Reasoning in Rough Description Logics with Multiple Indiscernibility Relations

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Abstract. Rough description logics (DLs) can express approximations of concepts by partitioning the interpretation domain into so-called granules by an indiscernibility relation. Admitting a family of indiscernibility relations yields multi-granular partitionings which can interact with each other. In this paper, we investigate reasoning in rough DLs with multiple indiscernibility relations. We focus on the extension of rough $\mathcal{EL}$ with linear multigranulation orders, where granulations are structured from finest to coarsest, and provide a polynomial-time procedure for deciding concept subsumption. We also study reasoning in the rough DL $\mathcal{SHI}(\mathsf{Self})$ w.r.t. arbitrary multi-granular partitionings, and show that the complexity of reasoning remains exponential, just as in classical $\mathcal{ALC}$.

Keywords: Rough description logics · multigranularity · reasoning

1 Introduction

Rough description logics [14] extend classical description logics (DLs) [**?**] by new concept constructors that, through the use of rough sets, add a qualitative notion of vagueness. In the context of rough sets, the domain is partitioned by a so-called *indiscernibility relation* ρ—formally an equivalence relation—whic groups indistinguishable elements into *granules*, i.e. the equivalence classes. Based on this granulation, each set M is associated with two additional sets: the *lower approximation* $\underline{M}$, which contains all elements whose granule is completely contained in M and the *upper approximation* $\overline{M}$, which contains all those elements that belong to a granule that overlap with M. As a concept constructor in rough DLs, the lower approximation $\underline{C}$ models the set of "typical" instances of a concept C, while the upper approximation $\overline{C}$ represents the set of elements that are at least "similar" to instances of C.

Several rough description logics, extending classical DLs ranging from $\mathcal{EL}$ to $\mathcal{ALC}$ (and beyond) have been defined, and their main reasoning tasks (deciding subsumption [5, 6, 10, 14] and answering conjunctive queries [11]) investigated. These rough DLs are well-behaved in the sense that reasoning in them is usually of the same complexity as in their classical counterparts. One limitation of the existing rough DL formalisms is that they admit only a single indiscernibility relation; and yet, whether objects are indiscernible or not may vary depending on the perspective taken. That is, concept members can be indiscernible or discernible w.r.t. different criteria. For example, patients can be indiscernible according to genetic factors or according to the symptoms they present. This can be represented by the use of two or more indiscernibility relations, leading to multi-granular rough sets [7, 12], which have been investigated in the rough set community over the last decade. So far, the incorporation of several indiscernibility relations has not been considered for rough DLs.

In this paper we introduce multigranular rough DLs, which admit upper and lower approximation constructors that use several different indiscernibility relations, and investigate the complexity of reasoning in them. As it turns out, multi-granular rough DLs have a rugged complexity landscape. Even for very inexpressive DLs, we show that allowing arbitrary sets of indiscernibility relations leads to an ExpTime-hard subsumption problem. On the other hand, if the set of indiscernibility relations is linearly ordered (forming coarser and coarser partitions), then reasoning is as hard as in the corresponding classical DL. A linear order on the indiscernibility relations may seem like a strong restriction, but the resulting rough DLs enable to vary the "degree" of indiscernibility. This, in turn, admits structuring the data into finer or coarser granules and thus considering the data on different levels of abstraction. For a more detailed discussion on this setting, see our preliminary study [9].

After introducing basic notions and defining multigranular rough DLs, we present the following results. In Section 4, we show that for the DL that offers only conjunction, reasoning in its multigranular extension with an arbitrary set of indiscernibility relations is already ExpTime-hard—which is a surprising result. Then we study linearly ordered sets of indiscernibility relations and show in Section 5 that deciding subsumption in multigranular $\mathcal{EL}_\bot$ remains polynomial by developing a subsumption algorithm and in Section 6 we show that reasoning in multigranular $\mathcal{SHI}(\mathsf{Self})$ remains in ExpTime. Some of the results from Section 5 appear in [9].

2 Preliminaries

We start by introducing the main notions of description logics and rough sets needed to understand this work. Specifically, we introduce the expressive DL $\mathcal{SHI}$ and some of its sublogics, followed by multigranular rough sets.

2.1 The Description Logic $\mathcal{SHI}$

$\mathcal{SHI}$ is a very expressive description logic which allows various constructors for concepts and roles. Syntactically, given mutually disjoint sets $\mathsf{N_C}$ of *concept names* and $\mathsf{N_R}$ of *role names*, a *role* is either a role name $r \in \mathsf{N_R}$ or an *inverse role* r^-, where $r \in \mathsf{N_R}$. The class of *concepts* is constructed via the syntactic rule

$$C ::= A \mid C \sqcap C \mid \neg C \mid \exists s.C \mid \forall s.C \mid \exists r.\mathsf{Self}$$

$$
\begin{aligned}
(r^-)^\mathcal{I} &:= \{(\delta,\eta) \mid (\eta,\delta) \in r^\mathcal{I}\}\\
(C \sqcap D)^\mathcal{I} &:= C^\mathcal{I} \cap D^\mathcal{I}\\
(\neg C)^\mathcal{I} &:= \Delta^\mathcal{I} \setminus C^\mathcal{I}\\
(\exists s.C)^\mathcal{I} &:= \{\delta \in \Delta^\mathcal{I} \mid \exists \eta \in C^\mathcal{I}.(\delta,\eta) \in r^\mathcal{I}\}\\
(\forall s.C)^\mathcal{I} &:= \{\delta \in \Delta^\mathcal{I} \mid \forall \eta \in \Delta^\mathcal{I}.(\delta,\eta) \in r^\mathcal{I} \Rightarrow \eta \in C^\mathcal{I}\}\\
(\exists r.\mathsf{Self})^\mathcal{I} &:= \{\delta \in \Delta^\mathcal{I} \mid (\delta,\delta) \in r^\mathcal{I}\}
\end{aligned}
$$

Fig. 1. Interpretation of complex concepts and roles in $\mathcal{SHI}$.

where $A \in \mathsf{N_C}$, s is a role, $r \in \mathsf{N_R}$, and Self is a designated symbol.

This logic uses an interpretation-based semantics. An *interpretation* is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where $\Delta^\mathcal{I}$ is a non-empty set called the *domain* and $\cdot^\mathcal{I}$ is the *interpretation function* which maps every $A \in \mathsf{N_C}$ to a set $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ and every $r \in \mathsf{N_R}$ to a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. This function is extended to arbitrary roles as concepts as shown in Figure 1.

Knowledge in this logic is expressed through a set of restrictions over the "meaningful" interpretations of the symbols. A knowledge base (KB) is a set of *axioms*, which can be *general concept inclusions* (GCI) of the form $C \sqsubseteq D$ where C, D are concepts; *role inclusions* (RI) $s \sqsubseteq t$ where s and t are roles; or *transitive axioms* of the form $\mathsf{tran}(r)$ where $r \in \mathsf{N_R}$. The interpretation $\mathcal{I}$ *satisfies* the GCI $C \sqsubseteq D$ or the RI $s \sqsubseteq t$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$ or $s^\mathcal{I} \subseteq t^\mathcal{I}$, respectively; it *satisfies* the transitive axiom $\mathsf{tran}(r)$ iff $r^\mathcal{I}$ is a transitive relation. $\mathcal{I}$ is a *model* of the KB $\mathcal{K}$ iff it satisfies all the axioms in $\mathcal{K}$.

The two main reasoning problems are *consistency*—that is, deciding whether there is at least one model for a given KB $\mathcal{K}$—and *concept subsumption*—deciding whether every model $\mathcal{I}$ of $\mathcal{K}$ also satisfies a given GCI $C \sqsubseteq D$. In this case, we denote it by $\mathcal{K} \models C \sqsubseteq D$.

