Rumitkumar Patel
MAE 598: Design Optimization
HW-5

2>

$$\dot{h}(t) = v(t)$$
$$\dot{v}(t) = -g + \frac{a(t)}{m(t)}$$
$$\cancel{\dot{m}(t) = -ka}$$

$$\dot{m}(t) = -ka(t).$$

→ given initial conditions are: $h(0) = h_0$
$v(0) = v_0$
$m(0) = m_0$

It is a fixed end, free time problem.

Assumptions are that: (1) height $h(t)$ cannot be negative so, $h(t) \geq 0$
(2) Mass $m(t)$ cannot be negative
and $m(t) \geq 0$.

→ We need to minimize fuel consumption. here we have considered τ as the terminal time. We can see accelaration $a(t) = \dot{m}(t)/k$,

Where k is constant fuel burning rate. So, if we can apply ~~min~~ minimum applied thrust, which will ultimately give us the minimal fuel consumption and maximum mass of moon lander.

$$\min_{a(t)} \; p(a) := \int_0^{\tau} a(t)\,dt = \frac{m_0 - m(\tau)}{K}$$

Initial mass (m_0); mass at terminal time ($m(\tau)$)

Subjected to $h(\tau) = 0$

$v(\tau) = 0$

τ = Terminal time.

$$f = \begin{bmatrix} v \\ -g + \frac{a}{m} \\ -Ka \end{bmatrix}, \quad l = a \text{ and } c = 0.$$

So, Hamiltonian

$$H = -L + \lambda^T f \Rightarrow -a + \lambda_1 v + \lambda_2\left(-g + \frac{a}{m}\right) + \lambda_3(-Ka)$$

$$H = \left(-1 + \frac{\lambda_2}{m} - K\lambda_3\right)a + \left(\lambda_1 v - \lambda_2 g\right)$$

We know that, we ~~no~~ need to minimize H to gain optimal control.

So, $a^* = \underset{a \in [0,1]}{\operatorname{argmax}}\; H$

So, $a^* = \underset{a \in [0,1]}{\operatorname{argmax}} \left[\left(-1 + \frac{\lambda_2}{m} - K\lambda_3\right)a + \left(\lambda_1 v - \lambda_2 g\right)\right]$

__/__/__

So,

$$a(t) = \begin{cases} 0 ; & b \leq 0 \\ 1 ; & b > 0 \end{cases}$$

here $b = -1 + \frac{\lambda_2}{m} - \lambda_3 k$

the guess policy $a(t) = \begin{cases} 0 & \text{for } t \in [0, t^*] \\ 1 & \text{for } t \in [t^*, T] \end{cases}$

We need to show that b is monotonically increasing or decreasing to prove the guess of the optimal policy.

So, $\dot{b} = \frac{\dot{\lambda}_2}{m} - \frac{\lambda_2 \dot{m}}{m^2} - \dot{\lambda}_3 k$

We already know that

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial h} = 0$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial V} = -\lambda_1$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial m} = \frac{\lambda_2 a}{m^2}$$

So, putting all the values in $\dot{b}$

$$\dot{b} = \frac{-\lambda_1}{m} - \frac{\lambda_2}{m^2}(-ka) - \left(\frac{\lambda_2 a}{m^2}\right)k$$

$$\Rightarrow \dot{b} = \frac{-\lambda_1}{m} \quad \text{(A)} \quad \text{(*)}$$

__/__/____

→ due to switching condition, we decided when to start or stop the thrust according to the guess.

because we want $V(c) = 0$.

We know that $\dot{b} = -\frac{\lambda_1}{m}$ (from ⊛)

→ Here, λ_1 is negative, then b is monotonically increasing function because $\dot{b}$ is positive

→ λ_1 is taken negative because we cannot use thrust before the switching point. ~~at~~ otherwise the velocity at the final time will not be zero.

$$a(t) = \begin{cases} 0 & \text{if } b<0 \text{ at } [0, t^*] \\ 1 & \text{if } b>0 \text{ at } [t^*, T] \end{cases}$$

→ The dynamics of the moon lander are:

$$\dot{h}(t) = v(t)$$
$$\dot{v}(t) = -g$$
$$\dot{m}(t) = 0$$
$$\dot{m}(t) = 0 \Rightarrow \int_0^{t^*} \dot{m}(t)\, dt = \int_0^{t^*} 0\, dt.$$

$$\Rightarrow m = 0 + C_1$$
$$\Rightarrow m = C_1$$

So, at the start we know that $m = m_0$

So, $C_1 = m_0$

__/__/____

So, $\boxed{m = m_0}$

→ $\dot{v} = -g$

$\int \dot{v} = \int -g$

$v = -gt + c_2$

at the time $t = 0$, $v = V_0$

So, $V_0 = -g\overset{=0}{t} + c_2$

$\boxed{c_2 = V_0}$

So, $\boxed{v = -gt + V_0}$ —— (1)

→ Now, $\dot{h} = v$

Now, from above eqn (1).

