Exercise 2.4: A spaceship travels from Earth in a straight line at relativistic speed v to another planet x light years away. Write a program to ask the user for the value of x and the speed v as a fraction of the speed of light c, then print out the time in years that the spaceship takes to reach its destination (a) in the rest frame of an observer on Earth and (b) as perceived by a passenger on board the ship. Use your program to calculate the answers for a planet 10 light years away with $v = 0.99c$.

Exercise 2.6: Planetary orbits

The orbit in space of one body around another, such as a planet around the Sun, need not be circular. In general it takes the form of an ellipse, with the body sometimes closer in and sometimes further out. If you are given the distance ℓ_1 of closest approach that a planet makes to the Sun, also called its *perihelion*, and its linear velocity v_1 at perihelion, then any other property of the orbit can be calculated from these two as follows.

a) Kepler's second law tells us that the distance ℓ_2 and velocity v_2 of the planet at its most distant point, or *aphelion*, satisfy $\ell_2 v_2 = \ell_1 v_1$. At the same time the total energy, kinetic plus gravitational, of a planet with velocity v and distance r from the Sun is given by

$$E = \tfrac{1}{2}mv^2 - G\frac{mM}{r},$$

where m is the planet's mass, $M = 1.9891 \times 10^{30}\,\mathrm{kg}$ is the mass of the Sun, and $G = 6.6738 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ is Newton's gravitational constant. Given that energy must be conserved, show that v_2 is the smaller root of the quadratic equation

$$v_2^2 - \frac{2GM}{v_1 \ell_1} v_2 - \left[v_1^2 - \frac{2GM}{\ell_1} \right] = 0.$$

Once we have v_2 we can calculate ℓ_2 using the relation $\ell_2 = \ell_1 v_1 / v_2$.

b) Given the values of v_1, ℓ_1, and ℓ_2, other parameters of the orbit are given by simple formulas can that be derived from Kepler's laws and the fact that the orbit is an ellipse:

$$\begin{aligned}
\text{Semi-major axis:} \quad & a = \tfrac{1}{2}(\ell_1 + \ell_2), \\
\text{Semi-minor axis:} \quad & b = \sqrt{\ell_1 \ell_2}, \\
\text{Orbital period:} \quad & T = \frac{2\pi ab}{\ell_1 v_1}, \\
\text{Orbital eccentricity:} \quad & e = \frac{\ell_2 - \ell_1}{\ell_2 + \ell_1}.
\end{aligned}$$

Write a program that asks the user to enter the distance to the Sun and velocity at perihelion, then calculates and prints the quantities ℓ_2, v_2, T, and e.

c) Test your program by having it calculate the properties of the orbits of the Earth (for which $\ell_1 = 1.4710 \times 10^{11}\,\mathrm{m}$ and $v_1 = 3.0287 \times 10^4\,\mathrm{m\,s^{-1}}$) and Halley's comet ($\ell_1 = 8.7830 \times 10^{10}\,\mathrm{m}$ and $v_1 = 5.4529 \times 10^4\,\mathrm{m\,s^{-1}}$). Among other things, you should find that the orbital period of the Earth is one year and that of Halley's comet is about 76 years.

Exercise 3.2: Curve plotting

Although the `plot` function is designed primarily for plotting standard xy graphs, it can be adapted for other kinds of plotting as well.

a) Make a plot of the so-called *deltoid* curve, which is defined parametrically by the equations

$$x = 2\cos\theta + \cos 2\theta, \qquad y = 2\sin\theta - \sin 2\theta,$$

where $0 \le \theta < 2\pi$. Take a set of values of θ between zero and 2π and calculate x and y for each from the equations above, then plot y as a function of x.

b) Taking this approach a step further, one can make a polar plot $r = f(\theta)$ for some function f by calculating r for a range of values of θ and then converting r and θ to Cartesian coordinates using the standard equations $x = r\cos\theta$, $y = r\sin\theta$. Use this method to make a plot of the Galilean spiral $r = \theta^2$ for $0 \le \theta \le 10\pi$.

c) Using the same method, make a polar plot of "Fey's function"

$$r = \mathrm{e}^{\cos\theta} - 2\cos 4\theta + \sin^5\frac{\theta}{12}$$

in the range $0 \le \theta \le 24\pi$.

