$$R\text{-alg} = \{(S,f) \mid S \text{ is a ring, } f: R \to S \text{ ring hom.}\}$$

$\varphi: S_1 \to S_2$ is a map of R-algs. if

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ S_1 \arrow[rr, "\varphi"'] & & S_2 \end{tikzcd} \qquad \varphi \circ f_1 = f_2$$

Every ring is a $\mathbb{Z}$-alg in a unique way.

$k[x_1, \dots, x_n]$ is a k-alg

$k \to$ degree 0 polys.

If S is an R-alg

R-mod $\to$ S-mod

$M \to M \otimes_R S$

$s \cdot (m \otimes s_0) := m \otimes s s_0$

$(s_1 + s_2) \cdot (m \otimes s_0)$

$= s_1 \cdot (m \otimes s_0) +$

$s_2 \cdot (m \otimes s_0)$

$\vdots$

$S\text{-mod} \to R\text{-mod}$

$M \to M$ $\qquad\qquad$ $r \cdot m = f(r)m$

If M is an R-module, N is an S-module,

then $\operatorname{Hom}_R(M, N) = \operatorname{Hom}_S(M \otimes_R S, N)$

Prop: Let U be a mult. closed subset of R. M an R-module. Then $M \otimes_R R[U^{-1}] \cong M[U^{-1}]$.

(This is as $R[U^{-1}]$-modules, $R[U^{-1}]$ is an R-alg. under $\varphi: R \to R[U^{-1}]$ given by $\varphi(r) = \frac{r}{1}$)

Proof:

$$M \otimes_R R[u^{-1}] \xrightarrow{\sim} M[u^{-1}]$$

$$m \otimes \frac{r}{u} \mapsto \frac{rm}{u}$$

$$m \otimes \frac{1}{u} \leftarrow \frac{m}{u} \quad \text{is well-defined}$$

If $\frac{m}{u} = \frac{m'}{u'}$, then $m \otimes \frac{1}{u} = m' \otimes \frac{1}{u'}$.

$$\frac{m}{u} = \frac{m'}{u'} \Rightarrow \underline{vum' = vu'm}$$

$$m \otimes \frac{1}{u} = \left(\frac{vu'}{vu'}\right)\left(m \otimes \frac{1}{u}\right)$$

$$= \underline{(vu'm)} \otimes \left(\frac{1}{vuu'}\right)$$

$$= (vum') \otimes \left(\frac{1}{vuu'}\right) = m' \otimes \frac{1}{u'}$$

$\varphi: R \to R[u^{-1}]$ by $\varphi(r) = \frac{r}{1}$.

Prop. a) The map $I \to \varphi^{-1}(I)R[u^{-1}]$ from {ideals of $R[u^{-1}]$} $\circlearrowleft$ is the identity.

b) The map $I \to \varphi^{-1}(I)$ from {ideals of $R[u^{-1}]$} to {ideals of R} is an injection that preserves inclusion, intersection, and primality.

$( )$ $J = \varphi^{-1}(I)$ for some $I \subset R[U^{-1}]$ $\Leftrightarrow$

$J = \varphi^{-1}(J \cdot R[U^{-1}])$ $\Leftrightarrow$

$\forall\, u \in U$, the image $\bar{u}$ in R/J is not a zero-divisor.

