has at most $3n$ edges, the maximum number of intersections is at most $\binom{3n}{2}$ which is quadratic. However, this approach is more sensitive to the shortest sequence. For example, if two near-triangulations differ by at most one flip, there is only one intersection between the edges of the triangulations, therefore, their algorithm would perform only one flip as opposed to a converting one of the near-triangulations into a canonical form such as the Delaunay triangulation. Hanke et al. [53] prove this result by showing that if one has two near-triangulations defined on the same point set, there always exists an edge flip in one of the two triangulations that reduces the total number of intersections by at least 1.

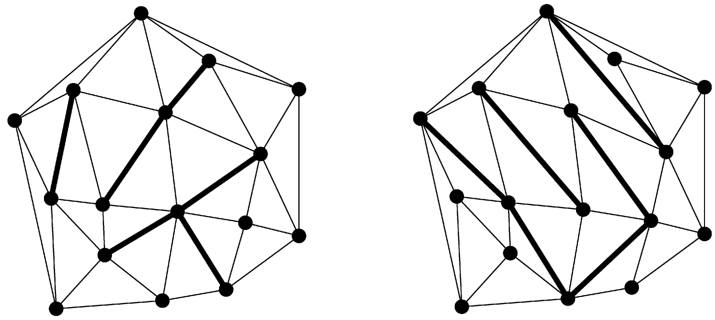


Fig. 7. The darkened edges in the near-triangulation on the left are simultaneously flipped to give the triangulation on the right.

This was taken a step further by Eppstein [44] who showed how to compute in polynomial time a lower bound on the flip distance between two triangulations. Moreover, he showed that for special classes of point sets, this bound is actually the true flip distance. Given a set P of n points in the plane, define the *quadrilateral graph* $QG(P)$ as a graph whose vertex set is all $\binom{n}{2}$ pairs of vertices, and two vertices ab and cd are joined by an edge provided that the four points a, b, c, d form a convex quadrilateral and the two segments ab and cd intersect. Given two triangulations T and T' defined on the same point set P, define a complete bipartite graph between the edges of T and the edges of T'. Next define a weight for each edge in this bipartite graph as the length of the shortest path in $QG(P)$ between these two edges. Eppstein [44] showed that the weight of a minimum weight perfect matching in this bipartite graph provides a lower bound on the number of flips needed to convert T to T'.

Another interesting question is to determine whether or not some of these bounds are sensitive to properties of the point set. In the case where the points are in convex position it is obvious that $n-3$ flips suffice to convert any triangulation in a *fan* from any prescribed vertex v (all the diagonals will have v as a common endpoint); this implies that at most $2n-6$ edge flips are sufficient to convert any triangulation of a set of n points in convex position to any other triangulation of the same point set. If a set of n points has k convex layers,[5] Hurtado et al. [64] show that $O(kn)$ edge flips are sufficient, and that for simple triangulated n-gons with k reflex vertices, $O(n+k^2)$ edge flips are sufficient. When studying the maximum number of edges that can be flipped in any near-triangulation of a set of n points in the plane, Hurtado et al. [64] prove that at least $\lceil (n-4)/2 \rceil$ edges are flippable. They show that this bound is tight by providing a construction that allows only $\lceil (n-4)/2 \rceil$ flippable edges. Several open questions remain in this area.

Open Problems. Can one find matching constants in the upper and lower bound on the number of edge flips? Can one find a better upper bound for the case where the point set has k convex layers or is $O(kn)$ the correct asymptotic answer? Is there a class of graphs that can be reached in fewer edge flips? For example, in the combinatorial setting, fewer flips were needed to convert a given triangulation into a Hamiltonian one. Is the same true in the geometric setting? Is there always a sequence of $o(n^2)$ flips that allows one to convert any near-triangulation into a Hamiltonian one?

In an n-vertex near-triangulation, since there are always $\lceil (n-4)/2 \rceil$ edges that can be individually flipped, Hurtado et al. [63] asked whether flipping several edges at the same time could help. They introduced the notion of a simultaneous geometric flip (this is similar to the notion of simultaneous flips discussed in the previous section. See Fig. 7.). Given an n-vertex near-triangulation T and an independent subset E of edges, the operation of a *simultaneous flip* consists of flipping each of the edges in E to produce a distinct near-triangulation T'. Such a set E of edges is said to be *simultaneously flippable*. Galtier et al. [47] showed that $O(n)$ simultaneous edge flips are sufficient to convert any n-vertex near-triangulation to any other near-triangulation on the same vertex set. They modified the construction in Hurtado et al. [64] to show that there exist pairs of near-triangulations that require $\Omega(n)$ simultaneous edge flips. For the restricted case where the points are in convex position, they showed that $O(\log n)$ simultaneous flips are sufficient and that there are pairs of near-triangulations that require $\Omega(\log n)$ simultaneous flips. Finally, they showed that every near-triangulation on n points has at least $(n-4)/6$ edges that can be flipped simultaneously and that there exist triangulations that have at most $(n-4)/5$ edges that can be flipped simultaneously. A number of questions remain unsolved: Although asymptotically, the bounds on the number of simultaneous flips are tight both in the general case and the case where the points are in convex position, in neither case is the constant tight. Can the gap between the $(n-4)/6$ lower bound and $(n-4)/5$ upper bound be closed? Can a smaller number of simultaneous flips allow one to convert any n-vertex near triangulation into a Hamiltonian one? What happens if one restricts their attention to only Delaunay flips?[6]

On a different direction, one may study a specific family of near-triangulations of a point set S instead of the whole set. For example, let E be any given set of non-crossing edges and consider the set of triangulations $\mathcal{T}_E(S)$ of S whose edge set

[5] The number of convex layers is the number of times the convex hull of a point set can be removed until the point set is empty.

[6] See Okabe et al. [83] for a comprehensive survey on Voronoi Diagrams and Delaunay triangulations.

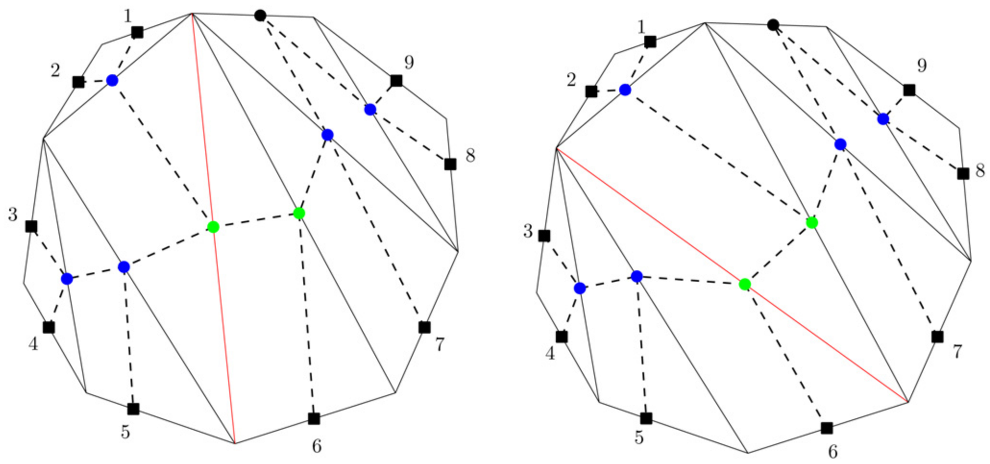


Fig. 8. Correspondence between binary trees and triangulations.

contains the set E. The usual edge flip operation in $\mathcal{T}_E(S)$ yields a connected flip graph. This can be proved using *constrained* Delaunay flips (see [72]) but admits a direct proof as well [60]. In the case where $|S|$ is even, Houle et al. [60] proved that the set of triangulations on top of S that contains perfect matchings is connected via edge flips. Another example is the case of near-triangulations that use bounded-order Delaunay edges. For $u, v \in S$, we say that the edge uv is an order-k Delaunay edge if there is a circle with u and v on its boundary that contains at most k points from $S \setminus \{u, v\}$ in its interior [1,52]. Order-0 Delaunay edges are the standard edges of the Delaunay triangulation, and order-k edges provide a generalization of the graph. Bounded-order graphs have also been considered for other structures like the order-k Gabriel Graph and the order-k Relative Neighborhood Graph [65]. Let $\mathcal{T}_k(S)$ be the set of near-triangulations of S that use only order-k Delaunay edges. Notice that unless there are degeneracies $\mathcal{T}_0(S)$ has only one element, namely the Delaunay triangulation. For larger k, the set $\mathcal{T}_k(S)$ has many elements. A natural question to consider is whether the flip graph of $\mathcal{T}_k(S)$ is connected under the usual edge flip operation. Abellanas et al. [1] have shown that the answer is always positive for the case $k = 1$, but there exist point sets for $k \geqslant 2$ where the flip graph of $\mathcal{T}_k(S)$ is not connected. We conclude this section by mentioning an open problem in the same spirit as the preceding ones.

