Homework 1

MATH 218: Differential Equations for Engineers

Due: September 25^{th} at 11:59 PM EST

This assignment is based primarily on the material in unit 2.

After you have completed the assignment, please save, scan, or take a photo of your work and upload it to crowdmark. Crowdmark accepts pdf, jpg, and png file formats.

As a reminder, it is fine to discuss the problems with other students. However, you must write up your own solutions.

Question 1 (8 points)**.** For each of the following differential equations, answer the following questions (with justification):

- Is the DE linear?
- Is the DE separable?
- Is the DE exact (without doing anything complicated, e.g. integrating factors)?
- If given the initial condition $y(0) = 1$, do the existence and uniqueness theorems guarantee a unique solution?

(a) $\dfrac{dy}{dx} = y^2 + x^2y^2$

(b) $\dfrac{dy}{dx} = x^3 + 3xy$

(c) $3y^2\dfrac{dy}{dx} = 3x^2 + y + x\dfrac{dy}{dx}$

(d) $\dfrac{dy}{dx} = x^2 + y^2$

Solution 1. We have the following solutions:

(a) $\dfrac{dy}{dx} = y^2 + x^2y^2$

- This DE is non-linear:

$$\frac{dy}{dx} = y^2 + x^2y^2 \neq A(x)y + B(x)$$

There is no such $A(x)$ and $B(x)$ for this equality to stand true therefore this DE is linear.

- This DE is separable:

$$\begin{aligned}
\frac{dy}{dx} &= y^2 + x^2y^2 \\
\frac{dy}{dx} &= (1+x^2)y^2 \\
\frac{dy}{dx} &= A(x)B(y) \quad \text{where} \quad A(x) = 1 + x^2 \quad \text{and} \quad B(y) = y^2
\end{aligned}$$

- This DE is not exact:

$$\begin{aligned}
\frac{dy}{dx} &= y^2 + x^2y^2 \\
0 &= -y^2 - x^2y^2 + \frac{dy}{dx} \\
0 &= M(x,y) + N(x,y)\frac{dy}{dx}
\end{aligned}$$

$$M(x,y) = -y^2 - x^2y^2 \qquad N(x,y) = 1$$

$$\frac{\partial M}{\partial y} = -2y - 2x^2y$$

$$\frac{\partial N}{\partial x} = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

- The existence-uniqueness theorem does guarantee a solution:

$$\frac{dy}{dx} = f(x,y)$$

$$f(x,y) = y^2 + x^2y^2$$

$$\frac{\partial f}{\partial x} = 2xy^2 \qquad \frac{\partial f}{\partial y} = 2y + 2x^2y$$

The function $f(x,y)$ is differentiable and we can see that its partials are continuous:

$$\therefore f \in C^1$$

Since we are given an IC (initial condition) and since $f \in C^1$, by the existence-uniqueness theorem, there is a unique solution to this DE.

(b) $\dfrac{dy}{dx} = x^3 + 3xy$

- This DE is linear:

$$\frac{dy}{dx} = x^3 + 3xy = A(x)y + B(x)$$
$$A(x) = 3x \qquad B(x) = x^3$$

- This DE is not separable:

$$\begin{aligned}\frac{dy}{dx} &= x^3 + 3xy \\ &= y\left(\frac{1}{y}x^3 + 3x\right) \\ &= x(x^2 + 3y) \\ &= x^3\left(1 + 3\frac{1}{x^2}y\right) \\ &\neq A(x)B(y)\end{aligned}$$

  There is no such $A(x)$ and $B(x)$ for this last equality to stand true.

- This DE is not exact:

$$\begin{aligned}\frac{dy}{dx} &= x^3 + 3xy \\ 0 &= -x^3 - 3xy + \frac{dy}{dx} \\ 0 &= M(x,y) + N(x,y)\frac{dy}{dx}\end{aligned}$$

$$M(x,y) = -x^3 - 3xy \qquad N(x,y) = 1$$
$$\frac{\partial M}{\partial y} = -3x$$
$$\frac{\partial N}{\partial x} = 0$$
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

- The existence-uniqueness theorem does guarantee a solution:

$$\frac{dy}{dx} = f(x,y)$$
$$f(x,y) = x^3 + 3xy$$
$$\frac{\partial f}{\partial x} = 3x^2 + 3y \qquad \frac{\partial f}{\partial y} = 3x$$

  The function $f(x,y)$ is differentiable and we can see that its partials are continuous:

$$\therefore f \in C^1$$

  Since we are given an IC (initial condition) and since $f \in C^1$, by the existence-uniqueness theorem, there is a unique solution to this DE.

(c) $3y^2\dfrac{dy}{dx} = 3x^2 + y + x\dfrac{dy}{dx}$

- This DE is non-linear:

$$\begin{aligned} 3y^2\frac{dy}{dx} &= 3x^2 + y + x\frac{dy}{dx} \\ \frac{dy}{dx}(3y^2 - x) &= 3x^2 + y \\ \frac{dy}{dx} &= \frac{3x^2 + y}{3y^2 - x} \\ &\neq A(x)y + B(x) \end{aligned}$$

There is no $A(x)$ and $B(x)$ for that equality to stand true.

