$V = k^n$

$\sigma \in S_n$

V has basis vectors $e_1, \dots, e_n$

$S_n \curvearrowright V : \sigma(e_i) = e_{\sigma(i)}$

$V^\vee : \{f : V \to k \text{ linear}\}$

$= \langle x_1, \dots, x_n \rangle$

$x_i(e_i) = 1$

$x_i(e_j) = 0 \quad \forall j \neq i$

$S_n \curvearrowright V^V$

$$\sigma(f)(v) = f(\sigma^{-1}(v))$$

$$\sigma_1(\sigma_2(f))(v) = \sigma_2(f)(\sigma_1^{-1}(v))$$

$$= f(\sigma_2^{-1} \cdot \sigma_1^{-1}(v))$$

$$= f((\sigma_1 \sigma_2)^{-1}(v))$$

$$= ((\sigma_1 \sigma_2) f)(v)$$

$$\sigma(x_i)(e_{\sigma(i)}) = x_i(\sigma^{-1} e_{\sigma(i)})$$

$$= x_i(e_i) = 1$$

$$\sigma(x_i) = x_{\sigma(i)}$$

$S_n \curvearrowright V^\vee$

$$\mathrm{Sym}^i(V^\vee) := \{ f : V^i \to k \mid f \text{ is linear in each coordinate}, \ f \text{ is symmetric} \}$$

$$\forall g \in S_i, \quad f(v_1, \dots, v_i) = f(v_{g(1)}, \dots, v_{g(i)})$$

$$\mathrm{Sym}^2(V^\vee) = \{\text{symmetric, billinear forms}\}$$

(inner products)

1) If $f \in \mathrm{Sym}^i(V^\vee)$, $v \in V$, $f(v) = f(v, v, \ldots, v)$

(If $(\, , \,)$ is an inner product, $Q(\underline{x}) = (x, x)$ is a quadratic form).

2) One may choose a basis of $\mathrm{Sym}^i(V^\vee)$ given by $\{ x_{j_1} \cdots x_{j_i} \mid j_1 \leq j_2 \leq \cdots \leq j_i \}$

$$(x_{j_1} \cdots x_{j_i})(v) = x_{j_1}(v) \cdot x_{j_2}(v) \cdot \cdots \cdot x_{j_i}(v).$$

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$\mathrm{Sym}^\bullet(V^\vee) = k[x_1, \ldots, x_n]$ w/ the standard grading.

$S_n \circlearrowright \mathrm{Sym}^i(V^\vee)$: $\sigma(x_{j_1} \cdots x_{j_i}) = x_{\sigma(j_1)} \cdots x_{\sigma(j_i)}$

What is $\mathrm{Sym}^i(V^\vee)^{S_n} := \{ f \in \mathrm{Sym}^i(V^\vee) \mid \sigma(f) = f \ \forall \sigma \}$?

$(\mathrm{Sym}^\bullet(V^\vee))^{S_n} = k[s_1, \ldots, s_n]$,

$$s_i = \sum_{j_1 < \cdots < j_i} x_{j_1} \cdots x_{j_i}$$

$$s_1 = x_1 + \cdots + x_n$$

$$\vdots$$

$$s_{n-1} = x_1 \cdots x_{n-1} + x_1 \cdots x_{n-2} x_n + \cdots + x_2 \cdots x_n$$

$$s_n = x_1 \cdots x_n$$

$$\mathrm{Sym}^\bullet(V^\vee) = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(V^\vee)$$

$$f_i \in \mathrm{Sym}^i(V^\vee),\ f_j \in \mathrm{Sym}^j(V^\vee) \Rightarrow f_i f_j \in \mathrm{Sym}^{i+j}(V^\vee)$$

$\mathrm{Sym}^\bullet(V^\vee)$ – functions on V

$\mathrm{Sym}^\bullet(V^\vee)^{S_n}$ – functions on V invariant under S_n.

Let G be a finite group, $|G|$ is invertible in k
V be a vector space $/k$,
G acts on k through linear transformations.

$$\mathrm{Sym}^\bullet(V^\vee)^G = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(V^\vee)^G = \bigoplus_{i=0}^{\infty} \{ f \in \mathrm{Sym}^i(V^\vee) \mid g(f) = f \ \forall g \in G \}$$

Thm: Let G, V be as above. Then $(\mathrm{Sym}^\bullet(V^\vee))^G$ is finitely generated as a k-alg.

Pf: 2 parts.

Part I) Let $R = (\mathrm{Sym}^\bullet(V^\vee))^{G}$, $S = \mathrm{Sym}^\bullet(V^\vee)$

Then the natural graded inclusion

$\iota : R \hookrightarrow S$ admits a right/left inverse

$\pi : S \to R$ s.t. $\pi(\iota(r)) = r$.

(π is a map of R-modules)

Want a $\pi: S \to R$ s.t. $\pi|_R = id$.

$$\pi(f) = \sum_{g \in G} g(f)$$

$$g_0(\pi(f)) = \sum_{g \in G} g_0 g(f) = \sum_{g \in G} g(f) = \pi(f)$$

If $f \in R$, $\pi(f) = \sum_{g \in G} g(f) = \sum_{g \in G} f = |G| \cdot f$

Define $\pi(f) = \frac{1}{|G|} \sum_{g \in G} g(f)$.

(For hiding $\mathbb{Z}/p^2\mathbb{Z}$ if $k = \mathbb{F}_p$)

π is R-linear:

Let $f \in R$, $h \in S$

$f\pi(h)$ vs $\pi(fh)$

$$\pi(fh) = \frac{1}{|G|} \sum_{g \in G} g(fh) = \frac{1}{|G|} \sum_{g \in G} g(f)\, g(h)$$

$$= \frac{1}{|G|} \sum_{g \in G} f\, g(h)$$

$$= f\pi(h)$$

Part II: Let $S = k[x_1, \dots x_n]$, $R \subset S$ a k-subalgebra graded w/ the same grading. Assume $\exists\ \varphi : S \to R$ R linear, $\varphi|_R = id$. Then R is f.g. as a k-algebra.

Pf. Let $\mathfrak{m} \subset R$ be the irrelevant ideal (i.e. $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$)

Then $\mathfrak{m}S \subset S$ is an ideal in S

$(\{\sum m_i s_i \mid m_i \in \mathfrak{m}, s_i \in S\})$

$\mathfrak{m}S = (f_1, \dots, f_s)$ f_i is a homogeneous ele.,

$f_i \in \mathfrak{m}$.

W.t.s. $R' = k[f_1, \dots, f_s]$, then $R' = R$.

$R'_i = R_i \quad \forall i$ by induction.

$i=0$ is clear: $R_0 = R'_0 = k$.

$i>0$, $R_j = R'_j \ \forall j<i$.

$R_i \supset R'_i$, so we need to show

$R_i \subset R'_i$.

Choose $f \in R_i$ $i > 0 \Rightarrow f \in \mathfrak{m}$

$f \in \mathfrak{m}S \Rightarrow f = \sum g_j f_j$ w/ $g_j \in S$

g_j's are homogeneous, degree $< i$

$f = \varphi(f) = \sum \varphi(g_j) f_j$

$\varphi(g_j) \in R$, homogeneous, $\deg(\varphi(g_j)) < i$

$\Rightarrow \varphi(g_j) \in R'$

$$f = \sum \varphi(y_j) \cdot f_j \Rightarrow f \in R'$$

in R' in R'

$$\Rightarrow f \in R'_i$$

$$\Rightarrow R'_i \supset R_i \Rightarrow R'_i = R_i$$