Solution to Q1 on Homework 1

Emma Knight

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There were a bunch of questions about question one, so I'm going to post my solution to this question.

Problem 1. Let $R = \mathbb{C}[x, y]$. R is a unique factorization domain but not a PID; this problem is about showing that basically everything in the argument for classification of modules over PIDs goes wrong for modules over R.

(a) Show that there are torsion modules generated by one element over R that are not of the form $R/(f)$ for some element $f \in R$.

(b) Show that there are fintie torsion modules over R that are not of the form $\oplus R/I$.

(c) Show that there are finite torsion-free modules over R that are not free.

(d) Give an example of two finite modules M, N over R, with N torsion-free, together with a surjection $M \twoheadrightarrow N$ that doesn't admit a section (i.e. a map $N \to M$ such that the composition $N \to M \to N$ is the identity).

Answer 1. Each part demands only one example, so I will do so. It is not too hard to see how to produce a bevy of examples from the ones I give though.

(a) This question is basically "give an example of an ideal that isn't principal." It is pretty straightforward to see that $R/(x, y)$ works for this.

(b) Let $I = (x, y)$, $J = (x^3, x^2y, xy^2, y^3)$, and $M = I/J$. This module is just the positive degree elements modulo the elements of degree three or more. Notice that $M = M_1 \oplus M_2$ where M_1 is the elements of degree 1 and M_2 is the elements of degree two. Additionally, notice that $\dim_{\mathbb{C}}(M) = 5$, with $\dim_{\mathbb{C}}(M_1) = 2$ and $\dim_{\mathbb{C}}(M_2) = 3$.

Assume that $M \cong \oplus R/I_i$ for some set of ideals I_i. Write e_i for the element that is 1 in R/I_i and 0 elsewhere. Let α_i be the image of e_i in M. I will show that there are both at least two e_is such that $(\alpha_i)_1 \neq 0$ and at most one with the same property.

Notice that, if $(\alpha_i)_1 \neq 0$, then α_i, $x\alpha_i$, and $y\alpha_i$ are linearly independent over $\mathbb{C}$, so one must have $\dim_{\mathbb{C}}(R/I_i) \geq 3$ (in fact it equals 3 but is at least 3 is enough). Since $\dim_{\mathbb{C}}(M) = 5$, there can't be two or more such e_is.

However, one must have that $\{(\alpha_i)_1\}$ span M_1 as a $\mathbb{C}$-vector space. Since $\dim_{\mathbb{C}}(M_1) = 2$, there must be at least two α_is such that $(\alpha_i)_1 \neq 0$.

This contradiction shows that M is not isomorphic to such a direct sum.

(c) Again, this basically comes down to "write down an ideal that isn't principal." In this case, again, (x, y) works.

(d) Consider the map $R^2 \to (x, y)$ given by sending (r_1, r_2) to $xr_1 + yr_2$. This map is almost by definition surjective.

Assume there is a section $s : (x, y) \to R^2$. One must have that $s(x) = (1+yf(x, y), -xf(x, y))$ for some polynomial $f(x, y) \in R$, and similarly, one must have $s(y) = (-yg(x, y), 1+xg(x, y))$ for some $g(x, y) \in R$.

Now, $xs(y) = s(xy) = ys(x)$, so we may compare the first coordinate of these two. One gets $y + y^2 f(x, y) = -xyg(x, y)$. However, the left hand side has nontrivial degree 1 component, wherease the right hand side is made up only of degree 2 and higher terms. This contradiction shows that there cannot be a section.