Hello!

Commutative algebra!

Rings & Modules / Rings.

Ring: Set w/ $+, \cdot, 0, 1, -$

Satisfy the obvious things $(a \cdot b = b \cdot a)$

(Field: Ring + multiplicative inverses)

Field - Numbers

Ring — Number-valued functions on a space.

In a field: $a/b$ is fine if $b \neq 0$

In a ring: $a/b$ is fine if $b$ is not 0 anywhere

Examples: 1) $\mathbb{C}[x_1, \dots, x_n]$ literally functions on $\mathbb{C}^n$

2) $\mathbb{Z}$ (interpretation in terms of functions)

3) $\mathbb{Z}_p$ - p-adic integers

4) $\mathbb{C}[[x_1, \dots, x_n]]$

$\to a_{-2}x^{-2} + \dots + a_3x^3$.

5) $\mathbb{C}[x, x^{-1}] \to \{ \sum_{a \le i \le b} a_i x^i \mid a_i \in \mathbb{C} \}$

6) $\mathbb{C}[x^2, x^3] \to \{ a_0 + a_2x^2 + a_3x^3 + \dots + a_ix^i \mid a_i \in \mathbb{C} \}$

$\mathbb{C}[x, x^{-1}]$ :

$$a_{-i}x^{-i} + a_{-i+1}x^{-i+1} + \dots + a_0 + a_1 x + \dots + a_j x^j$$

$i, j \geq 0$

Modules definition: vector space/rings.

$M/R$ $\quad m_1, m_2 \to m_1 + m_2 \in M$

$\quad m, r \to rm$

Examples: 1) $R$ is a module over $R$

If $R' \subset R$ is a subring,

$R$ is a module $/ R'$

2) If $R = \mathrm{End}(V)$ $V/_k$ is a f.d. vector space

$V$ is an $R$-module (module $R$ is not a ring)

3) $\{R\text{-submodules of } R\} - \{\text{ideals of } R\}$

$I \subset R$ is an ideal $\iff$

$i_1 + i_2 \in I$

$ri \in I$

4) $\mathbb{Z}$-module $-$ abelian groups

Writing the group operation additively,

$\forall$ $G$ ab gp. $n \cdot a = a + \cdots + a$ $n$ times

$-n \cdot a = -a + \cdots + -a$ $n$ times.

5) $R^n = \{(r_1, \ldots, r_n) \mid r_i \in R\}$ - free module of rank $n/R$

6) $R_1 \xrightarrow{f} R_2$ , $M/R_2$ is an $R_2$-module,

$M$ is an $R_1$-module: $r_1 \cdot m = f(r_1) \cdot m$

Module homomorphisms.

$f: M_1 \to M_2$ $\quad M_i$ are $R$-modules.

Then $f$ is a homomorphism if:

1) $f(m_1+m_2) = f(m_1) + f(m_2)$

2) $f(rm) = rf(m)$

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If $M_1 \subset M_2$ is a submodule, define

$M_1/M_2$

An $R$-module is <u>finite</u> (finitely presented)

1) $\exists \{m_1, \dots, m_k\} \in M$ s.t. $\forall m \in M$

$m = \sum r_i m_i$ for some $r_i \in R$

2) $R^k \xrightarrow{f} M$, then $\ker(f)$ is also finitely generated.

Examples: finite $\mathbb{Z}$-modules!

Is $\mathbb{Z}_p$ a finite $\mathbb{Z}$-module?

No: $\mathbb{Z}_p$ is uncountable as a set, A finite $\mathbb{Z}$-module must be at most countable

A ~~finitely generated~~ abelian group will be:
finite

Let $G$ be a finite abelian group:

$$G = \mathbb{Z}/p_1^{a_1} \oplus \mathbb{Z}/p_2^{a_2} \oplus \cdots \oplus \mathbb{Z}/p_n^{a_n}$$

Generators $\to$ Generators of the $\mathbb{Z}/p_i^{a_i}$'s.

$$\ker\left(\mathbb{Z}^n \to G\right) = \{(b_1, \dots, b_n) \mid p_i^{a_i} \mid b_i\}$$

$$(b_1, \dots, b_n) \mapsto \left(b_1 \pmod{p_1^{a_1}}, \dots, b_n \pmod{p_n^{a_n}}\right)$$

Gen by
$$(p_1^{a_1}, 0, \dots, 0)$$
$$(0, p_2^{a_2}, \dots, 0)$$
$$\vdots$$
$$(0, 0, \dots, p_n^{a_n})$$

Cokernel of $f: M_1 \to M_2$: $M_2 / \mathrm{im}(M_1)$

kernel: how $f$ misrepresents $M_1$,

cokernel: how $f$ misrepresents $M_2$

$\mathrm{Coker}(f)$

$G$ is finite abelian group, then

$G \times \mathbb{Z}^n$ is a finite $\mathbb{Z}$-module

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kernel of a homomorphism $f: M_1 \to M_2$:

$\{ m \in M_1 \mid f(m) = 0 \}$.

$$\mathbb{C}[x] \xrightarrow{f} \mathbb{C}[x]$$

$\cdot\, x(x-1)$

$$f(x) \mapsto x(x-1)f(x)$$

$$\ker(f) = 0$$

$$\operatorname{coker}(f) = \mathbb{C}[x]/(x(x-1))$$

$\operatorname{im}(f) = (x(x-1)) \in$ ideal gen. by $x(x-1)$

$\operatorname{coker}(f) = \mathbb{C} \oplus x\mathbb{C}$

$x \cdot (a + bx) = ((a+b)x)$