- This DE is not separable:

$$\begin{aligned} \frac{dy}{dx} &= \frac{3x^2 + y}{3y^2 - x} \\ &\neq A(x)B(y) \end{aligned}$$

It is impossible to separate functions of x and y through partial fractions or other techniques. Therefore, there is no $A(x)$ and $B(y)$ for this equality to be true.

- This DE is exact:

$$\begin{aligned} 3y^2\frac{dy}{dx} &= 3x^2 + y + x\frac{dy}{dx} \\ 0 &= -3x^2 - y + (3y^2 - x)\frac{dy}{dx} \\ 0 &= M(x,y) + N(x,y)\frac{dy}{dx} \end{aligned}$$

$$\begin{gathered} M(x,y) = -3x^2 - y \qquad N(x,y) = 3y^2 - x \\ \frac{\partial M}{\partial y} = -1 \\ \frac{\partial N}{\partial x} = -1 \\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \end{gathered}$$

- The existence-uniqueness theorem does guarantee a solution:

$$\begin{aligned} \frac{dy}{dx} &= f(x,y) \\ f(x,y) &= \frac{3x^2 + y}{3y^2 - x} \\ \frac{\partial f}{\partial x} &= \frac{\frac{\partial}{\partial x}(3x^2 + y)(3y^2 - x) - \frac{\partial}{\partial x}(3y^2 - x)(3x^2 + y)}{(3y^2 - x)^2} \\ &= \frac{6x(3y^2 - x) - (-1)(3x^2 + y)}{(3y^2 - 1)^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{18xy^2 - 3x^2 + y}{(3y^2 - x)^2} \\
\frac{\partial f}{\partial y} &= \frac{\frac{\partial}{\partial y}(3x^2 + y)(3y^2 - x) - \frac{\partial}{\partial y}(3y^2 - x)(3x^2 + y)}{(3y^2 - x)^2} \\
&= \frac{(3y^2 - x) - 6y(3x^2 + y)}{(3y^2 - x)^2} \\
&= \frac{-3y^2 - 18x^2y - x}{(3y^2 - x)^2}
\end{aligned}
$$

We see here that our partials are not continuous when $y(x) = \pm\sqrt{x/3}$ but we know that $y(0) = 1$, we test the initial condition:

$$
\begin{aligned}
y(x) &= \pm\sqrt{\frac{x}{3}} \\
y(0) &= \pm\sqrt{\frac{1}{3}} \\
1 &\neq \pm\sqrt{\frac{1}{3}}
\end{aligned}
$$

We see here that it is impossible for our partials to be discontinuous, and so:

$$\therefore f(x, y) \in C^1$$

Our initial condition confirmed that our partials are continuous and that $f \in C^1$. By the existence-uniqueness theorem, this means that there is a unique solution.

(d) $\dfrac{dy}{dx} = x^2 + y^2$

- This DE is non-linear:

$$
\begin{aligned}
\frac{dy}{dx} &= x^2 + y^2 \\
&\neq A(x)y + B(x)
\end{aligned}
$$

There is no $A(x)$ and $B(x)$ for that equality to stand true.

- This DE is not separable:

$$
\begin{aligned}
\frac{dy}{dx} &= x^2 + y^2 \\
&= x^2\left(1 + \frac{y^2}{x^2}\right) \\
&= y^2\left(\frac{x^2}{y^2} + 1\right) \\
&\neq A(x)B(y)
\end{aligned}
$$

There is no $A(x)$ and $B(y)$ for this equality to be true.

- This DE is not exact:

$$\begin{aligned}\frac{dy}{dx} &= x^2 + y^2\\ 0 &= -x^2 - y^2 + \frac{dy}{dx}\\ 0 &= M(x,y) + N(x,y)\frac{dy}{dx}\end{aligned}$$

$$M(x,y) = -x^2 - y^2 \qquad N(x,y) = 1$$
$$\frac{\partial M}{\partial y} = -2y$$
$$\frac{\partial N}{\partial x} = 0$$
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

- The existence-uniqueness theorem does guarantee a solution:

$$\begin{aligned}\frac{dy}{dx} &= f(x,y)\\ f(x,y) &= x^2 + y^2\\ \frac{\partial f}{\partial x} &= 2x\\ \frac{\partial f}{\partial y} &= 2y\end{aligned}$$

  The function $f(x,y)$ is differentiable and we can see that its partials are continuous:

$$\therefore f \in C^1$$

  Since we are given an IC (initial condition) and since $f \in C^1$, by the existence-uniqueness theorem, there is a unique solution to this DE.

Question 2 (6 points)**.** Sometimes, it's possible to deduce information about solutions to a differential equation without actually solving it. For both of the following equations, find all trivial solutions (i.e. solutions of the form $y(x) = C$ for some constant C). Additionally, find all inflection points not on trivial solutions. You do not need to solve the equations.