Open Problem. Let $\mathcal{T}_H(S)$ be the set of near-triangulations of S that contain some (possibly different) Hamiltonian cycle, is the flip graph of $\mathcal{T}_H(S)$ connected under edge flip?

3.2. Points in convex position

The case where the points are in convex position has received much attention. One of the main reasons is that there is a bijection between near-triangulations of convex n-gons and binary trees with $n - 2$ internal nodes. Diagonal flips in the near-triangulation correspond isomorphically to rotations in the tree. The bijection can be described as follows: take a fixed edge e of the polygon as root of the tree, and the two other sides of the triangle with base e as children of the root. Continue building the tree recursively (see Figs. 8 and 9).

Let us denote by $T_n(G)$ the flip graph of the convex n-gon. This graph can be realized as the skeleton of a convex $(n - 3)$-polytope (called the *associahedron*), as shown by Lee [71]. Lee also shows that the automorphism group of $T_n(G)$ is the dihedral group of symmetries of a regular n-gon. Sleator et al. [97] use 3-dimensional hyperbolic geometry to show that the diameter of $T_n(G)$ is $2n - 10$ for large values of n, i.e., $2n - 10$ edge flips are sufficient and occasionally necessary to convert any near-triangulation of a set of n points in convex position to any other near-triangulation on the same point set. In addition, Lucas [75] proved that $T_n(G)$ is a Hamiltonian graph by means of a particular encoding of binary trees. Several of these results were proved again in [62] under a unifying framework called the *tree of triangulations* which is an (infinite) hierarchy allowing inductive arguments to proceed in a natural manner.

It is worth mentioning in this section that the problem for points in convex position in higher dimensions is challenging even for the basic questions. For example, in dimension 3, the convex hull of any five points in convex position admits two decompositions into tetrahedra, one of them consists of two tetrahedra sharing a face, the other consists of three tetrahedra with a common edge and the natural flip operation is to switch between these two configurations (see Fig. 10). This flip may be used for transforming the tetrahedralizations of a 3-dimensional polytope with n-vertices, but it is not known whether the corresponding flip graph is connected.

The answer to this question is also unknown in dimension 4, but for dimensions greater than 4, the graph can be disconnected. Examples have been obtained by Santos in a series of papers in the context of *bistellar flips*. This notion is more powerful than the flips described here as it allows some insertions and deletions of vertices, yet their negative results apply in our setting. The interested reader is referred to the survey [91] on bistellar flips.

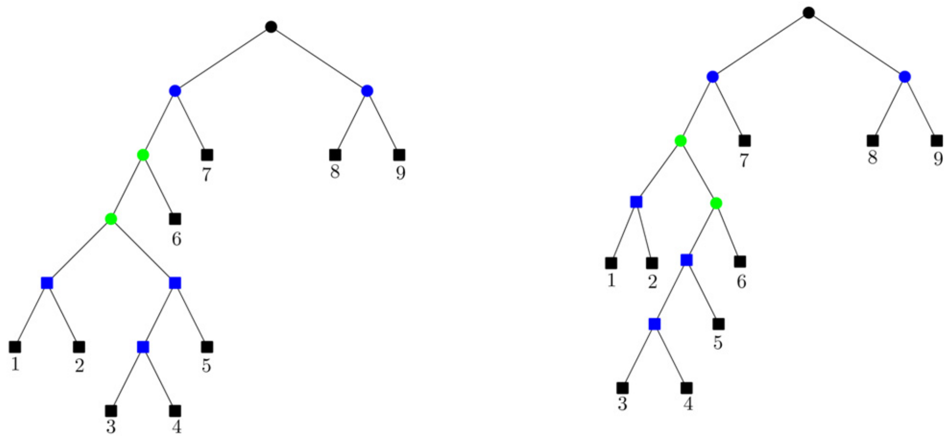


Fig. 9. Rotation between the two trees corresponds to the flip of the diagonal in Fig. 8.

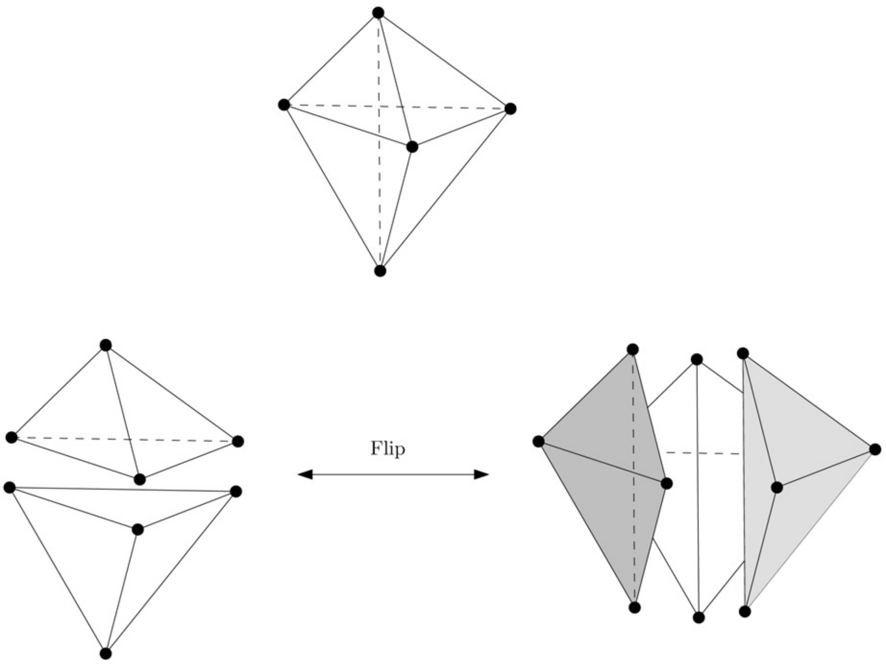


Fig. 10. Basic flip in dimension 3.

We conclude this section with the mention of a somehow surprising result for the convex case. A *signed triangulation* of a convex polygon is a triangulation in which each face has been assigned a $+$ sign or a $-$ sign. A *signed flip* consists of taking two adjacent faces having the same sign, flipping the edge they share, and giving to both new faces the opposite sign to the original ones. It was conjectured that any two triangulations T and T' of a convex n-gon admit sign assignments such that T and T' can be transformed into each other by a sequence of signed flips; this conjecture is true, as it has been proved [42,50,67] that the conjecture is *equivalent* to the 4-Color Theorem for planar graphs! More results in the same spirit are described in [43].