We consider the following sublanguages of $\mathcal{SHI}$. $\mathcal{ALC}$ is the sublogic obtained by removing inverse roles and Self from the concept constructors, and disallowing RIs and transitive axioms from appearing in the KBs. $\mathcal{EL}_\bot$ further restricts the language to exclude concept negations ($\neg$) and value restrictions ($\forall$), but introducing two new concepts: $\top$ and $\bot$, which are interpreted by $\top^\mathcal{I} = \Delta^\mathcal{I}$ and $\bot^\mathcal{I} = \emptyset$. The DL $\mathcal{HL}_\bot$ removes from $\mathcal{EL}_\bot$ the existential quantification ($\exists$). $\mathcal{ELI}$ instead extends $\mathcal{EL}_\bot$ with inverse roles, but disallows $\bot$. It is well-known that concept subsumption w.r.t. a KB can be decided in polynomial time in $\mathcal{EL}_\bot$ [1] and $\mathcal{HL}_\bot$, and is ExpTime-complete in $\mathcal{ELI}$ [3], $\mathcal{ALC}$ [13], and $\mathcal{SHI}$ [2].

2.2 Multigranular Rough Sets

Rough sets [8] allow for approximate descriptions of sets through an indiscernibility relation between elements of the universe U. Briefly, the elements of U are associated through an equivalence (i.e., transitive, symmetric, and reflexive) relation $\sim$. The equivalence classes of $\sim$ are often called *granules*. With the help of this $\sim$, we can define, for every $S \subseteq U$, its *lower* ($\underline{S}$) and *upper* ($\overline{S}$) approximations as the sets of elements that are: indiscernible only with elements of S; or indiscernible with at least one element of S, respectively. More formally, $s \in \underline{S}$ iff $[s]_\sim \subseteq S$ and $s \in \overline{S}$ iff $[s]_\sim \cap S \neq \emptyset$, where $[s]_\sim$ denotes the equivalence class of s w.r.t. $\sim$.

The idea is that elements in one equivalence class cannot be distinguished from a point of view that defines $\sim$. But they could still be distinguishable from a different perspective. This gives rise to the idea of multigranular rough sets, where multiple equivalence relations (and hence multiple upper and lower approximations) are considered [7,12]. As we see next, these notions can be used to approximate concepts (which semantically are sets of domain elements) by means of equivalence relations over the interpretation domain.

3 Multigranular Rough Description Logics

Let $\mathcal{L}$ be an arbitrary but fixed DL, and $n \in \mathbb{N}$. The multigranular rough extension of $\mathcal{L}$ (with n indiscernibility relations) $\mathcal{L}^{\sim,n}$ is obtained, syntactically, by allowing the new concept constructors $\underline{C}_i$ and $\overline{C}^i$, where $1 \leq i \leq n$. Concepts of the form $\underline{C}_i$ are called *lower approximation* of C w.r.t. $\sim_i$ and those of the form $\overline{C}^i$ are called *upper approximation* of C w.r.t. $\sim_i$. To interpret the new concepts, we extend the notion of an interpretation.

Definition 1. *A* rough interpretation *(with n indiscernibility relations) is a tuple $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I}, \sim_1, \ldots, \sim_n)$ where $(\Delta^\mathcal{I}, \cdot^\mathcal{I})$ is a (classical) interpretation and each $\sim_i$ is an equivalence relation over $\Delta^\mathcal{I}$. Given an element $\delta \in \Delta^\mathcal{I}$, $[\delta]_i$ denotes the equivalence class of δ w.r.t. $\sim_i$. The interpretation function is extended to rough concepts by:*

$$(\underline{C}_i)^\mathcal{I} := \{\delta \in \Delta^\mathcal{I} \mid [\delta]_i \subseteq C^\mathcal{I}\}$$
$$(\overline{C}^i)^\mathcal{I} := \{\delta \in \Delta^\mathcal{I} \mid [\delta]_i \cap C^\mathcal{I} \neq \emptyset\}.$$

We will sometimes index the indiscernibility relations with a finite set of names $\mathcal{N}$ and represent an interpretation as $(\Delta^\mathcal{I}, \cdot^\mathcal{I}, \{\sim_i \mid i \in \mathcal{N}\})$.

By construction, for every concept C, every $i, 1 \leq i \leq n$, and every interpretation $\mathcal{I}$ it holds that $(\underline{C}_i)^\mathcal{I} \subseteq C^\mathcal{I} \subseteq (\overline{C}^i)^\mathcal{I}$. Other important properties which combine the approximation concept constructors for each given indiscernibility relation are the following.

Proposition 2 (from [10]). *Let $\mathcal{L}$ be a DL, $i, 1 \leq i \leq n$, $\mathcal{K}$ a KB, and C, D, E three rough $\mathcal{L}$ concepts. The following properties hold:*

1. *$\mathcal{K} \models \overline{C}^i \sqsubseteq D$ iff $\mathcal{K} \models C \sqsubseteq \underline{D}_i$; and*
2. *if $\mathcal{K} \models C \sqsubseteq \overline{D}^i$ and $\mathcal{K} \models D \sqsubseteq \underline{E}_i$, then $\mathcal{K} \models C \sqsubseteq \underline{E}_i$; and*
3. *if $\mathcal{K} \models C \sqsubseteq \overline{D}^i$ and $\mathcal{K} \models \underline{D}_i \sqsubseteq E$, then $\mathcal{K} \models C \sqsubseteq \underline{E}_i$.*

These properties, which indicate how rough information is propagated within the same granulation, will be useful when we design a reasoning algorithm in Section 5.

Note that, as introduced in this section, the different equivalence relations do not have to preserve any relationship between them, which means that an analogue of Proposition 2 connecting different indiscernibility relations is impossible in this general setting. For instance, in general even $\overline{C}^i$ and $\overline{C}^j$ may be incomparable. As we see in the next section, this freedom leads to ExpTime-hardness of reasoning, even in the inexpressive logic $\mathcal{HL}_\bot$.

4 Multigranular Rough $\mathcal{HL}_\bot$ is Hard

We show that subsumption in $\mathcal{HL}_\bot^{\rho,n}$, which is a very inexpressive logic that does not use any roles, is ExpTime-hard. The proof is based on a reduction from $\mathcal{ELI}$, whose subsumption problem is known to be ExpTime-hard [1].

Let $\mathcal{K}$ be an $\mathcal{ELI}$ KB. Without loss of generality, we can assume that all its GCIs are in normal form; that is, all GCIs take one of the forms

$$A \sqcap B \sqsubseteq C, \qquad A \sqsubseteq \exists s.B, \qquad \exists s.B \sqsubseteq A,$$

where $A, B, C \in \mathsf{N_C} \cup \{\top\}$, and r is a role (i.e., a role name or the inverse of a role name). Following the ideas from [3, 4], we construct an $\mathcal{HL}_\bot^{\rho,n}$ KB $\mathcal{K}_\rho$ that preserves the same subsumptions of atomic concepts as $\mathcal{K}$.

