MODELING WEEK AND STUDY GROUP MEETING ON INDUSTRIAL PROBLEMS A REPORT PRESENTED TO THE SGMIP ON THE PROBLEM OF FISH-FEEDING

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MODELING WEEK AND STUDY GROUP MEETING ON INDUSTRIAL PROBLEMS

A REPORT PRESENTED TO THE SGMIP
ON THE PROBLEM OF FISH-FEEDING

SOUVIK ROY

18th March 2011

TEAM MEMBERS

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Abstract

One Growth Industry in Scotland over recent years has been fish farming. Fish are raised in large cages kept within sea inlets,estuaries or lakes.The fish are fed with food pellets which are scattered onto the water above the cages. Ideally the pellets sink within 15 seconds and can be eaten by the caged fish. Sometimes there have been problems at fish farms with the fish food floating for too long, whereas in simple laboratory experiments with the same batch of pellets, throwing a handful onto water in a bucket, the pellets were observed to sink quick. The aim is to understand why **??**

INTRODUCTION

As said in the abstract we wish to understand the role of surface tension, the difference between laboratory tests and farming experience, and what sort of balance between size, density and surface tension should lead to prompt sinking.

One hypothesis has been that the surface tension of the water plays a major role. So we would try to see how this effect can help us to determine whether or not an object, such as fish food pellet, will float on water or not.

When we consider a fish food pellet, we consider them as circular cylinders with lengths greater than their diameters. But human nature is always to look for simplicity. So before we consider such cylindrical pellets, we first consider a simple case with respect to the symmetry of the pellet. So we consider a spherical pellet. We start by writing down the standard equations for the free surface of water z = h(r) where z is measured vertically downwards from the undisturbed water level and r is the distance measured along the horizontal axis, in the plane of the pellet cross-section, measured from the centre of the circular cross- section i.e. on the water surface z = h(r) = 0 at r = $\pm\ \infty$.

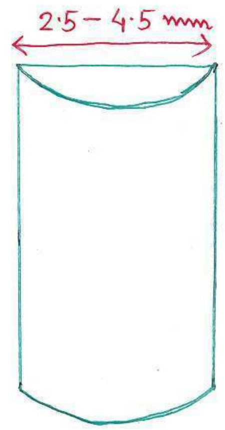


Fig:-1 Schematic Diagram of a vertical cross-section of a fish-food pellet.

SPHERICAL PELLET WITHOUT SURFACE TENSION

MODELLING THE PROBLEM

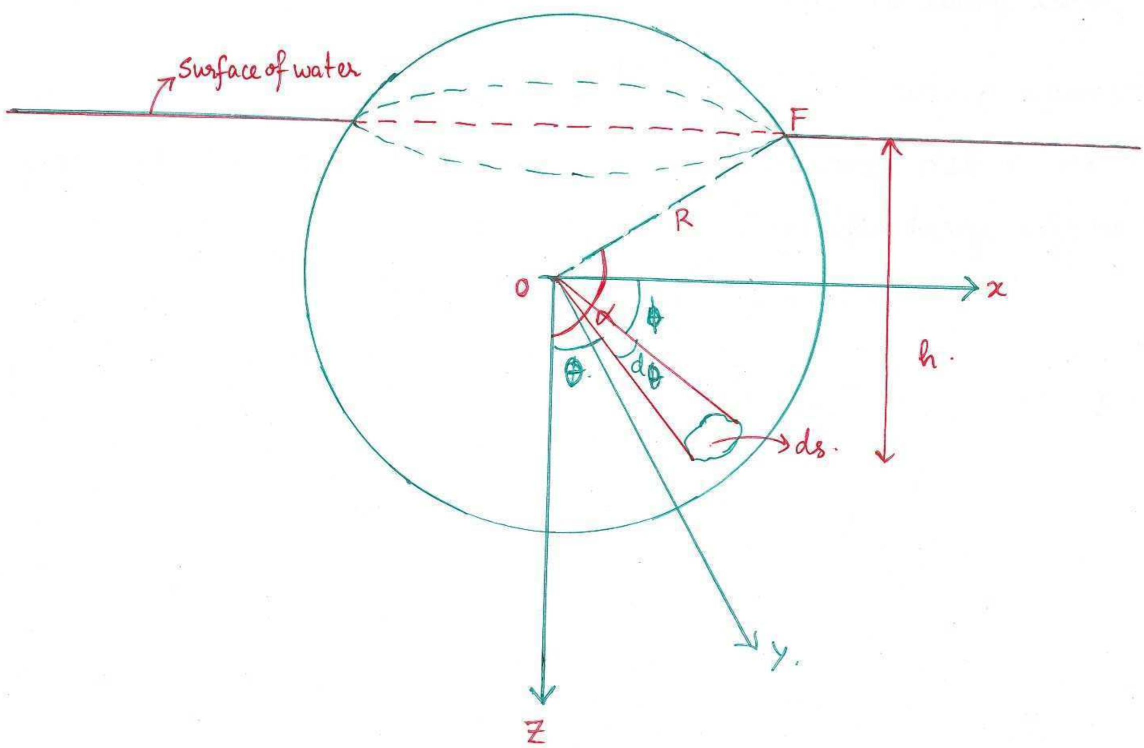


Fig:-2 Vertical Cross-Section of the spherical pellet with α being the angle between OZ & OF where F is the point where the free surface of water comes in contact with the pellet.