4. Extensions

4.1. Disparity between flips in plane graphs and planar graphs

As noted in the introduction, there is quite a disparity between the combinatorial setting of the problem and the geometric one. In the combinatorial setting, all the results are with respect to the class of triangulations whereas in the geometric setting, the transformations are restricted to a fixed point set. For example, Wagner [103] showed that *every* triangulation on n vertices can be transformed to every other triangulation via edge flips. On the other hand, in the geometric setting, Lawson [70] showed that only the near-triangulations that are defined on a specified point set can be attained via edge flips. This disparity initiated a new line of investigation. Namely, does there exist a set of local operations, in addition to edge flips, that permits the enumeration of all n-vertex triangulations in the geometric setting. In order to achieve this, it is essential to allow a point to be moved because given a set of n points in the plane, not all n-vertex triangulations can be straight-line embedded on the given point set. Abellanas et al. [2] defined a point move in an n-vertex triangulation embedded in the plane as simply the modification of the coordinates of one vertex of the graph. The point move is deemed valid provided that no edge crossings are introduced after the move. In this setting, Abellanas et al. [2] showed that $O(n)$ point moves and $O(n^2)$ edge flips are sufficient to transform any n-vertex triangulation embedded in the plane into any other n-vertex triangulation. Moreover, if the initial graph is embedded in an $n \times n$ grid, all point moves stay within a $5n \times 5n$ grid (i.e., the size of the coordinates in the move is bounded). Although Hurtado et al. [64] provide a pair of n-vertex near-triangulations that require $\Omega(n^2)$ edge flips to transform one into the other, this lower bound no longer holds in the presence of point moves. In fact, it can be shown that $O(n)$ point moves and edge flips are sufficient to transform between the two graphs in the lower bound construction. Therefore, the question becomes is there an $\Omega(n^2)$ lower bound on the number of point/edge moves required? If one removes the restriction on the size of the coordinates, Aloupis et al. [13] were able to show that with $O(n \log n)$ point moves and edge flips, one can convert any n-vertex straight-line embedded triangulation into any other. Is this best possible or can it be shown that a linear number of edge flips and point moves is sufficient? In the labelled setting, Abellanas et al. [2] showed that $O(n^2)$ point moves (with all moves restricted to the $5n \times 5n$ grid) and edge flips are sufficient. Aloupis et al. [13] proved that $O(n \log n)$ point moves and edge flips are sufficient when there are no restrictions on the size of the coordinates.

Open Problem. Are these bounds optimal?

4.2. Variations on geometric flips

Recall that a *geometric graph* is a graph drawn in the plane with its vertices represented by distinct points and its edges represented by straight-line segments connecting the corresponding points. The study of these graphs has been the focus of intensive research that has started quite recently yet yielded many deep results; we refer the interested reader to the surveys [27,84–86].

Let S be a set of n points in the Euclidean plane. For ease of description assume that S is in *general position*, i.e., that no three points from S are collinear. Denote by $K(S)$ the complete geometric graph with vertex set S. A *crossing-free* (or *non-crossing*) subgraph of $K(S)$ is any subgraph such that its edges are straight line segments that pairwise do not cross. Much research has been devoted to determining the number of such subgraphs. This vein of investigation was initiated with the crucial contribution by Ajtai et al. [10] that the number of plane graphs defined on any set of n points is bounded from above by some fixed exponential c^n, where $c \leqslant 10^{13}$. This has been successively improved with the current bound of $c \leqslant 344$ shown by Sharir and Welzl [95]. Sharper bounds have been obtained for some specific classes of graphs, such as near-triangulations, polygonizations,[7] perfect matchings or spanning trees, see [8,94,95] for references and recent results. When the point set S is in convex position most of these countings are well known or can be done exactly, see [45] for a generic framework.

Maximal planar subgraphs of $K(S)$ are precisely the near-triangulations of S. In the same way that we have considered flips for that set, it is natural to study whether similar elementary transformations can be defined for the other classes. Notice that for polygonizations or for non-crossing matchings, a proper edge flip is not possible because once an edge is removed there is no other edge that can be added while remaining in the same class. In such situations, the alternative is to define operators that are as local as possible and ideally involve a small subset of the edges. In fact, a basic generalization of the edge flip in a graph consists of the removal of a k-subset of the edges in the graph followed by the insertion of k edges (possibly including some of the edges just removed), in such a way that the resulting graph remains in the same graph class. This is often called a k-flip, and obviously a 1-flip is the single edge flip. For the operation to be "small" one would like k to be "small". We now describe flip properties for several geometric-graph classes and transformations. The special case of pseudotriangulations is deferred to Section 4.3.

The set $\mathcal{T}_S$ of non-crossing spanning trees is the class that has received much attention from the viewpoint of flips. The most natural flip operation is called the *edge move*, which is essentially the edge flip operation defined on trees. The nodes in the flip graph corresponding to two trees in the set $\mathcal{T}_S$ are adjacent if they differ by one edge. Avis and Fukuda [17] proved that the corresponding flip graph is connected and has diameter bounded above by $2n - 4$. A lower bound of $3n/2 - 5$ given by point sets in convex position was proved by Hernando et al. in [57], where it is also shown that the flip graph is Hamiltonian in this special case and achieves maximum connectivity.

The edge move inside $\mathcal{T}_S$ is a non-local operation because once an edge is removed, the number of possible replacements may be quadratic. In other words, the edge move is quite powerful and it is not surprising that the diameter is only linear. The flip graph remains connected when the edges are labelled and the replacing edge gets the label of the disappearing one [56] (see Fig. 11).

[7] A polygonization of a set of n points in the plane refers to the construction of a simple polygon whose vertices are the points.

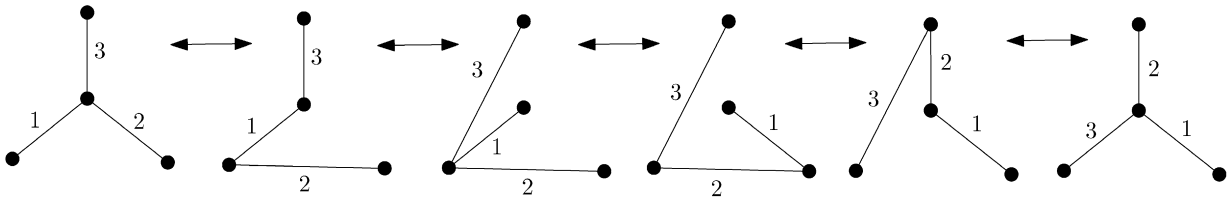


Fig. 11. A sequence of edge moves for trees with labelled edges. Notice that the first and the last ones would be just the same were it not for the labels.

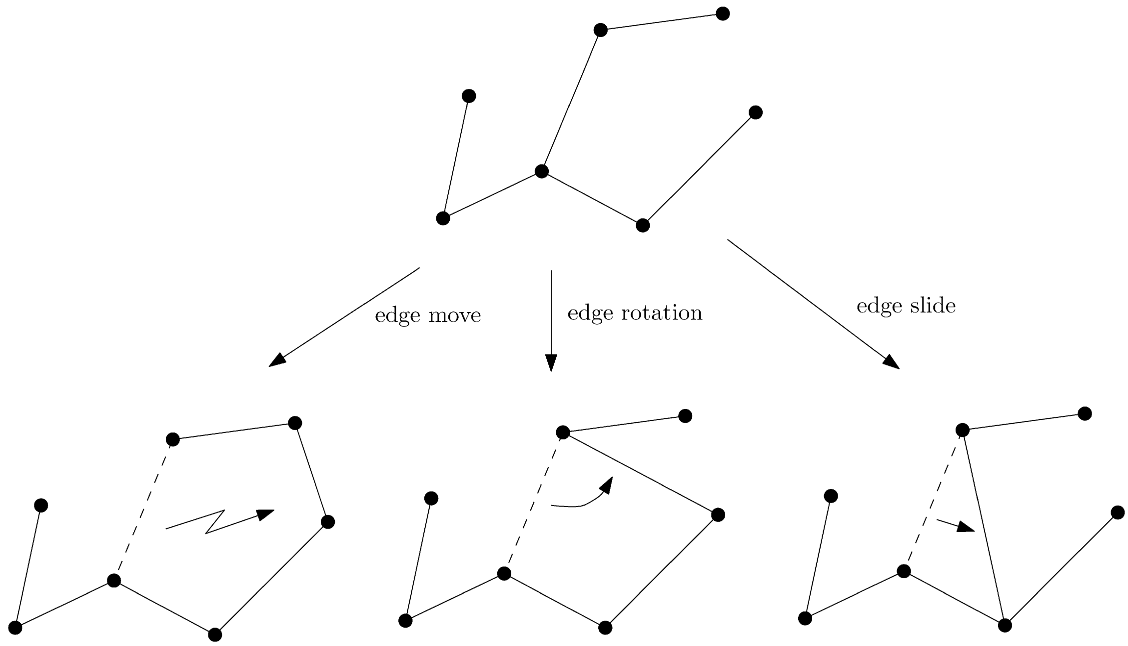


Fig. 12. Examples of edge move, edge rotation, and edge slide.

