NYU, Tandon School of Engineering
Extended Bridge to CS – Summer 2022
Homework #7

Rozalin Mohanty
rm6102@nyu.edu

Note: The greyed-out portions in the solutions are rough workings or my own additional notes. You need not mark them. Additionally, where the answers are too big, I have left the answer in factorial form instead of solving for the final answer.

Question 3

A. Exercise 8.2.2

Give complete proofs for the growth rates of the polynomials below. You should provide specific values for c and n_0 and prove algebraically that the functions satisfy the definitions for O and Ω.

b. $f(n) = n^3 + 3n^2 + 4$. Prove that $f = \Theta(n^3)$.

Step 1 • T(N) = O (f(n)) if and only if there are positive constants c and n_0 such that T(N) ≤ cf(N) when N ≥ n_0. • In this case, $n^3 + 3n^2 + 4 \leq 2n^3$ $for\ N \geq 4$. Thus, $n^3 + 3n^2 + 4 = O(n^3)$.
Step 2 • T(N) = Ω (g(n)) if and only if there are positive constants c and n_0 such that T(N) ≥ cg(N) when N ≥ n_0. • In this case, $n^3 + 3n^2 + 4 \geq n^3$ $for\ N \geq 0$. Thus, $n^3 + 3n^2 + 4 = \Omega(n^3)$.
Step 3 • T(N) = Θ (h(n)) if and only if T(N) = O(h(N)) and T(N) = Ω (N)). • Since $n^3 + 3n^2 + 4 = O(n^3)$ and $n^3 + 3n^2 + 4 = \Omega(n^3)$, in this case, we can conclude that $n^3 + 3n^2 + 4 = \Theta(n^3)$. ■

B. Exercise 8.3.5

The algorithm below makes some changes to an input sequence of numbers.

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MysteryAlgorithm

Input: a1, a2,...,an
        n, the length of the sequence.
        p, a number.
Output: ??

i := 1
j := n

While (i < j)
    While (i < j and ai < p)
        i := i + 1
    End-while
    While (i < j and aj ≥ p)
        j := j - 1
    End-while
    If (i < j), swap ai and aj
End-while

Return( a1, a2,...,an )
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a. Describe in English how the sequence of numbers is changed by the algorithm. (Hint: try the algorithm out on a small list of positive and negative numbers with p = 0)

The sequence of numbers gets rearranged such that all the numbers in the first half are lesser than p, while all the numbers in the first half are greater than or equal to p.

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b. What is the total number of times that the lines "i := i + 1" or "j := j - 1" are executed on a sequence of length n? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the two lines are executed.

The lines are executed n – 1 times for a sequence with length n.

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c. What is the total number of times that the swap operation is executed? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the swap is executed.

The total number of times the swap operation is executed depends on both the actual values of the numbers and the length of the sequence. It will be executed the greatest number of times ($\frac{n}{2}$ times for a sequence with length n) in the situation where all the numbers in the first half are greater than p and need to be swapped to bring them to the second half, and similarly all the numbers in the second half are greater than or equal to p and need to be swapped to bring them to the first half. This is for a situation where there are an equal number of numbers on either side of p.

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d. Give an asymptotic lower bound for the time complexity of the algorithm. Is it important to consider the worst-case input in determining an asymptotic lower bound (using Ω) on the time complexity of the algorithm? (Hint: argue that the number of swaps is at most the number of times that i is incremented or j is decremented).

$\Omega(n)$.

Incomplete – I wasn't sure how to explain this.

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e. Give a matching upper bound (using O-notation) for the time complexity of the algorithm.

$O(n)$.

Incomplete – I wasn't sure how to explain this.

Question 4

A. Exercise 5.1.2

Consider the following definitions for sets of characters:

- Digits = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }
- Letters = { a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z }
- Special characters = { *, &, $, # }

Compute the number of passwords that satisfy the given constraints.

b. Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.