Using cylindrical coordinates if we consider an elementary volume element with surface area dS, then the force exerted by the liquid column of height h on the elementary element dS in the radial direction is given by:-

dF = dS.h.ρ_w.g where:-

h = cosθ - cosα
θ is the angle made by the element with z-axis.
α is the angle indicated in the diagram.
ρ_w is the density of water.
g is the acceleration due to gravity.

Now the pellet being symmetric, the component of dF in the x-y plane cancel out each other. So the only contributing force on the body is the force acting on the element dS in the -ve z direction i.e. upwards.
That force df is given by

df = dS.h.ρ_w.g.cosθ.

$\therefore$ The total buoyant force acting on the pellet is given by
B = $\int$ dF

$$= \rho_w.g \int_{\theta=0}^{\alpha} \int_{\phi=0}^{2\pi} R^3 sin\theta(cos\theta - cos\alpha)cos\theta \, d\theta d\phi$$

$$= (\frac{\overline{M}}{\nu 4} cos2\alpha - \frac{\overline{M}}{4} + \frac{M}{3} - \frac{M}{3} cos^3\alpha), M = 2\pi R^3 \rho_w g$$

$$\overline{M} = 2\pi R^3 cos\alpha \rho_w g \qquad \longrightarrow (a)$$

R is the radius of the pellet

Now by the Archimedian Principle, total buyant force acting on the body is equal to the weight of liquid displaced when a body floats in a liquid.

So let V_i be the volume of the immersed part of the pellet and let V_B be the volume of the pellet.

Now B = ρ_wgV_i

$$\Rightarrow V_i = \frac{B}{\rho_w g} \qquad \longrightarrow (1)$$

Also

$$V_B = \frac{4}{3}\pi R^3 \qquad \longrightarrow (2)$$

Let $\frac{\rho_B}{\rho_W} = \rho \qquad \longrightarrow (3)$

$$\therefore \frac{W_B}{B} = \frac{\rho_B V_B g}{\rho_W V_i g} = \frac{\rho_B V_B}{\rho_W V_i}$$

For floatation $\frac{W_B}{B} = 1$

$$\Rightarrow \frac{\rho_B}{\rho_W} = \frac{V_i}{V_B}$$

$$\Rightarrow \frac{V_i}{V_B} = \rho \qquad \cdots By(3)$$

$\therefore$ Using (a),(1),(2) we get,

$$(\frac{3}{8}\cos 2\alpha - \frac{3}{8})\cos\alpha + \frac{1}{2} - \frac{1}{2}\cos^3\alpha = \rho$$
$$\Rightarrow 3\cos 2\alpha\cos\alpha - 3\cos\alpha + 4 - 4\cos^3\alpha = 8\rho$$
$$\Rightarrow 3(2\cos^2\alpha - 1)\cos\alpha - 3\cos\alpha - 4\cos^3\alpha + 4 - 8\rho = 0$$
$$\Rightarrow 2\cos^3\alpha - 6\cos\alpha + (4 - 8\rho) = 0 \qquad \longrightarrow (4)$$

Now if $\rho > 1$ i.e if the pellet density is heavier than water density we have 4 - 8 $\rho < 0$

So eq$\underset{..}{^n}$ (4) has atleast 1 positive root in cosα

But cosα = 1 $\Rightarrow$ in (4) 2 cos$^3\alpha$ - 6 cosα + (4 - 8ρ) < 0
& cosα = -1 $\Rightarrow$ in (4) 2 cos$^3\alpha$ - 6 cosα + (4 - 8ρ) < 0
& cosα = 3 $\Rightarrow$ in (4) 2 cos$^3\alpha$ - 6 cosα + (4 - 8ρ) < 0

So the only positive root for eq$\underset{..}{\underline{n}}$ (4) in cosα lies in (1,3)
Also the discriminant of the eq$\underset{..}{\underline{n}}$ is given by

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$$

where a = 2, b = 0, c = -6, d = 4-8ρ

= 6912(ρ-ρ^2)< 0 if ρ >1

So the eq$\underset{..}{\underline{n}}$ (4) has only 1 real root which is positive & lies in (1,3)

$\Rightarrow \cos\alpha \in (1,3) \longrightarrow\!\!\longleftarrow$

So the pellet will totally sink if $\rho > 1$ in absence of surface tension.

