NYU, Tandon School of Engineering
Extended Bridge to CS – Summer 2022
Homework #8

Rozalin Mohanty
rm6102@nyu.edu

Note: The greyed-out portions in the solutions are rough workings or my own additional notes. You need not mark them. Additionally, where the answers are too big, I have left the answer in factorial form instead of solving for the final answer.

Question 7

A. Exercise 6.1.5

A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

b. What is the probability that the hand is a three of a kind? A three of a kind has 3 cards of the same rank. The other two cards do not have the same rank as each other and do not have the same rank as the three with the same rank. For example, {4♠, 4♦, 4♣, J♠, 8♥} is a three of a kind.

Probability = $\frac{13C1 \times 4C3 \ \times 12C2 \times 4C1 \times 4C1}{52C5} = \frac{88}{4165}$

Thought process:

- There are 13 boxes with 4 cards of the same rank each. You pick one box and pick 3 out of 4 cards in the box.
- You pick 2 boxes out of the remaining 12 boxes. And then pick 1 out of 4 cards for each.

---

c. What is the probability that all 5 cards have the same suit?

Probability = $\frac{4C1 \times 13C5}{52C5} = \frac{33}{16660}$

Thought process: There are 4 boxes with 13 cards of the same suit each. You pick one box and pick 5 out of 13 cards in the box.

---

d. What is the probability that the hand is a two of a kind? A two of a kind has two cards of the same rank (called the pair). Among the remaining three cards, not in the pair, no two have the same rank and none of them have the same rank as the pair. For example, {4♠, 4♦, J♠, K♣, 8♥} is a two of a kind.

Probability = $\frac{13C1 \times 4C2 \ \times 12C3 \times 4C1 \times 4C1 \times 4C1}{52C5} = \frac{352}{833}$

B. Exercise 6.2.4

A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

a. The hand has at least one club.

P(at least 1 club) = 1 – P(no clubs)

$= 1 - \frac{39C5}{52C5}$

$= 1 - \frac{2109}{9520}$

$= \frac{7411}{9520}$

b. The hand has at least two cards with the same rank.

P(≥ 2 cards with the same rank) = 1 – P(all cards with different rank)

$$= 1 - \frac{13C5 \times 4C1}{52C5}$$

$$= 1 - \frac{13C5 \times (4C1)^5}{52C5}$$

$$= 1 - \frac{2112}{4165}$$

$$= \frac{2053}{4165}$$

c. The hand has <u>exactly</u> one club <u>or</u> exactly one spade.

Let P(C) be the probability of getting exactly one club
Let P(S) be the probability of getting exactly one spade

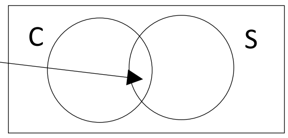
Required probability
$= P(C) + P(S) - P\,(C \cap S)$

$$= \frac{13C1\ \times 39C4}{52C5} + \frac{13C1\ \times 39C4}{52C5} - \frac{13C1\ \times 13C1 \times 26C3}{52C5}$$

$= \ 0.654\ (3\ s.f.)$

Thought process: There are 4 boxes with 13 cards of the same rank each. For both scenarios, you pick 1 out of 13 cards from the box containing cards with clubs only/spades only and pick the remaining 4 cards out of the remaining 39 cards. Probabilities are added together. You also need to subtract away the cases where there are both one club and one spade.

There are combinations where there could be one club and one spade. These need to be subtracted.



d. The hand has <u>at least</u> one club <u>or</u> at least one spade.

Required probability

$= 1 - \frac{26C5}{56C5}$

$= 0.983\ (3s.f.)$

Thought process: If the hand has to have at least one club, we want to exclude cases where there are 0 clubs. Similarly, if the hand has to have at least one spade, we want to exclude cases where there are 0 spades. Taking these together, we want to exclude cases with 0 clubs and 0 spades, i.e. those combinations where all 5 cards have diamonds and hearts only (total of 52/2 = 26 cards to choose from). This will give us the required probability.

Question 8

A. Exercise 6.3.2

The letters {a, b, c, d, e, f, g} are put in a random order. Each permutation is equally likely. Define the following events:

- A: The letter b falls in the middle (with three before it and three after it)
- B: The letter c appears to the right of b, although c is not necessarily immediately to the right of b. For example, "agbdcef" would be an outcome in this event.
- C: The letters "def" occur together in that order (e.g. "gdefbca")

---

a. Calculate the probability of each individual event. That is, calculate p(A), p(B), and p(C).