No. of possible passwords = $40^9 + 40^8 + 40^7$

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c. Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

No. of possible passwords = $14 \times (40^8 + 40^7 + 40^6)$

B. Exercise 5.3.2

a. How many strings are there over the set {a, b, c} that have length 10 in which no two consecutive characters are the same? For example, the string "abcbcbabcb" would count and the strings "abbbcbabcb" and "aacbcbabcb" would not count.

No. of strings = $3 \times 2^9 = 1536$

For the first place, you can choose any of the 3 letters a, b or c, but for the other 9 places, you can only choose from 2 letters that are not the same as the letter in the previous place.

C. Exercise 5.3.3

License plate numbers in a certain state consists of seven characters. The first character is a digit (0 through 9). The next four characters are capital letters (A through Z) and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form:

Digit-Letter-Letter-Letter-Letter-Digit-Digit

b. How many license plate numbers are possible if no digit appears more than once?

No. of possible license plate numbers = 10 x (4 x 26) x 9 x 8 = 74880

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c. How many license plate numbers are possible if no digit or letter appears more than once?

No. of possible license plate numbers = 10 x 26 x 25 x 24 x 23 x 9 x 8 = 157656

D. Exercise 5.2.3

Let B = {0, 1}. B^n is the set of binary strings with n bits. Define the set E_n to be the set of binary strings with n bits that have an even number of 1's. Note that zero is an even number, so a string with zero 1's (i.e., a string that is all 0's) has an even number of 1's.

a. Show a bijection between B^9 and E_{10}. Explain why your function is a bijection.

Incomplete – I wasn't sure how to explain this.

A function is bijective if it is both one-to-one and onto. A bijective function is called a bijection. A bijection is also called a one-to-one correspondence.

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b. What is $|E_{10}|$?

Incomplete.

Question 5

A. Exercise 5.4.2

At a certain university in the U.S., all phone numbers are 7-digits long and start with either 824 or 825.

a. How many different phone numbers are possible?

Possible no. of phone numbers = $10^4 + 10^4 = 20000$

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b. How many different phone numbers are there in which the last four digits are all different?

Possible no. of phone number = $2 \times 10 \times 9 \times 8 \times 7 = 10080$

B. Exercise 5.5.3

How many 10-bit strings are there subject to each of the following restrictions?

a. No restrictions.

No. of strings = $2^{10} = 1024$

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b. The string starts with 001.

No. of strings = $2^7 = 128$

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c. The string starts with 001 or 10.

No. of strings = $2^7 + 2^8 = 384$

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d. The first two bits are the same as the last two bits.

No. of strings = $4 \times 2^6 = 256$

Possible combinations for the first/last 2 bits are 00,01,10, 11.

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e. The string has exactly six 0's.

No. of strings = $\frac{10!}{6!4!} = 210$

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f. The string has exactly six 0's and the first bit is 1.

No. of strings = $\frac{9!}{6!3!} = 84$

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g. There is exactly one 1 in the first half and exactly three 1's in the second half.

No. of strings = $\frac{5!}{1!4!} \times \frac{5!}{3!2!} = 50$

C. Exercise 5.5.5

a. There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?

No. of ways = ${}^{30}C_{10}$ x ${}^{35}C_{10}$

D. Exercise 5.5.8

This question refers to a standard deck of playing cards. If you are unfamiliar with playing cards, there is an explanation in "Probability of an event" section under the heading "Standard playing cards." A five-card hand is just a subset of 5 cards from a deck of 52 cards.

c. How many five-card hands are made entirely of hearts and diamonds?

No. of five-card hands = ${}^{26}C_5$ = 65780

There are a total of 13 * 2 = 26 cards containing only hearts and diamonds.

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d. How many five-card hands have four cards of the same rank?

No. of five-card hands = ${}^{13}C_1$ x ${}^{48}C_1$ = 624

There are 13 different ranks with 4 cards of the same rank. You can pick any of these boxes of 4 cards and pick a random card from the 52-4 = 48 remaining cards, which will be of a different rank than those in the box.