For each role name $r \in \mathsf{N_R}$ appearing in $\mathcal{K}$, we introduce two equivalence relations $\sim_{r1}$, $\sim_{r2}$, and three concept names N_r, M_r, and V_r that do not appear in $\mathcal{K}$. We define the function h that maps each $\mathcal{ELI}$ GCI in normal form to an $\mathcal{HL}_\bot^{\rho,n}$ KB depending on its shape as follows:

$$\begin{aligned}
h(A \sqcap B \sqsubseteq C) &= \{A \sqcap B \sqsubseteq C\};\\
h(A \sqsubseteq \exists r.B) &= \{A \sqsubseteq \overline{A \sqcap N_r}^{r1},\ A \sqsubseteq \overline{V_r}^{r1},\ V_r \sqcap \overline{A}^{r1} \sqsubseteq \overline{V_r \sqcap \overline{B \sqcap M_r}^{r2}}^{r1}\} \cup\\
&\qquad \{N_r \sqcap V_r \sqsubseteq \bot,\ M_r \sqcap V_r \sqsubseteq \bot\};\\
h(A \sqsubseteq \exists r^-.B) &= \{A \sqsubseteq \overline{A \sqcap M_r}^{r2},\ A \sqsubseteq \overline{V_r}^{r2},\ V_r \sqcap \overline{A}^{r2} \sqsubseteq \overline{V_r \sqcap \overline{B \sqcap N_r}^{r1}}^{r2}\} \cup\\
&\qquad \{N_r \sqcap V_r \sqsubseteq \bot,\ M_r \sqcap V_r \sqsubseteq \bot\};\\
h(\exists r.B \sqsubseteq A) &= \{N_r \sqcap \overline{\overline{B \sqcap M_r}^{r2}}^{r1} \sqsubseteq A,\ \overline{N_r \sqcap A}^{r1} \sqsubseteq A\};\\
h(\exists r^-.B \sqsubseteq A) &= \{M_r \sqcap \overline{\overline{B \sqcap N_r}^{r1}}^{r2} \sqsubseteq A,\ \overline{M_r \sqcap A}^{r2} \sqsubseteq A\}.
\end{aligned}$$

Given an $\mathcal{ELI}$ KB $\mathcal{K}$, we define $\mathcal{K}_\rho := \bigcup_{\alpha \in \mathcal{K}} h(\alpha)$, and denote by $\mathsf{N_C}(\mathcal{K})$ and $\mathsf{N_R}(\mathcal{K})$ the set of all concept names and all role names appearing in $\mathcal{K}$, respectively. We see that this transformation preserves atomic subsumptions.

Theorem 3. *Let $\mathcal{K}$ be an $\mathcal{ELI}$ TBox in normal form, $\mathcal{K}_\rho$ the $\mathcal{HL}_\bot^{\rho,n}$ TBox obtained from $\mathcal{K}$, and A, B two concept names appearing in $\mathcal{K}$. $\mathcal{K} \models A \sqsubseteq B$ iff $\mathcal{K}_\rho \models A \sqsubseteq B$.*

Proof (sketch). Suppose that $\mathcal{K} \not\models A \sqsubseteq B$. Then, there must exist a model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of $\mathcal{K}$ such that $A^{\mathcal{I}} \not\subseteq B^{\mathcal{I}}$. Without loss of generality we can assume that $\mathcal{I}$ is a tree model with root node δ_0 such that $\delta_0 \in A^{\mathcal{I}} \setminus B^{\mathcal{I}}$. We define the $\mathcal{HL}_{\bot}^{\rho,n}$ interpretation $\mathcal{I}_\rho = (\Delta_\rho, \cdot^{\mathcal{I}_\rho}, \{\sim_{r1}, \sim_{r2} \mid r \in \mathsf{N_R}(\mathcal{K})\})$ with domain $\Delta_\rho := \Delta^{\mathcal{I}} \cup \{\eta_{r,\delta,\gamma} \mid (\delta,\gamma) \in r^{\mathcal{I}}\}$, where its interpretation function assigns for every $A \in \mathsf{N_C}$

$$A^{\mathcal{I}_\rho} := \begin{cases} A^{\mathcal{I}} \cup \{\eta_{r,\delta,\gamma} \in \Delta_\rho \mid \{\delta,\gamma\} \cap A^{\mathcal{I}} \neq \emptyset\} & \text{if } A \in \mathsf{N_C}(\mathcal{K}); \\ \{\eta_{r,\delta,\gamma} \in \Delta_\rho \mid \delta,\gamma \in \Delta^{\mathcal{I}}\} & \text{if } A = V_r; \\ \{\gamma \in \Delta^{\mathcal{I}} \mid (\delta,\gamma) \in r^{\mathcal{I}}\} & \text{if } A = M_r; \\ \{\delta \in \Delta^{\mathcal{I}} \mid (\delta,\gamma) \in r^{\mathcal{I}}\} & \text{if } A = N_r; \end{cases}$$

and $\sim_{r1}$ and $\sim r2$ are the transitive, reflexive, and symmetric closure of the relations $\{(\delta, \eta_{r,\delta,\gamma}) \mid \gamma \in \Delta^{\mathcal{I}}\}$ and $\{(\gamma, \eta_{r,\delta,\gamma}) \mid \delta \in \Delta^{\mathcal{I}}\}$, respectively. It remains to verify that $\mathcal{I}_\rho$ is indeed a model of $\mathcal{K}_\rho$ and that $\delta_0 \in A^{\mathcal{I}_\rho} \setminus B^{\mathcal{I}_\rho}$. Hence $\mathcal{K}_\rho \not\models A \sqsubseteq B$. We detail the main cases next.

If $\{A \sqsubseteq \overline{A \sqcap N_r}^{r1}, A \sqsubseteq \overline{V_r}^{r1}, V_r \sqcap \overline{A}^{r1} \sqsubseteq \overline{V_r \sqcap \overline{B \sqcap M_r}^{r2}}^{r1}\} \subseteq \mathcal{K}_\rho$, we know that $A \sqsubseteq \exists r.B \in \mathcal{K}$, and since $\mathcal{I} \models \mathcal{K}$, $A^{\mathcal{I}} \subseteq (\exists r.B)^{\mathcal{I}}$. Let $\chi \in A^{\mathcal{I}_\rho}$. If $\chi \in A^{\mathcal{I}}$, then there is a $\tau \in \Delta^{\mathcal{I}}$ with $(\chi,\tau) \in r^{\mathcal{I}}$ and by construction, $\chi \in N_r^{\mathcal{I}_\rho}$. Moreover, $\eta_{r,\chi,\tau} \in V_r^{\mathcal{I}_\rho}$ and $\chi \sim_{r1} \eta_{r,\chi,\tau}$, which means that $\chi \in \overline{A \sqcap N_r}^{r1}$ and $\chi \in \overline{V_r}^{r1}$. If $\chi \notin A^{\mathcal{I}}$, then χ must be of the form $\eta_{r,\delta,\gamma}$ for some $\delta,\gamma \in \Delta^{\mathcal{I}}$. Then, $\delta \sim_{r1} \chi$ and $\delta \in N_r^{\mathcal{I}_\rho}$. As a consequence $\chi \in \overline{A \sqcap N_r}^{r1}$ and $\chi \in \overline{V_r}^{r1}$. Overall, this means that $\mathcal{I}_\rho \models A \sqsubseteq \overline{A \sqcap N_r}^{r1}$ and $\mathcal{I}_\rho \models A \sqsubseteq \overline{V_r}^{r1}$. Let now $\chi \in (V_r \sqcap \overline{A}^{r1})^{\mathcal{I}_\rho}$. By construction, χ is of the form $\eta_{r,\delta,\gamma}$ for some $(\delta,\gamma) \in r^{\mathcal{I}}$ and $\delta \in A^{\mathcal{I}}$. As $\mathcal{I} \models A \sqsubseteq \exists r.B$, there is some $\gamma' \in B^{\mathcal{I}}$ with $(\delta,\gamma') \in r^{\mathcal{I}}$. Then $\eta_{r,\delta,\gamma'} \in \Delta_\rho$, $\chi \sim_{r1} \eta_{r,\delta,\gamma'}$, and $\eta_{r,\delta,\gamma'} \sim_{r2} \gamma'$ which implies that $\chi \in \left(\overline{V_r \sqcap \overline{B \sqcap M_r}^{r2}}^{r1}\right)^{\mathcal{I}_\rho}$. Thus, $\mathcal{I}_\rho$ is a model of the three axioms described. The other cases can be verified analogously.