SPHERICAL PELLET WITH SURFACE TENSION

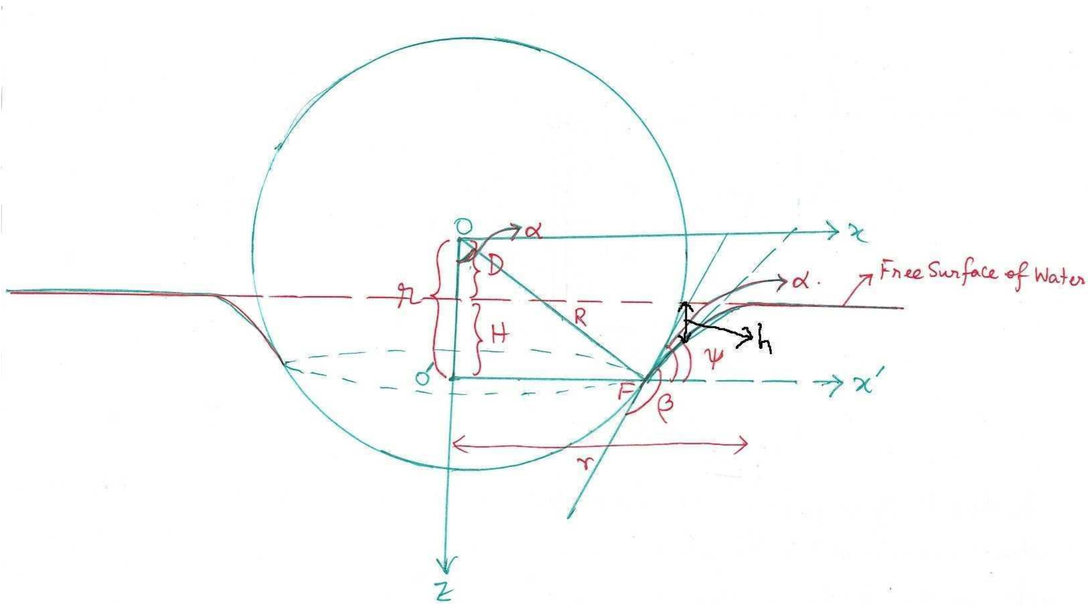


Fig:-3 Vertical Cross-Section of the spherical pellet with α being the same angle as before but now surface tension comes in picture.

MODELLING THE PROBLEM

Now we consider the case where the pellet floats on the surface of the liquid due to surface tension. We consider the axis O'Z in vertically downward dir$^{\underline{n}}$, where O' is the centre of the circle where the liquid surface comes in contact with the pellet, and O'x' is the horizontal axes. The angles will be measured with anti-clockwise direction being the positive direction.

Let ν be the surface tension which acts on the pellet in the tangential direction as shown in fig (4).

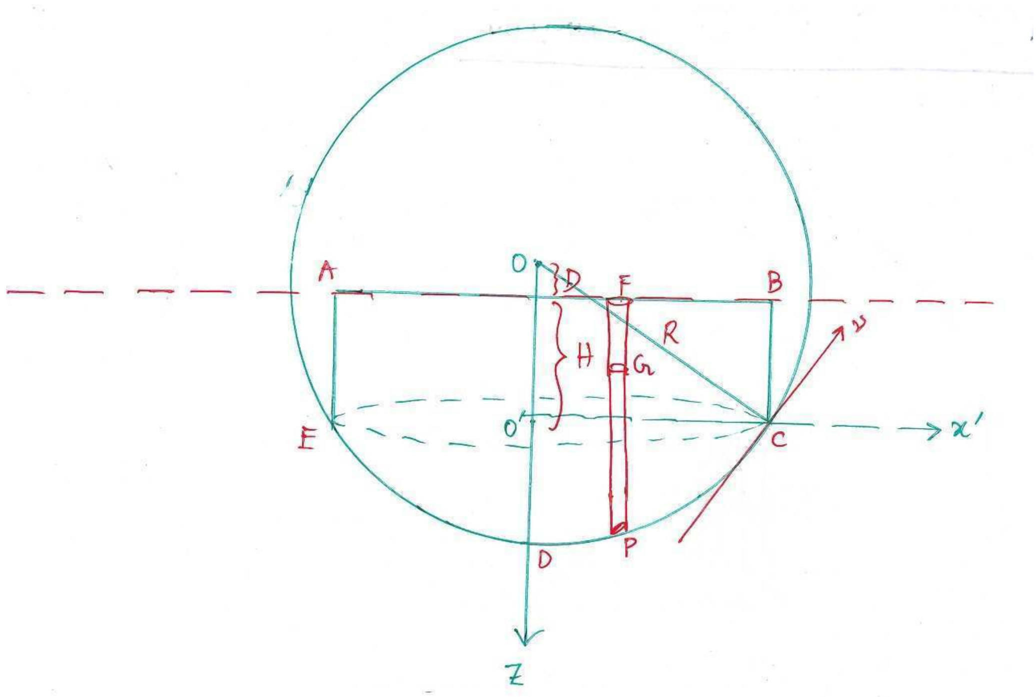


Fig:-4

So due to symmetry, the forces due to surface tension in the horizontal direction cancel out each other & so the resultant force due to surface tension acts on the body in the direction opposite to O'Z So the total surface tension force acting on the pellet is given by

STF = -ν sinψ 2πR sinα(ψ is in fig(4))

Now ψ = π - (α+β) where β is the contact angle.

