Bubble Growth Problem

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BUBBLE DRAG COEFFICIENT FORMULATION AND STABILITY ANALYSIS FOR MULTIPHASE-TURBOMACHINERY PROBLEMS (SHEAR FLOW / BREAK UP-GE2)

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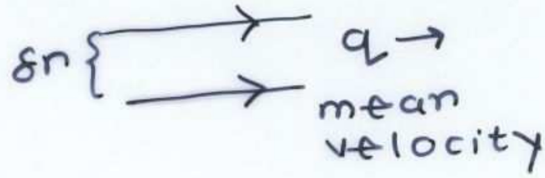
* ## Stream function

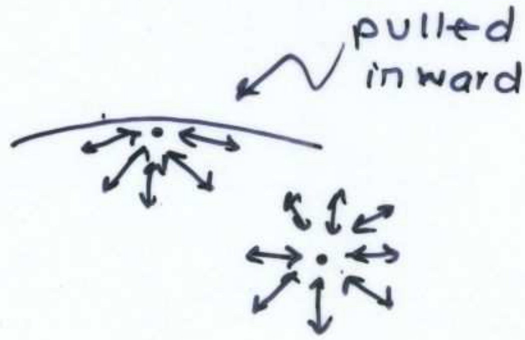
$\delta\psi = q\,\delta n$

$q = \dfrac{\partial \psi}{\partial n}$ (lim $\delta n \to 0$)



Surface Tension

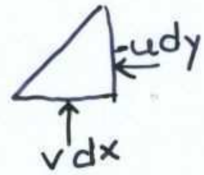
cohesive forces are responsible for surface tension.



This creates internal pressure & forces liquid to contract minimal area

Another viewpoint is minimizing no. of molecules at surface (i.e. molecules with higher energy)

* $\dfrac{\partial \psi}{\partial x} \cdot \partial x + \dfrac{\partial \psi}{\partial y} \cdot \partial y = 0$



$\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$ (continuity eq^n)

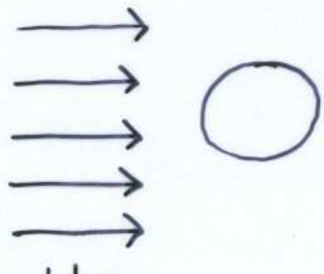
$u \equiv \partial\psi / \partial y$; $v \equiv -\dfrac{\partial \psi}{\partial x}$

In spherical coordinate,

$u_r = \dfrac{1}{r^2 \sin\theta} \dfrac{\partial \psi}{\partial \theta}$ $\quad u_\theta = -\dfrac{1}{r\sin\theta} \dfrac{\partial \psi}{\partial r}$

Nature Of Flow

1 UNIFORM FlOW

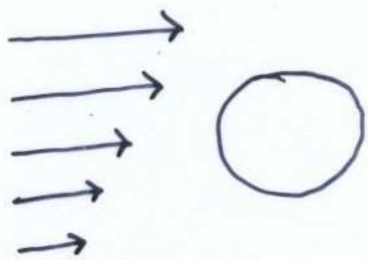


In Spherical Coordinate,

$$\psi \longrightarrow \frac{1}{2} U_\infty r^2 \sin^2\theta$$

as $r \rightarrow \infty$

2. SHEAR FLOW



Representation is

$$\psi \rightarrow \frac{U_\infty}{2} r^2 \sin^2\theta \cancel{+ \delta f_1(\theta, r, \phi) V_\infty}$$

as $r \rightarrow \infty$ (in Spherical coordinate)

$$\boxed{u = U_\infty + \delta f(r, \theta, \phi) V_\infty}$$

Assumptions

1) Fluid is incompressible.