For the converse direction, if $\mathcal{K}_\rho \not\models A \sqsubseteq B$, then there exists a model of $\mathcal{K}_\rho$ $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \{\sim_{r1}, \sim_{r2} \mid r \in \mathsf{N_R}(\mathcal{K})\})$ and a $\delta_0 \in \Delta^{\mathcal{I}}$ such that $\delta_0 \in A^{\mathcal{I}} \setminus B^{\mathcal{I}}$. We define the $\mathcal{ELI}$ interpretation $\mathcal{I}_0 = (\Delta^{\mathcal{I}_0}, \cdot^{\mathcal{I}_0})$ where $\Delta^{\mathcal{I}_0} = \Delta^{\mathcal{I}} \setminus \{V_r^{\mathcal{I}} \mid r \in \mathsf{N_R}(\mathcal{K})\}$ and the interpretation function is defined by $A^{\mathcal{I}_0} = A^{\mathcal{I}} \cap \Delta^{\mathcal{I}_0}$ for all $A \in \mathsf{N_C}$, and $r^{\mathcal{I}_0} = \{(\delta,\gamma) \in \Delta^{\mathcal{I}_0} \mid \exists \gamma \in \Delta^{\mathcal{I}}.\delta \sim_{r1} \eta \sim_{r2} \gamma\}$ for each $r \in \mathsf{N_R}$. Once again, it is easy to verify that $\mathcal{I}_0 \models \mathcal{K}$ and that $\delta \in A^{\mathcal{I}_0} \setminus B^{\mathcal{I}_0}$. Thus, $\mathcal{K} \not\models A \sqsubseteq B$. $\square$

We explain the intuition behind this construction with an example.

Example 4. Consider the $\mathcal{ELI}$ KB $\mathcal{K} = \{A \sqsubseteq \exists r.B, \exists r^-.A \sqsubseteq C, \exists r.C \sqsubseteq D\}$, which entails $A \sqsubseteq D$. We see how the $\mathcal{HL}_{\bot}^{\rho,n}$ KB $\mathcal{K}_\rho$ yields this result. Consider an object δ_0 belonging to A. The axioms in $h(A \sqsubseteq \exists r.B)$ enforce the existence of two (distinct) objects η_0 and γ_0 belonging to $A \sqcap N_r$ and to V_r, respectively, which are indiscernible through $\sim_{r1}$, and another object (δ_1) belonging to $B \sqcap M_r$ indiscernible from γ_0 through $\sim_{r2}$ (see Figure 2 (a)). Intuitively, N_r and M_r

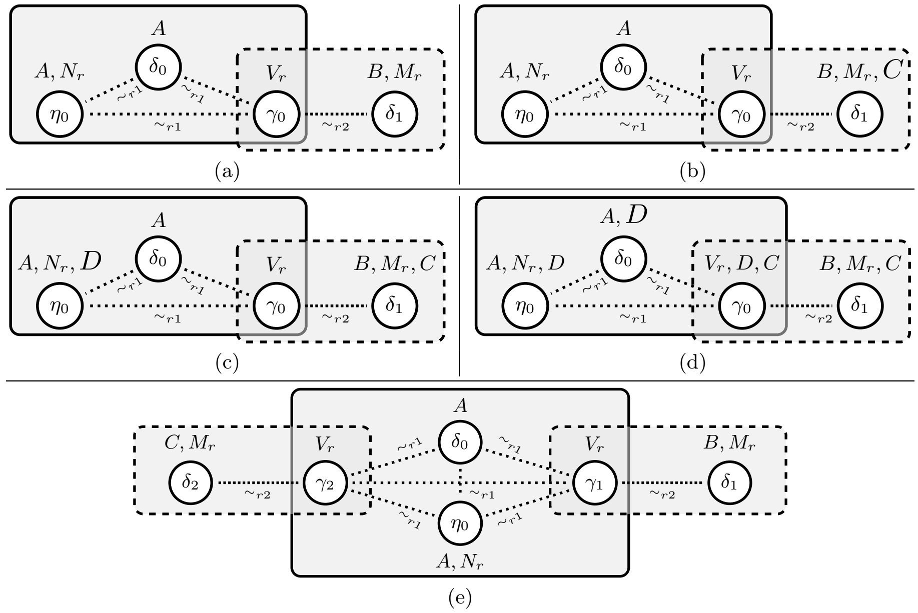

Fig. 2. Step-wise construction of a model for the $\mathcal{HL}_{\perp}^{\rho,n}$ reduction from the $\mathcal{ELI}$ KB in Example 4. Boxes represent equivalence classes: a continuous border refers to a class of $\sim_{r1}$, while dashed borders refer to $\sim_{r2}$.

represent the first and second element of an r-edge, respectively, and V_r is a "border" element, which connects $\sim_{r1}$ and $\sim_{r2}$ granules.

From $h(\exists r^{-}.A \sqsubseteq C)$, since δ_1 belongs to M_r and is connected through a $\sim_{r2} \circ \sim_{r1}$-path to an element in $A \sqcap N_r$, we conclude that δ_1 must also belong to C. Note that since $\sim_{r1}$ and $\sim_{r2}$ are reflexive, the concept names N_r, M_r are needed to avoid deducing (erroneously) that δ_0 and η_0 also belong to C (Figure 2 (b)). An analogous analysis, over $h(\exists r.C \sqsubseteq D)$ allows us to conclude that η_0 must belong to D (Figure 2 (c)). Yet, recall that we are trying to conclude that $A \sqsubseteq D$ which is not yet satisfied by δ_0. This is where the axiom $\overline{N_r \sqcap D}^{r1} \sqsubseteq D$ introduced by $h(\exists r.C \sqsubseteq D)$ comes into play. Indeed, at this point, δ_0 belongs to $\overline{N_r \sqcap D}^{r1}$, which the axiom forces to be subsumed by D, thus concluding that δ_0 belongs to D. The result of this construction appears in Figure 2 (d).

If we "zoom out" from this structure, we can think of each equivalence class as a domain element, and the border elements as bridges (edges) between these elements. Under this view, $[\delta_0]_{\sim_{r1}}$ belongs to A and D; $[\delta_1]_{\sim_{r2}}$ belongs to B and C; and there is an r-connection from $[\delta_0]_{\sim_{r1}}$ to $[\delta_1]_{\sim_{r2}}$. To emphasise this intuition, Figure 2 (e) depicts the construction arising from the KB $\{A \sqsubseteq \exists r.B, A \sqsubseteq C\}$.

Since deciding subsumption in $\mathcal{ELI}$ is ExpTime-hard, Theorem 3 shows the same exponential lower bound for $\mathcal{HL}_{\perp}^{\rho,n}$. The matching upper bound is a consequence of the upper bound for rough $\mathcal{SHI}$ in Section 6.

Corollary 5. *Deciding subsumption in $\mathcal{HL}_\bot^{\rho,n}$ is* ExpTime*-complete.*

To regain tractability, in the next section we impose a restriction on the set of indiscernibility relations, requiring them to form a linear order.

5 The Multigranular Rough Description Logic $\mathcal{EL}_\bot^{\rho/\mathsf{lin}}$

We now consider a variant of multigranular rough $\mathcal{EL}_\bot$, where the indiscernibility relations are totally ordered from the coarsest to the most finely-grained. More formally, given $n \geq 1$, we consider n equivalence relations $\sim_1, \dots, \sim_n$ such that $\sim_i \subseteq \sim_{i+1}$ for all $1 \leq i < n$. That is, $\sim_1$ is the most fine-grained relation, while $\sim_n$ is the coarsest. Note that $\sim_i$ partitions each equivalence class of $\sim_{i+1}$ into (possibly) smaller classes.

The new logic, which we call $\mathcal{EL}_\bot^{\rho/\mathsf{lin}}$, is $\mathcal{EL}_\bot^{\rho,n}$ exactly as defined in Section 3, with the only difference that models are *required* to interpret the n indiscernibility relations as linearly ordered. In particular, for every $\delta \in \Delta^\mathcal{I}$ and every i where $1 \leq i < n$, it holds that $[\delta]_i \subseteq [\delta]_{i+1}$, and hence also for all concepts C

$$(\underline{C}_{i+1})^\mathcal{I} \subseteq (\underline{C}_i)^\mathcal{I} \subseteq (\overline{C}^i)^\mathcal{I} \subseteq (\overline{C}^{i+1})^\mathcal{I}.$$

The following proposition is a consequence of these properties.