The edge move flip graph of the spanning trees of a combinatorial graph has been largely studied in graph theory (see Holzmann and Harary [58], and Liu [73]). Note that edge "crossings" are not an issue in the combinatorial setting. The flip graph in this case is always Hamiltonian, a fact showing the power of the edge move in this setting. The drawback is that as a measure of similarity for graphs of equal order and size, the edge move is a weak instrument which is why more constrained flip operations have been defined [49]. An example of such an operation is the *edge rotation*, where an edge uv can only be replaced by an edge of the type uw, incident to the previous edge. Another more stringent transformation is the *edge slide*, defined like the edge rotation with the additional requirement that w must be a neighbor of v (see Fig. 12).

It was proved in [4] that the flip-graph of $\mathcal{T}_S$ via edge slides is connected. However, the proof did not lead to any subexponential bound on the diameter of the flip graph. A bound of $O(n^2)$ is described in [9]. It is a curious fact that the result in [4] was a corollary of a theorem in the "opposite" direction, namely a parallel flip result: for every tree $T \in \mathcal{T}_S$ let $f(T)$ be the shortest tree (in the sense that the sum of the edge lengths is minimized) such that no edge of T is crossed by any edge of $f(T)$. It is proved in [4] that $f(T) = T$ if, and only if, T is the minimum spanning tree of S (the *fixed-tree theorem*), and that starting from any tree T there is a unique sequence $T, f(T), f^2(T), \ldots$ that always converges to the minimum spanning tree of S. At most $O(\log n)$ steps lead to the minimum spanning tree, which is a tight bound.

Besides the aforementioned fact that the flip-graph of $\mathcal{T}_S$ is connected via edge slides, the fixed-tree theorem yielded another interesting corollary. In the same way in which Delaunay flips are improving flips for near-triangulations, and that a sequence of them leads to the Delaunay near-triangulation, it is natural to wonder whether a suitable sequence of length-decreasing edge flips could lead to the minimum spanning tree of S starting from any tree in $\mathcal{T}_S$. Neither edge rotations [26] nor edge slides are sufficient, but it was shown in [4] that for improving (single) edge moves such a sequence always exists. At most $O(n \log n)$ length-decreasing edge moves are sufficient and given any starting tree, the corresponding sequence can be computed in $O(n \log^2 n)$ time.

While non-crossing trees and triangulations are the structures that have received the most attention in the literature, many other structures are of interest and their study lead to challenging problems. We describe four such examples that raise a question on which we elaborate more at the end of this section: when are *local* transformations possible?

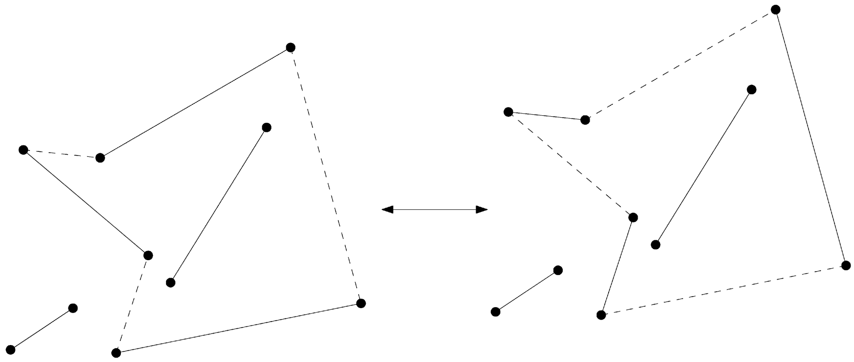


Fig. 13. Example of a 3-flip in a non-crossing matching.

Assume that the cardinality of the point set S is an even number $n = 2m$. We say that two non-crossing perfect matchings in S differ by a *k-flip* if their symmetric difference is a single non-crossing cycle of length $2k$ (see Fig. 13). Notice that for this problem the requirement that the cycle involved in the exchange be crossing-free is crucial (and natural) because the symmetric difference of two perfect matchings is always a set of alternating cycles. This problem is non-trivial only in the geometric setting. Houle et al. [60] proved that for geometric matchings, the corresponding flip-graph is connected provided that no bound is prescribed on the size of the flip. An $O(\log n)$ upper bound on the number of required flips is given in Aichholzer et al. [6] and an $\Omega(\log n/\log\log n)$ lower bound is described in Razen [87].

Open Problem. It is an open problem to decide whether the graph is still connected for some constant value of k. On the other hand, when points are in convex position, the flip graph based on 2-flips is connected. It has a Hamiltonian cycle when $m \geqslant 4$ is even and no Hamiltonian path for $m > 3$ odd.

One class of objects that is specially interesting is the set of polygonizations (or simple polygons or non-crossing Hamiltonian cycles) of a given point set S. As many algorithms for simple polygons have been developed in the field of computational geometry, it would be interesting to test them with "random" polygons, but how can one define and obtain such random objects? A natural approach that has been suggested is to generate n random points in the unit disk with uniform probability and then take a random polygonization of the point set with uniform probability among all the possible simple polygons; the latter step could be accomplished by starting with any polygonization and then walking at random inside the flip graph. After a large number of steps, the last position can represent a 'random polygon' (this method will be described more precisely in Section 5). A basic requirement for the success of this approach is that the flip graph should be connected. To date, it is still unknown whether the class of polygons on a set of n points is connected via any constant-size local transformation. 2-flips have been shown not to be sufficient [59]. Some polygon classes have been shown to be connected under a combination of 2-flips and 3-flips, such as monotone, x-monotone, star-shaped, edge visible and externally visible polygons, with the latter class being the most general [55]. An interesting new approach has recently been presented by Damian et al. [34]. However, their moves allow the loss of proper simplicity and the number of steps (or "twangs") is only proved to be bounded above by $O(n^n)$.

The related problem of obtaining a connected flip-graph for Hamiltonian crossing-free paths as opposed to cycles using "small-sized" transformations has also been considered. Again, for point sets in convex position, the problem is well understood and the flip graph is even known to be Hamiltonian for 1-flips [90,102]. We are unaware of any progress for the same problem on generic point sets.

Finally let us consider the case of *convex subdivisions* of a point set S. These are the decompositions of the convex hull of S into convex regions by means of edges that use all the vertices (see Fig. 14). For point sets in convex position the 1-flip is a powerful transformation that even yields a Hamiltonian flip graph [61]. However, for generic point sets, there exists an example of a convex decomposition of a set of $3n$ points such that the smallest possible flip requires that n edges are simultaneously replaced in order to get a new convex decomposition [77].

For generic point sets, this striking result makes it unclear whether local small-sized transformations are really possible for classes of crossing-free objects such as simple polygons, Hamiltonian paths or perfect matchings. This has also raised interesting considerations from the viewpoint of parallel computation [11].

4.3. Pseudotriangulations

A *pseudotriangle* is a simple polygon that has exactly 3 convex vertices which are often referred to as the corners of the pseudotriangle. Let e be an edge of a pseudotriangle. The edge e lies on the path between two corners of the pseudotriangle, which is often referred to as a *side chain*. The third corner is known as the *opposite* corner with respect to the edge e. Given a set P of n points in the plane, a *pseudotriangulation* of P is a plane graph whose vertex set is

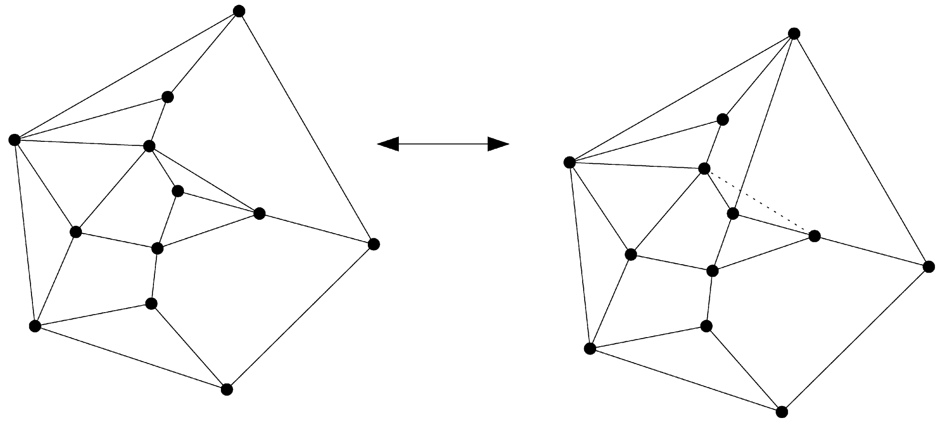


Fig. 14. Example of a 1-flip in a convex decomposition.

