MAE 598 Design Optimization

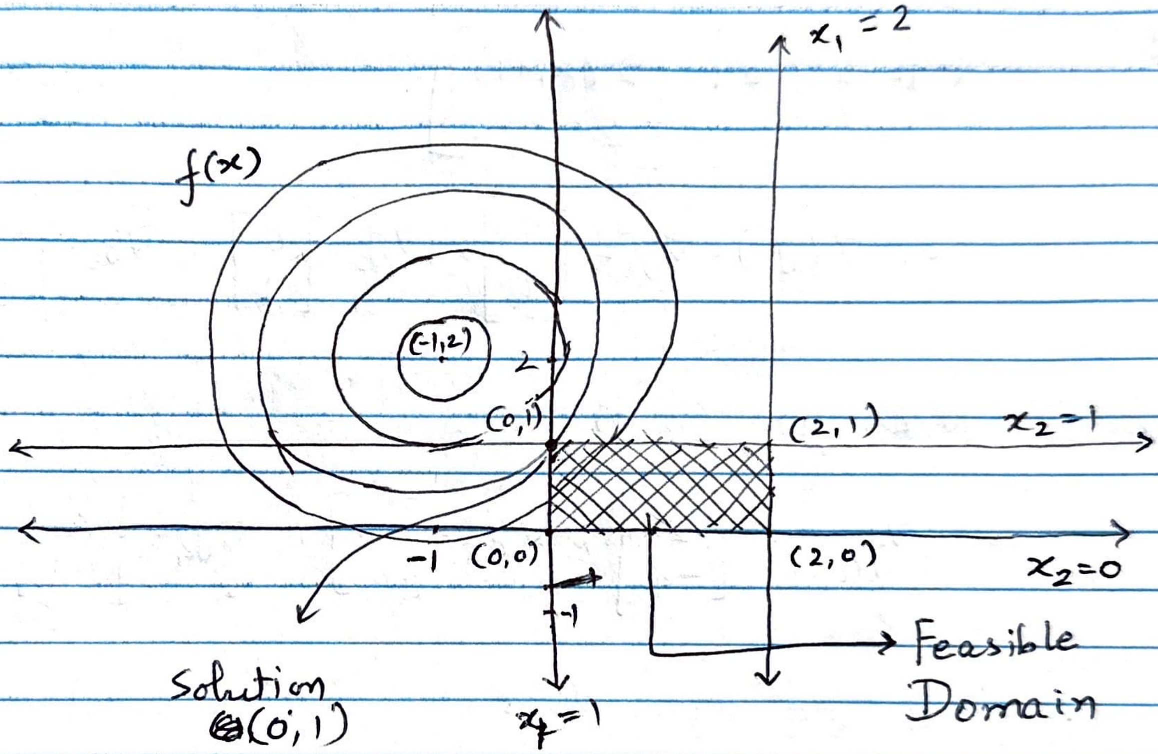
Assignment-4 Rumitkumar Patel

1>. $\min f(x) = (x_1+1)^2 + (x_2-2)^2$

s.t. $g_1 = x_1 - 2 \le 0$, $g_3 = -x_1 \le 0$,

$g_2 = x_2 - 1 \le 0$, $g_4 = -x_2 \le 0$.

Here, $f(x) = (x_1+1)^2 + (x_2-2)^2$ is a function of a circle with center $(-1, 2)$



Here, As we can see from the graph, Min. point as (0,1) graphically.

So, Lagrangian $L = (x_1+1)^2 + (x_2-2)^2 + l_1(x_2-2) + l_2(x_1-1) + l_3(-x_1) + l_4(-x_2)$

→ Conditions

if $x_2 - 2 = 0$, $l_1 > 0$, if $x_2 - 1 < 0$, $l_1 = 0$

if $x_1 - 1 = 0$, $l_2 > 0$, if $x_1 - 2 < 0$, $l_2 = 0$

if $-x_1 = 0$, $l_3 > 0$, if $-x_1 < 0$, $l_3 = 0$

if $-x_2 = 0$, $l_4 > 0$, if $-x_2 < 0$, $l_4 = 0$

~~→ for point (0,0), the active constraints are g_3 and g_4. So, $\nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$~~

Direction of feasible domain at the corner points of feasible domain:

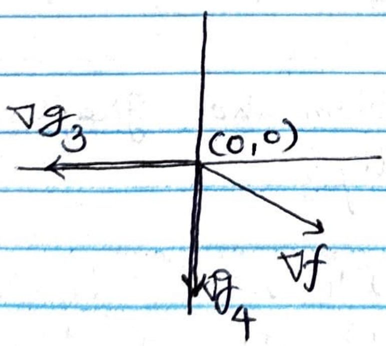
$$\nabla(f) = g(f) : \begin{bmatrix} 2(x_1+1) \\ 2(x_2-2) \end{bmatrix}$$

$$\text{so, } \nabla(g_1) = \nabla g_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \nabla g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

→ Now, for (0,0)

$$\nabla f_{(0,0)} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \quad \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

If, we draw this

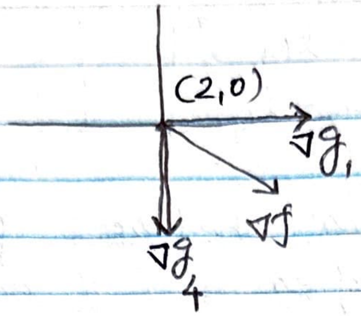


Here, Equilibrium of the point (0,0) is not possible

→ for (2,0),

$$\nabla f = \begin{bmatrix} 6 \\ -4 \end{bmatrix}, \quad \nabla g_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

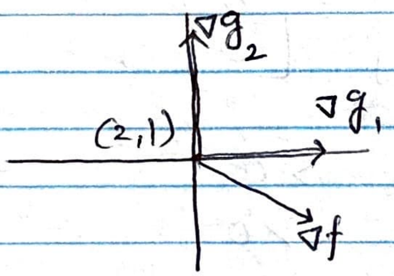
If, we draw this.



As, we can see from the figure, the equilibrium at point (2,0) is not possible.

for (2,1),

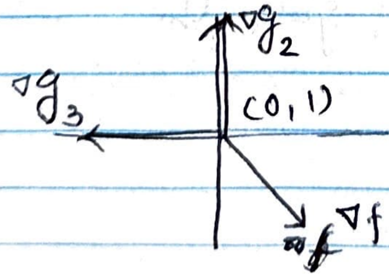
$$\nabla f = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, \quad \nabla g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \nabla g_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Equilibrium is not possible for point (2,1)

for (0,1)

$$\nabla f = \cancel{\begin{bmatrix} 2 \\ -4 \end{bmatrix}} \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \nabla g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Here, Equilibrium is possible at point (0,1)

Now, from KKT condition,

$$\nabla(L) = \begin{bmatrix} 2(x_1+1) + \mu_1 - \mu_3 \\ 2(x_2-2) + \mu_2 - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

from all the conditions before, if we put $(0,1)$, it satisfies, ~~So~~, g_2 and g_3. So,

$$\mu_1 = \mu_4 = 0, \quad \mu_3, \mu_2 > 0$$

Now, $x_1 = 0$, $x_2 = 1$

$$\nabla L = \begin{bmatrix} 2x_1 + 2 - \mu_3 \\ 2x_2 - 4 + \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mu_3 = 2x_1 + 2 = 2 > 0$$
$$\mu_2 = 4 - 2x_2 = 2 > 0$$

So, KKT condition is satisfied.