Proposition 6. *For all i, j with $1 \leq i \leq j \leq n$, all concepts C, and all interpretations $\mathcal{I}$ the following equivalences hold:*

1. (a) $\big(\underline{(\underline{C}_i)}_j\big)^\mathcal{I} = (\underline{C}_j)^\mathcal{I}$; (b) $\big(\underline{(\underline{C}_j)}_i\big)^\mathcal{I} = (\underline{C}_j)^\mathcal{I}$;
2. (a) $\big(\overline{(\overline{C}^i)}^j\big)^\mathcal{I} = (\overline{C}^j)^\mathcal{I}$; (b) $\big(\overline{(\overline{C}^j)}^i\big)^\mathcal{I} = (\overline{C}^j)^\mathcal{I}$;
3. $\big(\overline{(\underline{C}_j)}^i\big)^\mathcal{I} = (\underline{C}_j)^\mathcal{I}$; *and*
4. $\big(\underline{(\overline{C}^j)}_i\big)^\mathcal{I} = (\overline{C}^j)^\mathcal{I}$.

Proof. We prove only the Claims 1. and 3.; the other two can be shown analogously. For Claim 1.(a), $\delta \in (\underline{(\underline{C}_i)}_j)^\mathcal{I}$ iff $[\delta]_j \subseteq (\underline{C}_i)^\mathcal{I}$ iff (since $\sim_i \subseteq \sim_j$) $[\delta]_i \subseteq [\delta]_j \subseteq C^\mathcal{I}$ iff $\delta \in (\underline{C}_j)^\mathcal{I}$. Similarly for 1.(b), $\delta \in (\underline{(\underline{C}_j)}_i)^\mathcal{I}$ iff $[\delta]_i \subseteq (\underline{C}_j)^\mathcal{I}$ iff for every $\eta \in [\delta]_i$, it holds that $[\eta]_j \subseteq C^\mathcal{I}$ iff (since $\delta \sim_i \eta$ holds and implies that $\delta \sim_j \eta$ holds) $[\delta]_j \subseteq C^\mathcal{I}$ iff $\delta \in (\underline{C}_j)^\mathcal{I}$.

For Claim 3., $\delta \in (\overline{(\underline{C}_j)}^i)^\mathcal{I}$ iff $[\delta]_i \cap (\underline{C}_j)^\mathcal{I} \neq \emptyset$ iff there exists $\eta \in [\delta]_i$ such that $\eta \in (\underline{C}_j)^\mathcal{I}$ iff there is $\eta \in [\delta]_i$ with $[\eta]_j \subseteq C^\mathcal{I}$ iff (because $\delta \sim_i \eta$ holds and implies that $\delta \sim_j \eta$ holds) $[\delta]_j \subseteq C^\mathcal{I}$ iff $\delta \in (\underline{C}_j)^\mathcal{I}$. □

If $i = j$ claims 1 and 2 from Proposition 6 cover idempotence of both kinds of approximations. This affects the decision algorithm for subsumption, which will be based on extending the completion algorithm for $\mathcal{EL}_\bot$ [1] to propagate information w.r.t. a single relation $\sim_i$. Instead for relations at different levels of

Table 1. Normalisation rules. Where $A \in \mathsf{N_C} \cup \{\top\}$, C, D are complex concepts, and X is a new concept name (not previously appearing in the KB).

NF1	$F \sqcap C \sqsubseteq E$	$\longrightarrow$	$\{C \sqsubseteq X, F \sqcap X \sqsubseteq E\}$
NF2	$\exists r.C \sqsubseteq E$	$\longrightarrow$	$\{C \sqsubseteq X, \exists r.X \sqsubseteq E\}$
NF3	$\underline{C}_i \sqsubseteq E$	$\longrightarrow$	$\{C \sqsubseteq X, \underline{X}_i \sqsubseteq E\}$
NF4	$\overline{C}^i \sqsubseteq E$	$\longrightarrow$	$\{C \sqsubseteq \underline{E}_i\}$
NF5	$C \sqsubseteq D$	$\longrightarrow$	$\{C \sqsubseteq X, X \sqsubseteq D\}$
NF6	$A \sqsubseteq E \sqcap F$	$\longrightarrow$	$\{A \sqsubseteq E, A \sqsubseteq F\}$
NF7	$A \sqsubseteq \exists r.C$	$\longrightarrow$	$\{A \sqsubseteq \exists r.X, X \sqsubseteq C\}$
NF8	$A \sqsubseteq \underline{C}_i$	$\longrightarrow$	$\{A \sqsubseteq \underline{X}_i, X \sqsubseteq C\}$
NF9	$A \sqsubseteq \overline{C}^i$	$\longrightarrow$	$\{A \sqsubseteq \overline{X}^i, X \sqsubseteq C\}$
NF10	$\bot \sqsubseteq E$	$\longrightarrow$	$\emptyset$

granularity ($i < j$) the equivalences from Proposition 6 indicate how information is to be propagated or absorbed between different levels of roughness.

The rough DL $\mathcal{EL}^{\rho}_{\bot}$ [10] is the special case of $\mathcal{EL}^{\rho/\mathsf{lin}}_{\bot}$ where $n = 1$; that is, where only one indiscernibility relation is used. Since $\mathcal{EL}_{\bot}$ is a particular case of $\mathcal{EL}^{\rho}_{\bot}$, where the GCIs $A \sqsubseteq \underline{A}_1$ and $\overline{A}^1 \sqsubseteq A$ are satisfied for all $A \in \mathsf{N_C}$, $\mathcal{EL}^{\rho/\mathsf{lin}}_{\bot}$ is obviously a generalisation of the classical DL $\mathcal{EL}_{\bot}$. We are mainly interested in deciding subsumption between two concept names. Recall that $A \in \mathsf{N_C}$ is *subsumed* by $B \in \mathsf{N_C}$ w.r.t. the KB $\mathcal{K}$ ($\mathcal{K} \models A \sqsubseteq B$) iff every model of $\mathcal{K}$ also satisfies the GCI $A \sqsubseteq B$.

We develop a reasoning algorithm for solving this problem in $\mathcal{EL}^{\rho/\mathsf{lin}}_{\bot}$. As this logic is an extension of $\mathcal{EL}^{\rho}_{\bot}$, we extend the known completion algorithm [10] to handle the new cases required by the multiple indiscernibility relations available. As a first step, we require the KB to be in $\mathcal{EL}^{\rho/\mathsf{lin}}_{\bot}$ *normal form*; i.e. all the axioms should be of one of the forms

$$A_1 \sqcap A_2 \sqsubseteq C, \quad A \sqsubseteq \exists r.B, \quad \exists r.A \sqsubseteq C, \quad \underline{A}_i \sqsubseteq C, \quad A \sqsubseteq \underline{B}_i, \quad A \sqsubseteq \overline{B}^i,$$

where $A, B \in \mathsf{N_C} \cup \{\top\}$, $C \in \mathsf{N_C} \cup \{\top, \bot\}$, and $1 \leq i \leq n$.[3]

Any KB $\mathcal{K}$ can be transformed into normal form applying the rules from Table 1—where **NF1** uses the commutativity of conjunction—until no rule can be applied anymore. The resulting KB is a conservative extension of $\mathcal{K}$ which, importantly, is only polynomially larger than $\mathcal{K}$ as it is found after only a polynomial number of rule applications.