P(A) = $\frac{6!}{7!} = \frac{1}{7}$

Thought process: The position of b is fixed, but the other 6 letters can be arranged randomly.

P(B) = $\frac{1}{2}$

Thought process: The diagram below sums up all the possibilities. In any of these cases, b and c could easily be swapped. So half of the cases would have c to the right of b (not sure if there is a better way to calculate this.)

					b	c
				b		c
			b			c
		b				c
	b					c
b						c

P(C) = $\frac{5!}{7!} = \frac{1}{42}$

Thought process: Group d, e, and f into 1 unit. That gives us 5 units which can be arranged randomly.

---

b. What is P(A|C)?

$P(A|C)$

$= \frac{P(A \cap C)}{P(C)}$

$= \frac{2 \times 3!}{7!} \div \frac{1}{42}$

$= \frac{1}{10}$

Thought process: $P(A \cap C)$ refers to the probability that b is in the middle and d, e, f are together as one unit.

---

c. What is P(B|C)?

$P(B|C)$

$= \frac{P(B \cap C)}{P(C)}$

$= \frac{}{} \div \frac{1}{42}$

$= \frac{}{}$

Not sure how to calculate this. It's related to the issue I had above with calculating P(B).

Thought process: $P(B \cap C)$ refers to the probability that c appears to the left of b and d, e, f are together as one unit.

---

d. What is P(A|B)?

$P(A|B)$

$= \frac{P(A \cap B)}{P(B)}$

$= — \div \frac{1}{2}$

$= —$

Not sure how to calculate this. It's related to the issue I had above with calculating P(B).
Thought process: $P(A \cap B)$ refers to the probability that b is in the middle and c falls to the right of b.

---

e. Which pairs of events among A, B, and C are independent?

If two events X and Y are independent, then P(X|Y) = P(X). In this case,

Not sure how to calculate this. It's related to the issue I had above with calculating P(B). However, my approach to solving this would essentially be to check if the above relationship holds for any of the answers obtained for b, c and d above. If yes, then both events would be independent.

B. Exercise 6.3.6

A biased coin is flipped 10 times. In a single flip of the coin, the probability of heads is 1/3 and the probability of tails is 2/3. The outcomes of the coin flips are mutually independent. What is the probability of each event?

---

b. The first 5 flips come up heads. The last 5 flips come up tails.

Required probability = $\left(\frac{1}{3}\right)^5 \times \left(\frac{2}{3}\right)^5 = \frac{32}{59049}$

---

c. The first flip comes up heads. The rest of the flips come up tails.

Required probability = $\left(\frac{1}{3}\right)^1 \times \left(\frac{2}{3}\right)^9 = \frac{512}{59049}$

C. Exercise 6.4.2

a. Assume that you have two dice, one of which is fair, and the other is biased toward landing on six, so that 0.25 of the time it lands on six, and 0.15 of the time it lands on each of 1, 2, 3, 4 and 5. You choose a die at random, and roll it six times, getting the values 4, 3, 6, 6, 5, 5. What is the probability that the die you chose is the fair die? The outcomes of the rolls are mutually independent.

Let P(F) be the probability that the die is fair.
Let P(X) be the probability of the outcome described above.

$$P(F|X) = \frac{P(X|F)P(F)}{P(X|F)P(F) + P(X|\bar{F})P(\bar{F})}$$

$$= \frac{\left(\frac{1}{6}\right)^6 \times \frac{1}{2}}{\left(\frac{1}{6}\right)^6 \times \frac{1}{2} + ((0.15)^4 \times (0.25)^2) \times \frac{1}{2}}$$

$= 0.404\ (3\ s.f.)$

Question 9

A. Exercise 6.5.2

A hand of 5 cards is dealt from a perfectly shuffled deck of playing cards. Let the random variable A denote the number of aces in the hand.

a. What is the range of A?

Range = {0, 1, 2, 3, 4}

b. Give the distribution over the random variable A.