(a) $\dfrac{dy}{dx} = y^3 - y$

(b) $\dfrac{dy}{dx} = \sin(y)$

Solution 2. We have the following solutions:

(a) $\dfrac{dy}{dx} = y^3 - y$

We assume a trivial solution of the form:

$$\begin{aligned} y(x) &= C \\ \frac{dy}{dx} &= 0 \\ 0 &= C^3 - C \\ C &= C^3 \end{aligned}$$

By inspection this is only true when $C = -1, 0, 1$. The trivial solutions are therefore $y = -1, 0, 1$. We have inflection points when $\frac{d^2y}{dx^2} = 0$:

$$\begin{aligned} \frac{dy}{dx} &= y^3 - y \\ \frac{d^2y}{dx^2} &= 3y^2\frac{dy}{dx} - \frac{dy}{dx} \\ 0 &= (3y^2 - 1)(y^3 - y) \\ &\text{If the first term is 0:} \\ 0 &= 3y^2 - 1 \\ \frac{1}{3} &= y^2 \\ y &= \pm\frac{1}{\sqrt{3}} \\ &\text{If the second term is 0:} \\ 0 &= y^3 - y \\ y &= y^3 \\ y &= -1, 0, 1 \end{aligned}$$

We have inflection points at $y = \pm\frac{1}{\sqrt{3}}$ that are not on trivial solutions.

(b) $\dfrac{dy}{dx} = \sin(y)$

We assume a trivial solution of the form:

$$
\begin{aligned}
y(x) &= C \\
\frac{dy}{dx} &= 0 \\
0 &= \sin(C) \\
C &= \arcsin(0) \\
C &= n\pi \quad n \in \mathbb{Z}
\end{aligned}
$$

We see that our trivial solutions are $y = n\pi \quad n \in \mathbb{Z}$. We have inflection points when $\frac{d^2y}{dx^2} = 0$:

$$
\begin{aligned}
\frac{dy}{dx} &= \sin(y) \\
\frac{d^2y}{dx^2} &= \cos(y)\frac{dy}{dx} \\
0 &= \cos(y)\sin(y) \quad \cos(y) = 0 \text{ or } \sin(y) = 0 \\
y &= \arccos(0) \\
y &= \frac{(2n+1)\pi}{2} \quad n \in \mathbb{Z} \\
y &= \arcsin(0) \\
y &= n\pi \quad n \in \mathbb{Z}
\end{aligned}
$$

We have inflection points at $y = \frac{(2n+1)\pi}{2} \quad n \in \mathbb{Z}$ that are not on trivial solutions.

Question 3 (6 points)**.** Let $a > 0$ be a positive constant. Assume that $y(0) = 1$ and $\dfrac{dy}{dx} = y^a$. Clearly one has that $y(x)$ increases more and more rapidly as x increases. For which values of a does $y(x)$ get to infinity in finite time? That is, for which values of a does this function have a vertical asymptote and is thus not defined for all x values?

Solution 3.

$$\begin{aligned}
\frac{dy}{dx} &= y^a \\
y^{-a}dy &= dx \\
\int y^{-a}dy &= \int dx \\
\frac{y^{-a+1}}{-a+1} &= x + C \\
y(0) &= 1 \\
\frac{1^{-a+1}}{-a+1} &= C \\
C &= \frac{1}{1-a} \\
\frac{y^{1-a}-1}{1-a} &= x
\end{aligned}$$

We see first that when $a = 1$, we get an exception, we solve this case differently:

$$\begin{aligned}
\frac{dy}{dx} &= y \\
\frac{1}{y}dy &= dx \\
\int \frac{1}{y}dy &= \int dx \\
\ln y &= x + C \\
y &= e^{x+C} \quad e^C = A \\
y &= Ae^x \\
y(0) &= 1 \\
A &= 1 \\
y &= e^x
\end{aligned}$$

We continue with our original equation:

$$\begin{aligned}
\frac{y^{1-a}-1}{1-a} &= x \\
y &= ((1-a)x+1)^{\frac{1}{1-a}} \\
b &= \frac{1}{1-a} \\
y &= ((1-a)(x+b))^b
\end{aligned}$$

Here, we see that this function is well defined for any values of $b > 0$. On the other hand, the function is not defined when $b < 0$ and $x = -b$.

$$\begin{aligned} b < 0 &\iff \frac{1}{1-a} < 0 \\ &\iff 1 - a < 0 \\ &\iff a > 1 \end{aligned}$$

The solution reaches infinity at finite times when $a > 1$. This infinity is reached at $x = \frac{1}{1-a}$.

Question 4 (10 points)**.** You decide that you want to inflate a balloon with helium. The balloon has a volume of $10000cm^3$ when fully inflated, and currently has $1000cm^3$ of regular air in it. You hook it up to a helium pump, which will feed in $200cm^3$ of air that is 90% helium and 10% regular air per second. However, you don't attach the balloon tightly, so $100cm^3$ of air in the balloon leaks out every second. Assume that the air in the balloon (and hence the air that leaks out) is perfectly homogenized. How much helium is in the balloon when it's full?

