Problem Session 1 Problems

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Sept. 14-18

This is a collection of problems that are starting points for discussion in a problem session.

Question 1. Write down an example of the following:

(a) A linear, first order differential equation

$$\frac{dy}{dt} = \alpha y$$

(b) A nonlinear, first order differential equation

$$\frac{dy}{dt} = \alpha y^2$$

(c) A linear, higher order differential equation

$$\frac{d^2y}{dt} + \frac{dy}{dt} + y = 0$$

(d) A nonlinear, higher order differential equation'

$$\frac{d^2y}{dt} + \frac{dy}{dt} + \cos(y) = 0$$

Question 2. The equation $\frac{dy}{dx} = y^4$ has a solution of the form $y(x) = ax^b$ for some real numbers a and b. Find a and b.

Is the same true of $\frac{dy}{dx} = y^3$?

If you have any issues with this you can ask me (William Bell, your problem session TA), but basically it is separation of variables plus a decent guess at the right part.

Question 3. The equation $\frac{dy}{dx} = y$ has all solutions of the from $y(x) = Ce^x$ for some constant C. Can you find all solutions to $\frac{dy}{dx} = 3y$? $\frac{dy}{dx} = ky$ for any constant k?

The solution to $y' = 3y$ is $y(x) = Ce^{3x}$, and to $y' = ky$ as you might be able to pattern match it is $y(x) = Ce^{kx}$.

Question 4. Find some solutions to $\dfrac{d^2y}{dx^2} + y = 0$. If I tell you that I'm thinking of a solution to this equation that has $y(0) = 1$, can you tell me the solution I'm thinking of? If so, what is it? If not, what else would you need to know?

If you consider $\dfrac{d^2y}{dx^2} + y = 0$, we see there are two solutions, $y(x) = \sin(x)$ and $y(x) = \cos(x)$, but you can also make more solutions by any combination $y(x) = c_1 \sin(x) + c_2 \cos(x)$, we see we can satisfy the initial condition if $c_2 = 1$, but that still leaves c_1 as a free parameter representing multiple possible solutions. There are multiple pieces of information we could use to find c_1, such as $\frac{dy}{dt}$ at any point or $y(r)$, r some real number that isn't a multiple of two pi.

Question 5. Consider the equation $\dfrac{dy}{dx} = \cos(y)$. If $y(0) = 1$, what is the smallest lower bound you can give for $y(t)$?

The best lower bound is $\frac{\pi}{2}$, because starting from that initial condition the value will increase closer and closer to $\frac{\pi}{2}$ since $\frac{dy}{dt} > 0$ until that point. But since $\cos(\frac{\pi}{2}) = 0$, that is an equilibrium and it cannot pass through the equilibrium.

Question 6. Consider the differential equation $\dfrac{dy}{dx} = f(x)$ for some continuous function $f : \mathbb{R} \to \mathbb{R}$. What is the solution to this equation?

Its anti-derivative, this is what first-year calculus studies. Anti-derivatives are not always known closed form, but we can always simply write $y(x) = y(0) + \int_0^x f(x)\, dx$ which is better than for the vast majority of differential equations as far as either closed form or numerical solutions go.