Localization, Hom, and $\otimes$

Hom: Let M, N be R-modules. Define

$$\mathrm{Hom}_R(M, N) = \{f : M \to N \mid f \text{ is an } R\text{-mod homomorphism}\}$$

write $\mathrm{Hom}(M, N)$ if R is clear.

1) $\mathrm{Hom}_R(M, N)$ is an R-module

$$(f+g)(m) = f(m) + g(m)$$

$$0(m) = 0$$

$$(rf)(m) = rf(m)$$

2) $\mathrm{Hom}_R(R, M) \cong M$ (but $\mathrm{Hom}_R(M, R) \not\cong M$)

$f \mapsto f(1)$

$f(r) = rm \leftarrow m$

3) $f: M_1 \to M_2 \rightsquigarrow f^*: \mathrm{Hom}(M_2, N) \to \mathrm{Hom}(M_1, N)$

$(f^* h)(m_1) = h(f(m_1))$

$g: N_1 \to N_2$, $g_*: \mathrm{Hom}(M, N_1) \to \mathrm{Hom}(M, N_2)$

$g_* h(m) = g(h(m))$

4) If $0 \to A \to B \to C$ is exact, and M is any R-mod, then

$$0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$$

is as well. Similarly, if

$A \to B \to C \to 0$ is exact, and N is any R-mod, then

$$0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N)$$ is as well.

This is phrased as "Hom is left-exact."

$\mathrm{Hom}(M, -) : \{R\text{-modules}\} \to \{R\text{-modules}\}$

preserves exactness on the left. Alternatively, "Hom preserves kernels," as this is equivalent to, if $f: B \to C$ is a homomorphism, then $\ker(f_*) = \mathrm{Hom}(M, \ker(f))$

If $0 \to A \to B \to C \to 0$ is exact, then

$$0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C) \to 0$$

may not be.

$\left(0 \to \mathbb{C}[x] \xrightarrow{\cdot x} \mathbb{C}[x] \to \mathbb{C}[x]/(x) \to 0, \quad M = \mathbb{C}[x]/(x) \right)$

Categorical phrasing: $\mathrm{Hom}(M, -)$ is a covariant functor from R-Mod $\to$ R-mod, f_* is the functor applied to Hom sets.

wts. If $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact, then

$0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C)$ is too

Pf. First, lets show that f_* is injective.

Assume $h \in \mathrm{Hom}(M,A)$, $f_* h = 0$.

$f(h(m)) = 0 \quad \forall m \in M. \Rightarrow h(m) = 0 \ \forall m \in M$

$\Rightarrow h = 0.$

Now we need $\mathrm{Im}(f_*) = \ker(g_*)$

$\mathrm{Im}(f_*) \subset \ker(g_*)$: Choose $h \in \mathrm{Hom}(M, A)$.

$f_* h \in \ker(g_*)$, $g_* f_* h = 0$

$g(f(\underbrace{h(m)}_{A})) = 0 \quad \forall\, m \in M.$ ✓

$\mathrm{Im}(f_*) \supset \ker(g_*)$: Choose $h \in \mathrm{Hom}(M, B)$ s.t.

$g_*(h) = 0$.

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$h = f_*(h')$

$M \xrightarrow{h}$

Vrednings $A \subset B$

$A \subset B$

w.t.s. $h(m) \in A \quad \forall m \in M.$

$0 \to A \to B$ is exact,

$\Leftrightarrow A \hookrightarrow B$

$g_* h = 0$

$g(\underline{h(m)}) = 0 \quad \forall m \in M$

$\Uparrow$

$\mathrm{Im}(f)$

$\Rightarrow h(m) \in A \quad \forall m \in M.$

$\otimes$: Let M, N be R-mods.

Define $M \otimes_R N$ (or $M \otimes N$ if R is clear)

To be the module generated by symbols

$m \otimes n$ $\forall m \in M, n \in N$, quotiented out

by the following relations:

$r(m \otimes n) = (rm) \otimes n$ $\qquad (m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$

$r(m \otimes n) = m \otimes (rn)$ $\qquad m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$

$(m,n) \overset{\varphi}{\mapsto} (m \otimes n)$ is bilinear in M, N.

Universal property: There is a map

$\varphi : M \times N \to M \otimes N$ bilinear, and

s.t., if A is an R-mod w/

a bilinear map $f : M \times N \to A$, then

$\exists!\ g : M \otimes N \to A$ (linear) s.t.

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[dr, bend right, "f"'] & M \otimes N \arrow[d, "g"] \\ & A \end{tikzcd}$$

Commutes

"φ is the universal bilinear map out of $M \times N$"

$$g(m \otimes n) = f(m, n)$$

$$(M \otimes N) \otimes O \cong M \otimes (N \otimes O)$$

$$M \times N \times O \to A$$

$\Omega^k(X) = \{$alternating k-forms on $X\}$

$= \Lambda^k T^* X$

— $M = \mathbb{R}^m$, N be anything.

Then I claim $M \otimes N = N^m$

Write $e_1, \ldots, e_m$ for basis elts. of M.

Every elt. of $M \otimes N$ is of the form

$$\sum_{i=1}^{m} (e_i \otimes n_i) \quad \text{w/ } n_i \in N.$$

$m \otimes n$ is of this form.

$$m = \sum r_i e_i$$

$$\begin{aligned} m \otimes n &= \left(\sum r_i e_i\right) \otimes n \\ &= \sum \left((r_i e_i) \otimes n \right) \\ &= \sum e_i \otimes (r_i n) \end{aligned}$$