An important thing to notice here: the volume in the balloon isn't remaining constant over time. You need to account for that when setting up your differential equation.

Solution 4. This problem is very similar to the mixing problem on page 3 of the course notes. We start by denoting the change in volume of helium like this:

$$\frac{dm}{dt} = r_{in} - r_{out}$$

Where r_{in} and r_{out} are the rates at which helium gets in or out of the balloon. From the context we know that every second, $180cm^3$ of Helium flows into the balloon. Helium flows out of the balloon through the homogenized air where the concentration of helium depends on the volume, mathematically:

$$r_{out} = 100\frac{m(t)}{V(t)}$$

From the context, we see that every second, the balloon gains $200cm^3$ and loses $100cm^3$ of gas. The balloon also starts with an original volume of $1000cm^3$. We then define the volume:

$$\begin{aligned} V(t) &= 200t - 100t + 1000 \\ V(t) &= 100t + 1000 \end{aligned}$$

We put everything together:

$$\begin{aligned} \frac{dm}{dt} &= 180 - 100\frac{m}{100t + 1000} \\ \frac{dm}{dt} &= 180 - \frac{m}{t + 10} \end{aligned}$$

We can now solve using integrating factors:

$$\begin{aligned} &\frac{dm}{dt} + \frac{1}{t+10}m = 180 \\ &\frac{dm}{dt}(t+10) + m = 180(t+10) \\ &\frac{d}{dt}\left((t+10)m\right) = 180(t+10) \\ &\int \frac{d}{dt}\left((t+10)m\right) dt = \int 180(t+10)dt \\ &(t+10)m = 180\int (t+10)dt \\ &(t+10)m = 180\left[\frac{t^2}{2} + 10t + C\right] \end{aligned}$$

$$(t+10)m = 90(t^2+20t)+C$$
$$m(t) = 90\frac{t^2+20t}{t+10} + \frac{C}{t+10}$$

Where C is our constant of integration. We know that at $t = 0$, we have a volume of $m = 0$ of helium in the balloon:

$$0 = 90\frac{0}{10} + \frac{C}{10}$$
$$0 = C$$

Therefore, our equation for $m(t)$:

$$m(t) = 90\frac{t^2+20t}{t+10}$$

Using our equation for volume, we find the time at which the balloon is full, $V(t) = 10000$:

$$10000 = 100t + 1000$$
$$9000 = 100t$$
$$t = 90$$

We plug this into our equation:

$$m(90) = 90\frac{(90)^2+20(90)}{90+10}$$
$$m(90) = 90\frac{9900}{100}$$
$$m(90) = 8910$$

When the balloon is full, there is $8910cm^3$ of helium in it.

Question 5 (10 points)**.** *Newton's law of heating/cooling* states that the rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the ambient temperature, that is, if $T(t)$ is the temperature of the object and T_a is the ambient temperature, then $\frac{dT}{dt} = k(T - T_a)$ for some constant k (which is negative: objects tend towards the ambient temperature after all!). Assume that you have a piece of metal that has been heated to $650°C$ and it is now resting in a room that has an ambient temperature of $25°C$. After one minute, the temperature of the object is now $525°C$. How long does it take for the object to cool down to $50°C$?

Solution 5. We solve this separable DE:

$$\begin{aligned}
\frac{dT}{dt} &= k(T - T_a) \\
\frac{1}{T - T_a} dT &= k dt \\
\int \frac{1}{T - T_a} dT &= k \int dt \\
\ln(T - T_a) + C &= kt \\
\ln(T - T_a) &= kt - C \\
T - T_a &= e^{kt - C} \\
T &= e^{kt-C} + T_a \\
e^{-C} &= A \\
T(t) &= Ae^{kt} + T_a
\end{aligned}$$

We can find our value of A using the initial conditions given:

$$\begin{aligned}
T(0) = 650 &= Ae^0 + 25 \\
625 &= A \\
T(t) &= 625e^{kt} + 25
\end{aligned}$$

Using this equation and the values given in the question, we can find the value of k:

$$\begin{aligned}
T(1) = 525 &= 625e^k + 25 \\
500 &= 625e^k \\
\frac{500}{625} = \frac{4}{5} &= e^k \\
k &= \ln\left(\frac{4}{5}\right) \\
T(t) &= 625e^{\ln\left(\frac{4}{5}\right)t} + 25 \\
&= 625e^{\ln\left[\left(\frac{4}{5}\right)^t\right]} + 25 \\
&= 625\left(\frac{4}{5}\right)^t + 25
\end{aligned}$$

Now that we have solved for every unknown, we can find the time necessary for the object to cool down to $50°C$:

$$T = 50 = 625\left(\frac{4}{5}\right)^t + 25$$

$$
\begin{aligned}
25 &= 625\left(\frac{4}{5}\right)^t \\
\frac{25}{625} = \frac{1}{25} &= \left(\frac{4}{5}\right)^t \\
\ln\left(\frac{1}{25}\right) &= t\ln\left(\frac{4}{5}\right) \\
t &= 14.425
\end{aligned}
$$

It will take approximately 14 minutes and 26 seconds for the object to cool to this temperature.

