1 5.1: Mathematical induction

"Ladder"

Can we reach (prove) the first rung? (i.e., the smallest problem size; the basis step, base case, $P(1)$).

Inductive step: $k,\ k+1$ rungs.

$P(k) \to P(k+1)$. Assume $P(k)$, don't prove it.

Conclusion: since the basis step and induction step have been shown to be true, *by the principle of mathematical induction*, $\forall n P(n), n \in \mathbb{Z}^{+}$ is true.

$P(n) : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \forall n P(n), n \in \mathbb{Z}^{+}$.

Proof by mathematical induction.

Basis step: $P(1) : \sum_{i=1}^{n} i = 1$ The summation (LHS) yields 1.

$\frac{1(1+1)}{2} = 1$ The RHS (closed form) yields 1 when simplified.

Conclusion: since the sum is 1 and the closed form is 1, $P(1)$ is true.

Inductive step:

I will prove $P(k) \to P(k+1), k \in \mathbb{Z}^{+}$.

Assume $P(k) : \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Aside:
$1 + 2 + 3 + 4 + \cdots + k \to k(k+1)/2$
To add the first $k+1$ numbers, $1+2+3+4+\cdots+k+[k+1] \to k(k+1)/2+[k+1] = \sum_{i=1}^{k+1} i = \frac{(k+1)[(k+1)+1]}{2}$

$\sum_{i=1}^{k} i + [k+1] = \frac{k(k+1)}{2} + [k+1]$

$\sum_{i=1}^{k+1} i + [k+1] = \frac{k(k+1)+2[k+1]}{2}$ Simplify the summation, find common denominator.

$= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$

Note this is our $k+1$ statement. Hence, it does follow from our k statement. I have shown $P(k) \to P(k+1)$, completing the inductive step.

Conclusion: since the basis step and inductive step are both true, by the principle of mathematical induction, $P(n)$ is true for all positive integers.

$1 + 3 + 5 + 7 + \cdots = ?,\ n \in \mathbb{Z}^+$

The sum of the first element is 1.

The sum of the first 2 elements is 4.

The sum of the first 3 elements is 9.

It seems like the sum of the first n elements is n^2.

$P(n) : \sum_{i=1}^{n}(2i-1) = n^2,\ n \in \mathbb{Z}^+$ (conjecture)

Proof by mathematical induction.

Basis step: $P(1) : \sum_{i=1}^{1}(2i-1) = 1^2$

LHS: $2(1) - 1 = 2 - 1 = 1$

RHS: $1^2 + 1$

Since LHS = RHS, $P(1)$ is true.

Inductive step:

We will show $P(k) \to P(k+1), k \in \mathbb{Z}^+$

$P(k) : \sum_{i=1}^{k}(2i-1) = k^2$ Assume $P(k)$.

$\sum_{i=1}^{k}(2i-1) + [2(k+1)-1] = k^2 + [2(k+1)-1]$ Add next term to both

$\sum_{i=1}^{k+1}(2i-1) = k^2 + [2(k+1)-1]$ Clean up sum

$= k^2 + 2k + 2 - 1$ Simplify RHS

$= k^2 + 2k + 1$ Simplify

$= (k+1)^2$ Factor

$P(k)$ does lead to $P(k+1)$, being true, hence, $P(k) \to P(k+1)$

Conclusion: since the basis step and inductive step are both true, by the principle of mathematical induction, $P(n)$ is true for all positive integers.