Recall: $M \otimes N$: module generated by $m \otimes n$, s.t. $M \times N \to M \otimes N$
$(m,n) \mapsto m \otimes n$ is bilinear.

Properties of $\otimes$:

1) $M \otimes N \cong N \otimes M$
$m \otimes n \leftrightarrow n \otimes m$

2) $M \otimes R \cong M$

$m \otimes r \to rm$

$m \otimes 1 \leftarrow m$

3) If $f: M_1 \to M_2$, and N is an R-mod,

$f \otimes 1 : M_1 \otimes N \to M_2 \otimes N$

$(g: N_1 \to N_2 \quad , \quad f \otimes g : M_1 \otimes N_1 \to M_2 \otimes N_2)$

$$M \otimes (N_1 \oplus N_2) \cong (M \otimes N_1) \oplus (M \otimes N_2)$$

$$m \otimes (n_1, n_2) \to (m \otimes n_1, m \otimes n_2)$$

$$m \otimes (n_1, 0) \leftarrow (m \otimes n_1, 0)$$

$$m \otimes (0, n_2) \leftarrow (0, m \otimes n_2)$$

$$M \otimes R \cong M$$

$$m \otimes r \to rm$$

$$m \otimes 1 \leftarrow m$$

$M \to M \otimes R \to M$ is id

$$m \otimes r = r(m \otimes 1) = (rm) \otimes 1$$

$$m \otimes r \to rm \to rm \otimes 1 = m \otimes r$$

$M \otimes R \to M \to M \otimes R$ is id

$M \otimes R \to M$

$m_1 \otimes r + m_2 \otimes r - (m_1 + m_2) \otimes r \to 0$

$r m_1 + r m_2 - r(m_1 + m_2) = 0$

$m \otimes r_1 + m \otimes r_2 - m \otimes (r_1 + r_2) \to 0$

$M \to M \otimes R$

$m_1 + m_2 \to m_1 \otimes 1 + m_2 \otimes 1$

vs. $(m_1 + m_2) \otimes 1$

If $A \to B \to C \to 0$ is exact, and M is an R-mod, then

$A \otimes M \to B \otimes M \to C \otimes M \to 0$ is exact as well.

If M is such that, $\forall$

$0 \to A \to B \to C \to 0$ exact, then

$0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$ is also exact,

then M is <u>flat</u>.

Ex. $M = R^m$ is exact.

$0 \to A \to B \to C \to 0$ is exact implies

$0 \to A^m \to B^m \to C^m \to 0$ is exact

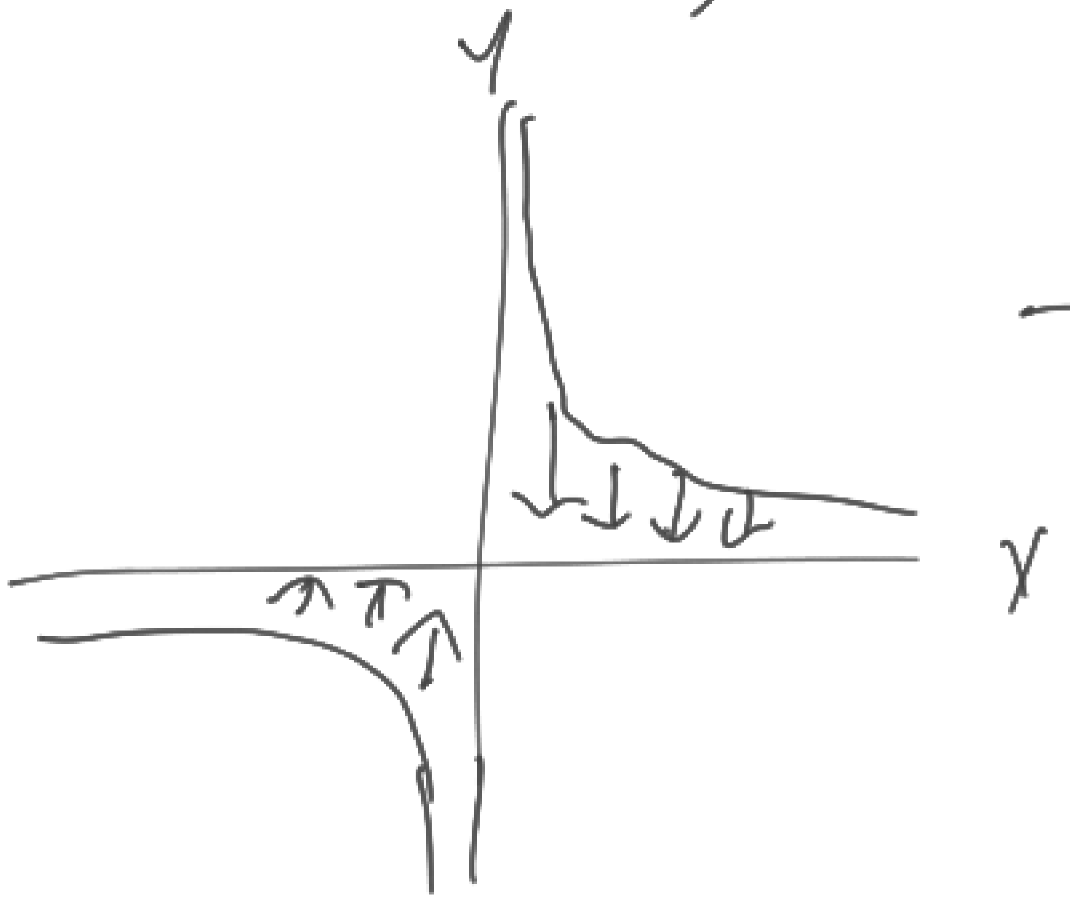
Localization:

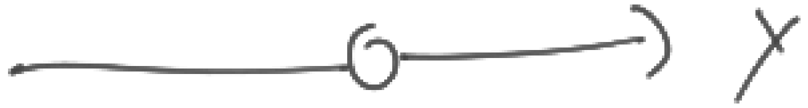
$k[x]$ is $\{$algebraic functions on $k\}$

What is {algebraic functions on $k \setminus \{0\}$}?

