Homework 1

MATH 218: Differential Equations for Engineers

Due: September 25^{th} at 11:59 PM EST

This assignment is based primarily on the material in unit 2.

After you have completed the assignment, please save, scan, or take a photo of your work and upload it to crowdmark. Crowdmark accepts pdf, jpg, and png file formats.

As a reminder, it is fine to discuss the problems with other students. However, you must write up your own solutions.

Question 1 (8 points)**.** For each of the following differential equations, answer the following questions (with justification):

- Is the DE linear?
- Is the DE separable?
- Is the DE exact (without doing anything complicated, e.g. integrating factors)?
- If given the initial condition $y(0) = 1$, do the existence and uniqueness theorems guarantee a unique solution?

(a) $\dfrac{dy}{dx} = y^2 + x^2y^2$

(b) $\dfrac{dy}{dx} = x^3 + 3xy$

(c) $3y^2\dfrac{dy}{dx} = 3x^2 + y + x\dfrac{dy}{dx}$

(d) $\dfrac{dy}{dx} = x^2 + y^2$

Question 2 (6 points)**.** Sometimes, it's possible to deduce information about solutions to a differential equation without actually solving it. For both of the following equations, find all trivial solutions (i.e. solutions of the form $y(x) = C$ for some constant C). Additionally, find all inflection points not on trivial solutions. You do not need to solve the equations.

(a) $\dfrac{dy}{dx} = y^3 - y$

(b) $\dfrac{dy}{dx} = \sin(y)$

Question 3 (6 points)**.** Let $a > 0$ be a positive constant. Assume that $y(0) = 1$ and $\dfrac{dy}{dx} = y^a$. Clearly one has that $y(x)$ increases more and more rapidly as x increases. For which values of a does $y(x)$ get to infinity in finite time? That is, for which values of a does this function have a vertical asymptote and is thus not defined for all x values?

Question 4 (10 points)**.** You decide that you want to inflate a balloon with helium. The balloon has a volume of $10000cm^3$ when fully inflated, and currently has $1000cm^3$ of regular air in it. You hook it up to a helium pump, which will feed in $200cm^3$ of air that is 90% helium and 10% regular air per second. However, you don't attach the balloon tightly, so $100cm^3$ of air in the balloon leaks out every second. Assume that the air in the balloon (and hence the air that leaks out) is perfectly homogenized. How much helium is in the balloon when it's full?

An important thing to notice here: the volume in the balloon isn't remaining constant over time. You need to account for that when setting up your differential equation.

Question 5 (10 points)**.** *Newton's law of heating/cooling* states that the rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the ambient temperature, that is, if $T(t)$ is the temperature of the object and T_a is the ambient temperature, then $\dfrac{dT}{dt} = k(T - T_a)$ for some constant k (which is negative: objects tend towards the ambient temperature after all!). Assume that you have a piece of metal that has been heated to $650°C$ and it is now resting in a room that has an ambient temperature of $25°C$. After one minute, the temperature of the object is now $525°C$. How long does it take for the object to cool down to $50°C$?

Question 6 (15 points)**.** While Waterloo has a population of around 113000, to make numbers easier, assume that Waterloo actually has a population of 100000 for this problem.

Assume that someone with a virus comes back to Waterloo. The logistic model says that the number of people that get infected is proportional to the number of currently infected people times the ratio of healthy people to all people in a population. Let $y(t)$ be the number of infected people as a function of t.

(a) Assuming that $y(0) = 1$, without doing any calculation, what should $\lim_{t\to\infty} y(t)$ be? What about $\lim_{y\to-\infty} y(t)$? Why?

(b) Write down a differential equation describing y. There will be a constant of proportionality in your equation.

(c) As in problem 2, find the trivial solutions to this differential equation, as well as the y value of the inflection point.

(d) Find all solutions to this eqation. You should have the constant of porportionality floating around, as well as another constant coming from all solutions of the differential equation.

(e) Choose a particular value for the constant in part (b), and graph a handful of solutions to the equation. I want you to think about what "intersting" solutions are, and how changing the constant that came out of solving the differential equation changes the graph.

(f) Now, assume that $y(0) = 1$. What does changing the constant in part (b) do? Plot some graphs with various choices of this constant.

N.B. Some of you may use computer graphing programs to do this problem. That's good! I want you to use tools to help you visualize solutions to problems. The goal of this problem is to think about what the graphs should look like, and then see if that aligns with what it does.

Question 7 (20 points)**.** Two chemicals (let's call them A and B) are being mixed to produce a third chemical (which we'll call C). Initially (at $t = 0$), one litre of A and one litre of B are mixed together. Assume that, for every unit of C is made up of a units of A and b units of B, where a and b are constants such that $a + b = 1$ and $a < b$ (if this much abstraction is difficult, significant partial credit will be awarded if you only think about the case where $a = \frac{1}{3}$ and $b = \frac{2}{3}$ in parts $(a) - (g)$). Additionally, assume that the amount of C produced at time t is porportional to the amount of A and the amount of B at time t. Write $A(t)$, $B(t)$, and $C(t)$ for the amount of A, B, and C at time t respectively.

(a) Before solving any differential equations, what should $\lim_{t\to\infty} C(t)$ be? Why?

(b) Solve for $A(t)$ and $B(t)$ in terms of $C(t)$.

(c) Write a differential equation that is satisfied by $C(t)$ that only involves $C(t)$ and $\frac{dC}{dt}$ (i.e. no higher derivitaves and no use of $A(t)$ or $B(t)$ in your final answer). There will be a constant of porportionality in your equation.

(d) Additionally, write down the initial condition that this equation satisfies.

(e) Solve for $C(t)$, keeping in mind both the constant of porportionality and the initial condition.

(f) Compute $\lim_{t\to\infty} C(t)$. Is this equal to the answer you got in part (a)?

(g) Assume that $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Additionally, assume at $t = 1$ minute, half a litre of C has been formed. At what value of t will 1 litre of C have formed?

(h) You may have noticed that everything breaks when $a = b = \frac{1}{2}$. Solve the equation in this case as well.