$\Rightarrow$ STF = -ν sin(β + α)2πR sinα

The total buoyant force acting on the pellet is the sum of the STF and the force due to the volume of the pellet given by ABCDE in fig(5). Let the volume be V. We consider an elementary volume element dV given by FGP in fig(5). So dV = $\sqrt{R^2 - r^2}$ - D rdrdϕ

where r is the distance of that element from O'Z'

$$\therefore V = \int dV$$

$$= \int_{\phi=0}^{2\pi} \int_{r=0}^{R\sin\alpha} \sqrt{R^2 - r^2} - D dr d\phi$$

$$= 2\pi R^2 [\frac{R}{3}(1 - \cos^3\alpha) - \frac{D}{2}\sin^2\alpha]$$

So force due to ABCDE =

$$2\pi R^2[\frac{R}{3}(1 - cos^3\alpha) - \frac{D}{2} sin^2\alpha]\rho_w g$$

So the total buoyant force B =

$$2\pi R^2[\frac{R}{3}(1 - cos^3\alpha) - \frac{D}{2} sin^2\alpha]\rho_w g + (-\nu sin(\beta + \alpha).2\pi R sin\alpha)$$

By Archimedian Principle, B = W_B

$$\Rightarrow R[\frac{R}{3}(1 - cos^3\alpha) - \frac{D}{2} sin^2\alpha] - l^2 sin(\alpha + \beta) sin(\alpha) = \frac{2}{3}R^2\rho$$

$$where \quad l^2 = \frac{\nu}{\rho_W g} \;\; \& \;\; \rho = \frac{\rho_B}{\rho_W} \longrightarrow (1)$$

Also we have H + D = R $\cos\alpha \longrightarrow$ (2)

So we have 3 unknowns H,D,α and 2 equations. To get the 3^{rd} equation we have to use the property of curvature of the liquid free surface. For this we use the
YOUNG-LAPLACE'S EQUATION which states that:

When all the forces are balanced, then

Δp = $\nu(\frac{1}{R_x} + \frac{1}{R_y}) \longrightarrow$ (3)

where
1) Δp is the pressure difference across the surface of the liquid.
2) ν is the surface tension.
3) R_x and R_y are the radii of curvature in each of the axes that are parallel to the surface.

Now $\frac{1}{R_x} = [\frac{h''}{(1+h'^2)^{\frac{3}{2}}}]$ & $\frac{1}{R_y} = [\frac{h'}{(1+h'^2)^{\frac{1}{2}} r}]$ & $h' = \frac{dh}{dr}$, $h'' = \frac{d^2h}{dr^2}$.

where h is as given in fig(3)measured from the free surface of water and r is the distance of the free surface of height h from O'Z as in fig(3).

So (3) gives hρ_Wg = ν [$\frac{h''}{(1+h'^2)^{\frac{3}{2}}}$ + $\frac{h'}{(1+h'^2)^{\frac{1}{2}}r}$]
$\Rightarrow$ l^2[$\frac{h''}{(1+h'^2)^{\frac{3}{2}}}$ + $\frac{h'}{(1+h'^2)^{\frac{1}{2}}r}$] = h.

with the conditions:-
h(∞) = 0 = h$'$(∞),
h(Rsinα) = H,
h$'$(Rsinα) = tan(ψ) = -tan(α+β).

It would have been really nice if we could solve (4) exactly but by the look of things it might not be possible. So we should consider an object which gives (4) in a simpler form. Since our original shape of the pellet was assumed to be a circular cylinder with a larger length in comparison to its radius, so we consider that case with and without surface tension.

LONG CYLINDRICAL PELLET WITHOUT SURFACE TENSION

MODELLING THE PROBLEM

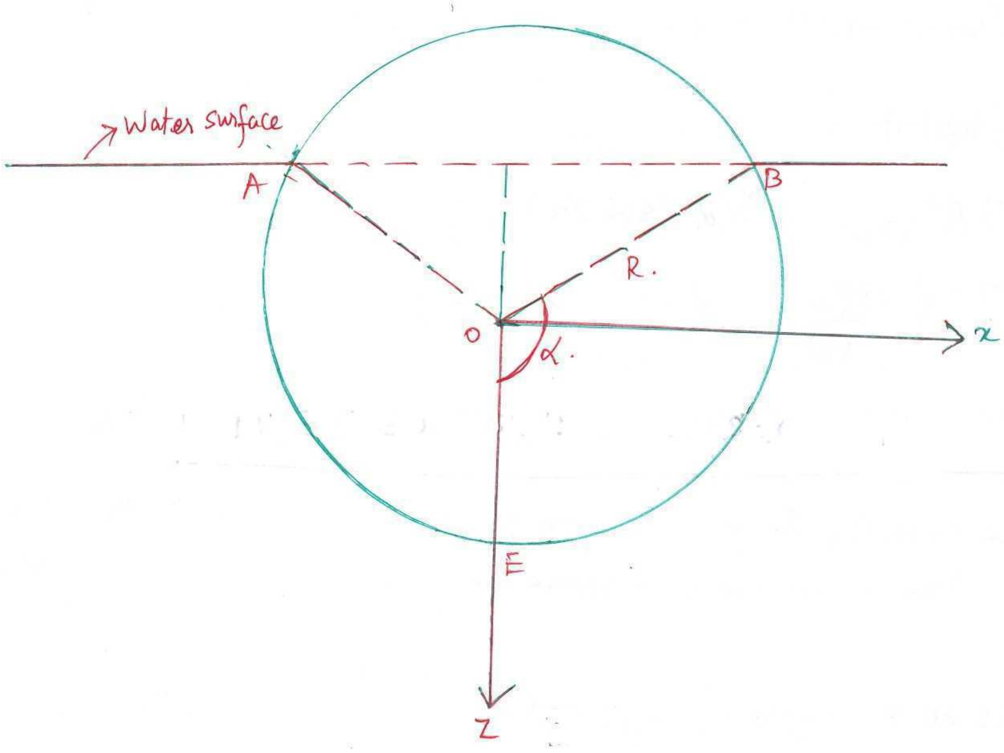


Fig:-5

We consider a 2-d vertical cross-section of the cylinder. In this case we do not have surface tension. So by Archimedian principle, total buoyant force B acting on the pellet equals to the weight of the immersed part of the pellet. Since we have a long cylinder, so we consider all forces per unit length of the cylinder.