Our completion algorithm extends the ideas introduced in [10] to handle lower and upper approximation concepts. Briefly, the completion algorithm for $\mathcal{EL}^{\rho}_{\bot}$ preserves, for each concept name A appearing in a normalised KB $\mathcal{K}$, a family of completion sets, which store the information of how the lower and upper approximations of other concept names relate to A. This information is needed for an adequate handling of the properties of these concept constructors. In the

[3] For brevity, we consider axioms of the form $A \sqsubseteq B$ as $\top \sqcap A \sqsubseteq B$.

present case, we must extend this idea to differentiate between the available indiscernibility relations.

More formally, for each $A \in \mathsf{N_C} \cup \{\top\}$ appearing in the normalised KB $\mathcal{K}$, and for each $i, 1 \leq i \leq n$ we preserve two sets called $\overline{S}^i(A)$ and $\underline{S}_i(A)$. In addition, we keep track of a set $S(A)$ and for each role name $r \in \mathsf{N_R}$ appearing in $\mathcal{K}$ a set $S(A, r)$. Hence, for each such A, we keep $2n + \ell + 1$ many such completion sets, where ℓ is the number of role names in $\mathcal{K}$. With polynomially many concept names in the normalised KB, the completion algorithm uses polynomially many completion sets.

The elements of each completion set all belong to $\mathsf{N_C} \cup \{\top, \bot\}$. The idea is that these sets represent subsumption relations among syntactically simple concepts that can be derived from subsumptions that were previously found, in a sound manner. Specifically, throughout the completion algorithm, the application of completion rules preserves the following invariants:

1. if $B \in \overline{S}^i(A)$ then $\mathcal{K} \models A \sqsubseteq \overline{B}^i$
2. if $B \in \underline{S}_i(A)$ then $\mathcal{K} \models A \sqsubseteq \underline{B}_i$
3. if $B \in S(A)$ then $\mathcal{K} \models A \sqsubseteq B$ and
4. if $B \in S(A, r)$ then $\mathcal{K} \models A \sqsubseteq \exists r.B$

for all $A \in \mathsf{N_C} \cup \{\top\}$, $B \in \mathsf{N_C} \cup \{\top, \bot\}$, $r \in \mathsf{N_R}$, and $1 \leq i \leq n$. These are essentially the same invariants that were used for $\mathcal{EL}^{\rho}_{\bot}$ in [10], but extended to consider the different indiscernibility relations.

The completion sets are initialized with obvious tautologies; that is, at the beginning of the algorithm the sets are defined as

$$S(A) = \overline{S}^i(A) := \{A, \top\}, \qquad \underline{S}_i(A) := \{\top\}, \qquad S(A, r) := \emptyset$$

for all $A \in \mathsf{N_C} \cup \{\top\}, r \in \mathsf{N_R}, 1 \leq i \leq n$. Clearly this initialization preserves the invariants mentioned above. These sets are extended through the application of the *completion rules* described in Table 2. As usual for these kinds of algorithms, the rules are only applied if they add an element to one of the sets involved; that is, if the concept to be added is not already present in the set. The completion algorithm applies rules until no rule is applicable anymore; at that point, we say that the algorithm is *saturated*.

Interestingly, this completion algorithm becomes saturated after at most polynomially many rule applications (in n and the size of $\mathcal{K}$). Indeed, there are $(2n + \ell + 1)m$ sets, where ℓ is the number of role names in $\mathcal{K}$ and m is the number of concept names in $\mathcal{K}$. Each of these sets contains at most $m + 2$ elements (the concept names in $\mathcal{K}$ plus $\top$ and $\bot$). Since each rule application adds one element to one of the sets, at most $(2n+\ell+1)(m+2)m$ rule applications are needed before reaching saturation. In addition, the conditions for the application of a rule require only a lookup between the sets and the GCIs in $\mathcal{K}$, which can also be performed in polynomial time. Thus, overall the algorithm needs only polynomial time until it becomes saturated.

The result of the completion algorithm can be used to decide all the atomic subsumption relations entailed by the KB $\mathcal{K}$. That is, for every $A, B \in \mathsf{N_C}$ we

Table 2. Completion rules for $\mathcal{EL}_{\bot}^{\rho/\mathsf{lin}}$.

CR1	if $\{B_1, B_2\} \subseteq S(A)$ and $B_1 \sqcap B_2 \sqsubseteq C \in \mathcal{K}$, then add C to $S(A)$
CR2	if $B \in S(A)$ and $B \sqsubseteq \exists r.C \in \mathcal{K}$, then add C to $S(A, r)$
CR3	if $B \in S(A, r), C \in S(B)$ and $\exists r.C \sqsubseteq D \in \mathcal{K}$, then add D to $S(A)$
CR4	if $\{B_1, B_2\} \in \underline{S}_i(A)$ and $B_1 \sqcap B_2 \sqsubseteq C \in \mathcal{K}$, then add C to $\underline{S}_i(A)$
CR5	if $B_1 \in \underline{S}_i(A)$, $B_2 \in \overline{S}^i(A)$ and $B_1 \sqcap B_2 \sqsubseteq C \in \mathcal{K}$, then add C to $\overline{S}^i(A)$
CR6	if $B \in \underline{S}_i(A)$ and $\underline{B}_i \sqsubseteq C \in \mathcal{K}$, then add C to $\underline{S}_i(A)$
CR7	if $B \in \overline{S}^i(A)$ and $B \sqsubseteq \underline{C}_i \in \mathcal{K}$, then add C to $\underline{S}_i(A)$
CR8	if $B \in \overline{S}^i(A)$ and $B \sqsubseteq \overline{C}^i \in \mathcal{K}$, then add C to $\overline{S}^i(A)$
CR9	if $B \in \underline{S}_i(A)$, then add B to $S(A)$
CR10	if $B \in S(A)$, then add B to $\overline{S}^i(A)$
CR11	if $B \in \underline{S}_j(A)$ and $i < j$, then add B to $\underline{S}_i(A)$
CR12	if $B \in \overline{S}^i(A)$ and $i < j$, then add B to $\overline{S}^j(A)$
CR13	if $B \in \underline{S}_i(A)$ and $C \in S(B)$, then add C to $\underline{S}_i(A)$
CR14	if $B \in \overline{S}^i(A)$ and $C \in \overline{S}^i(B)$, then add C to $\overline{S}^i(A)$
CR15	if $B \in \underline{S}_i(A)$ and $C \in \underline{S}_i(B)$, then add C to $\underline{S}_i(A)$
CR16	if $B \in S(A, r)$ and $\bot \in S(B)$, then add $\bot$ to $S(A)$
CR17	if $B \in \overline{S}^i(A)$ and $\bot \in \overline{S}^i(B)$, then add $\bot$ to $\underline{S}_i(A)$
CR18	if $\bot \in \overline{S}^i(A)$, then add $\bot$ to $\underline{S}_i(A)$

get that $\mathcal{K} \models A \sqsubseteq B$ iff $B \in S(A)$. In what follows we prove this claim. First, soundness is a consequence of the invariants described above.

Lemma 7. *The completion algorithm preserves the four invariants, throughout all rule applications.*

Proof. The proof is by induction on rule applications. The induction base is satisfied by the initialization. For rules without rough constructors (CR1-CR3 and CR16) soundness was shown already in [1].

For rules CR6 to CR15, CR17, and CR18 soundness is a consequence of Propositions 2 and 6. Since the rules CR11 and CR12 treat the interaction of different indiscernibility relations, we give a detailed proof of them. For CR11, suppose $\mathcal{K} \models A \sqsubseteq \underline{B}_i$ and $i < j$. For every model $\mathcal{I}$ and every $\delta \in \Delta^{\mathcal{I}}$, if $\delta \in A^{\mathcal{I}}$, then $\delta \in \underline{B}_j{}^{\mathcal{I}}$ and thus $[\delta]_j \subseteq B^{\mathcal{I}}$. Since from $i < j$ follows that $[\delta]_i \subseteq [\delta]_j$, we obtain $[\delta]_i \subseteq B^{\mathcal{I}}$ holds and thus $\delta \in \underline{B}_i{}^{\mathcal{I}}$. This implies $\mathcal{K} \models A \sqsubseteq \underline{B}_i$. The proof for CR12 is analogous.