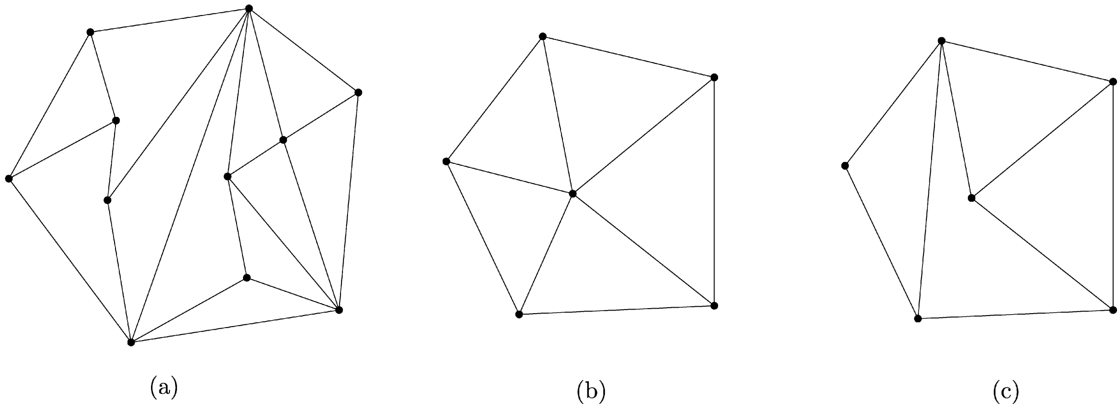


Fig. 15. Examples of pseudotriangulations: (a) pseudotriangulation, (b) minimal pseudotriangulation, (c) minimum pseudotriangulation.

P, edges are straight segments, the outerface is the convex hull of P and every other face is a pseudotriangle. From this definition, one can observe that pseudotriangulations are generalizations of triangulations since any triangulation of P is also a pseudotriangulation. A pseudotriangulation $\mathcal{PT}$ is *minimal* provided that there does not exist an edge e such that $\mathcal{PT} \setminus e$ is a pseudotriangulation. A pseudotriangulation $\mathcal{PT}$ is *pointed* provided that every vertex is reflex in one of its incident faces (including the outerface). A pseudotriangulation is *minimum* provided that no other pseudotriangulation on the same point set has fewer edges (see Fig. 15 for examples). Streinu [100] showed that a pseudotriangulation is minimum if and only if it is pointed. Minimal pseudotriangulations are not necessarily pointed or minimum as can be seen in Fig. 15(b). Euler's formula for planar graphs allows one to show that given n points in the plane, a minimum or pointed pseudotriangulation has exactly $2n - 3$ edges and $n - 1$ faces.

The flip operation needs to be generalized in the case of pseudotriangulations since faces are no longer triangles but pseudotriangles. Similar to edge flips in geometric triangulations, in the case of pseudotriangulations, only edges that are not on the convex hull can be flipped. Given an edge e in $\mathcal{PT}$ that is not on the convex hull, e is adjacent to two pseudotriangles pt_1 and pt_2. Let v_1, resp. v_2, be the corner opposite e in pt_1, resp. pt_2. The operation of flipping e is defined as removing e from $\mathcal{PT}$ and adding all the edges in the geodesic from v_1 to v_2 that are not already in $\mathcal{PT}$. We shall refer to this type of flip as a *pseudoflip* (see Fig. 16). It has been shown [5,28,100] that the graph resulting after a pseudoflip is still a pseudotriangulation of the given point set.

Depending on the type of pseudotriangulation, the pseudoflip can behave differently. For example, in some cases, a pseudoflip results in the deletion of an edge (see Fig. 17). For pointed or minimum pseudotriangulations, it has been shown [28,100] that a pseudoflip always results in the deletion and insertion of exactly one edge. The questions that have been asked about pseudoflips are similar to the questions studied for flips in triangulations. We review some of the results below.

In the case of minimum or pointed pseudotriangulations, Brönnimann et al. [28] showed that the pseudotriangulation flip graph is connected and that its diameter is $O(n^2)$. A vertex in the flip graph is a pseudotriangulation on a fixed set of n points in the plane. Two vertices are adjacent in the flip graph provided that the two pseudotriangulations representing the vertices of the flip graph differ by exactly one pseudoflip. The flip graph being connected with $O(n^2)$ diameter means that pseudoflips can transform any n vertex pointed pseudotriangulation to any other with at most $O(n^2)$ pseudoflips. The idea behind the proof is the following.

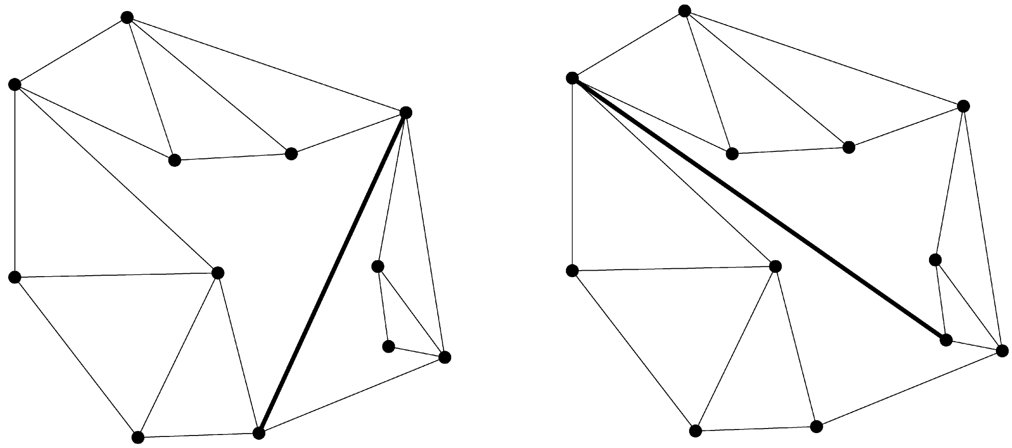
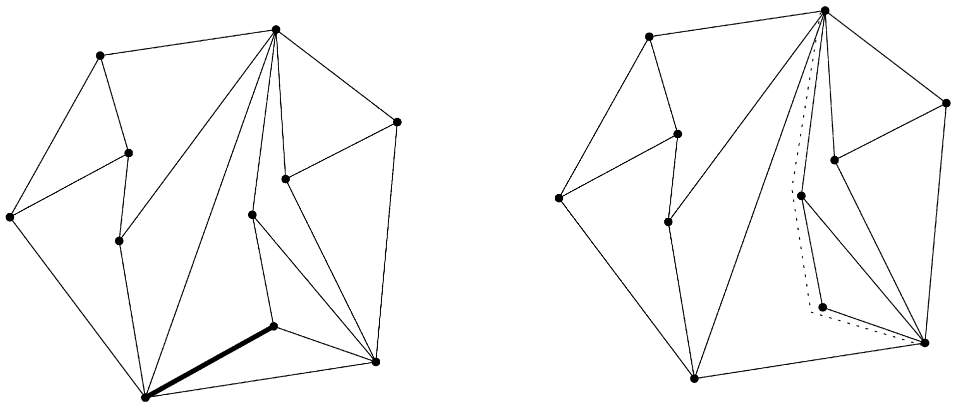


Fig. 16. Example of a pseudoflip in a pointed pseudotriangulation.



Edge to flip shown in bold. Geodesic between two opposite corners shown in dashed.

Fig. 17. Example of a pseudoflip that results in an edge deletion.

Let p be a convex hull vertex in P. A pointed pseudotriangulation where p is not incident to an interior edge is just a pointed pseudotriangulation of $P \setminus \{p\}$ together with p and its two incident edges. By induction, we can assume that all pseudotriangulations on $P \setminus \{p\}$ are connected using at most $(n-1)^2$ pseudoflips. Note that when p is a convex hull vertex, flipping an interior edge adjacent to p reduces the number of interior edges incident on p. Since p can be incident to at most $n-2$ interior edges, at most $n-2$ flips are needed to remove all interior edges incident on p. Since $(n-1)^2 + 2n - 4 \leqslant n^2$, the result follows.