So area of the immersed part of the pellet is the sum of the area of ΔAOB & area(AEB).

Now the area of Δ AOB = - Rcosα.Rsinα

$$= \text{-R}^2 \ \tfrac{\sin 2\alpha}{2}.$$

Area of AEB = 2.$\frac{\pi R^2}{2\pi}$.α = α R^2

$\therefore$ Total Area of the immersed portion of the liquid is R^2.(α-$\frac{\sin 2\alpha}{2}$)

$\therefore$ Force exerted by the liquid on the pellet per unit length
= R^2.(α-$\frac{\sin 2\alpha}{2}$)ρ_W.g

Weight / unit length of pellet = $\pi R^2 \rho_B g$.

So for floatation,

$$\pi R^2 \rho_B.g = R^2(\alpha - \sin 2\alpha)\rho_W.g$$

$$\Rightarrow \pi\rho = \pi\frac{\rho_B}{\rho_W} = \alpha - \frac{\sin 2\alpha}{2}$$

LONG CYLINDRICAL PELLET WITH SURFACE TENSION

MODELLING THE PROBLEM

When we consider surface tension then we follow the calculations as in case 2. The figures will be the same, only we do not consider volume element. Instead we consider area element.

So if dS be an area elementgiven by FGP, ds = $\sqrt{R^2 - r^2}$ - D dr

$\therefore$ Area of the pellet with water contact = $\int dS$
$= \int_{r=0}^{R\sin\alpha} \sqrt{R^2 - r^2} - D\,dr$
$= \frac{R^2 \sin 2\alpha}{2} + R^2\alpha - 2DR\sin\alpha$

The surface tension force / unit length = -2$\nu\sin(\alpha + \beta)$

W'_B = Weight of the cylinder / unit length = $\pi R^2 \rho_B g$

$\therefore$ B' = The total buoyant force acting on the pellet / unit length of the pellet

=[$\frac{R^2 \sin 2\alpha}{2}$ + R^2 α - 2DRsinα]ρ_Wg - 2νsin(α+β).

By Archimedian Principle, B' = W$_{B'}$

$\Rightarrow$ R[$\frac{R\sin 2\alpha}{2}$ + Rα - 2Dsinα] - 2l^2 sin(α+β) = πR$^2\rho$. $\longrightarrow$ (1)

Also we have H+D=Rcosα. $\longrightarrow$ (2)

Now again we have 3 unknowns α,H,D & 2 equations. So as in case(2), we use the **YOUNG-LAPLACE'S EQUATION** to get the 3rd equation. Since we have a 2-d figure so we have to determine the curvature in only 1 direction which is given by:-

K = $\frac{h''}{(1+h'^2)^{\frac{3}{2}}}$, where h,h$'$,h$''$,r have their usual meanings as in case(2).

So by Young-Laplace's Equation:-

$l^2[\frac{h''}{(1+h'^2)^{\frac{3}{2}}}] = h.$

with the conditions:-
$h(\infty) = 0 = h'(\infty)$,
$h(R\sin\alpha) = H$,
$h'(R\sin\alpha) = \tan(\psi) = -\tan(\alpha+\beta)$.

$\Rightarrow \frac{d(h'')}{(1+h'^2)^{\frac{3}{2}}} = \frac{2h}{l^2}$ dh

Using the first condition we get:-

$\frac{-2}{(1+h'^2)^{\frac{1}{2}}} = \frac{h^2}{l^2} - 2$.

Now $h(R\sin\alpha) = H$ & $h'(R\sin\alpha) = -\tan(\alpha+\beta)$.

$\Rightarrow \sqrt{\cos^2(\alpha+\beta)} = -\frac{H^2}{2l^2} + 1$.

When there is negligible surface tension, H = 0, $\alpha + \beta = \pi$.

$\Rightarrow -\cos(\alpha+\beta) = -1$.

$\therefore \sqrt{\cos^2(\alpha+\beta)} = -\cos(\alpha+\beta)$.

$\Rightarrow -\cos(\alpha+\beta) = -\frac{H^2}{2l^2} + 1$.

$\Rightarrow \frac{H^2}{2l^2} = 1 + \cos(\alpha+\beta)$.

$\Rightarrow H = \pm\, 2l \cos(\frac{\alpha+\beta}{2})$.

To determine the sign, when $H > 0$, $\frac{\alpha+\beta}{2} < \frac{\pi}{2} \Rightarrow \cos(\frac{\alpha+\beta}{2}) > 0$.