Exercise 5.4: The diffraction limit of a telescope

Our ability to resolve detail in astronomical observations is limited by the diffraction of light in our telescopes. Light from stars can be treated effectively as coming from a point source at infinity. When such light, with wavelength λ, passes through the circular aperture of a telescope (which we'll assume to have unit radius) and is focused by the telescope in the focal plane, it produces not a single dot, but a circular diffraction pattern consisting of central spot surrounded by a series of concentric rings. The intensity of the light in this diffraction pattern is given by

$$I(r) = \left(\frac{J_1(kr)}{kr}\right)^2,$$

where r is the distance in the focal plane from the center of the diffraction pattern, $k = 2\pi/\lambda$, and $J_1(x)$ is a Bessel function. The Bessel functions $J_m(x)$ are given by

$$J_m(x) = \frac{1}{\pi}\int_0^\pi \cos(m\theta - x\sin\theta)\,\mathrm{d}\theta,$$

where m is a nonnegative integer and $x \ge 0$.

a) Write a Python function `J(m,x)` that calculates the value of $J_m(x)$ using Simpson's rule with $N = 1000$ points. Use your function in a program to make a plot, on a single graph, of the Bessel functions J_0, J_1, and J_2 as a function of x from $x = 0$ to $x = 20$.

b) Make a second program that makes a density plot of the intensity of the circular diffraction pattern of a point light source with $\lambda = 500$ nm, in a square region of the focal plane, using the formula given above. Your picture should cover values of r from zero up to about 1 μm.

Hint 1: You may find it useful to know that $\lim_{x\to 0} J_1(x)/x = \frac{1}{2}$. Hint 2: The central spot in the diffraction pattern is so bright that it may be difficult to see the rings around it on the computer screen. If you run into this problem a simple way to deal with it is to use one of the other color schemes for density plots described in Section 3.3. The "`hot`" scheme works well. For a more sophisticated solution to the problem, the `imshow` function has an additional argument `vmax` that allows you to set the value that corresponds to the brightest point in the plot. For instance, if you say "`imshow(x,vmax=0.1)`", then elements in `x` with value 0.1, or any greater value, will produce the brightest (most positive) color on the screen. By lowering the `vmax` value, you can reduce the total range of values between the minimum and maximum brightness, and hence increase the sensitivity of the plot, making subtle details visible. (There is also a `vmin` argument that can be used to set the value that corresponds to the dimmest (most negative) color.) For this exercise a value of `vmax=0.01` appears to work well.

Exercise 5.21: Electric field of a charge distribution

Suppose we have a distribution of charges and we want to calculate the resulting electric field. One way to do this is to first calculate the electric potential ϕ and then take its gradient. For a point charge q at the origin, the electric potential at a distance r from the origin is $\phi = q/4\pi\epsilon_0 r$ and the electric field is $\mathbf{E} = -\nabla\phi$.

a) You have two charges, of $\pm 1\,\mathrm{C}$, 10 cm apart. Calculate the resulting electric potential on a $1\,\mathrm{m} \times 1\,\mathrm{m}$ square plane surrounding the charges and passing through them. Calculate the potential at 1 cm spaced points in a grid and make a visualization on the screen of the potential using a density plot.

b) Now calculate the partial derivatives of the potential with respect to x and y and hence find the electric field in the xy plane. Make a visualization of the field also. This is a little trickier than visualizing the potential, because the electric field has both magnitude and direction. One way to do it might be to make two density plots, one for the magnitude, and one for the direction, the latter using the "`hsv`" color scheme in `pylab`, which is a rainbow scheme that passes through all the colors but starts and ends with the same shade of red, which makes it suitable for representing things like directions or angles that go around the full circle and end up where they started. A more sophisticated visualization might use the arrow object from the `visual` package, drawing a grid of arrows with direction and length chosen to represent the field.

c) Now suppose you have a continuous distribution of charge over an $L \times L$ square. The charge density in Cm^{-2} is

$$\sigma(x,y) = q_0 \sin\frac{2\pi x}{L} \sin\frac{2\pi y}{L}.$$

Calculate and visualize the resulting electric field at 1 cm-spaced points in 1 square meter of the xy plane for the case where $L = 10\,\mathrm{cm}$, the charge distribution is centered in the middle of the visualized area, and $q_0 = 100\,\mathrm{Cm}^{-2}$. You will have to perform a double integral over x and y, then differentiate the potential with respect to position to get the electric field. Choose whatever integration method seems appropriate for the integrals.