Question 6 (15 points)**.** While Waterloo has a population of around 113000, to make numbers easier, assume that Waterloo actually has a population of 100000 for this problem.

Assume that someone with a virus comes back to Waterloo. The logistic model says that the number of people that get infected is proportional to the number of currently infected people times the ratio of healthy people to all people in a population. Let $y(t)$ be the number of infected people as a function of t.

(a) Assuming that $y(0) = 1$, without doing any calculation, what should $\lim_{t\to\infty} y(t)$ be? What about $\lim_{y\to-\infty} y(t)$? Why?

(b) Write down a differential equation describing y. There will be a constant of proportionality in your equation.

(c) As in problem 2, find the trivial solutions to this differential equation, as well as the y value of the inflection point.

(d) Find all solutions to this eqation. You should have the constant of porportionality floating around, as well as another constant coming from all solutions of the differential equation.

(e) Choose a particular value for the constant in part (*b*), and graph a handful of solutions to the equation. I want you to think about what "intersting" solutions are, and how changing the constant that came out of solving the differential equation changes the graph.

(f) Now, assume that $y(0) = 1$. What does changing the constant in part (*b*) do? Plot some graphs with various choices of this constant.

N.B. Some of you may use computer graphing programs to do this problem. That's good! I want you to use tools to help you visualize solutions to problems. The goal of this problem is to think about what the graphs should look like, and then see if that aligns with what it does.

Solution 6. We have the following solutions:

(a) We have that, as time goes to infinity, $y(t) = 100000$. Our model does not include any method of recovery and so, if people can get infected, all of them will eventually be infected. As time goes to negative infinity, we have that $y(t) = 0$.

(b) From the question, we have:

$$\frac{dy}{dt} \propto y\left(1 - \frac{y}{100000}\right)$$
$$\frac{dy}{dt} = \alpha y\left(1 - \frac{y}{100000}\right)$$

Where α is our constant of proportionality.

(c) We find the trivial solutions:

$$y = C$$

$$\frac{dy}{dt} = 0$$
$$\alpha C\left(1 - \frac{C}{100000}\right) = 0 \quad \alpha \neq 0$$
$$C\left(1 - \frac{C}{100000}\right) = 0$$
$$C = 0 \text{ or } C = 100000$$

The trivial solutions are $C = 0, 100000$. We find the inflection points:

$$\frac{dy}{dt} = \alpha y - \frac{\alpha y^2}{100000}$$
$$\frac{d^2y}{dt^2} = \alpha\frac{dy}{dx} - \frac{\alpha y}{50000}\frac{dy}{dx}$$
$$0 = \left(\alpha - \frac{\alpha y}{50000}\right)\left(\alpha y - \frac{\alpha y^2}{100000}\right)$$

In the case where the first term $= 0$

$$\alpha = \frac{\alpha y}{50000}$$
$$1 = \frac{y}{50000}$$
$$y = 50000$$

In the case where the second term $= 0$

$$0 = \alpha y\left(1 - \frac{y}{100000}\right) \quad \alpha \neq 0$$
$$0 = y\left(1 - \frac{y}{100000}\right)$$
$$y = 0, 100000$$

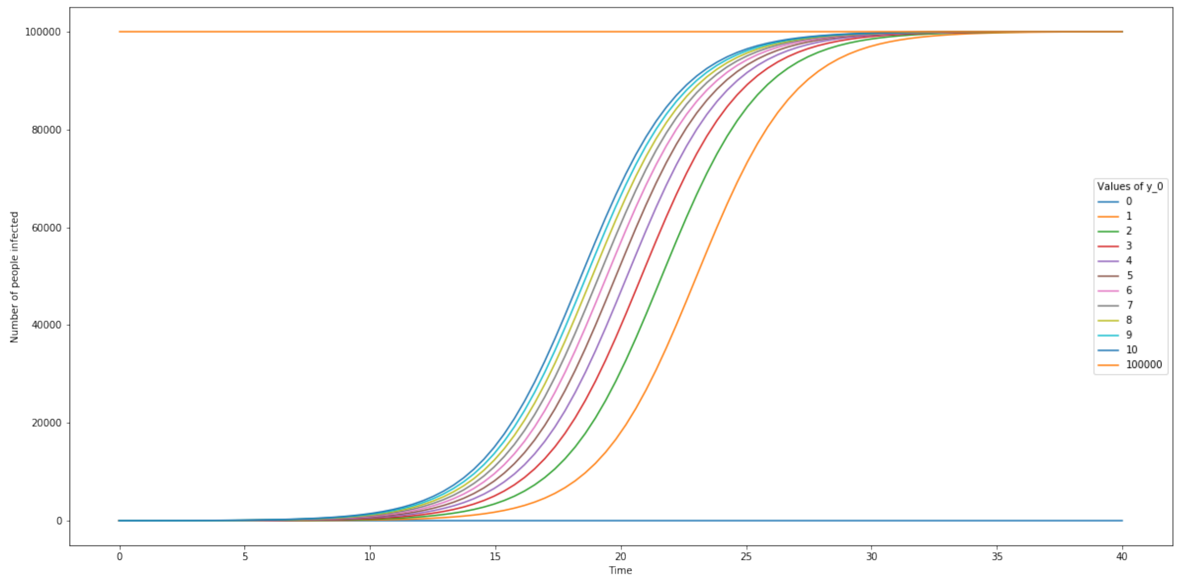
We have a inflection points when $y = 0, 50000, 100000$