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e. A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. For example, {queen of hearts, queen of spades, 8 of diamonds, 8 of spades, 8 of clubs}. How many five-card hands contain a full house?

No. of 5-card hands = ${}^{13}C_1$ x ${}^{4}C_3$ x ${}^{12}C_1$ x ${}^{4}C_2$ = 3744

Imagine there are 13 boxes containing 4 cards each of the same rank. You pick one of the boxes randomly and then pick 3 out of the 4 cards in the box. For the other 2 cards you have to choose, you have 12 boxes to randomly choose from.

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f. How many five-card hands do not have any two cards of the same rank?

No. of 5-card hands = ${}^{13}C_5$ x 4^5 = 1317888

In other words, all cards in the 5-card hand are of different ranks. Imagine that there are 13 boxes with 4 cards each of the same rank. You pick 5 out of the 13 boxes and pick 1 out of the 4 cards in each box, repeating this process 5 times.

E. Exercise 5.6.6

A country has two political parties, the Demonstrators and the Repudiators. Suppose that the national senate consists of 100 members, 44 of which are Demonstrators and 56 of which are Repudiators.

a. How many ways are there to select a committee of 10 senate members with the same number of Demonstrators and Repudiators?

No. of ways = ${}^{44}C_5$ x ${}^{56}C_5$

In other words, there must be 5 Demonstrators and 5 Repudiators in the committee of 10.

b. Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

No. of ways = ${}^{44}P_2 \times {}^{56}P_2 = 5827360$

Question 6

A. Exercise 5.7.2

A 5-card hand is drawn from a deck of standard playing cards.

a. How many 5-card hands have at least one club?

No. of possible 5-card hands with at least 1 club
= No. of possible 5-card hands - No. of possible 5-card hands with 0 clubs
= ${}^{52}C_5$ - ${}^{39}C_5$
= 2023203

A deck of standard playing cards contains a total of 52 cards. There are 13 ranks in each of the four French suits: clubs (♣), diamonds (♦), hearts (♥) and spades (♠).

No. of possible 5-card hands with 0 clubs → You can pick any 5 cards from the remaining 39 cards with no clubs.

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b. How many 5-card hands have at least two cards with the same rank?

No. of possible 5-card hands with at least 2 cards with the same rank
= No. of possible 5-card hands – No. of possible 5-card hands with no cards of the same rank
= ${}^{52}C_5$ - ${}^{13}C_5 \times 4^5$

No. of possible 5-card hands with no cards of the same rank → There are 13 different ranks with 4 cards of each rank. Imagine that there are 13 boxes, each with 4 cards of each rank. You pick 5 out of the 13 boxes at random. From each box, you then randomly draw 1 out of the 4 cards, and you repeat this for each of the 5 boxes you have chosen.

B. Exercise 5.8.4

20 different comic books will be distributed to five kids.

a. How many ways are there to distribute the comic books if there are no restrictions on how many go to each kid (other than the fact that all 20 will be given out)?

Number of ways = 5^{20}

How I thought of this is that there are 20 books, and setting aside the specifics of how the books will be distributed, there are 5 possible kids that each book might end up with across all the different ways the books may be distributed.

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b. How many ways are there to distribute the comic books if they are divided evenly so that 4 go to each kid?

Number of ways = ${}^{20}C_4 \times {}^{16}C_4 \times {}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4 = \frac{20!}{4!4!4!4!4!}$

The first child gets to choose 4 out of 20 books, the second child gets to choose 4 out of the remaining 16 books etc.

Question 7

How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a. 4

No. of one-to-one functions = 0.
There are fewer elements in the target than in the domain.

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b. 5

No. of one-to-one functions = ${}^5P_5$= 120

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c. 6

No. of one-to-one functions = ${}^6P_5$= 720

The first element in the domain can point to any 1 out of 6 elements in the target, the second element in the domain can point to any 1 out of the remaining 5 elements etc.

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d. 7

No. of one-to-one functions = ${}^7P_5$= 2520