Prop: Let R be noetherian, $U \subset R$ a mult. closed set containing 1. Then $R[U^{-1}]$ is noetherian as well.

Pf. $\varphi: R \to R[U^{-1}]$ $\varphi(r) = \frac{r}{1}$

Take $I \subset R[U^{-1}]$ an ideal. $I = (\varphi^{-1}(I))R[U^{-1}]$

If $r_1, \ldots, r_k$ generate $\varphi^{-1}(I)$, then $\varphi(r_1), \ldots, \varphi(r_k)$ generate I. ∎

Pf. 2: w.t.s. $R[u^{-1}]$ satisfies the ascending chain condition. If $I_1 \subset \cdots \subset I_k \subset \cdots$, $\exists n$ s.t. $I_k \subset I_n \ \forall k$.

φ^{-1} is an injection that preserves containment, to $\varphi^{-1}(I_1) \subset \cdots \subset \varphi^{-1}(I_k) \subset \cdots$

$\Rightarrow \exists n$ s.t. $\varphi^{-1}(I_n) \supset \varphi^{-1}(I_k) \ \forall k \Rightarrow$

$I_n \supset I_k \ \forall k$.

while being finitely generated / a field isn't closed under localization, being noetherian is. ($k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ is $\underline{\text{not}}$ f.g. over k, but still is noetherian.)

Prop: $R[u^{-1}]$ is flat as an R-mod.

(recall: M is flat as an R-mod iff

$0 \to A \to B \to C \to 0$ is exact $\Rightarrow$

$0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$ is exact,

sufficient to check $0 \to A \to B$

$\Rightarrow$ $0 \to A \otimes M \to B \otimes M$)

Pf: Lets assume $0 \to M \to N$ is exact

(eq. $M \overset{f}{\hookrightarrow} N$) w.t.s. $M \otimes R[u^{-1}] \hookrightarrow N \otimes R[u^{-1}]$

eq. $M[u^{-1}] \overset{f'}{\hookrightarrow} N[u^{-1}]$.

Assume $\frac{m}{u} \in \ker(f')$.

$$f'\left(\frac{m}{u}\right) = \frac{f(m)}{u} = 0 = \frac{0}{1}$$

$\Rightarrow \exists\, v$ s.t. $v f(m) = 0$

$\Leftrightarrow f(vm) = 0 \Rightarrow vm = 0$

$\Rightarrow \frac{m}{u} = \frac{0}{1}$ in $M[u^{-1}]$ $\blacksquare$

Cor: Let M be an R-mod, $M_1, \dots, M_n$ be submodules. Then $(\bigcap M_i)[u^{-1}] \cong \bigcap (M_i[u^{-1}])$

Pf: $\Delta : M \to \bigoplus_{i=1}^{n} M/M_i$

$$m \mapsto (m/M_1, \dots, m/M_n)$$

$$\ker(\Delta) = \bigcap_{i=1}^{n} M_i$$

$0 \to \bigcap_{i=1}^{n} M_i \to M \to \operatorname{im}(\Delta) \to 0$ is exact

$\Rightarrow \quad 0 \to \left(\bigcap M_i\right)[u^{-1}] \to M[u^{-1}] \to \operatorname{im}(\Delta)[u^{-1}] \to 0$ is too

$$\ker\left(M[u^{-1}] \to \operatorname{im}(\Delta)[u^{-1}]\right) = \bigcap_{i=1}^{n}\left(M_i[u^{-1}]\right)$$

N.B.: Argument only used $R[U^{-1}]$ is flat.

$R = k[x]$, $\quad U = \{1, x, \dots\}$

$M = R$, $\quad M_i = x^i R$

$\bigcap_{i=1}^{\infty} M_i = \{0\}$, but $M_i[U^{-1}] = M[U^{-1}]$,

so $\bigcap_{i=1}^{\infty} M_i[U^{-1}] = M$.

Thm

Let R be a ring, M an R-module.

a) $m \in M$ is 0 $\Leftrightarrow$ $\mathrm{im}(m) = 0$ in $M_{\mathfrak{m}}$ $\forall$ $\mathfrak{m}$ maximal ideals,

b) $M = 0$ $\Leftrightarrow$ $M_{\mathfrak{m}} = 0$ $\forall$ $\mathfrak{m}$ maximal.

$\left(R_{\mathfrak{m}} = R[u_{\mathfrak{m}}^{-1}], \ u_{\mathfrak{m}} = R \setminus \mathfrak{m} \right.$

$\left. M_{\mathfrak{m}} = M[u_{\mathfrak{m}}^{-1}] \right)$

Pf. Clearly a) $\Rightarrow$ b).

Just need to show a). Let $m \in M$.

Define $I = \text{ann}(m) = \{r \in R \mid rm = 0\}$.

Fact: $m = 0$ in $M_{\mathfrak{m}} \Leftrightarrow I \not\subset \mathfrak{m}$.

Pf: ($\Leftarrow$) Assume $I \not\subset \mathfrak{m}$. $\exists r \in I$, $r \notin \mathfrak{m}$. $\Rightarrow rm = 0 \Rightarrow \frac{m}{1} = 0$ in $M_{\mathfrak{m}}$

($r \in U_{\mathfrak{m}}$, $rm = 0 \Rightarrow \frac{m}{1} = 0$)

($\Rightarrow$) Assume $m = 0$ in $M_{\mathfrak{m}}$.

$\Rightarrow \exists\, r \in R \setminus \mathfrak{m}$ s.t. $rm = 0$

$\Rightarrow I \not\subseteq \mathfrak{m}$. $\checkmark$

Assume $\mathrm{im}(m) = 0$ in $M_{\mathfrak{m}}$ $\forall\, \mathfrak{m}$ maximal.

$I \not\subseteq \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$.

Zorn's lemma fact: Every $I \subset R$ is either equal to R or contained in a maximal ideal.

$\Rightarrow I = R \Rightarrow 1 \in \text{ann}(M) \Rightarrow 1m = 0 \Rightarrow m = 0$

Cor. Let $f: M \to N$ be a homomorphism. Then f is a monomorphism/epimorphism/isomorphism $\Leftrightarrow f_{\mathfrak{m}}: M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is $\forall \mathfrak{m}$.

monomorphism : fancy word for injection
epimorphism : —— " —— surjection.

Pf. of Cor: f is injective $\Leftrightarrow$ $\ker(f) = 0$
$\Leftrightarrow$ $\ker(f_{\mathfrak{m}}) = 0$ $\forall\, \mathfrak{m}$ $\Leftrightarrow$ $f_{\mathfrak{m}}$ is injective $\forall\, \mathfrak{m}$.

f is surjective $\Leftrightarrow$ $\operatorname{coker}(f) = 0$ $\Leftrightarrow$ …