Recall from last time

Thm: Let R be a PID, and M a finite R-module. Then $\exists! \, n, m \geq 0$, $a_i \in R$ $1 \leq i \leq m$, w/ $a_i \mid a_{i+1}$, s.t. $M \cong R^n \oplus \bigoplus_{i=1}^{m} R/(a_i)$.

Pf. outline: $0 \to M_{tors} \to M \to M_{tf} \to 0$

$M_{tors} = \{ m \in M \mid \exists a \in R, a \neq 0, am = 0 \}$.

$M_{tf} = M / M_{tors}$.

W.t.s.

1) M_{tf} is free

2) $\exists$ a section $M_{tf} \to M$, and

3) M_{tors} is of the right form.

Start w/ 2):

Lemma: Let N be any R-module, $f: N \twoheadrightarrow R^k$ be surj. Then f admits a section.

Pf. Let $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots 0) \dots$ & $e_k = (0, \dots, 0, 1)$. Choose $n_i \in N$ s.t. $f(n_i) = e_i$ $\forall i$.

$s(\sum r_i e_i) = \sum r_i n_i$ ✓

$\{$homomorphisms from $R^k \to N\}$

$= \{k$-tuples of elts. in $N\}$

N.B. This doesn't require R to be a PID.

1) $\underline{M_{tf} \text{ is free}}$.

a) M_{tf} is torsion free: $\nexists\, m \in M_{tf}$, $a \in R$

$m \neq 0$, $a \neq 0$ $am = 0$.

Pf. Assume $\exists\ n \in M_{tf}$, $a \in R$ s.t.

$am = 0$, $a \neq 0$, $m \neq 0$. Choose $\tilde{m} \in M$ mapping

to $m \in M_{tf}$. Then $a\tilde{m} \in M_{tors}$.

$\exists\, b \neq 0$ s.t. $ba\tilde{m} = 0$. $\Rightarrow \tilde{m} \in M_{tors}$

$\Rightarrow m = 0$ in M_{tf}. ⊘

N.B. This requires R to be a domain, but not a PID.

($R = k[x]/x^2$, $M = R$ as an R-mod, M/M_{tors} has torsion)

b) M_{tf} is a submodule of a free module.

any finite torsion-free module

Pf: Want to show $M_{tf} \hookrightarrow R^k$ for some $k \geq 0$.

Choose $m_1', \dots, m_k' \in M_{tf}$ s.t.

1) $\sum a_i m_i' = 0 \Rightarrow a_i = 0 \ \forall i$, and

2) $\{m_i'\}$ is maximal w.r.t. 1).

Let $m_1, \dots, m_n$ be a finite set of gens. of M_{tf}.

What do we know about $\{m_1', \dots, m_k', m_j\}$?

$\exists\, a_j, a_{ij} \in R$ s.t. $a_j m_j = \sum a_{ij} m_i'$, $a_j \neq 0$

Choose a s.t. $a_j | a$ $\forall j$. $(a_j b_j = a)$

$a m_j = \sum b_j a_{ij} m_i'$ $\forall j$.

$$M_{tf} \xrightarrow{\cdot a} \langle m_1', \dots, m_k' \rangle = R^k$$

$$\begin{tikzcd} M_{tf} \arrow[r, "\cdot a"] \arrow[dr, dotted] & M_{tf} \\ & \langle m_1', \dots, m_k' \rangle \arrow[u, phantom, "\cup"] \cong R^k \end{tikzcd}$$

$m \in \ker(\cdot a) \Rightarrow am = 0$

$\Rightarrow \ker(\cdot a) = 0$

$M_{tf} \hookrightarrow R^k$ ✓ (Note: this is true for any finite t.f. module/ a domain).

c) Any submodule of a finite free module over a PID is free.

$N \hookrightarrow R^k$, N finite.

Pf: Induction on k.

$\underline{k=1}$ $N \hookrightarrow R$, $N = I = (a) \Rightarrow N \overset{\text{module isomorphism}}{\cong} R$

$N = aR$

Assume $\forall M \subset R^{k-1}$, M is free.