So, solution is $(0,1)$

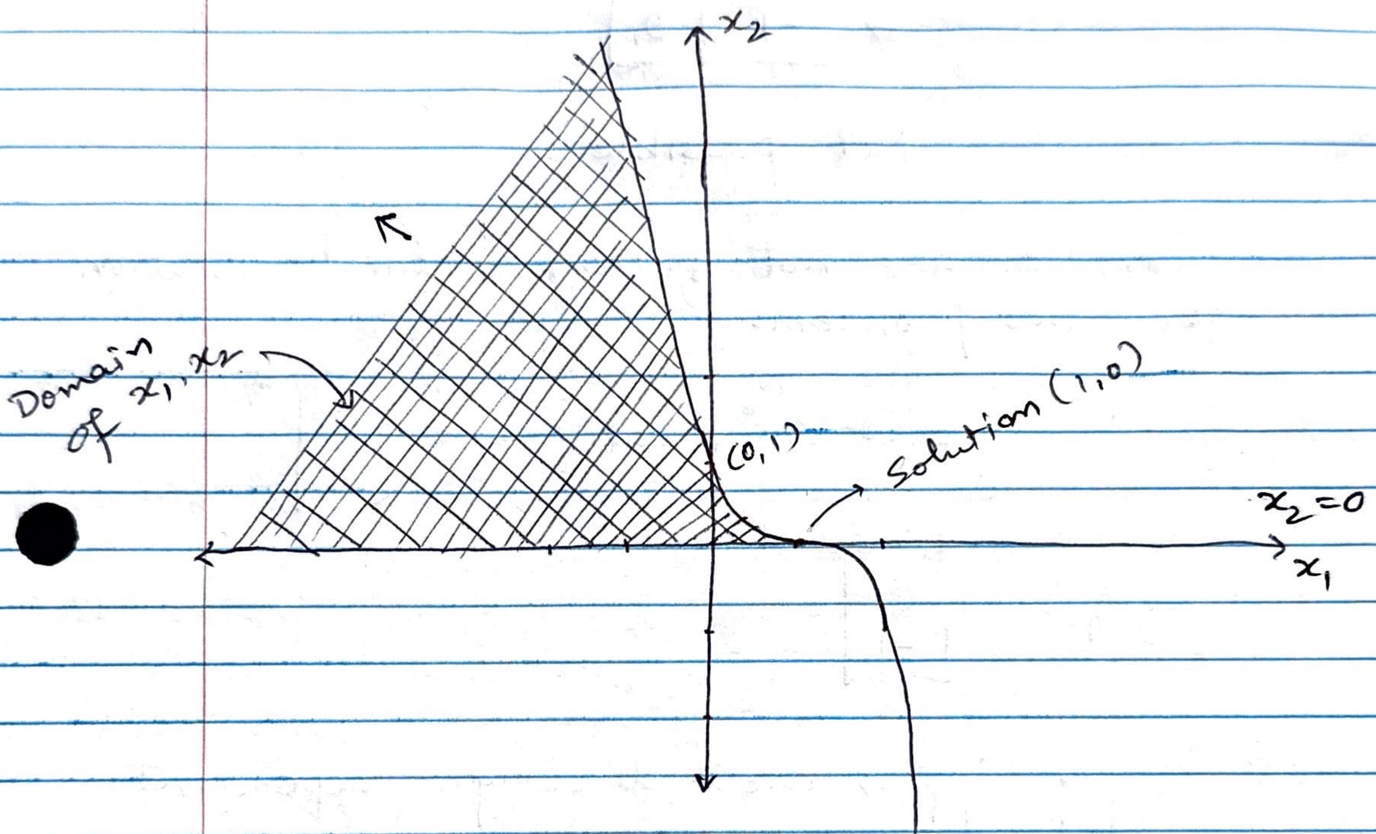
Also, graphically we got the same result. So, proved.

2).

$$\min_{x_1, x_2} \quad f = -x_1$$

$$g_1 = x_2 - (1-x_1)^3 \le 0 \text{ and } x_2 \ge 0.$$

$$g_2 = -x_2 \le 0$$



graphically we can say that min. value of f is $(1,0)$.

Now, let's consider KKT condition.

Lagrang : $L: -x_1 + l_1 (x_2 - (1-x_1)^3) + l_2(-x_2)$

$$g(L) = \begin{bmatrix} -1 + 3l_1(1-x_1)^2 \\ l_1 - l_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$l_1 = l_2$$

for

(1,0), $\mu_1 > 0$, $\mu_2 > 0$

at (1,0) $\Rightarrow$ $-1 + 3\mu_1 (1-1)^2 = 0$

$$\Rightarrow \boxed{-1 \neq 0}$$

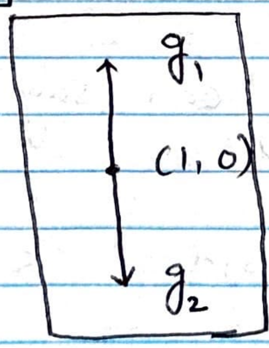
Not possible

So, we are not getting optimal solution for this problem.

$$g(g_1) = \begin{bmatrix} 3(1-x_1)^2 \\ 1 \end{bmatrix} \text{ at } (1,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$g(g_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{are linearly dependent.}$$



So, as it doesn't satisfy KKT condition, we cannot find solution analytically using KKT conditions.