Since a pseudotriangulation is also a triangulation, it was unclear whether or not there existed an $\Omega(n^2)$ lower bound in the worst case on the diameter of the flip graph as does in the in case of geometric triangulations. Aichholzer et al. [5] showed that in fact $O(n \log^2 n)$ pseudoflips were sufficient to transform any one pointed pseudotriangulation to another. This result was subsequently improved by Bereg [19] who showed that $O(n \log n)$ pseudoflips were sufficient to transform any one minimum pseudotriangulation of a given point set to any other. The main idea behind Bereg [19]'s proof is to define a balanced canonical pseudotriangulation. Let P be a set of n points in the plane and let p_0 be the lowest point on the convex hull of P. Sort the points of P radially around p and label them in clockwise order (i.e., p_1 is the clockwise neighbor of p_0 on the convex hull and p_{n-1} is the counter-clockwise neighbor). The canonical pseudotriangulation centered at p_0, denoted $T(p_0, P)$, is defined recursively. If $n = 3$, then $T(p_0, P)$ is a triangle. If $n > 3$, let $m = \lfloor n/2 \rfloor$. Let $P_1 = p_0, p_1, \ldots, p_m$ and $P_2 = p_0, p_{m+1}, \ldots p_{n-1}$. $T(p_0, P)$ is the union of $T(p_0, P_1)$ and $T(p_0, P_2)$ and the pseudotriangle that remains if the convex hull of P_1 and convex hull of P_2 is removed from the convex hull of P. Call this pseudotriangle a *splitting* pseudotriangle. Bereg [19] showed that given any pseudotriangulation, with a linear number of pseudoflips one could add the splitting pseudotriangle. Since the splitting pseudotriangle partitions the points in a balanced way, a simple analysis of the recursion shows that a total of $O(n \log n)$ pseudoflips are sufficient to reach the canonical pseudotriangulation.

Open Problem. An interesting open problem that remains is to close the gap between the trivial lower bound of $\Omega(n)$ pseudoflips and the $O(n \log n)$ upper bound.

So far, the results apply only to minimum or pointed pseudotriangulations. What if one wants to transform any one pseudotriangulation of a point set to any other (including triangulations of the point set). In this case, one needs to increase the repertoire of flips to include edge insertions, i.e., where an edge can be added to a pseudotriangulation as long as it remains a pseudotriangulation. This is essentially the inverse of the edge deletion operation shown in Fig. 17. With this additional operation in hand, the flexibility of pseudotriangulations helps circumvent the $\Omega(n^2)$ lower bound example constructed by Hurtado et al. [64]. Aichholzer et al. [5] showed that any two pseudotriangulations of a given point set P (including full triangulations) can be transformed into each other by applying $O(n \log n)$ pseudoflips or edge-inserting flips.

Open Problem. Closing the gap between the linear lower bound and the $O(n \log n)$ upper bound when only pseudoflips or edge-inserting flips are used remains an open problem.

Furthermore, if one allows vertex insertion and deletion operations, then the upper bound is linear. A vertex insertion operation adds a vertex inside a pseudotriangle and connects it to at least two of the three corners. Vertex deletion operation deletes a vertex of degree two or three provided that the resulting graph remains a pseudotriangulation. The results by Aichholzer et al. [5] are based on pseudoflips in simple polygons. Aichholzer et al. [5] show that one can transform a pseudotriangulation of a simple polygon (with points inside the polygon) into any other with at most a linear number of pseudoflips.

4.4. Flips on 2-dimensional surfaces

In this section, we briefly review several results for the basic edge flip on graphs embedded on surfaces other than the plane. We refer the interested reader to the books [76,81] and to the referenced papers for a more detailed description.

A triangulation of a closed surface is a graph with a given embedding in the surface where every face is a triangle. Drawing parallels from the situations described for the plane in Section 2 and Section 3, the embedding can be combinatorial (or *topological*) or geometric (or *metrical*). In the former case, the edges may be Jordan curves that join adjacent points. In the latter case, edges can only be arcs of geodesics.

In the topological setting, it has been shown that the flip graph of triangulations with n vertices on the sphere, the torus, the projective plane and the Klein bottle are connected [36,82,103]; the same result has been extended by Negami [80] to all closed surfaces provided that n is large enough (a value that depends on the Euler characteristic). The flip graph is also connected (again for n large enough) when one considers *outer-triangulations*, i.e., graphs embedded in such a way that all faces are triangles except the outerface, which contains all the vertices of the graph on its boundary [30,32,33]. These graphs generalize the notion of maximal outerplanar graphs in the plane.

In the geometric setting the situation is more complicated in several respects because the domain that is triangulated may be difficult to define, flips may be infeasible and proper triangulations may not exist. Let us consider the case of the cylinder; cutting along a generatrix, the surface can be represented as an infinite vertical strip (of width 2, say) where the bounding lines are identified and segments inside the strip that can be covered by a vertical strip of width 1 (taking into account the boundary identification) correspond to geodesic arcs on the cylinder. Now consider the quadrilateral *abcd* in Fig. 18: diagonal *ac* cannot be flipped inside the quadrilateral because the geodesic arc *bd* is external to the quadrilateral since *bd* does not fit in a vertical strip of width 1. Now, consider the point set in Fig. 19. When we keep adding segments that correspond to geodesic arcs we may end up with different triangulated domains; in other words, the correspondence that we had in the plane between maximal (planar) graphs defined on a point set and near-triangulations is no longer preserved. It has been shown that the preceding process on the cylinder, for any given point set that spans the whole strip, always ends with some *upper polygon* and some *lower polygon* wrapped around the cylinder. We get unbounded faces above and below these polygons, and the domain inside is triangulated. In addition, the flip graph of triangulations in the region bounded by a prescribed upper and lower polygon is always connected [51,76]. The same holds for the flip graph of *Euclidean polygons* which are the bounded regions delimited by a closed polygonal of geodesic segments.

For the flat torus, i.e., the torus as periodic quotient of the Euclidean plane, the situation is even more strange because there are point sets S such that some maximal sets of geodesic segments defined on S yield triangulations, while some other maximal sets of geodesic segments yield an arbitrarily large number of faces that are not triangles. In this respect one must realize the existence of polygons that do not admit any internal diagonal; an example (that we borrow from [76]) is shown in Fig. 20. Nevertheless, it has been proven that the flip graph of the triangulations of a polygon on the flat torus is either empty or connected [31]. Let us mention finally that for generic surfaces there exists always a metric and a polygon such that the corresponding flip graph is nonempty yet not connected [31].

We conclude this section with another topic that has a related yet different flavor. Let Q be a finite point set in 3-dimensional Euclidean space, and let $\mathcal{S}(Q)$ be the set of triangulated polyhedral surfaces, with vertex set Q, homeomorphic to a sphere. If in one of those surfaces we have an edge xy shared by two triangles axy and bxy, the edge flip operation replaces them by the triangles xab and yab as long as the resulting surface is not self-intersecting (Fig. 21). When surfaces are reconstructed from samples, repeated use of the edge flip operation has been considered as a strategy for optimizing criteria such as mean curvature or total absolute curvature. Unfortunately it has been shown in [3] that the flip graph of $\mathcal{S}(Q)$ under the edge flip operation as defined above may not be connected. In fact, there is even an example of

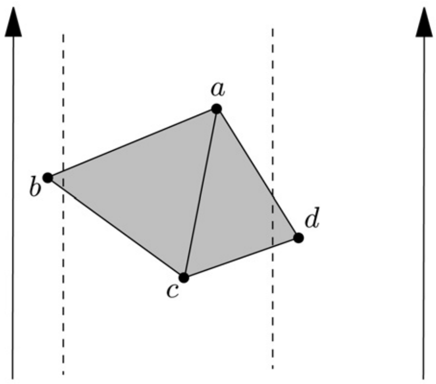


Fig. 18. Diagonal *ac* cannot be flipped inside the quadrilateral because the geodesic arc *bd* is external to the quadrilateral.