$\Rightarrow \rho = \text{constant}$

2) Re << 1 for bubble.

so, we can ignore inertial forces

from N-S equation

N-S equation reduces to

Dynamic Viscosity

$$\nabla p = \mu \nabla^2 u$$

Gradient of Pressure

Laplacian of velocity

Continuity Condition is given by

$$\nabla . u = 0$$

(Divergence free Velocity)

In spherical coordinate, N-S eq^n for a stream function ψ is given by

$$D^4\psi = 0$$

where $D^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}$

Boundary Conditions

1) As $r \to \infty$

$$\psi \to \frac{1}{2}U_\infty r^2 \sin^2\theta \;\cancel{+\delta f(\theta,r,\phi)\, V_\infty}$$

$$u \to u_\infty + V_\infty \delta f(r,\theta,\phi)$$

2) Velocity component of fluid particle normal to the bubble surface is zero.

$$\Rightarrow \underline{n}\cdot\underline{u} = 0 \quad \text{at} \quad r = a(1+\varepsilon f(\theta))$$

In terms of ψ, above equation gives

$$\frac{\partial\psi}{\partial\theta} + a\varepsilon f'(\theta)\frac{\partial\psi}{\partial r} = 0$$

$$\underline{n} = (1, -a\varepsilon f'(\theta))$$

$$u = (u_r, u_\theta)$$

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\;; \quad u_\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

3. $\underline{n} \cdot \tau \cdot \underline{t} = 0$

where, $\underline{t} = \left(\frac{\varepsilon a}{r} f'(\theta), 1 \right)$

$$\tau = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} \\ \tau_{r\theta} & \tau_{\theta\theta} \end{pmatrix}$$

$$\Rightarrow \tau_{r\theta} + f'(\theta) \cdot \frac{a\varepsilon}{r} (\tau_{rr} - \tau_{\theta\theta}) = 0$$

4. $\varepsilon \tau_{rr} = -(2K - K_0)$ —— (4)

where $K = \frac{K_1 + K_2}{2}$ (Mean Curvature)

K_1, K_2 are principle curvatures.

We have considered a uniform flow without pertuberation as an initial step.

$\Rightarrow \varepsilon = 0$ & $\delta = 0$

i.e. $D^4\psi = 0$ with boundary conditions

1. $\psi(\infty, \theta) = \frac{1}{2} U_\infty r^2 \sin^2\theta$

2. $\psi(a, \theta) = 0$ where a is radius of sphere

3. $\frac{\partial}{\partial r}\left(\frac{1}{r^2} \frac{\partial \psi}{\partial r}(a, \theta)\right) = 0$

With $\varepsilon = 0$ & $\delta = 0$, the problem is axisymmetric i.e. flow patterns are identical in all planes parallel to U_∞ & passing through center of sphere.

Solution to above system,

$\psi = \left(\frac{Ur^2}{2} - \frac{Uar}{2}\right)\sin^2\theta$ (Using variable separable method, $\psi = f(r)\sin^2\theta$)

Using N-S eq^n $\nabla p = \mu \nabla^2 u$

$$p = p_0 - \left(\frac{a\mu U \cos\theta}{r^2}\right)$$

$$\tau_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r} = \frac{3\mu U \cos\theta}{r^2} = \sigma_{rr}$$