(d) We solve by making a change of variable:

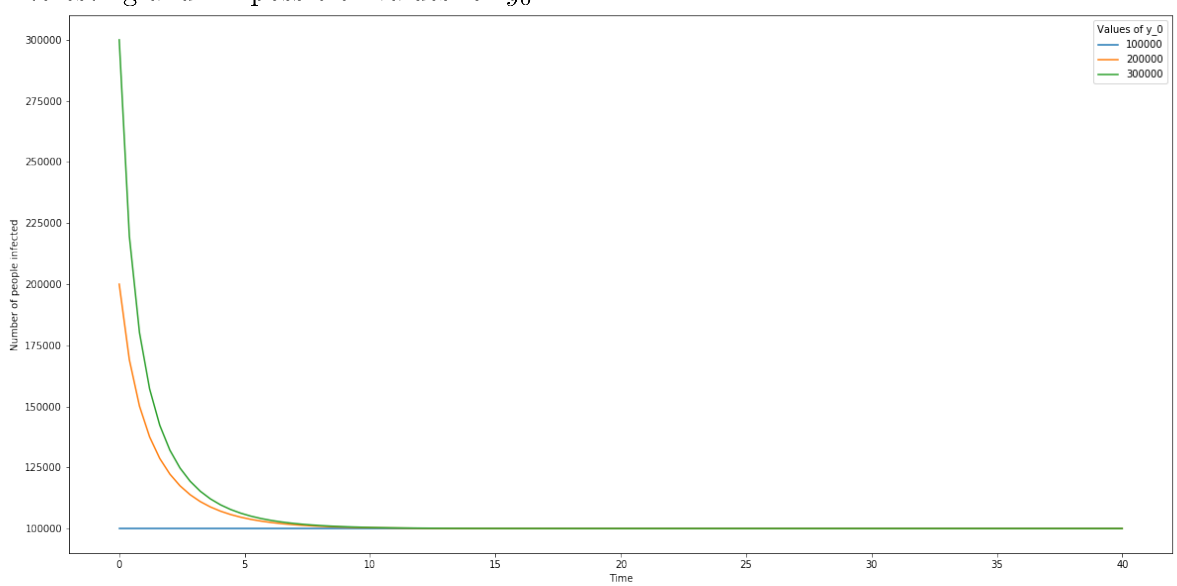
$$x = \frac{100000}{y}$$
$$\frac{dx}{dt}(x-1) = -\alpha(x-1)$$
$$y(t) = \frac{y_0 e^{\alpha t}}{1 - \frac{y_0}{100000} + \frac{y_0}{100000}e^{\alpha t}}$$

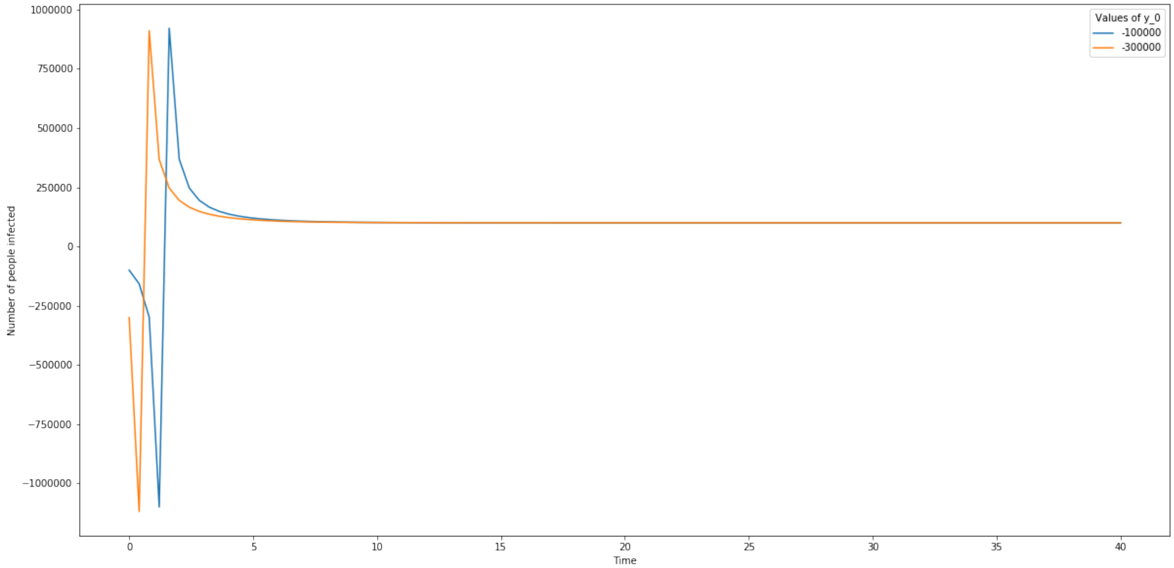
Where α is our constant of proportionality and y_0 is our constant that comes from every solution.