The only remaining rules are CR4 and CR5. For the rule CR4, suppose that $\mathcal{K} \models A \sqsubseteq \underline{B_1}_i$ and $\mathcal{K} \models A \sqsubseteq \underline{B_2}_i$. For every model $\mathcal{I}$ and every $\delta \in \Delta^{\mathcal{I}}$, if $\delta \in A^{\mathcal{I}}$ then $[\delta]_i \subseteq B_1^{\mathcal{I}} \cap B_2^{\mathcal{I}}$ and hence (as $\mathcal{I} \models B_1 \sqcap B_2 \sqsubseteq C$) $[\delta]_i \subseteq C^{\mathcal{I}}$, which implies $\mathcal{K} \models A \sqsubseteq \underline{C}_i$. Rule CR5 can be treated analogously. □

This shows that any atomic subsumption relation derived by the algorithm (in the form of $B \in S(A)$) is indeed a consequence of the KB.

For the converse direction—completeness—we follow the usual approach of building a sort of "canonical" model of $\mathcal{K}$ that serves as a counterexample for all the atomic subsumption relations which do *not* appear explicitly in the generated sets. In our case, the domain $\Delta^{\mathcal{I}}$ of the canonical model is composed of three kinds of elements. First, as usual for the $\mathcal{EL}$ family of DLs, it includes one domain element for each satisfiable concept name A appearing in $\mathcal{K}$, which stands for a standard instance representing that concept; i.e., it is a minimal representative of A. Hence, it will belong to each concept B that subsumes A w.r.t. $\mathcal{K}$. The two other kinds of domain elements handle the lower and upper approximations of named concepts in the interpretation domain. For the lower approximation, we include, for each $\sim_i$, with $1 \leq i \leq n$, an element $A_{\sim_i}$ that belongs to all concepts B such that $\mathcal{K} \models A \sqsubseteq \underline{B}_i$. In other words, $A_{\sim_i}$ keeps information about all the concept names B such that all objects indiscernible from instances of A are necessarily in B.

Dealing with the upper approximations requires a more nuanced construction, as a single element cannot fully witness the existence of indiscernible elements belonging to different concepts. Indeed, note that it could very well happen that $\mathcal{K} \models A \sqsubseteq \overline{B}^i$ and $\mathcal{K} \models A \sqsubseteq C$ while B and C are required to be disjoint concepts. Thus, we need different objects to handle different upper approximations. Specifically, for each concept name B such that $\mathcal{K} \models A \sqsubseteq \overline{B}^i$, we create an element $A^B_{\sim_i}$ which is a representative instance of B (i.e., belongs to B and all its subsumers), but exists only through its connection to the representative of A. To handle the indiscernibility relations, these elements A, $A_{\sim_i}$, and $A^B_{\sim_i}$ all belong to the same $\sim_i$-equivalence class. As this is not a trivial structure, we explain it in more detail here. Note that $\mathcal{K} \models A \sqsubseteq \overline{B}^i$ means that every element of A must be associated (via $\sim_i$) with some element of the concept B. In particular, the representative of A must have such an association as well. But we cannot connect A to the representative of B because the symmetry of $\sim_i$ would then entail that $B \sqsubseteq \overline{A}^i$, which is not necessarily a consequence of $\mathcal{K}$. We can also not choose only one representative, as we did for the lower approximations, because (again) we cannot guarantee that the representative belongs to other concepts that are not known subsumers of B. Figure 3 describes this intuition graphically. Each gray box is an equivalence class for $\sim_i$. There can be more elements than those shown in each class, but the figure zooms into some relevant elements of $[A]_{\sim_i}$, given by the derivations shown at the left of the figure. Since $A \sqsubseteq \underline{B}_i$, the object $A_{\sim_i}$ belongs to the concept B. On the other hand, since A is subsumed by $\overline{B}^i$, $\overline{C}^i$, and $\overline{D}^i$, we create the three objects $A^B_{\sim_i}$, $A^C_{\sim_i}$, and $A^D_{\sim_i}$, respectively. Importantly, these objects belong to the concepts B, C, and D (respectively), but not to $[B]_{\sim_i}$, $[C]_{\sim_i}$, or $[D]_{\sim_i}$, represented as the three boxes on the right. Also, since $A_{\sim_i}$ belongs to the concept B—which represents that $A \sqsubseteq \underline{B}_i$, all objects in $[A]_{\sim_i}$ belong to B as well.

Before formalising this construction, recall that $\bot$ requires a special treatment as a subsumer of a concept name. If $\mathcal{K} \models A \sqsubseteq \bot$, we know that every model makes A empty, and hence A is subsumed by all concepts. Rather than making all these relations explicit, we simply handle this special case separately.

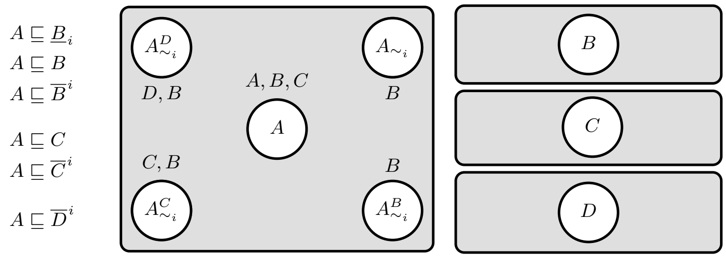

Fig. 3. The construction of the model for the proof of Lemma 8. Each gray box is an equivalence class for $\sim_i$. The details of $[A]_{\sim_i}$ are given, relative to the derived subsumptions depicted on the left.

Lemma 8. *Let A, B be two concept names appearing in the normalised $\mathcal{EL}_\perp^{\rho/\mathsf{lin}}$ KB $\mathcal{K}$, and $S(A)$ the set obtained after saturation of the completion algorithm. If $\{B, \perp\} \cap S(A) = \emptyset$, then $\mathcal{K} \not\models A \sqsubseteq B$.*

Proof. We build a model $\mathcal{I}$ of $\mathcal{K}$ such that $A^\mathcal{I} \not\subseteq B^\mathcal{I}$. The domain of this interpretation is

$$\Delta^\mathcal{I} := \{C, C_{\sim_i}, C^D_{\sim_i} \mid 1 \leq i \leq n, \text{ and } C, D \text{ are concept names appearing in } \mathcal{K}\}.$$

For each i, with $1 \leq i \leq n$, the equivalence relation $\sim_i$ is the transitive, symmetric, and reflexive closure of the relation

$$\{(C, C_{\sim_i}), (C, C^D_{\sim_i}) \mid C, D \text{ are concept names appearing in } \mathcal{K}\}.$$

Note that all objects in $\Delta^\mathcal{I}$ are of the form C, $C_{\sim_i}$, or $C^D_{\sim_i}$. By the definition of the equivalence relations $\sim_i$, there exists for every $\delta \in \Delta^\mathcal{I}$ some concept name C such that $\delta \sim_i C$. In particular, this means that every equivalence class of $\sim_i$ contains at least one concept name or, in other terms, that for every $\delta \in \Delta^\mathcal{I}$ there exists some $E \in \mathsf{N_C}$ such that $[\delta]_i = [E]_i$.