$$\tau_{r\theta} = 0 = \sigma_{r\theta}$$

Drag Force is given as

$$\iint \sigma_{rr} \cos\theta \, ds - \iint \sigma_{r\theta} \sin\theta \, ds$$

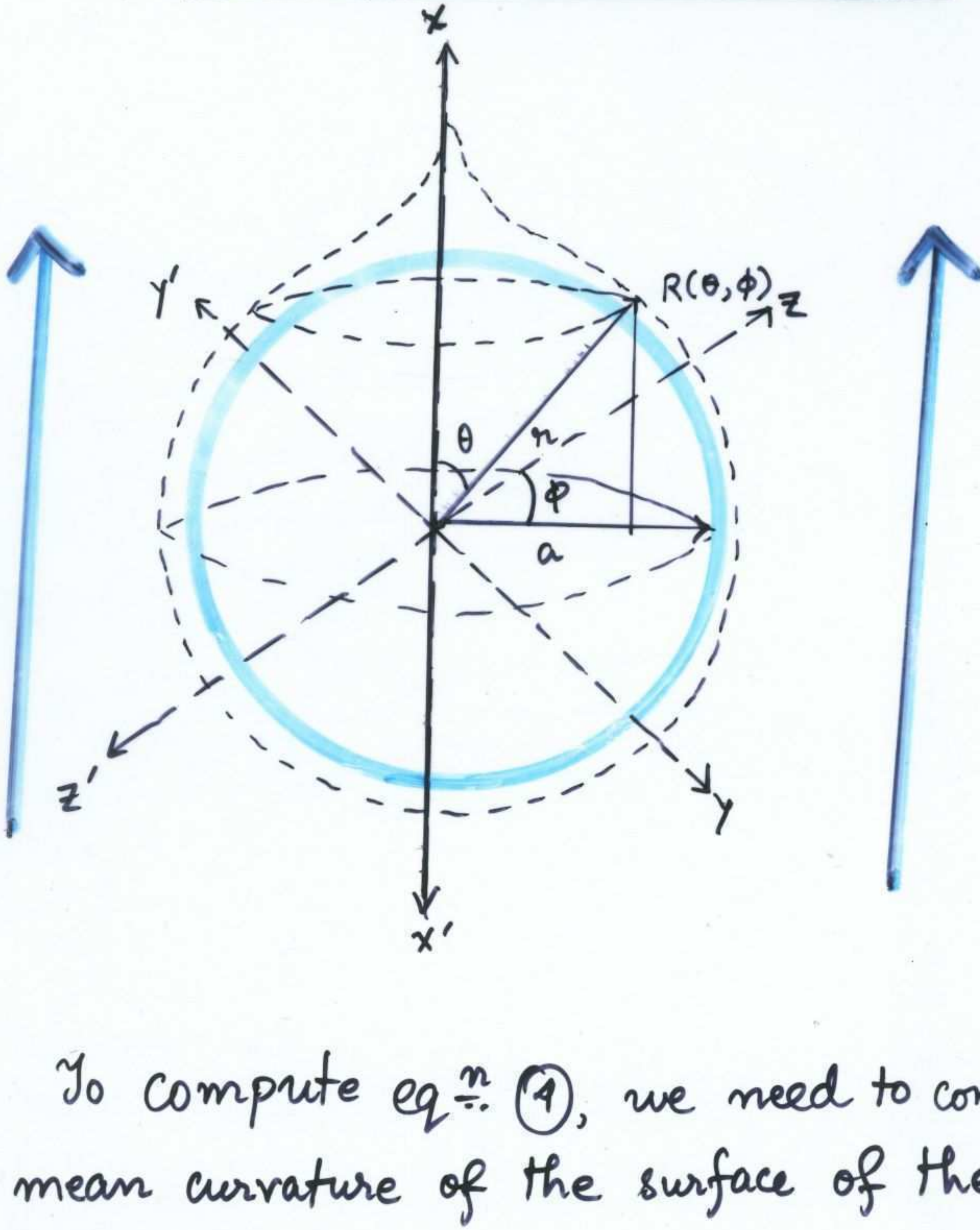
$$\Rightarrow F_{Drag} = 4\pi\mu U a$$

Ignoring the weight of air in bubble, force of buoyancy equals to drag force.

$$\Rightarrow 4\pi\mu U a = \frac{4}{3}\pi a^3 g \rho$$

$$U = \frac{1}{3}\frac{g a^3}{\nu} \quad (\nu = \mu/\rho)$$

COMPUTATION OF THE MEAN CURVATURE



To compute eq$\stackrel{n}{=}$ (1), we need to compute the mean curvature of the surface of the bubble. To do that we use spherical polar coordinates and the first & second fundamental forms. [For more, see the book on 'Elementary Diff$\stackrel{l}{=}$ Geometry' – Andrew Pressley].

Not going too much into the literature of the first & second fundamental forms, we purely do the analytical part.

Let $r = a(1+\epsilon f(\theta))$ where θ is indicated in the figure & a is the radius of the original sphere.

We parametrize our surface by considering any point on the surface as

$$R(\theta,\phi) = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)$$

where $r = a(1+\epsilon f(\theta))$

The <u>FIRST FUNDAMENTAL FORM</u> is given by:- $E\,d\theta^2 + 2F\,d\theta\,d\phi + G\,d\phi^2$

where $E = R_\theta \cdot R_\theta$

$F = R_\theta \cdot R_\phi$.

$G = R_\phi \cdot R_\phi$.

