This week's riddler is about marking a die:

Question 1. *I have an unlabeled, six-sided die. I make a single mark on the middle of each face that is parallel to one pair of that face's four edges.*

How many unique ways are there for me to mark the die? (Note: If two ways can be rotated so that they appear the same, then they are considered to be the same marking.)

Extra credit*: Suppose you can also mark a face along one of its two diagonals. Now how many unique ways are there to mark the die?*

This is a classic Burnside's Lemma problem. Let G be the group of rotations of the unit cube, and let X be the set of ways to mark the die not up to rotation (so there are $2^6 = 64$ elements of X). Then the number of orbits in X under the action of G is $\frac{1}{|G|}\sum_{g\in G}|X^g|$ where X^g is the set of elements of X that are fixed by g[1].

Then one can just list out what happens for each type of rotation:

Type of rotation	Number of such rotations	Size of stabilizer
Identity	1	64
90 degree face rotation	6	0
180 degree face rotation	3	16
120 degree corner rotation	8	4
180 degree edge rotation	6	8

This gives that the number of ways to cast a die is $\frac{64+48+32+48}{24} = 8$.

For the extra credit, the calculation is similar:

Type of rotation	Number of such rotations	Size of stabilizer
Identity	1	4096
90 degree face rotation	6	0
180 degree face rotation	3	256
120 degree corner rotation	8	16
180 degree edge rotation	6	64

This gives the number of such ways to cast a bonus die is $\frac{4096+768+128+384}{24} = 224$.

[1]The proof of this is almost immediate from the orbit stabilizer theorem: $|G|$ times the number of orbits is $\sum_{O\in G\backslash X}|G| = \sum_{x\in X}|Stab(x)| = \sum_{\substack{x\in X, g\in G\\ gx=x}} 1 = \sum_{g\in G}|X^g|$.