$\dot{h} = V_0 - gt$

$\frac{dh}{dt} = V_0 - gt$

$h = \int (V_0 - gt)\, dt$

$h = V_0 t \cancel{\phi} - \frac{gt^2}{2} + c_3$

at $t = 0$, $h = h_0$

So, $\boxed{h_0 = c_3}$

So, $\cancel{h = \frac{1}{2}}$ $\boxed{h = v_0 t - \frac{1}{2} g t^2 + h_0}$

→ For $t \in [t^*, \tau)$, $a = 1$

$$\dot{h}(t) = v(t)$$

$$\dot{v}(t) = -g + \frac{1}{m}$$

$$\dot{m}(t) = -k$$

$$\int \dot{m}(t) = \int -k \, dt$$

$$m = -kt + c_1$$

We know that $t = t^*$, $m = m_0$

So, $m_0 = -kt^* + c_1$

So, $c_1 = m_0 + kt^*$

$$m = m_0 + kt^* - kt$$

$$\boxed{m = m_0 + k(t^* - t)} \quad \text{—} \; (2)$$

$$\int \dot{v} = \int \left(-g + \left[\frac{1}{m_0 + k(t^* - t)}\right]\right) dt$$

$$v = -gt - \frac{\log_e (m_0 + k(t^* - t))}{k} + C_2$$

Now, at $t = T$, $v(T) = 0$.

$$0 = -g(T) - \frac{\log_e (m_0 + k(t^* - T))}{k} + C_2$$

$$C_2 = -gT + \frac{\log_e (m_0 + k(t^* - t))}{k} \cancel{+g(T)}$$

So. $$v = \left\{ -gt - \frac{\log_e (m_0 + k(t^* - t))}{k} + g(T) + \frac{\log_e (m_0 + k(t^* - T))}{k} \right.$$

$$v = g(T - t) + \frac{1}{k} \log_e \left[\frac{m_0 + k(t^* - T)}{m_0 + k(t^* - t)}\right]$$

3

Now, we know that derivative of height

$\dot{h} = v$

$$h = \int g(\tau - t) + \frac{1}{k} \log \left[\frac{m_0 + k(t^* - \tau)}{m_0 + k(t^* - t)}\right]$$

$$\rightarrow g\left(\tau t - \frac{t^2}{2}\right) + \frac{1}{k} \int \log_e \left[\frac{m_0 + k(t^* - \tau)}{m_0 + k(t^* - t)}\right]$$

Now, let's take $\int \log_e \left[\frac{m_0 + k(t^* - \tau)}{m_0 + k(t^* - t)}\right]$

using an online calculator such as wolframalpha I got.

$$\cancel{= \left(\frac{\tau - t}{k} - \frac{1}{k^2} \log_e\right(}$$

$$= \left(\frac{-\tau - t}{k} - \frac{1}{k^2} \log_e \left[\frac{m_0 + k(t^* - \tau)}{m_0 + k(t^* - t)}\right] (m_0 + k(t^* - t)\right)$$

$$- \frac{1}{2} g \tau^2$$

So, $$h = g\left(\tau t - \frac{t^2}{2}\right) - \frac{(\tau + t)}{k} - \frac{1}{k^2} \log \left(\frac{m_0 + k(t^* - \tau)}{m_0 + k(t^* - \tau)}\right)$$

$$\times (m_0 + k(t^* - t) - \frac{1}{2} g \tau^2$$

_/_/__

Now, if we put $t = t^*$ in ~~above~~ equation.

~~when~~ ① and ③. we get. $V = V^*$

a=1 $v^* = \cancel{gt^*} - g(\tau - t^*) + \frac{1}{k} \log_e \left[\frac{m_0 + k(t^* - \tau)}{m_0}\right]$

a=0 $v^* = V_0 - g t^*$

a=1 $h^* = -g(t^* - \tau^2) + \frac{t^* - \tau}{k} - \frac{m_0}{k^2} \log_e\left(\frac{m_0 + k(t^* - \tau)}{m_0}\right)$

a=0 $h^* = -\frac{1}{2} g t^{*2} + V_0 t^* + h_0$

a=1 $m^* = m_0$

a=0 $m^* = m_0$

Now, by using above equations equating all the equations when a=0 to a=1. we can find t^* and τ.

The system can be visualized as