Poisson Image Deblurring

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Overview

1. Motivation
   - Make local changes during image editing can be difficult
   - Traditional methods take a lot of time and leave noticeable seams
   - Laplacian pyramid approach can blend the edges but have color mismatch
2. Approach
   - Change in gradient domain
   - Reconstruct image from gradient
3. Mathematical Background
   - Definition
   - Basic Idea

Motivation

- Make local changes during image editing can be difficult
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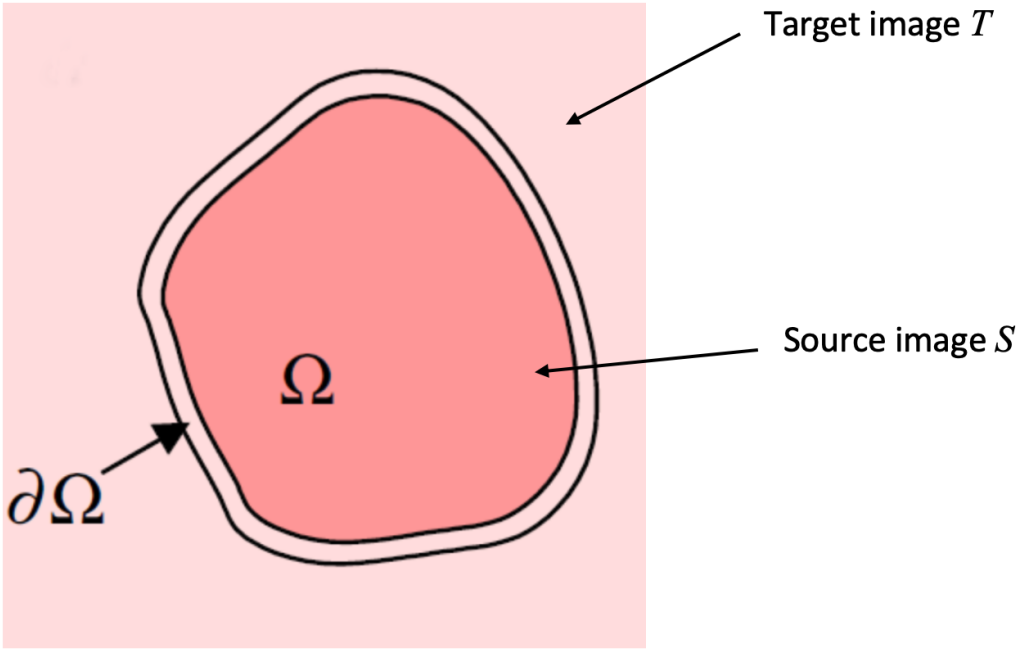
Approach

- Change in gradient domain
- Reconstruct image from gradient

Definition

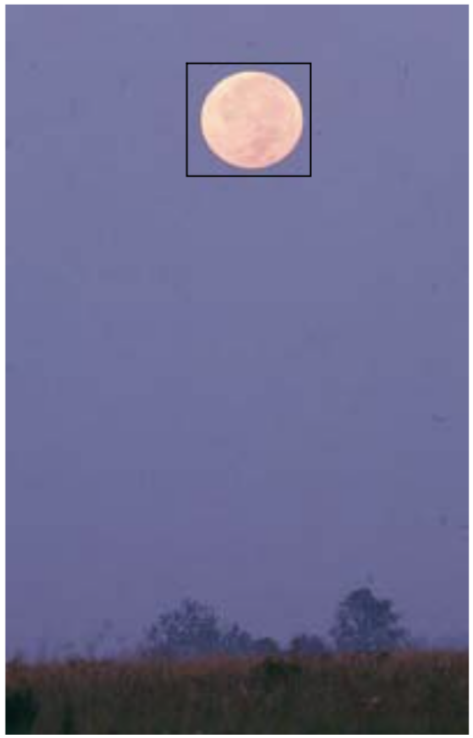


- The source image $S(x, y)$ defined over a closed region Ω
- The boundary of this region denoted as $\partial\Omega$
- The target image $T(x, y)$ defined some rectangular region in $\mathbb{R}^2$