& when $H < 0$, $\frac{\alpha+\beta}{2} > \frac{\pi}{2} \Rightarrow \cos(\frac{\alpha+\beta}{2}) < 0$, for $\frac{\alpha+\beta}{2}$ sufficiently close to $\frac{\pi}{2}$

$\Rightarrow H = 2l\cos(\frac{\alpha+\beta}{2}) \longrightarrow (3)$

So (1),(2),(3) $\Rightarrow$

α - sinαcosα + 4L sinαcos$(\frac{\alpha+\beta}{2})$ - 2L^{2}sin$(\alpha+\beta)$ = $\pi\rho \longrightarrow$ (4)

where L = $\frac{l}{R} = \frac{1}{dimensionless\, size\, of\, pellet}$

So when there is negligible surface tension, H = 0 & $\alpha + \beta = \pi$

$\therefore$ (4) $\Rightarrow$ α - sinαcosα = $\pi\rho$ which gives equation(1) in case(3).

NUMERICAL CALCULATION FOR DENSITY

Now for given L & β we get ρ in terms of α from (4). But this cannot be solved explicitly.So we resort to numerical methods to find the solution for $\rho = \rho(\alpha)$. We take L = 0,1,2 & for each value of L, we take β, the contact angle, $= 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$.Then we plot ρ vs α where we discretise the interval of α i.e. [0,π] into 50 parts.

