Exact Sequences

$$\cdots \to M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \to \cdots$$ is exact if

$$f_i : M_i \to M_{i+1} \qquad \boxed{\ker(f_{i+1}) = \operatorname{im}(f_i)}$$

A short exact sequence is an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$

Examples / Properties:

$0 \to M_1 \xrightarrow{f} M_2$ is exact $\Leftrightarrow$ f is injective

$im(0 \to M_1) = 0 \Rightarrow ker(f) = 0$

$M_1 \xrightarrow{f} M_2 \to 0$ is exact $\Leftrightarrow$ f is surjective

$ker(M_2 \to 0) = M_2 \Rightarrow Im(f) = M_2$

$0 \to M_1 \to M_2 \to 0$ Exact $\iff f: M_1 \xrightarrow{\sim} M_2$

2) $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$

3) $R = \mathbb{C}[x, y]$

$$0 \to R \to R^2 \to R \to R/(x,y) \to 0$$

$$f \mapsto (yf, -xf)$$

$$(g, h) \mapsto (xg, yh)$$

$0 \to R \to R^2 \longrightarrow R \to R/(x,y) \to 0$

$(g,h) \to xg + yh$

$xg + yh = 0 \Rightarrow xg = -yh$

$g = yf \qquad h = -xf$

$\ker\left(R \to R/(x,y)\right) =$

The ideal (x,y)

$\mathrm{im}(R^2 \to R)$

$= (x,y)$

$$\begin{tikzcd}
 & 0 \arrow[dr] & & & & & & 0 \\
 & & \mathrm{im}(f_1) \arrow[dr] & & & & \mathrm{im}(f_3) \arrow[ur] & \\
\cdots \arrow[r] & M_1 \arrow[ur] \arrow[rr] & & M_2 \arrow[rr] \arrow[dr] & & M_3 \arrow[rr] \arrow[ur] & & \cdots \\
 & & & & \mathrm{im}(f_2) \arrow[ur] \arrow[dr] & & & \\
 & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$\vdots$

$$0 \to \mathrm{im}(f_1) \to M_2 \to \mathrm{im}(f_2) \to 0$$

$$0 \to \mathrm{im}(f_2) \to M_3 \to \mathrm{im}(f_3) \to 0$$

$\vdots$

$f: M_1 \twoheadrightarrow M_2$ f is surjective.

A <u>section</u> of f is a map $s: M_2 \to M_1$

st $M_2 \xrightarrow{s} M_1 \xrightarrow{f} M_2$ is the id.

$\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ doesn't admit a section.

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

If there is a section $M_3 \xrightarrow{s} M_2$,

$$M_2 \cong M_1 \oplus M_3$$

Pf. Giving a map from $M_1 \oplus M_3 \to M_2$ is the same as a map from $M_1 \to M_2$ & a map from $M_3 \to M_2$.

Example of this: $R = \mathbb{C}[x]$

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] & R^2 \arrow[r] & R \arrow[r] \arrow[l, bend right, "e \mapsto (e,0)"'] & 0 \end{tikzcd}$$

$$f \mapsto (f, -f)$$

$$(g, h) \mapsto g + h$$

$M_1 \xrightarrow{f} M_2$

$M_3 \xrightarrow{s} M_2$

$M_1 \oplus M_3 \to M_2$

$(m_1, m_3) \to (f(m_1) + s(m_3))$

(s, t) is 1-1. Equivalently

$\ker((s,t)) = 0$

Assume $(f,s)(m_1, m_3) = 0$

$f(m_1) + s(m_3) = 0$

$g(f(m_1) + s(m_3)) = 0$

$g(\underbrace{f(m_1)}_{0}) + s(\underbrace{s(m_3)}_{m_3}) = 0$

$\Rightarrow m_3 = 0$

$f(m_1) = 0$

$\Rightarrow m_1 = 0$ ✓

Onto:

choose $m_2 \in M_2$ want to find m_1, m_3 s.t.

$$f(m_1) + s(m_3) = m_2$$

$$g(m_2) := m_3$$

$$g(m_2 - s(m_3)) = g(m_2) - m_3$$
$$= 0$$

$m_2 - s(m_3) \in \ker(g) = \operatorname{im}(f)$. $\exists\, m_1$ s.t.

$$f(m_1) = m_2 - s(m_3)$$

$$f(m_1) + s(m_3) = m_2 - s(m_3) + s(m_3) = m_2$$

Some properties of rings:

R is a <u>Domain</u> if $ab=0 \Rightarrow a=0$ or $b=0$

A domain is <u>factorial</u> if every nonzero elt. can be written as the product of irreds. in an essentially unique way.

Equivalent to every irred. is prime

($p \mid ab \Rightarrow p \mid a$ or $p \mid b$)

A domain is a <u>PID</u> if every ideal is generated by one element.

$\mathbb{Z}$ is a PID

$k[x]$ is a PID (k is a field)

$\mathbb{Z}[i]$ is a PID

$k[x_1, \dots, x_n]$ is factorial, not a PID $n \geq 2$.

Thm: Let R be a PID, M/R a finite R-module. Then M is isomorphic to a module of the form $R^n \oplus \bigoplus_{i=1}^{m} R/a_i$

$n, m \geq 0$, $a_i \mid a_{i+1}$.

Pf. (outline)

Define $M_{tors} = \{ m \in M \mid \exists a \in R, a \neq 0, am = 0 \}$

$0 \to M_{tors} \to M \to M_{tf} \to 0$ $\qquad M_{tf} := M/M_{tors}$

wts. 1) M_{tf} is free ($M_{tf} \cong R^n$ for some n)

2) $\exists$ section $M_{tf} \to M$.

3) M_{tors} is of the desired form. $\boxtimes$