$k[x, \frac{1}{x}] \rightsquigarrow (x) = \{$ functions vanishing at $0 \}$

$k[x][y]/(xy-1)$





$f(x_1, \dots, x_n)$ on k^n

$X = \{ \underline{x} \mid f(\underline{x}) = 0 \}$

$\{\text{functions on } k^n \setminus X\} = k[x_1, \dots, x_n][y] / (yf - 1)$

$\underline{x} \notin X$, then $f(\underline{x}) \neq 0 \Rightarrow \exists\ y(\underline{x})$

s.t. $f(\underline{x})\, y(\underline{x}) = 1$

"germ of a function":

$\underline{0} \in k^n$. A germ of a function at $\underline{0}$ is a pair (U, f): U is an open set containing $\underline{0}$, and $f: U \to k$.

$(U_1, f_1) \sim (U_2, f_2)$ if $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$

Algebraically: Say f is defined in a nbd. of $\underline{0}$ if $\exists$ $X \subset k^n$ which is the zero set of a function h and f is defined on $k^n \setminus X$.

Let R be a ring, $U \subset R$ be a mult. closed set. Assume $1 \in U$.

Define $R[U^{-1}] = \{r/u\}/\sim$

$$\frac{r_1}{u_1} \sim \frac{r_2}{u_2} \iff v(r_1u_2 - r_2u_1) = 0 \text{ for some } v \in U.$$

Define $\frac{r_1}{u_1} + \frac{r_2}{u_2} = \frac{r_1u_2 + r_2u_1}{u_1u_2}$, $\frac{r_1}{u_1} \cdot \frac{r_2}{u_2} = \frac{r_1r_2}{u_1u_2}$

$0 = \frac{0}{1}$, $1 = \frac{1}{1}$.

Assume $\frac{r_1}{u_1} \sim \frac{r_1'}{u_1'}$

Check that $\frac{r_1}{u_1} + \frac{r_2}{u_2} \sim \frac{r_1'}{u_1'} + \frac{r_2}{u_2}$

$$\frac{r_1 u_2 + r_2 u_1}{u_1 u_2} \quad \text{vs.} \quad \frac{r_1' u_2 + r_2 u_1'}{u_1' u_2}$$

$\exists v$ s.t. $v\left((r_1 u_2 + r_2 u_1)(u_1' u_2) - (r_1' u_2 + r_2 u_1') u_1 u_2\right)$

$= 0$

$$v(r_1 u_2^2 u_1' - r_1' u_2^2 u_1) = 0$$

$$u_2^2(v(r_1 u_1' - r_1' u_1)) = 0$$

If M is an R-module, Define

$$M[u^{-1}] = \{m/u\}/\sim$$

$M[u^{-1}]$ is an $R[u^{-1}]$-module.

Ex: 1) $R = \mathbb{Z}$, $U = R \setminus \{0\}$

what is $R[U^{-1}]$? $\mathbb{Q}$.

2) $R = k[x]$, $U = \{1, x, x^2, \ldots\}$

$R[U^{-1}] = k[x, x^{-1}]$

3) R, $U = \{1, 0\}$ $R[U^{-1}] = 0$

$\frac{r_1}{u_1} \sim \frac{r_2}{u_2}$ $\exists ?\, v \in U$ s.t. $v(r_1 u_2 - r_2 u_1) = 0$?

$v = 0$

4) Let $P \subset R$ be a prime ideal. $U := R \setminus P$

If $a, b \notin P$, then $ab \notin P$.

$R[U^{-1}] = R_P$

$R = k[x_1, \dots, x_n]$, $P = (x_1, \dots, x_n)$, then

$R_P = \{$ germs of functions about $\underline{0} \}$

If u' is a subset of R, and let $u =$ multiplicative closure of u', then

$R[u'^{-1}] \stackrel{?}{=} R[u^{-1}]$

If $f \in R$, $U = \{1, f, f^2, \dots\}$

$R[U^{-1}] = R_f$

$R_f \neq R_{(f)}$

$R = \mathbb{Z}/6\mathbb{Z}$, $U = \{1, 3\}$

Recall: Let R, S be rings. S is an R-algebra if $\exists\ R \xrightarrow{f} S$.

If M is an R-module, $M \otimes_R S$ is an S-module

$$s(m \otimes s_0) = m \otimes (s s_0)$$

$r(m \otimes s) = m \otimes (f(r)s)$

R-mod $\to$ S-mod

$M \to M \otimes_R S$

($R[u^{-1}]$ is an R-alg. under $r \mapsto r/1$)

If M is an R-module, $M[u^{-1}]$

$\cong M \otimes_R R[u^{-1}]$