(e) In the following graph, I have plotted 10 different values for y_0 assuming $\alpha = 0.5$:

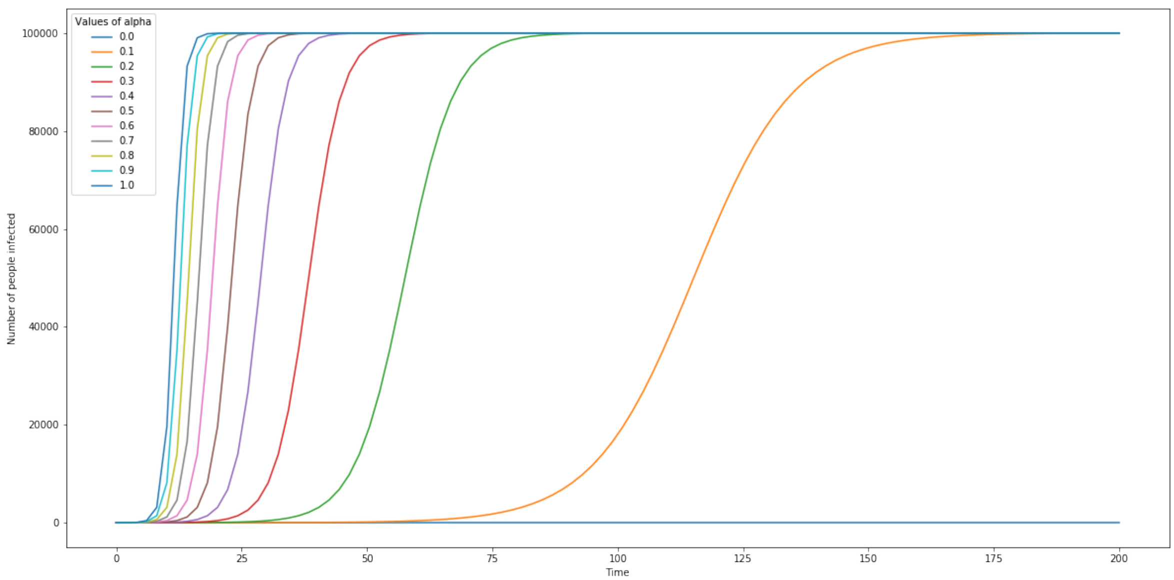


We see that as we increase our value of y_0, the spread of the virus becomes faster, this makes sense as this is the same as increasing the number of initially infected people in our situation, which will inevitably result in a faster growth of the virus. A couple interesting solutions are when $y_0 = 0$: no one gets infected as no one is infected initially and $y_0 = 100000$ where no one gets infected as everyone was initially infected. Additionally here are a couple more interesting and "impossible" values for y_0:





(f) In the following graph, I have plotted 10 different values for α assuming $y_0 = 1$:



Notice the change in scale for time compared to the previous graph. We see in this graph that the value of α corresponds to the exponential speed of the spread of the virus. In 'real life' this α is actually called R_0 and represents the ability of a population to restrain a virus, or in other words, how fast is the virus transmitted from person to person.

Question 7 (20 points)**.** Two chemicals (let's call them A and B) are being mixed to produce a third chemical (which we'll call C). Initially (at $t = 0$), one litre of A and one litre of B are mixed together. Assume that, for every unit of C is made up of a units of A and b units of B, where a and b are constants such that $a + b = 1$ and $a < b$ (if this much abstraction is difficult, significant partial credit will be awarded if you only think about the case where $a = \frac{1}{3}$ and $b = \frac{2}{3}$ in parts $(a) - (g)$). Additionally, assume that the amount of C produced at time t is porportional to the amount of A and the amount of B at time t. Write $A(t)$, $B(t)$, and $C(t)$ for the amount of A, B, and C at time t respectively.

(a) Before solving any differential equations, what should $\lim_{t\to\infty} C(t)$ be? Why?

(b) Solve for $A(t)$ and $B(t)$ in terms of $C(t)$.

(c) Write a differential equation that is satisfied by $C(t)$ that only involves $C(t)$ and $\frac{dC}{dt}$ (i.e. no higher derivitaves and no use of $A(t)$ or $B(t)$ in your final answer). There will be a constant of porportionality in your equation.

(d) Additionally, write down the initial condition that this equation satisfies.

(e) Solve for $C(t)$, keeping in mind both the constant of porportionality and the initial condition.

(f) Compute $\lim_{t\to\infty} C(t)$. Is this equal to the answer you got in part (a)?

(g) Assume that $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Additionally, assume at $t = 1$ minute, half a litre of C has been formed. At what value of t will 1 litre of C have formed?

(h) You may have noticed that everything breaks when $a = b = \frac{1}{2}$. Solve the equation in this case as well.