3).

$$\max \quad f = x_1x_2 + x_2x_3 + x_1x_3$$
$$\text{s.t.} \quad h = x_1 + x_2 + x_3 - 3 = 0$$

$$\min_{x_1,x_2,x_3} \quad f = -(x_1x_2 + x_2x_3) + x_1x_3)$$
$$\text{s.t.} \quad h: x_1 + x_2 + x_3 - 3 = 0.$$

since '=' constraint we can say that the problem is not convex problem.

we can use Reduced Gradient Method.

Decision variables = $n - m = 3 - 1 = 2$
State variables = $m = 1$

Let $d = (x_1, x_2)$
$s = x_3$

$$\frac{df}{dd} = \frac{\partial f}{\partial d} + \frac{\partial f}{\partial s}\frac{ds}{dd}$$

$$\frac{df}{dd} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\left(\frac{\partial h}{\partial d}\right)$$

↳ Reduced gradient.

$d \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $s \rightarrow x_3$, $f = -(x_1x_2 + x_2x_3 + x_1x_3)$

$\frac{\partial f}{\partial d} = \begin{bmatrix} -(x_2 + x_3) \\ -(x_1 + x_3) \end{bmatrix}$, $\frac{\partial f}{\partial s} = -(x_2 + x_1)$

$$h = x_1 + x_2 + x_3 - 3 = 0$$

$$\frac{\partial h}{\partial s} = 1, \quad \left(\frac{\partial h}{\partial s}\right)^{-1} = 1 \qquad \frac{\partial h}{\partial d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\cancel{\frac{\partial f}{\partial d}} \quad \frac{df}{dd} = -\begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \end{bmatrix} + (x_2 + x_1)(1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\cancel{x_2} - x_3 + \cancel{x_2} + x_1 \\ -\cancel{x_1} - x_3 + x_2 + \cancel{x_1} \end{bmatrix} = \begin{bmatrix} -x_3 + x_1 \\ -x_3 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As. $\frac{df}{dd} = 0.$

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = x_3 \\ x_2 = x_3 \end{matrix} \Rightarrow x_1 = x_2 = x_3$$

from the constraint,

$$x_1 + x_2 + x_3 = 3 \Rightarrow 3x_1 = 3 \Rightarrow \boxed{x_1 = 1}$$

$$\boxed{x_1 = x_2 = x_3 = 1}$$

Using Lagrangian Approach: (To verify).

$$L = -(x_1 x_2 + x_2 x_3 + x_3 x_1) + \lambda(x_1 + x_2 + x_3 - 3)$$

$$\frac{\partial L}{\partial x} = \nabla_x L = \begin{bmatrix} -x_2 - x_3 + \lambda \\ -x_1 - x_3 + \lambda \\ -x_1 - x_2 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 3 = 0.$$

$\lambda = x_2 + x_3$

$\Rightarrow \lambda = x_1 + x_3 \Rightarrow x_2 + x_3 = x_1 + x_3 \Rightarrow x_1 = x_2$

$\Rightarrow \lambda = x_1 + x_2 \Rightarrow x_2 + x_3 = x_1 + x_2 \Rightarrow x_1 = x_3$

$\boxed{x_1 = x_2 = x_3}$

Now As we did before

$x_1 + x_2 + x_3 = 3$

$3x_1 = 3$

$\boxed{x_1 = 1}$ So, $\boxed{x_1 = x_2 = x_3 = 1}$

Sufficient condition: $\partial x^T \frac{\partial^2 L}{\partial x^2} \partial x = 0$

$$\Rightarrow \begin{bmatrix} \partial x_1 & \partial x_2 & \partial x_3 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 L}{\partial x^2} \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \partial x_3 \end{bmatrix} \neq$$

$$\frac{\partial^2 L}{\partial x^2} = \text{Hessian of Lagrangian} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

Now, if we check the eigen values $\lambda_1 = -2$, $\lambda_2 = 1$, $\lambda_3 = 1$. All the eigen values are not positive.

$$\partial x^T \frac{\partial^2 L}{\partial x^2} \partial x = \begin{bmatrix} \partial x_1 & \partial x_2 & \partial x_3 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \partial x_3 \end{bmatrix}$$

$$= -2\partial x_1 \partial x_2 - 2\partial x_1 \partial x_3 - 2\partial x_2 \partial x_3$$

∂x to be feasible, the feasible perturbation is $\left(\frac{\partial h}{\partial x}\right)\partial x = 0$.