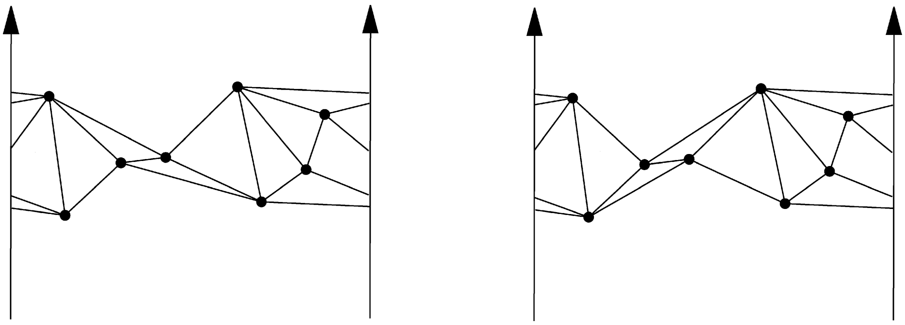


Fig. 19. Maximal graphs on the same point set on the cylinder that yield different triangulated domains.

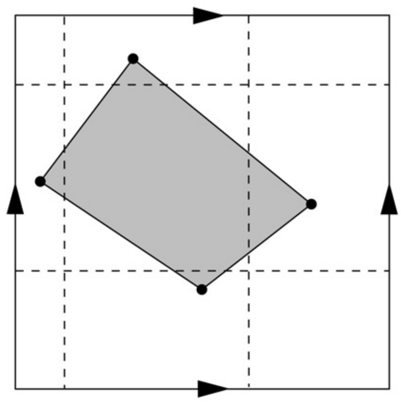


Fig. 20. In the torus, obtained from a square with opposite sides identified, this polygon admits no internal diagonal: the geodesic segments that would join the corners are external.

a polyhedron P with a set Q of 10 vertices, all of which are in convex position, such that no sequence of edge flips allows the transformation of P into the convex hull of Q; this polyhedron is shown in Fig. 22 [3].

5. Applications

Since we are dealing with variations of edge flips in planar graphs (both in the combinatorial and geometric setting), the main applications of edge flips fall under two main categories: enumeration and optimization. Both these application areas are closely related. Flips have played a fundamental role in the enumeration of different types of planar graphs as well as in the computation of planar graphs where some criterion of the graph is optimized. In some cases, when attempting to optimize some criterion, the globally optimal planar graph happens to be the only locally optimal solution. In such cases, flip operations provide a method to compute the global optimal. In other cases, there exist many locally optimal solutions attained via flip operations. In such cases, if locally optimal solutions are suitable for the application at hand, the flip

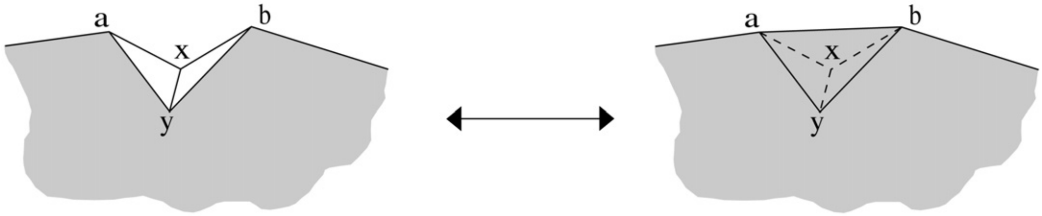


Fig. 21. Edge flip on a polyhedral surface.

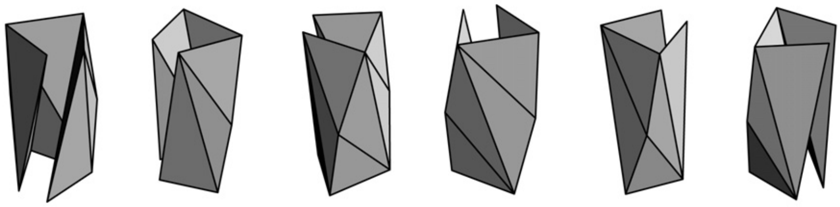


Fig. 22. Several views of the polyhedral surface that cannot be transformed to its convex hull via edge flips.

operation is considered to be a *heuristic* that works well in practical situations. We outline some of the main results in both these application areas.

5.1. Enumeration

Flips have been used as a fundamental tool to enumerate different types of planar graphs. The key connection between edge flips and enumeration is the flip graph itself. Notice that the enumeration of all the distinct planar graphs of a given size and type amounts to the enumeration of the vertices of the flip graph. Thus the approach often taken to enumerate all planar graphs of a given size and type is to traverse the flip graph in some systematic fashion. The technique which is most prevalent in this regard is *reverse search*, a technique proposed by Avis and Fukuda [16] in their seminal paper whose application to the enumeration of various combinatorial structures is highlighted in [17].

We briefly outline the reverse search paradigm as it applies to the flip graph.[8] Let G be the flip graph defined in a particular setting. Let us use triangulations of a given set P of n points in general position with the Delaunay edge flip operation as a running example.[9] Recall that an edge in a triangulation admits a Delaunay flip provided that the union of the two triangles adjacent to the edge forms a convex quadrilateral and the sum of the two angles opposite the edge is at least π. We already noted that enumerating all triangulations of P in the given setting is equivalent to enumerating all the vertices of the flip graph since each vertex of the flip graph represents a distinct triangulation. The reverse search technique is simply an efficient way to visit all the vertices of a particular spanning tree of the flip graph.

The spanning tree of the flip graph is defined and explored in the following way. First, given a vertex v in G, one should be able to enumerate all vertices in G adjacent to v. In our example, this is easy. Let $T(P)$ be a triangulation of P represented by vertex v in G. Each neighbor of v in G is the triangulation that results by applying a Delaunay edge flip (or the reverse of a Delaunay edge flip) to one edge in $T(P)$. Next, one vertex of the flip graph is selected to be the root of the spanning tree which we denote by v_r. In our example, we let v_r be the Delaunay triangulation of P. The root must be unique so assume for simplicity that no three points are collinear since that makes the Delaunay triangulation unique. Next, a function f must be defined on the vertices of G such that $f(v_r) = v_r$ and for any other vertex $v \in G$, there exists some integer $k > 0$ such that $f^k(v) = v_r$ (i.e., repeated applications of f starting at v generates a path from v to v_r in G). This function is called the *local search function*. Label each of the $\binom{n}{2}$ edges defined by pairs of points in P by e_i where i is the rank of the distance pair in increasing order, with ties broken arbitrarily. Thus, e_1 represents a closest pair of points in P and $e_{\binom{n}{2}}$ represents a furthest pair. In our example, the local search function is defined as follows. Let $v \neq v_r$ be an arbitrary vertex of G representing triangulation $T(P)$. Since $T(P)$ is not the Delaunay triangulation, it must have at least one edge that admits a Delaunay flip. Of all these edges, flip the edge of lowest rank and call this triangulation $T'(P)$ represented by vertex v' in G. Define $f(v) = v'$.

With the ability to enumerate the neighbors of a vertex in the flip graph as well as a local search function defined, reverse search traverses a spanning tree of the flip graph. The algorithm starts at the root in a depth-first fashion by following the edges defined by the local search function in *reverse order*, from whence the name of the technique is derived. So in our example, one would start with the Delaunay triangulation. Consider all edges of the triangulation that admit a

[8] The interested reader can refer to the tutorial by Avis [15] highlighting the salient points of the technique.
[9] Full details of this example can be found in [17].

reverse Delaunay flip, i.e., the union of the two triangles adjacent to the edge forms a convex quadrilateral and sum of the two angles opposite the edge is *at most* π. Of all these edges, select the one of lowest rank. Flip it and report the resulting triangulation. This is the first step in the depth-first search of the spanning tree of the flip graph.

As noted above, there are two conditions that need to be satisfied in order to apply reverse search on the flip graph. First, one needs to provide a mechanism to enumerate all the neighbors of a given vertex in the flip graph. In our example, this was easy because when one flips an edge out of the Delaunay triangulation, they have a triangulation that is distinct from the Delaunay triangulation. However, in the combinatorial setting, this is no longer as simple since by flipping a given edge of a triangulation, one may get another triangulation that is isomorphic to the original. Therefore, this is not an edge in the flip graph if we discount self loops. The second component is to define a unique global optimum for the local search function.