Distribution of A $= \left\{ \left(0, \frac{48C5}{52C5}\right), \left(1, \frac{13C1 \times 48C4}{52C5}\right), \left(2, \frac{13C2 \times 48C3}{52C5}\right), \left(3, \frac{13C3 \times 48C2}{52C5}\right), \left(4, \frac{13C4 \times 48C1}{52C5}\right) \right\}$

B. Exercise 6.6.1

a. Two student council representatives are chosen at random from a group of 7 girls and 3 boys. Let G be the random variable denoting the number of girls chosen. What is E[G]?

$E[G] = 0 \times \frac{3C2}{10C2} + 1 \times \frac{7C1 \times 3C1}{10C2} + 2 \times \frac{7C2}{10C2} = \frac{7}{15} + \frac{14}{15} = \frac{7}{5}$

Thought process: In this case, there might be 0 girls, 1 girl or 2 girls. We have to consider each of these 3 cases.

C. Exercise 6.6.4

a. A fair die is rolled once. Let X be the random variable that denotes the square of the number that shows up on the die. For example, if the die comes up 5, then X = 25. What is E[X]?

$E(X) = (1 + 4 + 9 + 16 + 25 + 36) \times \frac{1}{6} = \frac{91}{6}$

Thought process: The outcomes are 1, 4, 9, 16, 25, and 36, all of which are equally likely for a fair die.

b. A fair coin is tossed three times. Let Y be the random variable that denotes the square of the number of heads. For example, in the outcome HTH, there are two heads and Y = 4. What is E[Y]?

$E(Y) = 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} = 3$

D. Exercise 6.7.4

a. A class of 10 students hang up their coats when they arrive at school. Just before recess, the teacher hands one coat selected at random to each child. What is the expected number of children who get his or her own coat?

Not sure how to calculate this.

Question 10

A. Exercise 6.8.1

The probability that a circuit board produced by a particular manufacturer has a defect is 1%. You can assume that errors are independent, so the event that one circuit board has a defect is independent of whether a different circuit board has a defect.

---

a. What is the probability that out of 100 circuit boards made exactly 2 have defects?

Required probability = $100C98 \times (0.99)^{98} \times (0.01)^2$ = 0.185 (3 s.f.)

---

b. What is the probability that out of 100 circuit boards made at least 2 have defects?

Required probability = $1 - 100C100 \times (0.99)^{100} - 100C99 \times (0.99)^{99} \times (0.01)^1$ = 0.264 (3 s.f.)

Thought process: Exclude the situations with 0 and 1 defects.

---

c. What is the expected number of circuit boards with defects out of the 100 made?

Required probability = $100 \times 0.01 = 1$

---

d. Now suppose that the circuit boards are made in batches of two. Either both circuit boards in a batch have a defect or they are both free of defects. The probability that a batch has a defect is 1%. What is the probability that out of 100 circuit boards (50 batches) at least 2 have defects? What is the expected number of circuit boards with defects out of the 100 made? How do your answers compared to the situation in which each circuit board is made separately?

Probability that a batch has a defect = 0.01

Probability that at least 2 have defects
= 1 – Probability that 1 batch has defects – Probability that 0 batches have defects
= $1 - 50C1 \times (0.99)^{49} \times (0.01)^1 - 50C0 \times (0.99)^{50} \times (0.01)^0$
= 0.089

E(X) = $50 \times 0.01 =$ 0.5

The probability that at least 2 have defects is reduced in the case where the boards are made with batches of 2 as compared to when where they are made individually.

B. Exercise 6.8.3

A gambler has a coin which is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3. Without knowing which coin he is using, you ask him to flip the coin 10 times. If the number of heads is at least 4, you conclude that the coin is fair. If the number of heads is less than 4, you conclude that the coin is biased.

---

b. What is the probability that you reach an incorrect conclusion if the coin is biased?

You would reach an incorrect conclusion that the coin is biased if the number of heads is more than or equal to 4.

Let X be the number of heads.

Required probability
$= P\ (X \geq 4)$
$= 1 - P(X = 1) - P(X = 2) - P(X = 3)$
$= 1 - (0.7)^{10} - 10 \times (0.3) \times (0.7)^{9} - 10C2 \times (0.3)^{2} \times (0.7)^{8} - 10C3 \times (0.3)^{3} \times (0.7)^{7}$
$= 0.350$ (3 s.f.)