$M \hookrightarrow R^k$

$0 \to R^{k-1} \to R^k \to R \to 0$

$(a_1, \dots, a_{k-1}) \mapsto (a_1, \dots, a_{k-1}, 0)$

$(a_1, \dots, a_k) \mapsto a_k$

Let $M' = M \cap R^{k-1}$, $\overline{M} = M / M'$.

$$
\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\wr"] & M \arrow[r] \arrow[d, "\wr"] & \overline{M} \arrow[r] \arrow[d, "\wr"] & 0 \\
 & R^{k-1} & R^{k} & R &
\end{tikzcd}
$$

Induction $\Rightarrow$ M' is free,
Base case $\Rightarrow$ $\overline{M}$ is free.

2) $\Rightarrow$ $\exists$ section $\overline{M} \to M$

wcd $\Rightarrow$ $M \cong M' \oplus \bar{m}$

$\Rightarrow$ M is free.

Crucially relies on R being a PID.

$\mathbb{Q}$ - $\mathbb{Z}$.mod Torsion free $\not\Leftrightarrow$ $\oplus \mathbb{Z}$

(over a PID, f.g. $\Leftrightarrow$ finite)

3) Let T / R be a finite torsion R-mod.

Let $p \in R$ be a prime elt.

Define $T_p = \{t \in T \mid p^i t = 0 \text{ for some } i \geq 0\}$.

Lemma: $T = \bigoplus_{p \text{ prime}} T_p$.

Pf. Choose $t \in T$. $\exists\, d \in R$ s.t. $dt = 0$.

$d = p_1^{a_1} \cdots p_k^{a_k}$

$d_i = d / p_i^{a_i}$ $\qquad (d_1, \ldots, d_k) = (1)$ $\qquad t_i = d_i t$

$t_i \in T_{p_i}$

$\Rightarrow \exists\, b_i$ s.t. $\sum b_i d_i = 1$.

$(p_i^{a_i} t_i = p_i^{a_i} d_i t = d t = 0)$

$t = \sum b_i d_i t = \sum b_i t_i$

$\bigoplus_{p \text{ prime}} T_p \twoheadrightarrow T$

$(t_p) \to \sum t_p$

$\ker\left(\bigoplus T_p \to T\right) = 0$

Assume $(a_p) \to 0$

$$\sum_{p \text{ prime}} a_p = 0$$

$a_{p_1} \neq 0$

$$a_{p_1} + \underbrace{\sum_{q \neq p_1} a_q}_{= a'_{p_1}} = 0$$

a'_{p_1} is killed by some r w/ $(r, p_1) = 1$

$p_1^{b_1} a_1 = 0$

$(p_1^{b_1}, r) = (1) \Rightarrow \exists\, c, d \in R$ s.t. $c p_1^{b_1} + dr = 1$

$a_{p_1} \in T_{p_1}, \quad \in \operatorname{Im}\left(\bigoplus_{q \neq p_1} T_q\right)$

$$1 \cdot a_{p_1} = c p_1^{b_1} a_{p_1} + d r a_{p_1}$$

$$= c p_1^{b_1} a_{p_1} - d r a'_{p_1} = 0 - 0 = 0$$

$a_{p_1} + a'_{p_1} = 0$ $\exists\, p_1^{b_1} \cdot a_{p_1} = 0$

$a_{p_1} \in T_{p_1}$, $a'_{p_1} \in \sum_{q \neq p_1} T_q$

$\exists\, r \in R$, $(r, p_1) = (1)$, s.t.

$r \cdot a'_{p_1} = 0$

$\exists\, c, d \in R$ s.t. $c p_1^{b_1} + d r = 1$

$$1\cdot a_{p_1} = (c p_1^{b_1} + dr)\, a_{p_1}$$

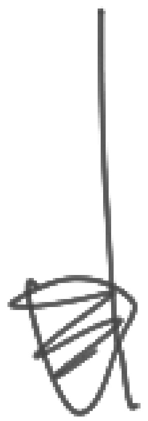
$$= c p_1^{b_1} a_{p_1} + dr\, a_{p_1}$$

$$= c p_1^{b_1} a_{p_1} - dr\, a'_{p_1}$$

$$= 0 - 0$$

$\Rightarrow a_{p_1} = 0 \;\Rightarrow\Leftarrow$, so $\ker(\oplus T_p \to T) = 0$

$$T \cong \bigoplus_p T_p$$



Chinese Remainder Theorem for modules