We outline some of the types of planar graphs that are successfully enumerated using the flip graph. Avis and Fukuda [17] use flips and reverse search to enumerate all triangulations of a point set,[10] all spanning trees of a graph, all straight-line plane spanning trees of a set of n-points and all connected subgraphs of a graph. Bespamyatnikh [23] enumerates all near-triangulations on a point set more efficiently. Avis [14] enumerates all rooted 2-connected and 3-connected triangulations without repetition using the flip graph and reverse search. A triangulation is rooted provided that the outerface is labelled and the order of the vertices around the outerface remains fixed. Aichholzer et al. [7] show how to enumerate plane straight-line graphs, plane spanning trees and connected plane straight-line graphs using flips and Gray codes. Brönnimann et al. [28] show how to enumerate all pointed pseudotriangulations defined on a point set using flips and Bespamyatnikh [24] improves on the efficiency by using reverse search. Avis et al. [18] generalizes this to planar Laman graphs, which are closely related to pseudotriangulations.

Given the flip graph defined for a particular class of planar graphs and a type of edge flip, one can also use the flip graph to generate a random instance of a planar graph in the given class by performing a random walk on the flip graph. In particular, if a flip graph G_f has maximum degree Δ, consider a random walk in which transitions are from node u to an adjacent node with uniform probability λ/Δ, while we would stay at node u with probability $1 - \lambda \cdot \mathit{degree}(u)/\Delta$, where $\lambda < 1$ is any positive constant. Then, if G_f is connected, the corresponding Markov chain has uniform stationary distribution. In other words, starting at any vertex in G_f, this simple random walk strategy would allow us to reach every vertex in G_f with the same limit probability, $1/|G_f|$. This fact emphasizes the importance that the flip operation yields a connected flip graph.

Markov chains techniques are used to analyze how many steps in the random walk on the flip graph are sufficient to approach the stationary distribution; this corresponds to the *rate of convergence* of the Markov chain. Chains that converge quickly are said to have the *rapidly mixing* property. The interested reader should consult the treatises on this topic like [74,96,99]. It has been proved that the flip graph of the triangulations of a convex polygon has the rapid mixing property [78]; nevertheless, in most instances, it is difficult to prove anything about the convergence rate, however, this does not prevent one from using a random walk on the flip graph as a heuristic.

5.2. Optimization

So far we have discussed how flips have been used to enumerate different types of planar graphs as well as to generate a random instance of a planar graph. The next natural direction to explore is how to generate a planar graph that optimizes some given criterion. Much of the work in this area has taken place in the context of mesh generation (i.e., the geometric setting). The reader may consult the surveys by Bern [20], Bern and Eppstein [22].

For a given quality measure, a *good flip* in a near-triangulation consists of flipping a diagonal of a convex quadrilateral Q when the new diagonal gives a better triangulation of Q. This is called a *local improvement*, and for criteria such as maximizing the minimum angle one can start at any triangulation and keep performing local improvements. The sequence always leads to the optimum which is the Delaunay triangulation [70].

The result by Lawson [70] is the seminal paper in this area where he showed how to use flips to convert any near-triangulation of a given point set into the Delaunay triangulation with $O(n^2)$ flips. In fact, flips can be used to compute the *Constrained* Delaunay triangulation (CDT) of a point set [29,72,93,104]. Given a point set P and a disjoint set of line segments S whose endpoints are in P, the CDT is a near-triangulation of P such that $S \subset$ CDT and every edge xy in CDT that is not in S has the property that there exists a circle with x and y on the boundary such that no point of P inside the circle is visible to the edge xy. We say that a point z is visible to an edge ab with respect to S provided that there is a point w on ab such that the segment wz does not intersect any of the line segments in S. The Delaunay triangulation and the Constrained Delaunay triangulation optimize many different criteria such as minimizing the largest circumcircle, minimizing the angle vector,[11] and maximizing the minimum angle to name a few. In fact, the third criterion is implied by the second one. See Okabe et al. [83] for a comprehensive list of all the criteria that these triangulations are known to possess.

The success of edge flips for optimizing the criteria possessed by the Delaunay and Constrained Delaunay triangulations have led to the use of flips to optimize other criteria such as vertex degree [46], maximum angle [54], or total edge length

[10] As noted above using the Delaunay triangulation.

[11] The angle vector of a triangulation is the list of all the angles of the triangles listed in sorted order from smallest to largest.

[105]. Unfortunately, it seems that the Delaunay triangulation is an anomaly with respect to optimization. All the criteria for which the DT is the optimal, the criterion possesses the property that no other triangulation is a local optimal solution. The only local optimal solution is the DT which is also the global optimal solution. For optimizing other criteria, edge flipping in near-triangulations has very often the limitation that the local rule for optimization causes cycles or gets stuck in a local optimum. This may be overruled by combining the flip with other techniques, like simulated annealing, or by using more powerful operators.

An example of the latter is the *edge insertion* technique: a new edge e is tentatively introduced; all the edges it crosses are removed and the cleared-out area greedily retriangulated by adding ears; if this gives a better triangulation, iterate until no more improvements are possible, else back out and try again—a more detailed description is given in the survey [22]. Obviously this method generalizes the edge flip and is less prone to get stuck, but leads in general to slower running times. In particular, this method allows the construction of the near-triangulation that minimizes the maximum angle, as shown by Edelsbrunner et al. [41]; this also holds for several natural criteria, such as maximizing the minimum height of the triangles or minimizing the maximum distance of circumcentre to triangle, as proved in a subsequent paper by Bern et al. [21].

We conclude this section with a brief mention of yet another domain in which edge flips have been used for the purpose of optimization, namely, surface interpolation. In general, the goal is to construct a surface that contains a given set of data points and values, a situation that arises in many applied scientific fields such as terrain modelling, computer-aided geometric design or medical imaging. The interested reader can check the vast literature on multivariate approximation and interpolation (for example [40]). One possible approach to address this problem is as follows: Given a set of planar data points (x_i, y_i) with given values z_i, construct a near-triangulation of the data points and lift the resulting triangles to the corresponding points (x_i, y_i, z_i); in this way we obtain what is called a *piecewise linear interpolation*. It is clear that in this situation the quality of the lifted triangles and the way in which they connect together to form a surface can be more important than the quality of the triangles that constitute the mesh on the plane. This is why these kinds of triangulation methods are referred to as *data dependent triangulations*. Several criteria can be considered, from the angle between normals (with min-max, min-sum, or min-sum of squares) to several functionals associated to discrete curvature values. A widely used approach for optimizing many criteria is to start with some triangulation and use the edge flip as a local improvement strategy, until a locally optimal triangulation is found [12,37–39,89]; in general that will not be the best global solution, but these heuristics work quite well and can again be combined with techniques for escaping local optima. It is worth mentioning here a surprising result proved by Rippa [88] that regardless of the data elevation values, the Delaunay triangulation of the data points always yields a surface that minimizes a certain energy value, the integral of the gradient squared, called the *roughness*. Therefore for this criterion, the edge flip strategy leads to a global optimum, a situation that is quite exceptional. Finally, let us mention that other methods like the edge insertion technique have also been used in the context of interpolation. For example, Bern et al. [21] proved that their paradigm gives a polynomial time algorithm for finding the minimum slope interpolating surface, i.e., the one that minimizes the maximum slope among the lifted triangles, taken for each one in the direction of the steepest descent.

6. Concluding remarks

We have presented an overview of the edge flip operation and some of its variants as they pertain to planar graphs in different settings. There are many open problems remaining in this area that have been highlighted throughout the article. We believe that a fundamental open issue that requires further study is to compute shortest paths in the flip graph. Currently, almost all approaches for transforming a given planar graph into another via some type of flip operation performs the transformation via a canonical planar graph in the given class. Such an approach will never lead to the computation of the minimum number of flips required to complete the transformation. Notable exceptions are the results by Hanke et al. [53] and Eppstein [44], both of which make progress towards the resolution of this issue in the geometric setting.

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