By performing necessary calculations,

$$E = a^2(1+\epsilon f)^2 + a^2\epsilon^2 f'^2 \quad \text{where } f' = \frac{df}{d\theta}.$$

$$F = 0$$

$$G = a^2(1+\epsilon f)^2 \sin^2\theta$$

The <u>SECOND FUNDAMENTAL FORM</u> is given by:-

$$L\,d\theta^2 + 2M\,d\theta\,d\phi + N\,d\phi^2$$

where

$$L = R_{\theta\theta} \cdot \tilde{N}$$

$$M = R_{\theta\phi} \cdot \tilde{N}$$

$$N = R_{\phi\phi} \cdot \tilde{N}$$

&

$$\tilde{N} = \frac{R_\theta \times R_\phi}{\|R_\theta \times R_\phi\|}$$

Again performing some tedious but necessary calculations we get:-

$$L = \frac{a(1+\epsilon f)^2 - a\epsilon f''(1+\epsilon f) + 2a\epsilon^2 f'^2}{D}$$

$$M = 0$$

$$N = \frac{a(1+\epsilon f)^2 \sin^2\theta + a\epsilon(1+\epsilon f) f' \sin\theta \cos\theta}{D}$$

where $D = \sqrt{(1+\epsilon f)^2 + f'^2 \epsilon^2}$ & $f'' = \dfrac{d^2 f}{d\theta^2}$

Now we form the matrices

$$\mathcal{F}_1 = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \ \& \ \mathcal{F}_{11} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

& we calculate $\mathcal{F}_1^{-1} \mathcal{F}_{11}$.

We get $\mathcal{F}_I^{-1}\mathcal{F}_{II} =$

$$\begin{bmatrix} \frac{1}{aD}\left\{1-\frac{\epsilon f''}{(1+\epsilon f)}+\frac{2\epsilon^2 f'^2}{(1+\epsilon f)^2}\right\} & 0 \\ 0 & \frac{1}{aD}\left\{1-\frac{\epsilon f'}{(1+\epsilon f)}\cot\theta\right\} \end{bmatrix}$$

The eigen values of this matrix gives us the principle curvatures.

$$\text{so } K_1 = \frac{1}{aD}\left\{1-\frac{\epsilon f''}{(1+\epsilon f)}+\frac{2\epsilon^2 f'^2}{(1+\epsilon f)^2}\right\}$$

$$\& \; K_2 = \frac{1}{aD}\left\{1-\frac{\epsilon f'}{1+\epsilon f}\cot\theta\right\} \text{ where } D=\sqrt{(1+\epsilon f)^2+f'^2\epsilon^2}$$

$$\therefore \text{ Mean Curvature } \bar{K} = \frac{1}{2aD}\left\{2+\epsilon\left(-\frac{f'\cot\theta}{1+\epsilon f}-\frac{f''}{1+\epsilon f}\right)+\epsilon^2\frac{2f'^2}{(1+\epsilon f)^2}\right\}$$

$$\& \text{ Total Curvature } K = \frac{1}{aD}\left\{2+\epsilon\left(-\frac{f'\cot\theta}{1+\epsilon f}-\frac{f''}{1+\epsilon f}\right)+\epsilon^2\frac{2f'^2}{(1+\epsilon f)^2}\right\}$$

<u>Special Case</u> :- when $\epsilon \to 0$, $K = \frac{2}{a}$ & $\bar{K} = \frac{1}{a}$.

Axisymmetric Case

Boundary condition on bubble $r = a(1+f(\theta))$ $f \ll 1$

$$\varepsilon T_{nn} = K - K_0 = K - 2/a \qquad (1)$$

Since LHS is multiplied by ε can approximate T_{nn} by expression for spherical bubble

ie $T_{nn} = T^0_{rr}\big|_{r=a} +$ smaller terms.

$$= \frac{3\mu U_\infty \cos\theta}{a}$$

Assuming $f \ll 1$ and substituting into (1) using formula for K

$$\boxed{\begin{aligned}\varepsilon \frac{3\mu U_\infty}{\cancel{a}} \cos\theta = &-(f'' + f'\cot\theta + 2f) + (2ff'' + f'^2 + 2f^2 + 2ff'\cot\theta) \\ &+ (-2f^3 - 3ff'^2 - 3f^2f'' + \tfrac{1}{2}f'^2f'' + \tfrac{1}{2}f'^3\cot\theta - 3f^2f'\cot\theta) \\ &+ \cdots\end{aligned}} \quad (*)$$

needs checking

(1) if $f = O(\varepsilon)$, put $f = \varepsilon\hat{f}$

$$\hat{f}'' + \hat{f}'\cot\theta + 2\hat{f} = -3\mu U_\infty \cos\theta$$

But this has no solution s.t. $\hat{f}$ is finite at $x = 0, \pi$

Consider $\mathcal{L}\hat{f} = \hat{f}'' + \hat{f}'\cot\theta + 2\hat{f} = G$ (2)