RESULTS OBTAINED

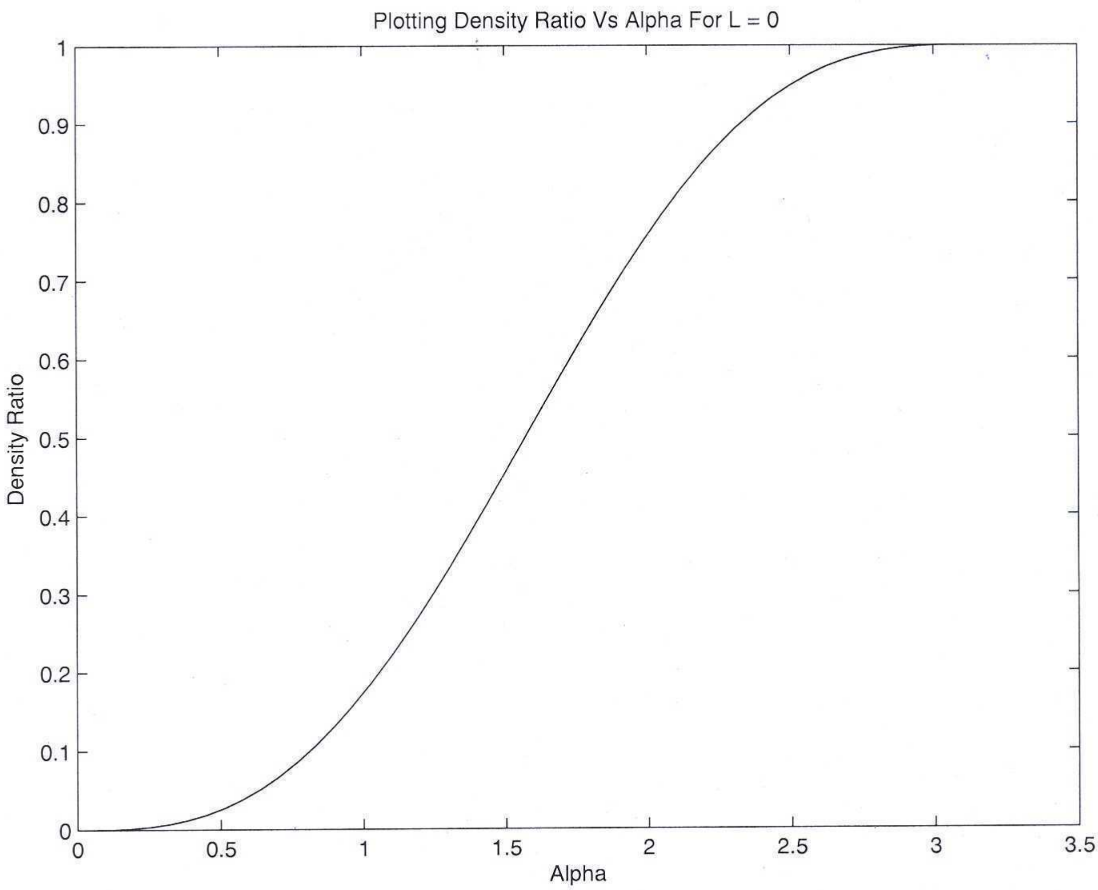


Fig:-6

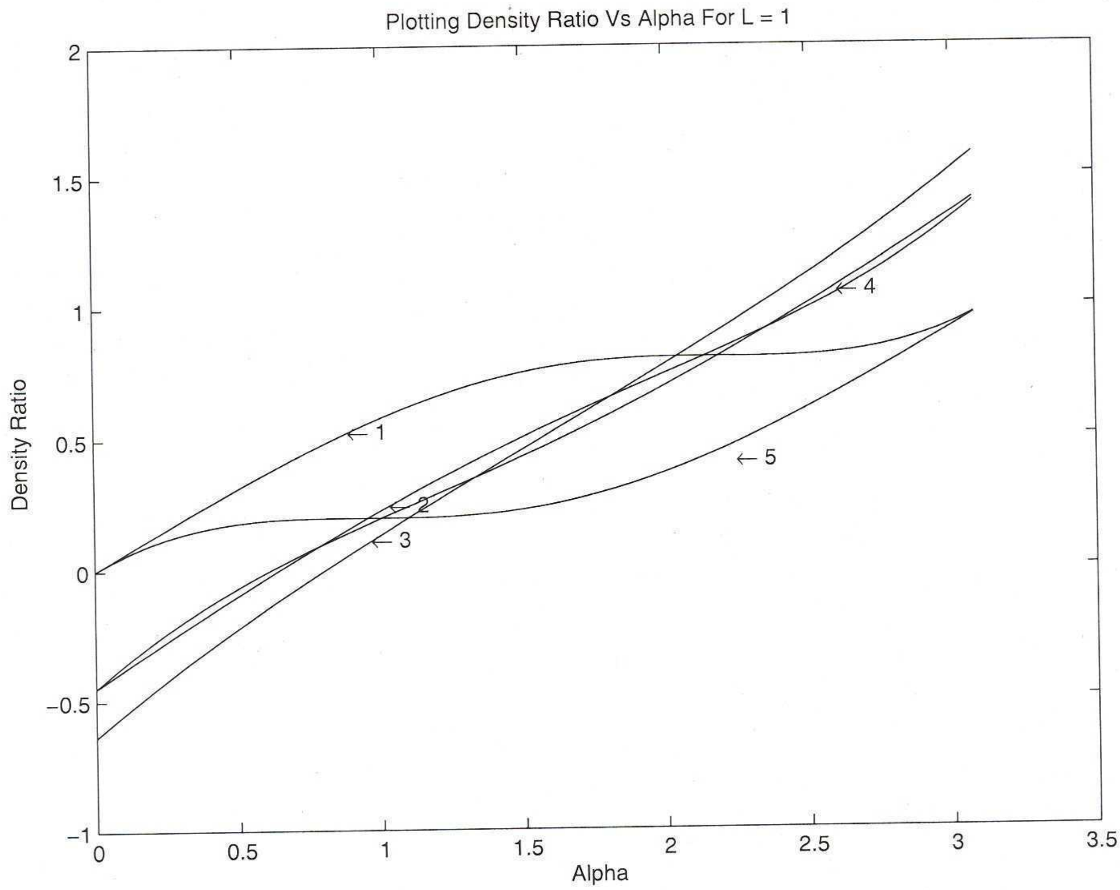
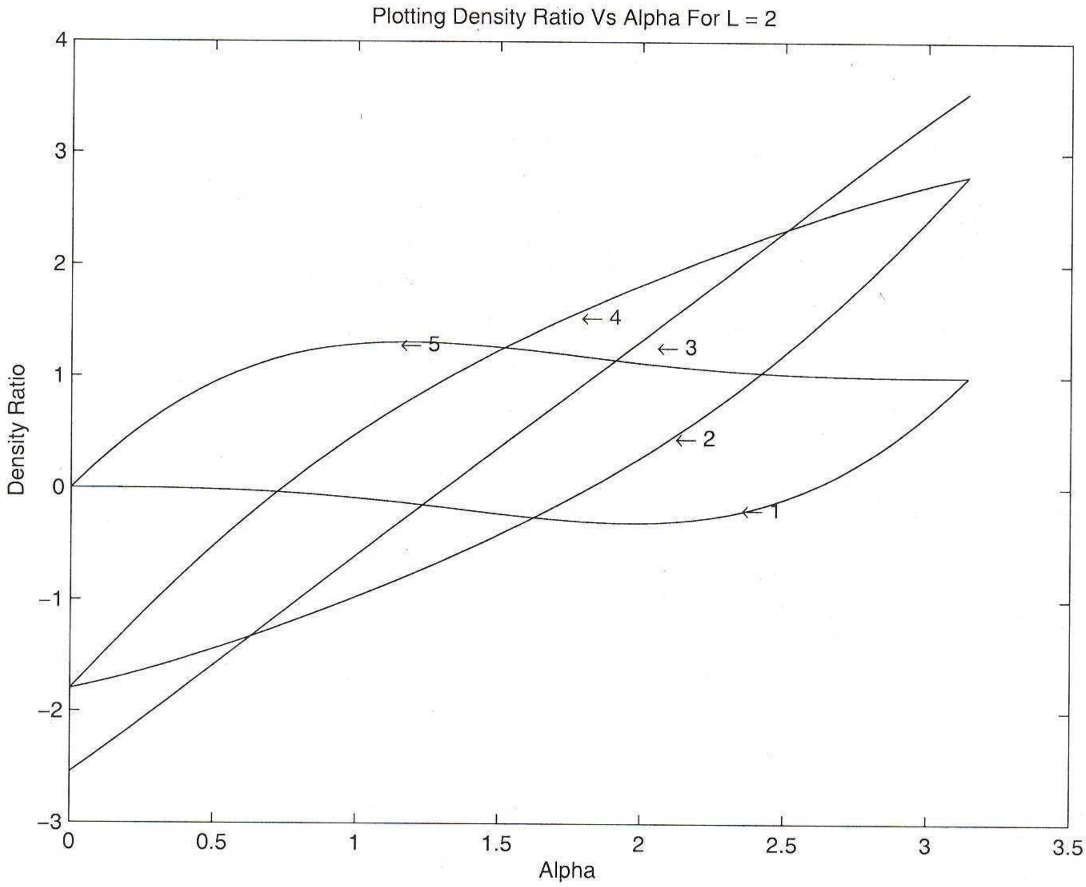