Solution 7. Here are the solutions:

(a) We have that as time tends to infinity, $C(t)$ tends to $\frac{1}{b}$. This makes sense because, there will be at least 1 liter of final product, and the additional product will be equal to $\frac{a}{b}$ since all of B will react and $\frac{a}{b}$ of A will react:

$$\begin{aligned}1 + \frac{a}{b} &= 1 + \frac{1-b}{b} \quad a = 1 - b \\ &= 1 + \frac{1}{b} - \frac{b}{b} \\ &= \frac{1}{b}\end{aligned}$$

(b) The solution for $A(t)$ and $B(t)$ are:

$$\begin{aligned}A(t) &= 1 - aC(t) \\ B(t) &= 1 - bC(t)\end{aligned}$$

(c) From the context, we have that:

$$\frac{dC}{dt} = kA(t)B(t)$$

Where k is our constant of proportionality. We input our answers from b):

$$\frac{dC}{dt} = k(1 - aC(t))(1 - bC(t))$$

(d) The initial conditions that this satisfies are:

$$\begin{aligned} C(0) &= 0 \\ A(0) &= 1 \\ B(0) &= 1 \end{aligned}$$

(e) We solve for C(t):

$$\begin{aligned}
\frac{1}{(1 - aC(t))(1 - bC(t))} dC &= kdt \\
\int \frac{1}{(1 - aC(t))(1 - bC(t))} dC &= \int kdt \\
\frac{1}{b-a} \ln\left(\frac{aC-1}{bC-1}\right) &= kt + h \\
\frac{aC-1}{bC-1} &= e^{(b-a)kt+h} \\
C(t) &= \frac{He^{(b-a)kt} - 1}{bHe^{(b-a)kt} - a} \\
C(0) &= 0 \\
C(0) &= \frac{He^{0} - 1}{bHe^{0} - a} \\
0 &= H - 1 \\
H &= 1 \\
C(t) &= \frac{e^{(b-a)kt} - 1}{be^{(b-a)kt} - a}
\end{aligned}$$

(f) We find the limit using l'Hospital's rule:

$$\begin{aligned}
\lim_{t\to\infty} \frac{e^{(b-a)kt} - 1}{be^{(b-a)kt} - a} &= \lim_{t\to\infty} \frac{\frac{d}{dt}\left(e^{(b-a)kt} - 1\right)}{\frac{d}{dt}\left(be^{(b-a)kt} - a\right)} \\
&= \lim_{t\to\infty} \frac{(b-a)e^{(b-a)kt}}{b(b-a)e^{(b-a)kt}} \\
&= \lim_{t\to\infty} \frac{1}{b} \\
&= \frac{1}{b}
\end{aligned}$$

This is what we found in (a).

(g) We have the following:

$$\begin{aligned}
b - a &= \frac{1}{3} \\
C(t) &= \frac{3e^{kt/3} - 3}{2e^{kt/3} - 1} \\
C(1) &= 0.5 \\
0.5 &= \frac{3e^{k/3} - 3}{2e^{k/3} - 1} \\
e^{k/3} - 0.5 &= 3e^{k/3} - 3 \\
2e^{k/3} &= 2.5 \\
e^{k/3} &= \frac{5}{4} \\
k &= \ln\left(\frac{125}{64}\right)
\end{aligned}$$

And so we have the following equation in our circumstances:

$$\begin{aligned}
C(t) &= \frac{3\left(\frac{5}{4}\right)^t - 3}{2\left(\frac{5}{4}\right)^t - 1} \\
1 &= \frac{3\left(\frac{5}{4}\right)^t - 3}{2\left(\frac{5}{4}\right)^t - 1} \\
2\left(\frac{5}{4}\right)^t - 1 &= 3\left(\frac{5}{4}\right)^t - 3 \\
\left(\frac{5}{4}\right)^t &= 2 \\
t\ln\left(\frac{5}{4}\right) &= \ln(2) \\
t &= \frac{\ln(2)}{\ln\left(\frac{5}{4}\right)} \\
t &\approx 3.11
\end{aligned}$$

There will be 1 liter of C at approximately 3 minutes and 7 seconds.

(h) We have:

$$\begin{aligned}
\frac{dC}{dt} &= k\left(1 - \frac{1}{2}C\right)^2 \\
\frac{1}{\left(1 - \frac{1}{2}C\right)^2} dC &= kdt \\
\int \frac{1}{\left(1 - \frac{1}{2}C\right)^2} dC &= \int kdt \\
-\frac{4}{C - 2} &= kt + H \\
-4 &= (C - 2)(kt + H)
\end{aligned}$$

$$
\begin{aligned}
-4 &= Ckt - 2kt + CH - 2H \\
C(kt+H) &= 2(kt+H) - 4 \\
C(t) &= \frac{2(kt+H)-4}{kt+H} \\
C(0) &= 0 \\
0 &= \frac{2H-4}{H} \quad H \neq 0 \\
4 &= 2H \\
H &= 2 \\
C(t) &= \frac{2kt}{kt+2}
\end{aligned}
$$