Last time: we reduced the torsion case to the p-power torsion case.

T - finite module / PID, every elt. in T is killed by a power of p (prime).

If $t \in T$, $\mathrm{ord}(t)$: smallest power of p (p^e)

s.t. $p^e t = 0$.

(e.g. in $k[x]/x^n$, $\mathrm{ord}(x^i) = x^{n-i}$)
$0 \leq i \leq n-1$

$\exists\, p^e$ s.t. $\forall t \in T$, $p^e t = 0$, (choose p^e smallest

$\exists\, y \in T$ s.t. $\mathrm{ord}(y) = p^e$.

$0 \to \langle y \rangle \to T \to \overline{T} \to 0$

$\overline{T} := T/\langle y \rangle.$

Lemma: If $\overline{x} \in \overline{T}$ has $\mathrm{ord}(\overline{x}) = p^f$, $\exists\, x \in T$ mapping to $\overline{x}$ s.t. $\mathrm{ord}(x) = p^f$.

Pf: Choose $x' \to \overline{x}$. $p^f x' \to 0$ in $\overline{T}$

Thus, $p^f x' \in \langle y \rangle$

$p^f x' = ay \qquad a \in R.$

. .

$$\underbrace{p^{e-f}\, p^f x'}_{=\,0} = p^{e-f} a y$$

$$(p^{e-f} a)\, y = 0 \implies p^e \mid p^{e-f} a$$

$$\implies p^f \mid a$$

$$a = p^f b$$

$$x = x' - by$$

$$p^f x = p^f y' - p^f b y$$

$$= ay - ay = 0$$

$$\Rightarrow \operatorname{ord}(x) \mid p^f \Rightarrow \operatorname{ord}(x) = p^f$$

$$x \to \overline{x} - b0 = \overline{x}$$

Lemma: Assume T is a finite p-power torsion module. Then $\exists$ $0 < e_1 \leq e_2 \leq \cdots \leq e_m$ s.t.

$$T \cong R/p^{e_1} \oplus R/p^{e_2} \oplus \cdots \oplus R/p^{e_m}.$$

Moreover, this is unique, and $m =$ smallest number of gens. of T.

Pf. Write $T = \langle y_1, \ldots, y_m \rangle$. assume m is minimal, and y_m has maximal order.

$$0 \to \langle y_m \rangle \to T \to \overline{T} \to 0$$

$\overline{T}$ gen by $\langle \overline{y_1}, \ldots, \overline{y_{m-1}} \rangle$

$$\overline{T} \cong R/p^{e_1} \oplus R/p^{e_2} \oplus \cdots \oplus R/p^{e_{m-1}}$$

$$\overline{z_1}, \quad \overline{z_2}, \ldots, \overline{z_{m-1}}$$

want to find a section $\overline{T} \to T$.

$$\{f : R/p^e \to T\} = \{t \in T \mid p^e t = 0\}$$

$$f \to f(1)$$

$$f(r) = rt \leftarrow t$$

$\exists\, z_1, \dots, z_{m-1}$ s.t. $z_i \to \overline{z}_i$, $\operatorname{ord}(z_i) = p^{e_i}$. Thus, $\exists\, i : \overline{T} \to T$, so $T = \langle y_m \rangle \oplus \overline{T}$

Thus, $T \cong R/_{p^{e_1}} \oplus \cdots \oplus R/_{p^{e_{m-1}}} \oplus R/_{p^{e_m}}$

Uniqueness: $f_1 = \dim_{R/p}(T/pT)$ $\qquad f_1 \geq f_2 \geq \cdots \geq 0$

$f_2 = \dim_{R/p}(pT/p^2T)$

$\vdots \qquad \vdots$

$f_j = \dim_{R/p}(p^{j-1}T/p^jT)$

$\vdots \qquad \vdots$

$f_j = \#\{e_i\text{'s} \geq j\}$

e_i's are determined by $(f_j)_{j=1}^{\infty}$

$\Rightarrow$ e_i's are determined by T.

we have proved:

M is a finite module over a PID,

Then $M \cong M^n \oplus \bigoplus_{i=1}^{m} M/r_i$, n, m unique,

$r_i \in R$, w/ $r_i \mid r_{i+1}$.

R/I is a field $\Leftrightarrow$ I is maximal.

Is $p \in R$ is prime (R is a PID),

$p \neq 0$

Then If $\exists (I) \supsetneq I' \supsetneq (p)$

$I' = (q) \Rightarrow p$ is not prime $\Rightarrow \Leftarrow$.

Def$^{\underline{n}}$: Let R be a ring. Then R is noetherian

$\iff \forall I \subset R$, I is f.g..

If M is an R-module, M is noetherian

$\iff \forall M' \subset M$, M' is f.g..

(R is noetherian $\iff$ R is noetherian as an R-mod).

Th$\underline{\underline{m}}$ (Exercise 1.1)

TFAE:

1) M is noetherian

2) If $M_1 \subset M_2 \subset \cdots \subset M_i \subset \cdots \subset M$ is an increasing chain of submodules of M, then $\exists\, k$ s.t. $M_i \subset M_k \ \forall i$

3) Every set of submodules of M contains maximal elts. under inclusion.

Pf. 1) $\Rightarrow$ 2)

Let M_i be an increasing chain of submodules of M.

$M' = \bigcup M_i$ is a submodule of M

Let $y_1, \dots, y_i$ generate M'

Every $y_j \in M_{k_j}$, choose k largest s.t.

$y_i \in M_k$ $\forall k$, $M \subset M_k$ ✓

$2) \Rightarrow 1)$

Let $M' \subset M$ be a submodule.

Choose $y_1 \in M'$, $y_2 \in M' \setminus \langle y_1 \rangle$, ...

$y_k \in M' \setminus \langle y_1, \dots, y_{k-1} \rangle$, ...

$M_i := \langle y_1, \dots, y_i \rangle$

$M_1 \subset M_2 \subset \dots \subset M_{k-1} \subset M_k \subset \dots$

$\exists\, M_k$ s.t. $M_\ell \subset M_k \;\forall \ell$.

$M_{k+1} = \langle y_1, \dots, y_k, y_{k+1} \rangle$, $y_{k+1} \in \langle y_1, \dots, y_k \rangle$

$\Rightarrow \Leftarrow$.

$\not\exists\, M_{k+1}$ so M' is gen by $\{y_1, \dots, y_k\}$.

$2) \Leftrightarrow 3)$ is fairly clear

Thm (Exercise 1.3):

Let M_1, M_2 be noetherian modules:

1) Any sub of M_1 is noetherian
2) Any quotient of M_1 is noetherian
3) $M_1 \oplus M_2$ is noetherian.

Cor: 1) I.I. R is noetherian, f.g. & finite are the same

Pf: I.I. M is f.g., $R^m \to M$, $N = \ker(R^m \to M)$, $N < R^m \Rightarrow N$ is f.g.

$\Rightarrow M$ is finite.

Cor: If R is noetherian, then finite R-modules are noetherian as well.

(Problem 1): $\mathbb{C}[x,y]$ is noetherian.

Thus, finite & f.g. are the same)

Def$\underline{\underline{n}}$: S is an R-alg. if S is a ring together w/ $R \to S$