Plotting Density Ratio Vs Alpha For L = 1
Density Ratio
Alpha
2
1.5
1
0.5
0
−0.5
−1
0
0.5
1
1.5
2
2.5
3
3.5
← 1
← 2
← 3
← 4
← 5

Fig:-7



Plotting Density Ratio Vs Alpha For L = 2
Density Ratio
Alpha
4
3
2
1
0
−1
−2
−3
0
0.5
1
1.5
2
2.5
3
3.5
← 5
← 4
← 3
← 2
← 1

Fig:-8

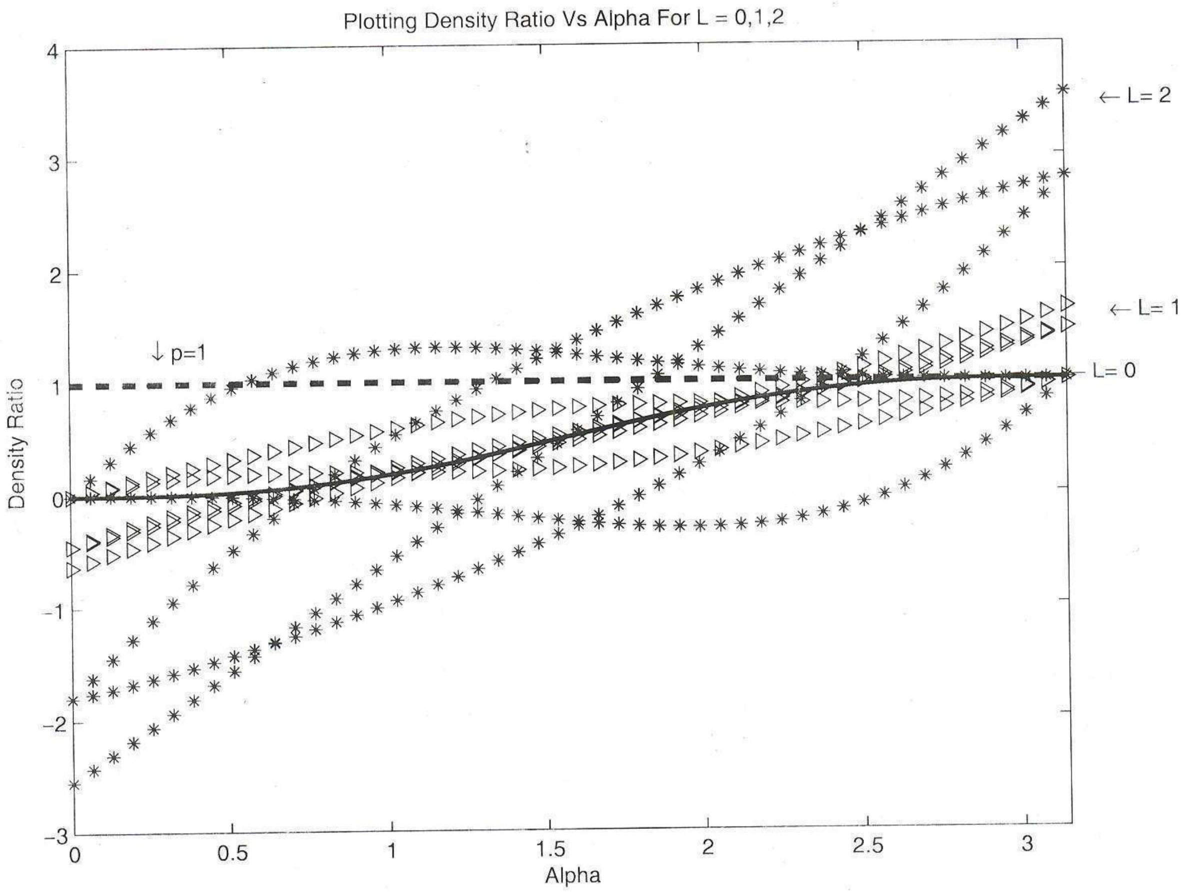


Fig:-9

From the figures of plot of ρ vs α we see that L α floatation of bodies for different densities. The more the value for L, the more chance that a body much heavier than water will float on it. On the graph 1,2,3,4,5 represents different values of β in increasing order.

Since L α $\frac{1}{R}$, where R is the radius of cylinder, so to increase chances of floatation we should decrease the radius of the cylinder as other factors in L like ν, β cannot be changed by much.

<u>CONCLUSIONS</u>

Before starting the problem we had asked 3 questions:-
1) role of surface tension.
2) Balance between size,density & surface tension leading to prompt sinking.
3) Difference between laboratory and farming experiences.

So far we have answered questions (1) & (2). To answer (3), we note that R α sinking. So when we throw the pellets into a bucket of water, we usually have the pellets thrown concentrated at some point due to the lesser dimensions of the bucket. So it effectively results in formation of a single mass of greater R leading to prompt sinking whereas in real life farming, we have a huge tank with higher dimensions than that of the bucket which leads to the pellets being at a far away distance from each other. So being spread off at a far away distance the presence of one pellet does not effect any other pellet and so R is small and hence floatation.

So, as a suggestion to find compatibility of the results in both the situations, we should always perform laboratory tests with a handful number of pellets.

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