Basic Idea

- Want to construct the composite image, $I(x, y)$, which should agree with $T(x, y)$ and look like $S(x, y)$
- $I(x, y)$ should exactly agree with $T(x, y)$
- $I(x, y)$ "look like" $S(x, y)$ inside Ω
- If we place directly S over T and smooth over the edges, the result maybe unacceptable, due to color mismatch

Basic Idea



source image

target image

Basic Idea

Figure: Simple cloning, background of source does not agree with the target

Basic Idea

source images

simple cloning

Basic Idea

source images

Poisson seamless cloning

How to Solve

- Make the interior of Ω "look like" $S(x, y)$
- Also want the color to match between $S(x, y)$ and $T(x, y)$

How to Solve

- Transfer the "edges" of the $S(x, y)$ to into Ω
- Computer colors inside Ω as close as possible with pixels from $T(x, y)$ surrounding Ω
- We want the gradient of the composite image, $\nabla I(x, y)$, inside Ω as close possible to $\nabla S(x, y)$
- Subject to the constraint the result must match existing values of $T(x, y)$ on the boundary $\partial\Omega$

How to solve

Miminize:

$$\min_{I(x,y)\in\Omega} \iint_{\Omega} ||\nabla I(x,y) - \nabla S(x,y)||^2 dxdy \tag{1}$$

Subject to:

$$I(x,y) = T(x,y), \quad (x,y) \in \partial\Omega \tag{2}$$

Euler-Lagrange Equation

A fundamental equation of **Calculus of Variations** states that, if J is defined by an integral of the form:

$$J = \int F(x, f, f_x)dx$$

Then J has a stationary value if the following differential equation is satisfied:

$$\frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial f_x} = 0$$

Euler-Lagrange Equation

In our case, we have:

$$\begin{aligned} F(x,y) &= ||\nabla I(x,y) - \nabla S(x,y)||^2 \\ &= \left(\frac{\partial I}{\partial x} - \frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial I}{\partial y} - \frac{\partial S}{\partial y}\right)^2 \end{aligned} \tag{3}$$

$I(x,y)$ that solves Equation 1 is a solution of the **Euler-Lagrange equation**:

$$\frac{\partial F}{\partial I} - \frac{d}{dx}\frac{\partial F}{\partial I_x} - \frac{d}{dy}\frac{\partial F}{\partial I_y} = 0, \quad (x,y) \in \Omega \tag{4}$$

Possion Equation

Plugging in Equation 3 into Equation 4:

$$2\left(\frac{\partial^2 I}{\partial x^2} - \frac{\partial S^2}{\partial x^2}\right) + 2\left(\frac{\partial^2 I}{\partial y^2} - \frac{\partial^2 S}{\partial y^2}\right) \tag{5}$$

We have:

$$\nabla^2 I(x, y) = \nabla^2 S(x, y), \quad (x, y) \in \Omega \tag{6}$$

Where:

$$\nabla^2 I = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}$$

Subject to:

$$I(x, y) = T(x, y), \quad (x, y) \in \partial\Omega \tag{7}$$

Note, an equation of the form 6 is called a **Possion equation**, and a constraint of the form of Equation 7 is called a **Dirichlet boundary condition**.

Discrete Poisson solver

- In Equation 6 the Laplacian of the new image $I(x, y)$ is equal to the Laplacian of the source image $S(x, y)$ inside Ω.
- More powerful to mim. the difference using a *guidance vector field* $\vec{\mathbf{S}} = (S_x(x, y), S_y(x, y))$ at every pixel (x, y), Equation 6 change to:

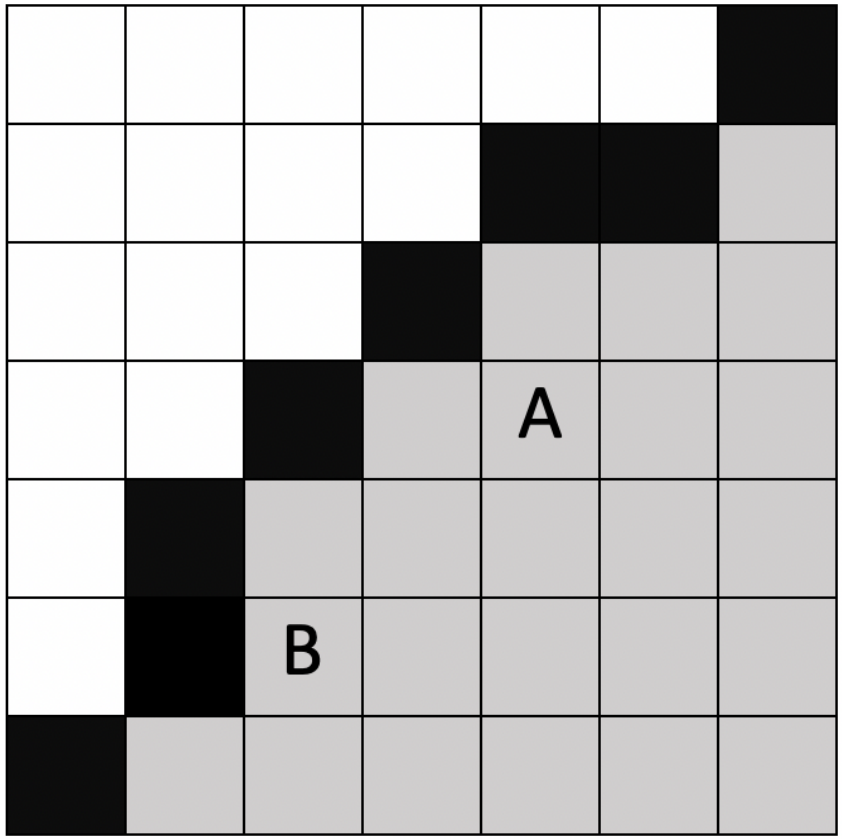
$$\nabla^2 I(x, y) = div \begin{bmatrix} S_x(x, y) \\ S_y(x, y) \end{bmatrix} = \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y}, \quad (x, y) \in \Omega \qquad (8)$$

- Create discrete version of Equation 6 and 7
- Use 4 neighborhood operator to approximiate the Laplacian operator

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Discrete Poisson solver

- Each pixel $p = (x, y)$ in Ω generates a linear equation for unknown values in $I(x, y)$
- Two cases, for the 4-neighborhood $N(p)$:



Discrete Poisson solver

1. $N(p) \in \Omega$ The neighborhoods of p are fully contained in Ω and the Laplacian:
   $I(x+1,y) + I(x-1,y) + I(x,y+1) + I(x,y-1) - 4I(x,y) =$
   $S(x+1,y) + S(x-1,y) + S(x,y+1) + S(x,y-1) - 4S(x,y)$
2. $N(p) \notin \Omega$ Not all neighborhoods of p are contained in Ω, at least one neighborhood is in $\partial\Omega$:

$$\left(\sum_{q \in N(p) \cap \Omega} I(q)\right) + \left(\sum_{q \in N(p) \cap \partial\Omega} T(q)\right) - 4I(x,y) =$$
$$S(x+1,y) + S(x-1,y) + S(x,y+1) + S(x,y-1) - 4S(x,y)$$

Discrete Poisson solver

- This reduces to the sparse linear systems:

$$\begin{pmatrix} 1 & 1 & -4 & 1 & 1 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 1 & -4 & 1 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 1 & -4 & 1 & 1 & \cdots & 0 \\ \vdots & & & & & & & & \vdots \\ \vdots & & & & & & & & \vdots \\ \vdots & & & & & & & & \vdots \\ 0 & & & & & & & & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b_1 \\ b_2 \\ 0 \\ \vdots \\ b_3 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}$$

The End