Recall: R is noetherian iff every ideal in R is finitely generated.

Thm (Hilbert Basis Theorem): Let R be noetherian. Then $R[x]$ is as well.

Pf: If $a_n x^n + \cdots + a_0 = f(x) \in R[x]$, then the leading term of f is $a_n x^n$, and the leading coefficient is a_n.

Let $I \subset R[x]$ be an ideal.

Choose $f_1, \dots, f_n, \dots \in I$ s.t.

$f_i \in I \setminus (f_1, \dots, f_{i-1})$ and is of least degree

among all such polys. If this sequence

terminates, then I is f.g.

Write a_i for the leading coefficient of f_i,

Choose k s.t. $(a_1, \dots, a_i, \dots)$ is gen by $a_1, \dots, a_k$.

Prop. $f_1, \dots, f_k$ generate I.

If not, then $\exists f_{k+1}$. $a_{k+1} = \sum_{i=1}^{k} r_i a_i$

$g = \sum r_i f_i x^{\deg(f_{k+1}) - \deg(f_i)} \in (f_1, \dots, f_k)$

leading term of g is $\sum r_i a_i x^{\deg(f_i)} x^{\deg(f_{k+1}) - \deg(f_i)}$

$= a_{k+1} x^{\deg(f_{k+1})}$

$\deg(f_{k+1} - g) < \deg(f_{k+1})$

$f_{k+1} - g \in I \Rightarrow f_{k+1} - g \in (f_1, \dots, f_k)$

$\Rightarrow f_{k+1} - g + g = f_{k+1} \in (f_1, \dots, f_k) \to \Leftarrow.$ $\blacksquare$

Cor: If k is a field, $k[x_1, \dots, x_n]/I$ is Noetherian.

Same is true for any PID (or anything you knew was noetherian already).

Graded ring: Let R be a ring. Then R is a <u>graded ring</u> if $\exists R_0, R_1, \ldots \subset R$, s.t.

1) R_0 is a ring under the natural operations.
2) R_i is an R_0-module $\forall i$.
3) $R = \bigoplus_{i=0}^{\infty} R_i$, and

4) $r_i \in R_i,\ r_j \in R_j \Rightarrow r_i r_j \in R_{i+j}$

Examples!

k is a field, then $k[x_1, \dots, x_n]$ is a graded ring

$R_i = \{$homogeneous degree i polys$\}$.

k is field, V is a f.d. v.s. $/k$, then

$$Sym^*(V^\vee) = \bigoplus_{i=0} Sym^i(V^\vee)$$

$\mathrm{Sym}^i(V^\vee) \times \mathrm{Sym}^j(V^\vee) \to \mathrm{Sym}^{i+j}(V^\vee)$

Polynomial functions on V.

- $k[x^2, x^3]$ $\quad R_0 = k,\ R_1 = 0,\ R_2 = x^2k,\ R_3\, x^3k, \ldots$

- Let $I \subset R$, R a graded ring. I is <u>homogeneous</u> $\Leftrightarrow$ I is generated by elts. in various R_is.

In $k[x,y]$

$I = (x+y, xy)$ ✓

$I = (x+y^2)$ ✗

Def: If R is a graded ring, the <u>irrelevant ideal</u> is $\bigoplus_{i=1}^{\infty} R_i$

Fun exercise: Let R be graded, I be an ideal in R. Then R/I is graded w/ the inherited grading $((R/I)_i = \text{im}(R_i)) \iff I$ is homogeneous.

Proposition: Let $I = (f_1, \dots, f_k)$ be a homogeneous ideal ($f_1, \dots, f_k$ are homogeneous eles.). If $f \in I$ is homogeneous, then we may write $f = \sum g_i f_i$ w/ g_i homogeneous.

Pf. If $g \in R$, write $(g)_i$ = degree i part of g.

$f \in I \Rightarrow f = \sum G_i f_i \quad G_i \in R.$

$$\deg(f) = d \qquad \deg(f_i) = d_i \qquad g_i := (G_i)_{d-d_i}$$

If $d_i > d$, $g_i = 0$

$$(f - \sum g_i f_i)_d \quad \text{v.s.} \quad (f - \sum G_i f_i)_d = 0$$

$$(G_i)_{d-d_i}$$

$$(f - \sum g_i f_i)_{d'} = 0 \quad d' \neq d$$

$$(f - \sum g_i f_i)_{d'} = 0 \quad \forall d' \quad \Rightarrow \quad f - \sum g_i f_i = 0$$

$$\Rightarrow f = \sum g_i f_i$$

[Sometimes, these rings are called $\mathbb{N}$-graded rings. $\mathbb{N}^i$ graded ring: elt. has degree $(n_1, \dots, n_i)$, n_j's ≥ 0

$\mathbb{Z}$-graded ring: $R = \bigoplus_{i=-\infty}^{\infty} R_i$]

[Laurent Polys: $k[x, x^{-1}]$ — $\mathbb{Z}$-graded ring.

$k[[x]]$ is not graded, because

$R_i = x^i k$ $\quad R = \prod_{i=0}^{\infty} R_i$, not $\bigoplus_{i=0}^{\infty} R_i$

$k((x))$ is somehow $\prod_{i=0}^{\infty} x^i k \oplus \bigoplus_{i=-\infty}^{-1} x^i k$

Let G be a finite group.

k a f of char. 0.

(ρ, V) is a rep. of G, i.e. $\rho : G \to GL(V)$

$G = S_n$, $V = k^n$, $\sigma(x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$

$\sigma = (123)$, $\sigma(3,4,5) = (4,5,3)$?

$$\left(\mathrm{Sym}^i(V^\vee)\right)^G := \{ f \in \mathrm{Sym}^i(V^\vee) \mid gf = f \ \forall g \in G \}$$

- G-invariant degree i forms on V.

$G = S_n \curvearrowright k^n$

1 $\quad \mathrm{Sym}^1(V^\vee)^G = \langle x_1 + x_2 + \cdots + x_n \rangle$

$\mathrm{Sym}^2(V^2)^G = \langle (x_1 + x_2 + \cdots + x_n)^2, \sum_{i<j} x_i x_j \rangle$

$$(\mathrm{Sym}^\bullet(V^\vee))^G \cong k[s_1, \dots, s_n]$$

s_i of degree i

$$s_1 = x_1 + \dots + x_n$$

$$s_2 = \sum_{i<j} x_i x_j$$

$$s_3 = \sum_{i<j<k} x_i x_j x_k$$

$$\vdots$$

$$s_n = x_1 \cdots x_n$$

What can be said about $(\mathrm{Sym}^\bullet(V^\vee))^G$?

Thm (Hilbert): This ring is finitely gen. as a ring over k.

$V^\vee$ - V dual,

$\mathrm{Sym}^i(W) = \{(w_1, \dots, w_i)\} / (w_1, \dots, w_i) \cdot (w_{\sigma(1)}, \dots, w_{\sigma(i)})$

If $W = \langle x_1, \dots, x_n \rangle$

$\mathrm{Sym}^2(W) \langle x_1 x_1, x_1 x_2, \dots, x_1 x_n, x_2 x_2, \dots, \dots \rangle$

$\otimes^i W$, $\mathrm{Sym}^i(W)$, $\bigwedge^i(W)$