$$\begin{bmatrix} \frac{\partial h}{\partial x_1}, & \frac{\partial h}{\partial x_2}, & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \partial x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \partial x_3 \end{bmatrix} = \{ 0 \quad \rightarrow \quad \partial x_1 + \partial x_2 + \partial x_3 = 0.$$

$$\therefore \partial x_1 = -\partial x_2 - \partial x_3$$

So putting this into the equation before we get,

$$= -2(-\partial x_2 - \partial x_3)\partial x_2 - 2(-\partial x_2 - \partial x_3)\partial x_3 - 2\partial x_2 \partial x_3$$

$$= 2(\partial x_2^2 + \partial x_2 \partial x_3 + \partial x_3^2)$$

$$= 2(\partial x_2 + \frac{1}{2}\partial x_3)^2 + \frac{3}{4}(\partial x_3)^2 \geq 0.$$

→ if we want $\partial x^T \frac{\partial^2 L}{\partial x^2} \partial x = 0$, then ∂x_1 must be 0.
which means $\partial x = 0$, is not possible.
So, $\partial x^T \frac{\partial^2 L}{\partial x^2} \partial x > 0$.

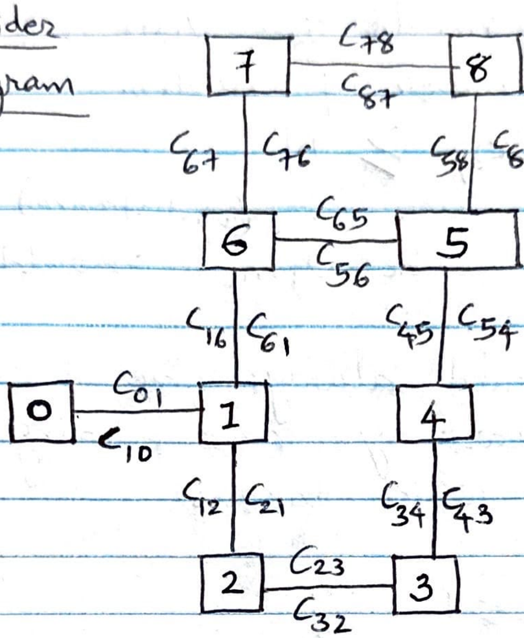
→ So, $x_1 = x_2 = x_3 = 1$ is global maximum to the original problem. if we put all the variables in the function we get $\boxed{f = 3}$

For Problem-4. Look at the python file.

5) cost from i to j is c_{ij}

Let's Consider movement x_{ij}

Consider Diagram



Let's say forward movement ~~$x_{ij} = \begin{cases} x : i \\ 0 \end{cases}$~~

$$x_{ij} = \begin{cases} 1, & \text{if } i \text{ is connected to } j \\ 0, & \text{if } i \text{ is not connected to } j \end{cases}$$

Similarly for backward movement

$$x_{ji} = \begin{cases} 1, & \text{if } j \text{ is connected to } i \\ 0, & \text{if } j \text{ is not connected to } i \end{cases}$$

→ If we include cost then, forward

$$x_{ij} = \begin{cases} c_{ij}, & \text{if } i \text{ is connected to } j \\ \infty, & \text{if } i \text{ is not connected to } j \end{cases}$$

→ Similarly including cost: Backward

$$x_{ji} = \begin{cases} c_{ij}, & \text{if } j \text{ is connected to } i \\ \infty, & \text{if } j \text{ is not connected to } i \end{cases}$$

→ So, Now, to find constraints,

The truck needs to visit all the nodes. where N is number of nodes.

So, $\sum_i \sum x_{ij} \geq N$

(or distance)

→ $\sum x_{ij} = \sum x_{ji}$ The time should be equal.

→ Now, There should always be a connection between the starting and ending node to one other node which means.

start, $\sum x_{oj} \geq 1 \quad \forall j$

Ending, $\sum x_{jo} \geq 1, \forall j$

So, the problem formulation is,

$$\min_{x_{ij}} \sum_{ij}^{N} (x_{ij} c_{ij})$$

$$\sum x_{ij} \geq N$$

$$\sum x_{ij} = \sum_{ji}$$

$$\sum x_{oj} \geq 1, \forall j$$

$$\sum x_{jo} \geq 1, \forall j$$