Pf. a) $I = \varphi^{-1}(I) \cdot R[u^{-1}]$

$I \supset \varphi^{-1}(I) \cdot R[u^{-1}]$

w.t.s. $I \subset \varphi^{-1}(I) \cdot R[u^{-1}]$

$\frac{r}{u} \in I \Rightarrow \frac{r}{1} \in I$

$\Rightarrow r \in \varphi^{-1}(I)$

$\Rightarrow \frac{r}{u} \in \varphi^{-1}(I) \cdot R[u^{-1}]$

$\frac{r}{u}$ is an elt. of $\varphi^{-1}(I)$ times an elt. of $R[u^{-1}]$

b) $I \to \varphi^{-1}(I)$

$JR[u^{-1}] \leftarrow J$

$$\{\text{ideals of } R[u^{-1}]\} \rightleftarrows \{\text{ideals of } R\}$$

$I \to \varphi^{-1}(I)$ is an injection

$J \to J \cdot R[u^{-1}]$ is a surjection

- Preserving finite intersection and inclusion: $\varphi: R \to S$

If $S_1 \subset S_2 \subset S$, then $\varphi^{-1}(S_1) \subset \varphi^{-1}(S_2)$

$\varphi^{-1}(S_1 \cap S_2) = \varphi^{-1}(S_1) \cap \varphi^{-1}(S_2)$ is true on sets.

If $\varphi: R \to S$, $I \subset S$ is an ideal

$R/\varphi^{-1}(I) \hookrightarrow S/I$. [I] I is prime, S/I is a domain, $R/\varphi^{-1}(I)$ is as well, so $\varphi^{-1}(I)$ is prime.

✓

$f: R \to S$. If $\exists\, g: S \to R$ s.t.

$g \circ f = id$, then f is injective and

g is surjective.

$f(a) = f(b) \Rightarrow g(f(a)) = g(f(b)) \Rightarrow a = b$

If $r \in R$, $g(f(r)) = r$, $\exists\ z = f(r) \in S$

s.t. $g(z) = r$.

c) $J = \varphi^{-1}(I)$ for some I $\Leftrightarrow$ $J = \varphi^{-1}(J R[u^{-1}])$

$\Leftarrow$ is clear

Assume $J = \varphi^{-1}(I)$ for some I.

$$I \supset J \cdot R[u^{-1}]$$

$$J = \varphi^{-1}(I) \supset \varphi^{-1}(J \cdot R[u^{-1}]) \supset J$$

$$\Rightarrow J = \varphi^{-1}(J \cdot R[u^{-1}])$$

$J = \varphi^{-1}(JR[U^{-1}]) \iff \bar{u} \in R/J$ is a non zero-divisor $\forall u \in U$.

Assume $J = \varphi^{-1}(JR[U^{-1}])$

$R/J \hookrightarrow R[U^{-1}]/JR[U^{-1}]$

($\bar{u}$ must be a nonzero divisor) $\Leftarrow$ [$\bar{u} \in U$ is a unit]

$$\bar{u}\cdot\bar{x} = 0 \quad \text{in} \quad R/J \quad \Rightarrow$$

$$\bar{u}.\bar{x} = 0 \quad \text{in} \quad R[u^{-1}]/JR[u^{-1}]$$

$$\bar{u}^{-1}\bar{u}\,\bar{x} = 0 \quad \text{in} \quad ''$$

$$\bar{x} = 0 \quad \text{in} \quad ''$$

$$\bar{x} = 0 \quad \text{in} \quad R/J.$$

Assume $u \in U$ never reduces to a zero-divisor in R/J

Choose $r \in \varphi^{-1}(JR[U^{-1}])$

$\frac{r}{1} \in JR[U^{-1}] \Rightarrow \frac{r}{1} = \frac{j}{u} \quad j \in J,\ u \in U$

$\Rightarrow \exists v$ s.t. $vur = vj$ in R

$\bar{v}\bar{u}\bar{r} = \bar{v}\bar{j} = 0$ in R/J

$\bar{v}\,\bar{u}\,\bar{r} = 0$ in R/J

$\Rightarrow \bar{r} = 0$ in R/J

$\Rightarrow r \in J$

$$\Rightarrow J \supset \varphi^{-1}(JR[u^{-1}]) \qquad \Rightarrow J = \varphi^{-1}(JR[u^{-1}])$$

$$J \subset \varphi^{-1}(JR[u^{-1}])$$

$\blacksquare$

Cor.: If R is noetherian, then $R[u^{-1}]$ is as well.