1. If $n = ab$ then $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ with $n, a, b \in \mathbb{Z}^+$

Proof by contraposition:

Step	Rule
$\neg(a \leq \sqrt{n} \vee b \leq \sqrt{n})$	Assume $\neg q$
$a \leq \sqrt{n} \wedge b \leq \sqrt{n}$	DeMorgan's
$a > \sqrt{n} \wedge b > \sqrt{n}$	Definiton of $\leq$
$ab > \sqrt{n}\sqrt{n}$	Multiply LHSs + RHSs
$ab > n$	Multiply

Therefore, $ab \neq n$ by definition of $>, =$. Note this is $\neg p$.

Conclusion: I have shown that, assuming $\neg q$ leads to $\neg p$, hence $p \to q$.

2. If m and n are perfect squares, then mn is a perfect square.

Definition: The integer x is a perfect square if $x = a^2$, with $x, a \in \mathbb{Z}$.

Direct proof:

Step	Rule
m is a perfect square. n is a perfect square.	Assume p.
$m = a^2, a \in \mathbb{Z}$	
$n = b^2, b \in \mathbb{Z}$	
$mn = a^2 b^2$	Multiply LHSs + RHSs
$mn = (ab)^2$	Math.

Hence mn is a perfect square, by definition, where $mn = c^2$, $c = ab$, with $c \in \mathbb{Z}$. This is our q proposition.

Conclusion: Assuming p leads to q, hence $p \to q$.

3. $\forall x \exists y \exists z (x = y^2 + z^2)$ with $y, z \in \mathbb{Z}$ and $x \in \mathbb{Z}^+$

Proof by counterexample (used to prove a claim is false.)

See $x = 3$.

$3 \in \mathbb{Z}^+$ and yet no sum of y^2 and z^2 will yield 3.

Let's look at the smallest perfect squares.

$0^2 = 0$

$-1^2 = 1^2 = 1$

$-2^2 = 2^2 = 4$

And these continue in increasing order.

A sum of 3 could allow some pair of 0 or 1. Yet,

$0 + 0 = 0$

$0 + 1 = 1$

$1 + 0 = 1$

$1 + 1 = 2$

Any larger sum overshoots 3.

Conclusion: $x = 3$ fails, so this claim is false.

4.

Proof by contradiction

Instead of starting from $p \to q$, we'll start from the following:

$\neg(p \to q)$

Material implication: $\neg(\neg p \vee q)$

Assume $p \wedge \neg q$. If we find $p \wedge \neg q \wedge q$, there's a contradiction. Likewise for $\neg p \wedge p \wedge \neg q$.