To define the interpretation function $\cdot^\mathcal{I}$, we set for each concept name C appearing in $\mathcal{K}$

$$\begin{aligned} C^\mathcal{I} := {} & \{D \mid C \in S(D)\} \cup \{D_{\sim_i} \mid C \in \underline{S}_i(D)\} \cup \\ & \{D^E_{\sim_i} \mid C \in S(E), E \in \overline{S}^i(D)\} \cup \{D^E_{\sim_i} \mid C \in \underline{S}_i(D), E \in \mathsf{N_C}\} \end{aligned}$$

and for each role name r

$$\begin{aligned} r^\mathcal{I} := {} & \{(C, D) \mid D \in S(C, r)\} \cup \{(C_{\sim_i}, D) \mid D \in S(E, r), E \in \underline{S}_i(C)\} \cup \\ & \{(C^E_{\sim_i}, D) \mid D \in S(E, r), E \in \overline{S}^i(C)\} \cup \\ & \{(C^E_{\sim_i}, D) \mid D \in S(F, r), F \in \underline{S}_i(C), E \in \mathsf{N_C}\}. \end{aligned}$$

By construction $A \in A^\mathcal{I}$ and since $B \notin S(A)$ we know that $A \notin B^\mathcal{I}$. It remains to show that this is indeed a model of $\mathcal{K}$. This is shown through a case distinction over the possible types of axioms admitted in the normal form. We show only the cases involving rough constructors.

[Case $\underline{C}_i \sqsubseteq D$] If $\delta \in (\underline{C}_i)^{\mathcal{I}}$, then by definition $[\delta]_i \subseteq C^{\mathcal{I}}$. Let $E \in \mathsf{N_C}$ be such that $[\delta]_i = [E]_i$. Then $E_{\sim_i} \in C^{\mathcal{I}}$ and hence $C \in \underline{S}_i(E)$. As the algorithm has finished, the rule CR6 is not applicable, this means that $D \in \underline{S}_i(E)$ and by CR9 $D \in S(E)$. Consider now an arbitrary $E^F_{\sim_i} \in [E]_i$. Since $D \in \underline{S}_i(E)$, by construction we know that $E^F_{\sim_i} \in D^{\mathcal{I}}$. Overall, this means that $\delta \in [E]_i \subseteq D^{\mathcal{I}}$, which proves the result.

[Case $C \sqsubseteq \underline{D}_i$] If $\delta \in C^{\mathcal{I}}$ and $[\delta]_i = [E]_{\sim_i}$ for some $E \in \mathsf{N_C}$, then by the rules CR9, CR10, and CR14 it follows that $C \in \overline{S}^i(E)$ which, by rule CR7 implies that $D \in \underline{S}_i(E) \subseteq S(E) \subseteq \overline{S}^i(E)$. Then, $[\delta]_i = [E]_i \subseteq D^{\mathcal{I}}$; that is, $\delta \in (\underline{D}_i)^{\mathcal{I}}$.

[Case $C \sqsubseteq \overline{D}^i$] As in the last case, if $\delta \in C^{\mathcal{I}}$ with $[\delta]_i = [E]_i$, then $C \in \overline{S}^i(E)$. Rule CR8 then implies that $D \in \overline{S}^i(E)$ and hence $E^D_{\sim_i} \in D^{\mathcal{I}}$. By construction, $E^D_{\sim_i} \in [\delta]_i$, which implies that $[\delta]_i \cap D^{\mathcal{I}} \neq \emptyset$, and hence $\delta \in (\overline{D}^i)^{\mathcal{I}}$. $\square$

Thus we have a decision procedure for subsumption in $\mathcal{EL}_\bot^{\rho/\mathsf{lin}}$. Overall, we get tractability for reasoning in this logic.

Theorem 9. *Subsumption between concept names w.r.t. $\mathcal{EL}_\bot^{\rho/\mathsf{lin}}$ KBs can be decided in polynomial time.*

Note that the completion algorithm can be used also to check KB consistency and concept satisfiability. For the latter, we have from Lemma 8 that A is unsatisfiable w.r.t. $\mathcal{K}$ iff $\bot \in S(A)$. For the former, we can add the GCI $\top \sqsubseteq X$ and check whether X is unsatisfiable. The results presented in this section appear in [9].

Anni
OK to put it like this?

6 Multigranular Rough $\mathcal{SHI}(\mathsf{Self})$

Rafael
I think it is ok, but we can also remove it (we have it in the introduction already). Up to you

We have seen that for the rather inexpressive DL $\mathcal{EL}_\bot$, extending the language to include linearly ordered multigranular rough set semantics does not affect tractability of reasoning, by providing a polynomial-time decision algorithm for subsumption between concepts. Conversely, if the multiple indiscernibility relations can be freely interpreted, then this problem becomes EXPTIME-hard, even if existential restrictions are not allowed. A remaining question is how is the complexity affected by adding rough constructors to expressive DLs.

As observed already in [14], the upper and lower approximation operators behave as an existential and universal restriction over the relation $\sim$ (seen as a role), if $\sim$ is interpreted as an equivalence relation. That is, $\overline{C}^i$ corresponds to $\exists{\sim_i}.C$ and $\underline{C}_i$ corresponds to $\forall{\sim_i}.C$. Hence, they show in [14] that any DL which can restrict roles to be transitive, reflexive, and symmetric can express rough concepts natively. In particular, in the DL $\mathcal{SHI}(\mathsf{Self})$ we can restrict $\sim_i$ to be (i) transitive through the axiom $\mathsf{tran}(\sim_i)$; (ii) reflexive via the GCI $\top \sqsubseteq \exists{\sim_i}.\mathsf{Self}$; and (iii) symmetric through the RI $\sim_i \sqsubseteq \sim_i^-$.

In other words, $\mathcal{SHI}(\mathsf{Self})^{\rho,n}$ is exactly as expressive as $\mathcal{SHI}(\mathsf{Self})$, and transforming a $\mathcal{SHI}(\mathsf{Self})^{\rho,n}$ KB into a $\mathcal{SHI}(\mathsf{Self})$ one incurs in only a linear blow-up. As a consequence, reasoning in this multigranular rough logic remains ExpTime-complete.

Theorem 10. *Reasoning in* $\mathcal{SHI}(\mathsf{Self})^{\rho,n}$ *is* ExpTime-*complete.*

Note that this reduction into classical DLs requires at least to express existential and value restrictions and symmetric roles. As even $\mathcal{EL}$ extended with symmetric roles and the DL that allows value restrictions are known to be ExpTime-hard [1], in any logic that natively expresses the rough constructors has exponential time reasoning problems.

7 Conclusions and Future Work

In this paper we introduced multigranular rough DLs, which extend the class of rough DLs to multiple indiscernibility relations. These logics admit reasoning w.r.t. several granulations, which can be obtained in applications, for instance, by the use of different clusterings of the data. We investigated the complexity of reasoning regarding (i) the expressivity of the underlying DL and (ii) whether either arbitrary sets or linearly ordered sets of indiscernibility relations are admitted. We showed that for arbitrary sets of indiscernibility relations, reasoning is already ExpTime-hard even if only conjunction is admitted in the DL. This is a somewhat severe and unexpected result.

In contrast, if the set of indiscernibility relations is linearly ordered, the complexity of reasoning in multigranular rough DLs does not increase for $\mathcal{EL}_\bot^{\rho/\mathsf{lin}}$ and $\mathcal{SHI}(\mathsf{Self})^{\rho,n}$ compared to their classical versions. Specifically, we have shown that testing subsumption $\mathcal{EL}_\bot^{\rho/\mathsf{lin}}$ by developing a decision procedure for subsumption and for the expressive Boolean- complete $\mathcal{SHI}(\mathsf{Self})^{\rho,n}$ we have shown ExpTime-completeness by extending the reduction from [14].

There are many open questions to address. With the investigated multigranular DL being either P- or ExpTime-complete, it is not clear whether there are multigranular DLs that admit PSpace-complete reasoning. Furthermore, since our reduction from Section 4 uses only the upper approximation constructor, it is uncertain whether admitting only lower approximations would lead to lower complexity. Finally, it would be interesting to extend reasoning in rough DLs in general and in multigranular DLs in particular to ABox reasoning.

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