Homework 2

Emma Knight

Due Febuary 22 at 5PM EST

Problem 1. This problem is some practice with universal properties.

(a) Let M_i be modules over a ring. Recall that $\prod M_i$ admits a map $\pi_n : \prod M_i \to M_n$ for all n. Show that $\prod M_i$ is universal in the sense that, if M is a module together with a map $f_n : M \to M_n$ for all n, then there exists a unique map $f : M \to \prod M_i$ such that $f_i = \pi_i \circ f$.

(b) Keeping M_i as modules over a ring, show that $\oplus M_i$ admits a similar universal property to the one above but with all arrows reversed.

(c) Let $f : M \to N$ be a map of modules. Show that f is surjective if and only if for all modules P together with maps g_1 and $g_2 : N \to P$ such that $g_1 \circ f = g_2 \circ f$, one must have that $g_1 = g_2$ (a map that satisfies this condition is called an *epimorphism*). Construct a similar criterion for injectivity (a map satisfying that condition is called a *monomorphism*).

Problem 2. Below is some practice with the tensor product.

(a) Verify that $\otimes$ is right-exact.

(b) Given an example of modules over a ring to show that $\otimes$ is not always exact. That is, give an exact sequence of modules $0 \to M_1 \to M_2 \to M_3 \to 0$ and a module N such that $0 \to M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$ is not exact.

(c) Show that $M \otimes R/I \cong M/IM$ (here, IM is the submodule of M generated by elements of the form im with $i \in I$ and $m \in M$).

(d) Let R and S be rings, and assume that S is an R-algebra (i.e. there is a map $R \to S$). Let M be an R-module and N be an S-module. Show that $\mathrm{Hom}_R(M, N) = \mathrm{Hom}_S(M \otimes_R S, N)$ as R-modules, where in the first Hom we view N as an R-module and in the second one we view it as an S-module[1].

Problem 3. Do exercise 2.4 in Eisenbud.

[1] If you're familiar with category theory, then this problem is almost saying that the functor $\mathcal{F} : R-\mathrm{mod} \to S-\mathrm{mod}$ given by sending $M \to M \otimes_R S$ is adjoint to the functor from $S-\mathrm{mod}$ to $R-\mathrm{mod}$ given by sending N to N (where one forgets about the action of all of S and just uses the action of R; all that one needs to show to complete adjointness is naturality of the isomorphism.

Problem 4. Let k be a field of characteristic 0. Let $R = k[x]$, $M_1 = R/(x^4 - x^2)$, $M_2 = k[x]/x^3 + 1$, $U_1 = \{1, x, x^2, \ldots\}$ and $U_2 = R \backslash (x)$. Compute $M_i[U_j^{-1}]$ for all i and j.

Problem 5. Let R be a PID, and $\mathfrak{p} \subset R$ a nonzero prime ideal, and choose a element p such that $(p) = \mathfrak{p}$. Let $S = R_\mathfrak{p}$ and K be the field of fractions of R. For all $x \in K$, define $v_\mathfrak{p}(x)$ by $v_\mathfrak{p}(0) = \infty$ and $v_\mathfrak{p}(p^a \frac{x}{y}) = a$ where $a \in \mathbb{Z}$ and x and y are coprime to p. Show that

(a) For all $x \in K$, at least one of x or x^{-1} is in S.

(b) $v_\mathfrak{p}(xy) = v_\mathfrak{p}(x)v_\mathfrak{p}(y)$, $v_\mathfrak{p}(x+y) \geq \min(v_\mathfrak{p}(x), v_\mathfrak{p}(y))$, and $v_\mathfrak{p}(x+y) = v_\mathfrak{p}(x)$ if $v_\mathfrak{p}(y) < v_\mathfrak{p}(x)$[2].

(c) Show that $I_n = \{s \in S | v_\mathfrak{p}(s) \geq n\}$ is an ideal for any nonnegative integer n.

(d) Show that every non-zero ideal of S is of the form I_n for some n.

Problem 6. Do exercise 2.9 of Eisenbud.

Problem 7. Do exercise 2.10 of Eisenbud. Additionally, explain why the "truly trivial" statement is, indeed, truly trivial.

[2]Something you can do if you are interested: show that, if y is a real number between 0 and 1, then the function defined by $|x| := y^{-v_\mathfrak{p}(x)}$ satisfies all the properties you want for an norm on a field (i.e. the distance function $d(x_1, x_2) = |x_1 - x_2|$ is a metric on K).