$\mathcal{L}y = 0$ is satisfied by $\cos\theta$ so put $\hat{f} = \cos\theta\, v$

$$\cos\theta\,(v'') + \left(\frac{\cos^2\theta}{\sin\theta} - 2\sin\theta\right)v' = G$$

Integrating factor is $\sin\theta\cos\theta$ so we should be able to solve it provided G is suitable from 2, multiply by $\sin\theta$

$$\sin\theta \hat{f}'' + \cos\theta \hat{f}' + 2\hat{f}\sin\theta = G\sin\theta$$

$$\frac{d}{d\theta}\left(\sin\theta \hat{f}'\right) + 2\hat{f}\sin\theta = G\sin\theta.$$

$\times\cos\theta$ and integrate from $\theta = 0$ to π.

$$\int_0^\pi \cos\theta\,(\sin\theta \hat{f}')' + 2\hat{f}\sin\theta\cos\theta\,d\theta = \int_0^\pi G\sin\theta \cos\theta\,d\theta$$

But LHS $= \left[\cos\theta\sin\theta\hat{f}' + \sin^2\theta\hat{f}\right]_0^\pi = 0$ if $\hat{f}$ is bded at $0, \pi$.

$\therefore$ necessary condition for a solution is

$$\boxed{\int_0^\pi G\sin\theta\cos\theta\,d\theta = 0}$$

NB if $G = \cos\theta$ $\quad \int G\sin\theta\cos\theta\,d\theta = \left[-\frac{1}{3}\cos^3\theta\right]_0^\pi = 2/3 \neq 0.$

$\therefore$ Need to put $f = \varepsilon^{1/3}f_1 + \varepsilon^{2/3}f_2 + \varepsilon f_3 + \cdots$

Then equate coeffs of powers of ε in (*)

$O(\varepsilon^{1/3})$ $\qquad 0 = f_1'' + f_1'\cot\theta + 2f_1 \qquad \therefore f_1 = A\cos\theta$, A not yet determin

$O(\varepsilon^{2/3})$ $\qquad 0 = f_2'' + f_2'\cot\theta + 2f_2 + A^2(1-3\cos^2\theta)$

this can be solved for f_2

$O(\varepsilon)$ $\quad 3\mu U_\infty \cos\theta = -(f_3'' + f_3'\cot\theta + 2f_3) + A^3(4\cos^3\theta - 4\sin^2\theta\cos\theta)$

$+\left(\begin{array}{l}\text{terms in } f_2 f_1 \text{ which may be important} \\ \text{but have not yet been calculated.}\end{array}\right) = H$, say

this equation for f_3 will be soluble if

$$\int_0^\pi \left(-3\mu U_\infty \cos\theta + A^3(4)(2\cos^3\theta - \cos\theta) + H\right) \sin\theta\cos\theta \, d\theta = 0$$

[If we neglect. H. $\Rightarrow$ $\quad +3\mu U_\infty \left[\frac{\cos^3\theta}{3}\right]_0^\pi + A^3.4\left[\frac{2}{5}\cos^5\theta - \frac{\cos^3\theta}{3}\right]_0^\pi = 0$]

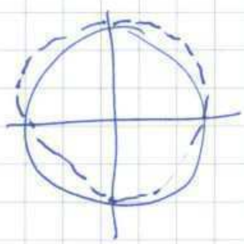
which determines A.

Then we need to solve for f_3 to find <u>shape</u>

of bubble.

[ignoring f_2, then $\quad f_3'' + f_3'\cot\theta + 2f_3 = 6C\cos\theta - 10C\cos^3\theta$

$\Rightarrow \quad f_3 = C\cos\theta\sin^2\theta$



pear shaped]